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Radiation and generalized uncertainty principle

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ABSTRACT

This paper tries to investigate the effect of Generalized Uncertainty Principle (GUP) on one of the physical concepts. In this study which is based on our previous work Moayedi et al. (2013) [25], the radiating systems in the framework of GUP is investigated. We obtain the modified electric dipole fields and the total power radiated in the presence of a minimal length scale. Also, the magnetic dipole and electric quadrupole fields in the framework of GUP is found. We show that in the limit $\hbar\sqrt{2\beta} \to 0$, all of the modified electric dipole fields and modified quadrupole fields become the usual forms of them. We also, estimate the upper bound on the deformation parameter β.

1. Introduction

On the most difficult duties in theoretical physics is unification between general theory of relativity and quantum mechanics due to quantum gravity theories that are ultraviolet divergent and therefor non-renormalizable [1,2]. Various studies such as string theory, loop quantum gravity and quantum geometry have been made to emphasize that introducing a fundamental length scale of the order of Planck length is essential [3–5]. The minimal length appears due to a modification of the Heisenberg uncertainty principle. Today, the modified uncertainty principle is known as generalized uncertainty principle (GUP) [6–8]. By considering an extension of Heisenberg uncertainty relation, we will obtain the generalized uncertainty as follows:

$$\triangle X \triangle P \geq \frac{\hbar}{2}\left[1 + a_1(\frac{L_p}{\hbar})^2(\Delta P)^2\right], \tag{1}$$

where L_p is the Planck length and a_1, is a positive numerical constant [9,10]. This generalized relation implies a minimum uncertainty of $(\triangle X)_{min} = \sqrt{a_1}L_p$. It seems that the GUP can have an effect on modifying fundamental physical concepts and analyzing the effects of gravity on the basic physical quantities. During recent years, many papers have been devoted to the gravity and quantum field theory in the framework of GUP [11–24]. In our previous work [25], we investigated formulation of a magnetostatic field with an external current density in the presence of a minimal length scale. In this work, we study radiating systems in the presence of a minimal length. Kempf and his collaborators showed that finite resolution of length can be found from the generalized Heisenberg algebra [26–28]. The Kempf algebra leading to the existence of a minimal length in a D-dimensional space is characterized by following deformed commutation relations

$$\left[X^i, P^j\right] = i\hbar[(1+\beta\mathbf{P}^2)\delta^{ij} + \beta' P^i P^j], \tag{2}$$
$$\left[X^i, X^j\right] = i\hbar\,\frac{(2\beta-\beta') + (2\beta+\beta')\beta\mathbf{P}^2}{1+\beta\mathbf{P}^2}(P^i X^j - P^j X^i),$$
$$\left[P^i, P^j\right] = 0,$$

where β and β' are two positive deformation parameters. In Eq. (2), X^i and P^i are position and momentum operators in the GUP framework. This paper is organized as follows: In Sec. 2, we study the electric dipole fields and radiation in the framework of GUP whereas the position operators commute to the first-order in β. Also, in an example, we obtained the modified total power radiated of an electric dipole radiator. In Sec. 3, the magnetic dipole and electric quadrupole fields in the presence of a minimal length are investigated. Our conclusions are presented in Sec. 4. We use SI units throughout this paper.

2. Electric dipole fields and radiation in the framework of GUP

In the present section, we discuss the emission of radiation by localized systems of oscillating charge and current densities in the

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framework of GUP. So that we must introduce the representation of modified position and momentum operators which satisfy Kempf algebra. Stetsko and Tkachuk introduced the approximate representation fulfilling the Kempf algebra in the first order over the deformation parameters β and β' [29]

$$X^i = x^i + \frac{2\beta - \beta'}{4}(\mathbf{p}^2 x^i + x^i \mathbf{p}^2), \quad (3)$$
$$P^i = p^i(1 + \frac{\beta'}{2}\mathbf{p}^2),$$

where the operators x^i and p^i satisfy the canonical commutation relation and $\mathbf{p}^2 = \sum_{i=1}^{D} p^i p^i$. It is interesting to note that in the special case of $\beta' = 2\beta$, the position operators commute in linear approximation over the deformation parameter β, i.e. $[X^i, X^j] = 0$. The following representations, which satisfy Kempf algebra in the special case of $\beta' = 2\beta$, was introduced by Brau [30]

$$X^i = x^i, \quad (4)$$
$$P^i = p^i(1 + \beta\mathbf{p}^2).$$

2.1. A brief review of fields and radiation of a localized oscillating source

We can make a Fourier analysis of the time dependence for a system of charges and currents varying in time. Thus, if we consider the potentials, fields and radiation from a localized system of charges and currents, we won't lose generality [31]. As we know, the vector potential $\mathbf{A}(\mathbf{x}, t)$ in the Lorenz gauge is [31]

