OSTERWALDER-SCHRADER POSITIVITY IN CONFORMAL INVARIANT QUANTUM FIELD THEORY*

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In the present notes we will describe an attempt to gain some insight into the nature of the axiomatic positivity constraint in local quantum field theory. It has recently become clear that it is advantageous to look at quantum field theory in terms of its Euclidean Green functions, also called Schwinger functions. As Osterwalder and Schrader [2], and also Glaser [3], have pointed out, spectrum condition and positivity of the Wightman functions (= vacuum expectation values of products of local fields) are equivalent, modulo other axioms, to a new type of positivity condition which must be satisfied by the Euclidean Green functions. We call this the OS-positivity. After some explanation of the preceding remarks in Sec. 1 we will restrict our attention to exactly conformal invariant quantum field theories. Such theories will hopefully describe the short distance behavior of more realistic theories, as was explained elsewhere [1]. In any case, they are interesting as a laboratory, because they can be analyzed to a remarkable extent by nonperturbative technique. The present note will further exemplify this, for it turns out that in conformal invariant theories the OS-positivity condition can be analyzed by group theoretical methods and that surprisingly simple sufficient conditions can be found for its validity. We think that there is a good chance that these conditions are also necessary; it will be explained in Sec. 5 why that is so.

The work reported here is still in progress at the time of this writing and is published here for the first time. It is the purpose of

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* The present notes present the content of part of the lectures which the author presented at the Bonn summer school 1974. The rest of the material covered there can be found in Reference 1.

these notes to get the readers interested in this new development at an early stage, and we will concentrate here on the main ideas. We are confident however that the analysis can be made rigorous and complete by working out all the technical details (such as growth properties of Q^χ- functions in χ , equivalence relations at integer points, etc.).

1. Euclidean Green Functions

Let us consider a local quantum field theory which satisfies the usual postulates [4] (Wightman axioms): locality, spectrum condition, positivity, Poincaré invariance, uniqueness of the vacuum and temperedness (i.e. some distribution theoretic properties).

For simplicity consider a theory of one hermitian scalar fundamental field $\phi(x)$. Thus we are given a Hilbert space $\mathcal{H}$ of physical states, a vacuum Ω in $\mathcal{H}$ and a field $\phi(x)$ which becomes an operator in $\mathcal{H}$ after smearing with test functions.

Poincaré invariance means that there exists a representation of the Poincaré group by unitary operators $U(a,\Lambda)$ such that

$$\mathcal{U}(a,\Lambda)\,\phi(x)\,\mathcal{U}(a,\Lambda)^{-1} = \phi(\Lambda x + a)\,; \quad \mathcal{U}(a,\Lambda)\Omega = \Omega$$

with $\{a,\Lambda\}$ standing for a Lorentz transformation by Λ followed by a translation by a.

Locality says that the commutator $[\phi(x),\phi(y)] = 0$ if $(x-y)^2 < 0$.

Positivity is the statement that all nonzero state vectors have positive norm, so that $(\Psi,\Psi) \geqslant 0$ in general. This is true if $\mathcal{H}$ is a Hilbert space.

Spectrum condition requires that the spectrum of the Hamiltonian (= generator of time translations) is positive semidefinite. It follows that

$$\int d^4x\, e^{-ipx}\, \mathcal{U}(x)\Psi = 0 \qquad \text{unless} \qquad p \in \overline{V}_+$$

with $\bar{V}_+$ the closed forward lightcone and $U(x) = U(x,1)$.

In addition there are distribution theoretic requirements as we said, we shall not go into them (see [14]).

