WHAT IS...

the Amplituhedron?

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The amplituhedron is a space whose geometric structure conjecturally determines scattering amplitudes in a certain class of quantum field theories. When it was introduced in 2013 by Arkani-Hamed and Trnka [1], *Quanta Magazine*[1] reported that "Physicists have discovered a jewel-like geometric object that dramatically simplifies calculations of particle interactions and challenges the notion that space and time are fundamental components of reality..." In this short article, we will not get as far as challenging the notions of space and time, but we will give some explanation as to the nature and significance of the amplituhedron both in physics and mathematics—where it arises as a natural generalization of the subject of total positivity for Grassmannians.

Scattering amplitudes are the bread and butter of quantum field theory.

Let us begin with the origin of these ideas in the physics of scattering amplitudes. For any particular quantum field theory, the probability for some number of particles to 'scatter' and produce some number of others is encoded by a function called the *(scattering) amplitude* for that process. These functions depend on all the observable numbers labeling the states involved in the process—in particular, momenta and helicity. Amplitudes can (often) be determined perturbatively according to the Feynman 'loop' expansion. At leading order, they are rational functions of the momenta; and at the 'ℓ loop' order of perturbation, they are integrals over rational forms on the space of ℓ internal loop momenta.

Scattering amplitudes are the bread and butter of quantum field theory; but they are notoriously difficult to compute using the Feynman expansion. Moreover, individual Feynman diagrams depend on many unobservable parameters which can greatly obscure the (often incredible) simplicity of amplitudes' mathematical structure.

A major breakthrough in physicists' understanding of (and ability to compute) scattering amplitudes came in 2004 with the discovery of recursion relations for tree-level ($\ell = 0$) amplitudes by Britto, Cachazo, Feng, and Witten [3]. They were first described graphically as in Figure 1:

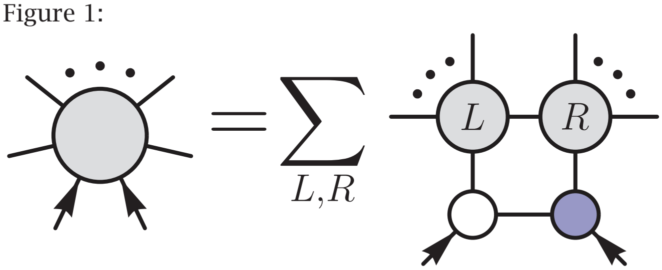

Figure 1. Recursion relations for amplitudes, encoded in terms of bi-colored, trivalent graphs.

The precise meaning of these graphs will not be important to us, but a few aspects are worth mentioning. In Figure 1, amplitudes are represented by grey circles, and any two legs may be chosen for the recursion (indicated by arrows). Thus, the BCFW recursion relations do not provide any particular representation of an amplitude in terms of graphs, but lead to a wide variety of inequivalent representations according to which two legs are chosen at every subsequent order of the recursion. The variety of possible recursive formulae obtained was reminiscent of different 'triangulations' of a polytope—an analogy that would be made sharp by the amplituhedron.

For several years after their discovery, the primary importance of the recursion relations followed from

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[1] *Natalie Wolchover, A Jewel in the Heart of Quantum Physics,* Quanta Magazine, *September 2013.*

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the fact that they yield incredibly more compact—and physically more transparent—representations for tree-level amplitudes. Processes that would have required more Feynman diagrams than atoms in the universe could now be represented by a single diagram. Another surprising feature of these relations was the discovery that individual terms enjoy much more symmetry than anyone had anticipated. These new symmetries were shown to hold for amplitudes, and were later identified with a Yangian Lie algebra.

Another key breakthrough came in 2009, when physicists discovered that the graphs arising in Figure 1 could be related to volume-forms on certain sub-varieties of $\mathrm{Gr}_{\geq 0}(k,n)$—the "nonnegative" portion of the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$, to be defined momentarily. Here, n, k are determined by the graph: n is the number of external edges, and $k = 2n_B + n_W - n_I - 2$ for a graph with n_B, n_W blue and white vertices, respectively, and n_I internal edges. (The correspondence between these graphs and the totally nonnegative Grassmannian was fully understood by mathematicians, such as Postnikov, considerably before physicists stumbled upon it independently.) The physical implications of this correspondence, together with its generalization to all orders of perturbation, can be found in [2].