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\mathbf{J}(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|}\delta(t' + \frac{|\mathbf{x} - \mathbf{x}'|}{c} - t). \quad (5)$$

By considering the sinusoidal time dependence of current density ($J(x, t) = J(x)\exp(-i\omega t)$), the vector potential becomes

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \mathbf{J}(\mathbf{x}')\frac{\exp(ik|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}, \quad (6)$$

where $k = \frac{\omega}{c}$ is the wave number and a sinusoidal time dependence is understood. The magnetic field is given by

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A}, \quad (7)$$

while, outside the source, the electric field is

$$\mathbf{E} = \frac{iZ_0}{k}\nabla \times \mathbf{H}, \quad (8)$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ is the impedance of free space. We can assume three spatial area of interest. If the source dimensions are from order d, and if $d \ll \lambda = \frac{2\pi c}{\omega}$, we have

The-near-zone: $d \ll r \ll \lambda$

The-intermediate-zone: $d \ll r \sim \lambda$

The-far-zone: $d \ll r \ll \lambda$

In the case ($kr \gg 1$), the exponential in Eq. (6) oscillators rapidly and determines the behavior of the vector potential. So, using the following approximate feels suitable

$$|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{n}\mathbf{x}', \quad (9)$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{x}$. If we insert Eq. (9) into Eq. (6), we will obtain the vector potential in the following form

$$\lim_{kr\to\infty}\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp ikr}{r}\int \mathbf{J}(\mathbf{x}')\exp(-ik\mathbf{n}.\mathbf{x}')d^3x'. \quad (10)$$

If the source dimensions are small compared to a wavelength, it is appropriate to expand the integral in Eq. (10) in powers of k [31]:

$$\lim_{kr\to\infty}\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\sum_n \frac{(-ik)^n}{n!}\int J(x')(\mathbf{n}.\mathbf{x}')^n d^3x'. \quad (11)$$

2.2. The modified radiation of electric dipole fields in the framework of GUP

By keeping the first term of Eq. (11), the vector potential can be found as follows:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp(r)}{r}\int \mathbf{J}(\mathbf{x}')d^3x'. \quad (12)$$

If we use integration by parts and continuity equation we will obtain the integral in the following form

$$\int \mathbf{J}(\mathbf{x}')d^3x' = -\int x'(\nabla' \cdot \mathbf{J})d^3x' = -i\omega\int x'\rho(x')d^3x'. \quad (13)$$

Hence the vector potential is

$$\mathbf{A}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\Pi\frac{\exp(ikr)}{r}, \quad (14)$$

where Π is the electric dipole moment. From Eqs. (7) and (8), the electric dipole fields are

$$\mathbf{H} = \frac{ck^2}{4\pi}(\mathbf{n} \times \Pi)\frac{\exp(ikr)}{r}(1 - \frac{1}{ikr}), \quad (15)$$
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\{k^2(\mathbf{n} \times \Pi) \times \mathbf{n}\frac{\exp(ikr)}{r} + [3\mathbf{n}(\mathbf{n} \cdot \Pi) - \Pi](\frac{1}{r^3} - \frac{ik}{r^2})\exp ikr\}.$$

Now, for obtaining the fields in the framework of GUP, we need to find the vector potential in the presence of a minimal length scale. So according to our previous work [25], in Eq. (52), the modified vector potential was achieved in the following form (see appendix A)

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{1 - \exp(\frac{-|\mathbf{x} - \mathbf{x}'|}{a})}{|\mathbf{x} - \mathbf{x}'|}\mathbf{J}(\mathbf{x}')d^3x', \quad (16)$$

where $a := \hbar\sqrt{2\beta}$ is Podolsky's characteristic length. If we consider Eq. (5) and the sinusoidal time dependence of current density, the vector potential in the framework of GUP becomes

$$\mathbf{A}_{ML}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\frac{(1 - \exp\frac{-|\mathbf{x} - \mathbf{x}'|}{a})}{|\mathbf{x} - \mathbf{x}'|}\exp(ik|\mathbf{x} - \mathbf{x}'|)d^3x'. \quad (17)$$