Consider now state vectors obtained by applying the field to the vacuum

$$\Psi(x_1 \dots x_n) = \phi(x_1) \dots \phi(x_n)\,\Omega \tag{1.1}$$

they should belong to $\mathcal{H}$ after smearing with test functions. The Wightman functions are defined by

$$W(x_1 \dots x_n) = (\Omega, \phi(x_1) \dots \phi(x_n)\Omega) = (\Omega, \Psi(x_1 \dots x_n)) \tag{1.2a}$$

Because of hermiticity of the field $\phi(x)$, one has the more general relation

$$(\Psi(x_1' \dots x_m'), \Psi(x_1 \dots x_n)) = W_{m+n}(x_m' \dots x_1', x_1 \dots x_n) \tag{1.2b}$$

Moreover, by translation invariance.

$$\Psi(x_1 \dots x_n) = U(x_1)\phi(0)U(x_2 - x_1)\phi(0) \dots U(x_n - x_{n-1})\phi(0)\,\Omega$$

Consider now the Fourier transform $\tilde{\Psi}(p;q_1 \dots q_{n-1})$ of this, considered as a function of x_1; $x_2 - x_1, \dots x_n - x_{n-1}$. Clearly, because of the spectrum condition as stated above,

$$\tilde{\Psi}(p; q_1 \dots q_{n-1}) = 0 \text{ unless } p \in \bar{V}_+,\ q_i \in \bar{V}_+ \quad (i = 1 \dots n-1) \tag{1.3}$$

The inverse Fourier transform gives back

$$\Psi(x_1 \dots x_n) = (2\pi)^{-4n} \int dp\, dq_1 \dots dq_{n-1}\, \tilde{\Psi}(p; q_1 \dots q_{n-1})\, \exp i \{ p x_1 + \Sigma q_i (x_{i+1} - x_i) \} \tag{1.4}$$

We will now pause for a moment to recall the notion of vector valued holomorphic function. Let $\mathcal{H}$ a normed space so that for every nonzero element $a \in \mathcal{H}$ its norm $\|a\| > 0$ is defined and positive. For a Hilbert space $\mathcal{H}$, $\|a\| = (a,a)^{1/2}$. A mapping f of a domain $D \subset \mathbb{C}$ into $\mathcal{H}$ is called a vector valued holomorphic function on D if for every z_0, f(z) can be expanded in a power series around z_0 with a non-zero radius ρ of absolute convergence. That is $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathcal{H}$, $\Sigma \|a_n\| z^n < \infty$ whenever $|z - z_0| < \rho$.

Such functions share all the properties of complex holomorphic functions. In the standard text book in analysis, ref. [10] the theory of holomorphic functions is developed right away for functions with values in an arbitrary normed space; the notion generalizes readily to functions of several variables.

Consider now the generalization of expression (1.4) to complex arguments z_1, viz

$$\Psi(z_1 \ldots z_n) = (2\pi)^{-4n} \int dp\, dq_1 \cdots dq_{n-1}\, \hat{\Psi}(p; q_1 \ldots q_{n-1})\, \exp i\{ p z_1 + \sum q_i (z_{i+1} - z_i) \}$$

Because of the support property (1.3) of $\hat{\Psi}$

Ψ $(z_1 \ldots z_n)$ is defined and holomorphic for arguments $z_j = x_j + iy_j$

with $y_1 \in \bar{V}_+$ and $y_{j+1} - y_j \in \bar{V}_+$ for all $j = 1 \ldots n-1$

(1.5)

Points of special interest are the socalled Euclidean points, viz $x_j^0 = 0$ and $\underset{\sim}{y}_j = 0$ (imaginary time and real space coordinates). Define the "Euclidean state vectors" Ψ^E,

$$\Psi^E(\vec{x}_1 \ldots \vec{x}_n) = \Psi(z_1 \ldots z_n) \qquad \text{for } z = (i x_k^4, \underset{\sim}{x}_k), \quad \vec{x} = (\underset{\sim}{x}, x^4)$$

defined and real analytic for $x_n^4 > x_{n-1}^4 > \cdots x_1^4 > 0$ (1.6)