Given the correspondence between the terms generated by BCFW (and its all-orders generalization) and the Grassmannian, it was natural to wonder if a more intrinsically geometric picture existed. Specifically, can the recursion relations be literally viewed as 'triangulating' some region in the Grassmannian? And if so, can this space be defined *intrinsically*—without reference to how it should be triangulated? In 2013, Arkani-Hamed and Trnka proposed the amplituhedron as the answer to both questions for the case of amplitudes in planar, maximally supersymmetric Yang-Mills theory [1].

Let us now turn to giving a precise mathematical definition of the amplituhedron, in the 0 loop "tree" case. (The ℓ loop amplituhedron for $\ell > 0$ has a definition which is similar but somewhat more complicated; we will not discuss it further.) Let $M \in \mathrm{Mat}(k \times n)$ be a $k \times n$ matrix of real numbers, with $k \leq n$. As we recall from linear algebra, the rows of M determine a subspace of $\mathbb{R}^n$ called its *row space*. If the row space is k-dimensional (the maximum possible), we say that M has full rank. Two full-rank matrices M_1 and M_2 have the same row space if for some K in the space of $k \times k$ invertible matrices, $K.M_1 = M_2$. Clearly, any k-dimensional subspace of $\mathbb{R}^n$ arises as the row space of some full-rank matrix; so the collection of all k-dimensional subspaces can be described as $\mathrm{Mat}_{\mathrm{full}}(k \times n)/GL(k)$. This is the *Grassmannian* $\mathrm{Gr}(k,n)$.

The next ingredient required to define the amplituhedron is the notion of positivity in the Grassmannian. A maximal minor of a matrix $M \in \mathrm{Mat}(k \times n)$ is the determinant of the $k \times k$ submatrix obtained from taking a subset of k of the columns, in the order they appear in the matrix. The totally positive Grassmannian, denoted $\mathrm{Gr}_{>0}(k,n)$, is the subspace of the Grassmannian for which all maximal minors have the same sign. The totally nonnegative Grassmannian, $\mathrm{Gr}_{\geq 0}(k,n)$, is defined in the same way, except that we also allow some minors to vanish.

The totally nonnegative Grassmannian was studied by Postnikov as a concrete instance of Lusztig's theory of total positivity for algebraic groups. It is topologically equivalent to a ball, and has a beautiful cell structure that can be indexed by decorated permutations (permutations of S_n with one extra bit of information for each fixed point). Importantly, cells in this 'positroid' stratification can be endowed with cluster coordinates and a natural volume form.

We now have everything required to define the tree amplituhedron. It is indexed by three integers, n, k, and d, and a totally positive $n \times (d+k)$ matrix P. From the point of view of physics, n is the number of particles, k, defined above in the context of graphs, encodes the total helicity flow, d is the space-time dimension, and P encodes the momenta and helicities of the particles. Let $M \in \mathrm{Gr}_{\geq 0}(k,n)$ vary. The amplituhedron $\mathcal{A}$, as defined by Arkani-Hamed and Trnka [1] is simply the total image locus,

$$\mathcal{A} = M.P \,, \tag{1}$$

inside $\mathrm{Gr}(k, d+k)$. Notice that $M.P$ is a $k \times (d+k)$ matrix, automatically having full rank, so $M.P$ really does define a point in $\mathrm{Gr}(k, d+k)$.

To help build intuition, it is useful to consider some special cases. If $n = d + k$, then the fact that P is totally positive implies it is invertible, so we may as well take it to be the identity matrix; in this case, the amplituhedron is simply $\mathrm{Gr}_{\geq 0}(k, d+k)$. Another simple case arises when $k = 1$, for which M is a totally nonnegative $1 \times n$ matrix—*i.e.* a vector of nonnegative numbers, not all zero. In this case, $\mathcal{A}$ consists of all non-negative linear combinations of the rows of P, and is thus a convex polytope in projective space. The fact that P is totally positive means that $\mathcal{A}$ is what is called a cyclic polytope; these have many interesting extremal properties. In this case, repeatedly applying the BCFW recursion literally triangulates this polytope. A point Y in projective space determines a hyperplane Y^* in the dual projective space, and similarly, the amplituhedron, being in this case a polytope in projective space, determines a dual polytope $\mathcal{A}^*$. The volume form on $\mathcal{A}$ at a point Y is given by the volume of $\mathcal{A}^*$ when Y^* is the hyperplane at infinity.

In general, the volume form on $\mathcal{A}$ can conjecturally be defined by the fact that it has logarithmic singularities on the boundaries of $\mathcal{A}$. It can also be calculated explicitly by pushing forward the volume form on BCFW cells. However, it would be desirable to have an explicit, intrinsic description, as given above for $k = 1$. This would presumably require an answer to the question, "What is the dual amplituhedron?"—something that we must leave for the future.

References

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