By using the approximation of Eq. (9), the modified vector potential can be written as

$$\mathbf{A}_{ML}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\{\int \mathbf{J}(\mathbf{x}')\exp(-ik\mathbf{n} \cdot \mathbf{x}')d^3x' - \exp(-\frac{r}{a})\int \mathbf{J}(\mathbf{x}')\exp(\mathbf{n} \cdot \mathbf{x}')(\frac{1}{a} - ik)d^3x'\}. \quad (18)$$

If we expand the integral in the above equation, we have

$$\mathbf{A}_{ML}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}\{\Sigma_n\frac{(-ik)^n}{n!}\int \mathbf{J}(\mathbf{x}')(\mathbf{n} \cdot \mathbf{x}')^n d^3x' - \exp(-\frac{r}{a})\sum_n \frac{(-ik)^n}{n!}\int \mathbf{J}(\mathbf{x}')[\mathbf{n} \cdot \mathbf{x}'(\frac{1}{-ika} + 1)]^n d^3x'\}. \quad (19)$$

For finding the electric dipole fields in the presence of a minimal length, we only keep the first term of Eq. (19). Hence, we obtain

$$\mathbf{A}_{ML}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}[\int \mathbf{J}(\mathbf{x}')d^3x' - \exp(-\frac{r}{a})\int \mathbf{J}(\mathbf{x}')d^3x']. \tag{20}$$

Using Eq. (13), the modified vector potential can be found as follows:

$$\mathbf{A}_{ML}(\mathbf{x}) = -\frac{i\mu_0\omega}{4\pi}\Pi\frac{\exp(ikr)}{r}(1-\exp(-\frac{r}{a})). \tag{21}$$

The term $\frac{i\mu_0\omega}{4\pi}\Pi\frac{\exp(ikr)}{r}\exp(-\frac{r}{a})$ in Eq. (21) can be considered as a minimal length effect. On the other hand, based on Eq. (4), the deformed position and derivative operators are

$$x^i \longrightarrow X^i = x^i, \tag{22}$$

$$\nabla \to \mathbf{D} := (1-\frac{a^2}{2}\nabla^2)\nabla. \tag{23}$$

Therefore the modified magnetic field is given by

$$\begin{aligned}\mathbf{H}_{ML} &= \frac{1}{\mu_0}\mathbf{D}\times\mathbf{A}_{ML} = \frac{1}{\mu_0}[(1-\frac{a^2}{2}\nabla^2)\nabla]\times\mathbf{A}_{ML} \\ &= \frac{1}{\mu_0}\frac{\partial}{\partial r}(\mathbf{n}\times\mathbf{A}_{ML}) \\ &\quad - \frac{a^2}{2\mu_0}(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\mathbf{L}^2}{\hbar^2 r^2})\frac{\partial}{\partial r}(\mathbf{n}\times\mathbf{A}_{ML}).\end{aligned} \tag{24}$$

If we substitute Eq. (21) into Eq. (24), we obtain the following magnetic field in the framework of GUP

$$\begin{aligned}\mathbf{H}_{ML} &= -\frac{ck^2}{4\pi}(\mathbf{n}\times\Pi)\{\frac{\exp(ikr)}{r}(1-\frac{1}{ikr}) - \alpha + \eta \\ &\quad + \gamma + \lambda + \lambda' - \frac{a^2}{2r^3\hbar}l(l+1)[\exp(ikr)(1-\frac{1}{ikr}) + r\alpha\hbar]\},\end{aligned} \tag{25}$$

where we have

$$\begin{aligned}\alpha &= \frac{\exp[(ika-1)\frac{r}{a}]}{r}[\frac{(ika-1)}{a} + (1-k^2a^2)], \\ \eta &= \frac{a^2}{2}(\frac{ik^3\exp(ikr)}{r})[1-\frac{3}{ikr}-\frac{6}{k^2r^2}+\frac{6}{ik^3r^3}], \\ \gamma &= \frac{a^2}{2}(ik-\frac{1}{a})^3(\frac{\exp[(ika-1)\frac{r}{a}]}{r}) \\ &\quad \times[1-\frac{3}{r(ik-\frac{1}{a})}+\frac{6}{r^2(ik-\frac{1}{a})^2}-\frac{6}{r^3(ik-\frac{1}{a})^3}], \\ \lambda &= \frac{(ika-1)^2}{a^2r^3}\exp[(ika-1)\frac{r}{a}][r^2-ra^2(ik-1)+2] \\ \lambda' &= \frac{ik\exp(ikr)}{r^3}[ik(r^2-r)-r+2].\end{aligned} \tag{26}$$

Also, the modified electric field is

$$\begin{aligned}\mathbf{E}_{ML} &= \frac{iZ_0}{k}\mathbf{D}\times\mathbf{H}_{ML} = \frac{iZ_0}{k}(1-\frac{a^2}{2}\nabla^2)\nabla\times\mathbf{H}_{ML} \\ &= \frac{iZ_0}{k}[\frac{\partial}{\partial r}(\mathbf{n}\times\mathbf{H}_{ML}) - (\frac{a^2}{2}\nabla^2)\frac{\partial}{\partial r}(\mathbf{n}\times\mathbf{H}_{ML})].\end{aligned} \tag{27}$$

If Eq. (25) is inserted into Eq. (27), the electric field in the framework of GUP can be found as follows:

$$\begin{aligned}\mathbf{E}_{ML} &= \frac{1}{4\pi\epsilon_0}\{k^2(\mathbf{n}\times\Pi)\times\mathbf{n}\frac{\exp(ikr)}{r} \\ &\quad + \exp(ikr)[3\mathbf{n}(\mathbf{n}\cdot\Pi)-\Pi](\frac{1}{r^3}-\frac{ik}{r^2})\} \\ &\quad - \frac{k^2(\mathbf{n}\times\Pi)\times\mathbf{n}}{4\pi\epsilon_0}\{-ik\alpha'+ik\eta'+\frac{2\gamma}{a^2}+\frac{2\eta}{a^2} \\ &\quad - \frac{a^2l(l+1)\exp(ikr)}{2r^4\hbar}[(ikr-3)-(ik-\frac{4}{r})+\frac{2r\alpha}{\hbar}]\} \\ &\quad - \frac{k^2(\mathbf{n}\times\Pi)\times\mathbf{n}}{4\pi\epsilon_0}[-ik(\frac{a^2}{2})(\varepsilon)-ik\frac{a^2}{2}(\varepsilon')],\end{aligned} \tag{28}$$

where

$$\begin{aligned}\alpha' &= \frac{\exp[(ika-1)\frac{r}{a}]}{r}[\frac{(ika-1)^2}{a^2}-\frac{(ika-1)}{ra}-\frac{(1-k^2a^2)}{r} \\ &\quad + \frac{(ika-1)(1-k^2a^2)}{a}], \\ \eta' &= (\frac{a^2}{2})[\exp(ikr)(-\frac{k^4}{r^4}+4\frac{k^3}{r^2}+12\frac{k^2}{r^3}+\frac{1}{r^4}(18ik-\frac{6}{ik})-\frac{24}{r^5})], \\ \varepsilon &= \frac{a^2}{2}\exp(ikr)[\frac{-ik^3}{r}+4\frac{k^2}{r^2}+12\frac{ik}{r^3}-\frac{24}{r^4}+\frac{24}{ikr^5}], \\ \varepsilon' &= \frac{\exp[(ika-1)\frac{r}{a}]}{r}[(ik-\frac{1}{a})^3-3\frac{(ik-\frac{1}{a})^2}{r}+6\frac{(ik-\frac{1}{a})}{r^3}-\frac{6}{r^4}].\end{aligned}$$

It should be noted that we neglect terms of β^2 and higher in the above equation. In the radiation zone the modified fields take on the limiting forms

$$\begin{aligned}\mathbf{H}_{ML} &= \frac{ck^2}{4\pi}(\mathbf{n}\times\Pi)\frac{\exp(ikr)}{r} + i\frac{ck^4}{4\pi}(\frac{a^2}{2})(\mathbf{n}\times\Pi)\frac{\exp(ikr)}{r}, \\ \mathbf{E}_{ML} &= \frac{1}{4\pi\epsilon_0}\{k^2(\mathbf{n}\times\Pi)\times\mathbf{n}\frac{\exp(ikr)}{r} \\ &\quad - (\frac{a^2}{2})k^4(\mathbf{n}\times\Pi)\times\mathbf{n}\frac{\exp(ikr)}{r}\}.\end{aligned} \tag{29}$$

In the near zone, the modified fields approach

$$\begin{aligned}\mathbf{H}_{ML} &= \frac{i\omega}{4\pi}(\mathbf{n}\times\Pi)\frac{1}{r^2} + \frac{3i\omega}{2\pi}(\frac{a^2}{2})(\frac{\mathbf{n}\times\Pi}{r^4}), \\ \mathbf{E}_{ML} &= \frac{1}{4\pi\epsilon_0}\{[3\mathbf{n}(\mathbf{n}\cdot\Pi)-\Pi]\frac{1}{r^3} \\ &\quad + [3\mathbf{n}(\mathbf{n}\cdot\Pi)-\Pi](\frac{a^2}{2})(30ik-24)\frac{1}{r^5}\}.\end{aligned} \tag{30}$$

It should be emphasized that in the limit $a=\hbar\sqrt{2\beta}\longrightarrow 0$, the modified fields in Eqs. (29) and (30) become the usual fields.

The time-averaged power radiated per unit solid angle by the oscillating dipole moment Π is

$$\frac{dP_{Rad}}{d\Omega} = \frac{1}{2}Re[r^2\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*]. \tag{31}$$

If we insert the usual form of fields $\mathbf{E}$ and $\mathbf{H}$ from Eq. (29), in Eq. (31) we will find

$$\frac{dP_{Rad}}{d\Omega} = \frac{c^2Z_0}{32\pi^2}k^4|(\mathbf{n}\times\Pi)\times\mathbf{n}|^2. \tag{32}$$

By considering the components of Π all have the same phase, the angular distribution and the total power radiated will be achieved as

$$\frac{dP_{Rad}}{d\Omega}=\frac{c^2Z_0}{32\pi^2}k^4|\Pi|^2\sin^2(\theta), \tag{33}$$

$$P_{Rad}=\frac{c^2Z_0k^4}{12\pi}|\Pi|^2. \tag{34}$$

On the other hand, power radiated per unit solid angle by the dipole moment Π in the presence of a minimal length scale can be obtained as follows

$$(\frac{dP_{Rad}}{d\Omega})_{ML}=\frac{1}{2}Re[r^2\mathbf{n}\cdot\mathbf{E}_{ML}\times\mathbf{H}^*_{ML}]. \tag{35}$$

If we substitute Eq. (29) into Eq. (35), we have

$$(\frac{dP_{Rad}}{d\Omega})_{ML}=\frac{c^2k^4Z_0}{32\pi^2}|\mathbf{n}\times\Pi)\times\mathbf{n}|^2 - \frac{c^2k^6Z_0}{32\pi^2}(\frac{a^2}{2})|\mathbf{n}\times\Pi)\times\mathbf{n}|^2. \tag{36}$$

The term $-\frac{c^2k^6Z_0}{32\pi^2}(\frac{a^2}{2})|\mathbf{n}\times\Pi)\times\mathbf{n}|^2$ in Eq. (36) shows the effect of GUP corrections. By assuming the components of Π all have the same phase, the modified total power radiated will be found as follows:

$$(P_{Rad})_{ML}=\frac{c^2Z_0}{12\pi}k^4|\Pi|^2(1-k^2(\frac{a^2}{2})). \tag{37}$$

In the limit $a:=\hbar\sqrt{2\beta}\to 0$, the modified total power radiated smoothly becomes the usual form in Eq. (34).

2.3. Example

Let us investigate a simple example of an electric dipole radiator in the framework of GUP. Assuming the antenna is oriented along the z axis, extending from $z=(\frac{d}{2})$ to $z=-(\frac{d}{2})$ with a thin gap at the center. The current is the same direction in each half of the antenna and the function of current is

$$I(z)\exp(-i\omega t)=I_0(1-\frac{2|z|}{2})\exp(-i\omega t). \tag{38}$$

From the continuity equation the following linear-charge density ρ' and then the dipole moment are found

$$\Pi=\int_{-\frac{d}{2}}^{\frac{d}{2}} z\rho'(z)dz=\int_{-\frac{d}{2}}^{\frac{d}{2}} z(\frac{2iI_0}{\omega d})dz=\frac{iI_0d}{2\omega}. \tag{39}$$

Now, according to Eq. (36) the angular distribution of radiated power in the frame work of GUP and also the modified total power radiated are obtained as follows:

$$(P_{Rad})_{ML}=\frac{Z_0I_0^2(kd)^2}{128\pi^2}\sin^2(\theta)[1-k^2(\frac{a^2}{2})], \tag{40}$$

$$(P_{Rad})_{ML}=\frac{Z_0I_0^2(kd)^2}{48\pi^2}[1-k^2(\frac{a^2}{2})]. \tag{41}$$

It must be noted that, in the limit $a\to 0$, Eqs. (40) and (41) becomes the usual form of the angular distribution of radiated power and the total radiated respectively.

3. Magnetic dipole and electric quadrupole field in the framework of GUP

The purpose of this section is obtaining the magnetic dipole and electric quadrupole fields in the presence of a minimal length. If we consider the next term of Eq. (19), the modified vector potential leads to

$$\mathbf{A}_{ML}(\mathbf{x})=[\frac{\mu_0}{4\pi}\frac{\exp(ikr)}{r}(\frac{1}{r}-ik)\int\mathbf{J}(\mathbf{x}')(\mathbf{n}\cdot\mathbf{x}')d^3x'] \times[1-\exp(-\frac{r}{a})]. \tag{42}$$

We can write the modified vector potential in Eq. (42), in two parts: one gives a transverse magnetic induction and the other gives a transverse electric field. Hence, the integral of Eq. (42), can be written as the sum of part symmetric in $\mathbf{J}$ and $\mathbf{x}'$ and a part that is antisymmetric. Therefore

$$(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}=\frac{1}{2}[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}+(\mathbf{n}\cdot\mathbf{J})\mathbf{x}']+\frac{1}{2}(\mathbf{x}'\times\mathbf{J})\times\mathbf{n}. \tag{43}$$

The second, antisymmetric part is recognizable as the magnetization due to the current $\mathbf{J}$:

$$\mathbf{M}=\frac{1}{2}(\mathbf{x}\times\mathbf{J}). \tag{44}$$

We will show how symmetric term are related to the electric quadrupole moment density. If we consider only the magnetization term, the modified vector potential will be found as follows:

$$\mathbf{A}(\mathbf{x})=\frac{ik\mu_0}{4\pi}(1-\exp(-\frac{r}{a}))[(\mathbf{n}\times\mathbf{m})\frac{\exp(ikr)}{r}(1-\frac{1}{ikr})], \tag{45}$$

where $\mathbf{m}$ is the magnetic dipole moment

$$\mathbf{m}=\int\mathbf{M}d^3x=\frac{1}{2}\int(\mathbf{x}\times\mathbf{J})d^3x. \tag{46}$$

We can easily determine the fields if we find the relationship between the modified vector potential in Eq. (45) and the modified magnetic field in Eq. (25) for an electric dipole. This means that the modified magnetic field for the present magnetic dipole source will be equal to $\frac{1}{Z_0}$ times. The modified electric field for the electric dipole, by substituting $\Pi\to\frac{\mathbf{m}}{c}$, will change to the following form

$$\mathbf{H}_{ML}=\frac{1}{4\pi}\{k^2(\mathbf{n}\times\mathbf{m})\times\mathbf{n}\frac{\exp(ikr)}{r} +[3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}](\frac{1}{r^3}-\frac{ik}{r^2})\exp(ikr) -ik\alpha'+ik\eta'-ik(\frac{a^2}{2})\varepsilon-ik(\frac{a^2}{2})\varepsilon'\}. \tag{47}$$

Also, the modified electric field for a magnetic dipole source is the negative of Z_0 of times the modified magnetic field for an electric dipole. So, we find

$$\mathbf{E}_{ML}=-\frac{Z_0}{4\pi}k^2(\mathbf{n}\times\mathbf{m})[\frac{\exp(ikr)}{r}(1-\frac{1}{ikr})-\alpha+\eta+\gamma]. \tag{48}$$

If we use an integration by parts and some simplification the modified integral of symmetric term in Eq. (42) can be written as

$$\frac{1}{2}[1-\exp(-\frac{r}{a})]\int[(\mathbf{n}\cdot\mathbf{x}')\mathbf{J}+(\mathbf{n}\cdot\mathbf{J})\mathbf{x}']d^3x' =-\frac{i\omega}{2}[1-\exp(-\frac{r}{a})]\int\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(x')d^3x' \tag{49}$$

By inserting Eq. (49) into Eq. (42) we will find the following modified vector potential

$$\mathbf{A}_{ML}(\mathbf{x})=-\frac{\mu_0ck^2}{8\pi}\frac{\exp(ikr)}{r}(1-\frac{1}{ikr})[1-\exp(-\frac{r}{a})] \times\int\mathbf{x}'(\mathbf{n}\cdot\mathbf{x}')\rho(x')d^3x'. \tag{50}$$

The modified fields in the radiation zone are defined as below [31]

$$\mathbf{H}_{ML} = ik\mathbf{n} \times \times \frac{\mathbf{A}_{ML}}{\mu_0}, \tag{51}$$

$$\mathbf{E}_{ML} = ikZ_0(\mathbf{n} \times \mathbf{A}_{ML}) \times \frac{n}{\mu_0}.$$

Therefore from Eqs. (50) and (51), the magnetic field in the framework of GUP is

$$\mathbf{H}_{ML} = -\frac{ick^3}{8\pi}\frac{\exp(ikr)}{r}[1 - \exp(-\frac{r}{a})] \times \int (\mathbf{n} \times \mathbf{x}')(\mathbf{n} \cdot \mathbf{x}')\rho(x')d^3x'. \tag{52}$$

Beside on the definition of the quadrupole moment tensor, the integral in Eq. (52) can be written as follows

$$\mathbf{n} \times \int \mathbf{x}'(\mathbf{n} \cdot \mathbf{x}')\rho(x')d^3x' = \frac{1}{3}\mathbf{n} \times \mathbf{Q}(\mathbf{n}), \tag{53}$$

where the vector $\mathbf{Q}(\mathbf{n})$ is defined as having components

$$Q_\alpha = \Sigma_\beta Q_{\alpha\beta} n_\beta. \tag{54}$$

With these definitions the magnetic induction and time-average power radiated per unit solid angle in the framework of GUP can be obtained as follows

$$\mathbf{H}_{ML} = -\frac{ick^3}{24\pi}\frac{\exp(ikr)}{r}[1 - \exp(-\frac{r}{a})](\mathbf{n} \times \mathbf{Q}(\mathbf{n})), \tag{55}$$

$$(\frac{dP_{Rad}}{d\Omega})_{ML} = \frac{c^2 Z_0}{1152\pi^2}k^6(1 - \exp(-\frac{r}{a}))|\mathbf{n} \times \mathbf{Q}(\mathbf{n}) \times \mathbf{n}|^2.$$

The final result for the modified total power radiated by a quadrupole source is obtained as follows:

$$(P)_{ML} = \frac{c^2 Z_0 k^6}{1440\pi}(1 - \exp(-\frac{r}{a}))\Sigma_{\alpha,\beta}|Q_{\alpha\beta}|^2. \tag{56}$$

It should be mentioned that for $a \longrightarrow 0$, the modified Eqs. (55) and (56) changes to their usual forms. According to the example part, if we consider the first term is the usual total power radiated in Eq. (41) and the second term is the modified total power radiated then we can obtain the following relative modification of total power radiated

$$\frac{(\Delta P_{Rad})}{(P_{Rad})_0} = k^2(\frac{a^2}{2}) = k^2\frac{(\hbar\sqrt{2\beta})^2}{2}. \tag{57}$$

Using the experimental precision of the total power radiated, we can estimate the upper bound on deformation parameter β. By considering the experimental values of parameters in Eq. (41), for example at frequency $f = 10$ MHz ($\lambda = 300$ m), the current $I_0 = 1$ A, the impedance of free space $Z_0 = 377\ \Omega$ and $d = 0.03$ m, the total power radiated will be found as $P_{Rad} = 10^{-7}$. So, according to Eq. (57), we have

$$10^{-7} \simeq 10^{-70}\beta. \tag{58}$$

Based on Eq. (58), the following upper bound can be found for the deformation parameter β:

$$\beta < 10^{63}. \tag{59}$$

This bound is far weaker than that found by electroweak measurements but it is near to the Lamb shift [32].

4. Conclusions

Many theoretical physicists believe that the introduction of such a minimal length scale leads to a divergenceless quantum field theory [33]. We know that the existence of a minimal length scale leads to a generalization of Heisenberg uncertainty principle. Based on the our previous work [25], we had investigated radiating systems in the framework of GUP. The electric dipole fields in the presence of a minimal length was obtained. The modified total power radiated, independent of the relative phases of the components of electric dipole had been found. Also, we had studied the special example in the framework of GUP. The magnetic dipole and electric quadrupole fields in the framework of GUP had been found. Based on the modified quadrupole fields, we had obtained the modified total power radiated by it. We had shown that in the limit $\hbar\sqrt{2\beta} \to 0$, the modified electric dipole fields and quadrupole fields became the usual forms of them. Using the experimental precision of the total power radiated we had estimated the upper bound on deformation parameter β. It is necessary to note that the upper bound on deformation parameter β in Eq. (57), was far weaker than that found by electroweak measurements but it was near to the Lamb shift.

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Appendix A

In this section we want to explain how to find the modified vector potential in Eq. (16). By assuming the Lagrangian density for a magnetostatic field with an external current density $\mathbf{J}(\mathbf{x})$ as follows [31]

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{ij}(\mathbf{x})F^{ij}(\mathbf{x}) + J^i(\mathbf{x})A^i(\mathbf{x}), \tag{60}$$

where $F_{ij}(\mathbf{x})$ is the electromagnetic field tensor and $A(\mathbf{x})$ is the vector potential. Incase, at first we use the Euler–Lagrange equation for the components of the vector potential, we will find the following field equation

$$\partial_l F^{lk} = \mu_0 J^k(\mathbf{x}), \tag{61}$$

and then by using the definition of three dimensional magnetic induction vector $\mathbf{B}(\mathbf{x})$ ($F_{ij} = -\epsilon_{ijk}B^k$, $F^{ij} = \epsilon^{ijk}B_k$) we have

$$\nabla \times \mathbf{B}(x) = \mu_0\mathbf{J}(x), \tag{62}$$

$$\nabla \cdot \mathbf{B}(x) = 0. \tag{63}$$

Now, let us obtain the Lagrangian density for a magnetostatic field in the presence of a minimal length scale. Based on Eqs. (22) and (23), if we replace the ordinary position and derivative operators in the Lagrangian density of Eq. (60), we will obtain

$$\mathcal{L} = -\frac{1}{4\mu_0}F_{ij}(\mathbf{x})F^{ij}(\mathbf{x}) - \frac{1}{4\mu_0}a^2\partial_n F_{ij}(\mathbf{x})\partial_n F^{ij}(\mathbf{x}) + J^i(\mathbf{x})A^i(\mathbf{x}). \tag{64}$$

It should be emphasized that in Eq. (64), the total derivative term and the terms of order β^2 and higher are neglected. We can easily find the following field equation for the magnetostatic field in the deformed space, if we insert the Lagrangian density in Eq. (60) into modified Euler–Lagrange equation

$$\partial_l F^{lk}(\mathbf{x}) + a^2\,\nabla^2 \partial_l F^{lk}(\mathbf{x}) = \mu_0\, J^k(\mathbf{x}). \tag{65}$$

The vector form of Eq. (65) can be written as follows

$$(1 - a^2\nabla^2)\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(x), \tag{66}$$

$$\nabla \cdot \mathbf{B}(x) = 0. \tag{67}$$

According to Eq. (67), the magnetostatic field $\mathbf{B}(\mathbf{x})$ can be found as

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}). \tag{68}$$

If we substitute Eq. (68), into Eq. (66) and after simplifying, we have

$$(1 - a^2\nabla^2)[\nabla(\nabla \cdot \mathbf{A}(\mathbf{x})) - \nabla^2 \mathbf{A}(\mathbf{x})] = \mu_0 \mathbf{J}(x). \tag{69}$$

In the Coulomb gauge ($\nabla \cdot \mathbf{A}(\mathbf{x}) = 0$), Eq. (69) becomes as follows:

$$(1 - a^2\nabla^2)\nabla^2 \mathbf{A}(\mathbf{x}) = -\mu_0 \mathbf{J}(\mathbf{x}). \tag{70}$$

The solution of Eq. (70) in terms of the Green's function, $G(\mathbf{x}, \mathbf{x}')$ is given by

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \frac{\mu_0}{4\pi} \int G(\mathbf{x}, \mathbf{x}')\mathbf{J}(\mathbf{x}')d^3x', \tag{71}$$

where $\mathbf{A}_0(\mathbf{x})$ and $G(\mathbf{x}, \mathbf{x}')$ satisfy the equations

$$(1 - a^2\nabla^2)\nabla^2 \mathbf{A}_0(\mathbf{x}) = 0, \tag{72}$$

and

$$(1 - a^2\nabla_x^2)\nabla_x^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\,\delta(x - x'). \tag{73}$$

If we write $G(\mathbf{x}, \mathbf{x}')$ and $\delta(x - x')$ in terms of Fourier integrals and insert them into Eq. (73), we will obtain the following particular solution of Eq. (70), which vanishes at infinity

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{1 - \exp\left(\frac{-|\mathbf{x}-\mathbf{x}'|}{a}\right)}{|\mathbf{x} - \mathbf{x}'|}\mathbf{J}(\mathbf{x}')d^3x'. \tag{74}$$

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