We are now ready to define the Euclidean Green functions $G(\vec{x}_1 \ldots \vec{x}_n)$, viz

$$G_n(\vec{x}_1 \ldots \vec{x}_n) = (\Omega, \Psi^E(\vec{x}_1 \ldots \vec{x}_n)) \tag{1.7}$$

in analogy with (1.2a). As defined here and throughout this paper, G_n is always the full disconnected Green function, this must be kept in mind. To start with, it is defined for arguments as specified in (1.6). However, the restriction $x_1^4 > 0$ is unnecessary because G_n depends on its arguments only through their differences. Indeed the same is true for the Wightman function W_n by translation invariance, and G_n is the analytic continuation of the Wightman function W_n because Ψ^E is the analytic continuation of $\Psi(x_1 \ldots x_n)$.

Let us now introduce the Euclidean time reversal operator Θ, which reverses x^4,

$$\Theta(\underset{\sim}{x}, x^4) = (\underset{\sim}{x}, -x^4) \tag{1.8}$$

Θ is really a complex conjugation of the complex variable z, because

$$z = (i x^4, \underset{\sim}{x}) \quad \text{implies} \quad \bar{z} = (-i x^4, \underset{\sim}{x}) = (i \Theta x^4, \Theta \underset{\sim}{x}).$$

Consider now the scalar product of two Euclidean state vectors. We find from Eq. (1.2b) by antianalytic continuation in the first m arguments and analytic continuation in the last n arguments that

$$(\psi^E(\vec{x}'_1 \dots \vec{x}'_m), \psi^E(\vec{x}_1 \dots \vec{x}_n)) = G_{m+n}(\Theta\vec{x}'_m \dots \Theta\vec{x}'_1 \vec{x}_1 \dots \vec{x}_n) \qquad (1.9)$$

for arguments

$$\theta x'^4_m < \Theta x'^4_{m-1} < \dots < \Theta x'^4_1 < 0 < x^4_1 < \dots < x^4_n$$

Suppose now that there is given a finite sequence (f) of test functions $f_0 \in \mathbb{C}$, $f_1(x_1) \dots f_N(x_1 \dots x_N)$ with support in the domain of definition of ψ^E, i.e. $0 < x^4_1 < \dots < x^4_k$ or else $f_k(\vec{x}_1 \dots \vec{x}_k) = 0$. Then

$$\psi^E(f) \equiv \sum_k \int d^{4k}x\, f_k(\vec{x}_1 \dots \vec{x}_k)\, \psi^E(\vec{x}_1 \dots \vec{x}_k) \qquad (1.10)$$

is an element of the Hilbert space $\mathcal{H}$ of physical states and thus must have nonnegative norm $(\psi^E(f), \psi^E(f)) \geqslant 0$. By (1.9) this norm is expressible in terms of the Euclidean Green functions. Thus

$$\sum_{k,\ell} G_{k+\ell}(\Theta f^*_k \times f_\ell) \geqslant 0 \qquad (E.2)$$

where

$$(\Theta f^*_k \times g_\ell)(\vec{x}'_1 \dots \vec{x}'_k \vec{x}_1 \dots \vec{x}_\ell) \equiv \bar{f}_k(\Theta\vec{x}'_k \dots \Theta\vec{x}'_1)\, g_\ell(\vec{x}_1 \dots \vec{x}_\ell)$$

and $G_n(h) = \int d^{4n}x\, h(\vec{x}_1 \dots \vec{x}_n)\, G_n(\vec{x}_1 \dots \vec{x}_n)$, integration being over Euclidean space. Inequality (E.2) is the OS-positivity condition, it is required to hold for finite sequences of test functions $f_k(x_1 \dots x_k)$ that vanish unless $0 < x_1^4 < \dots < x_k^4$.

In the special case that only $f_2 \not\equiv 0$, inequality (E.2) reads explicitly

$$\int dx_1 \dots dx_4\, \bar{f}_2(\Theta\vec{x}_2, \Theta\vec{x}_1)\, G_4(\vec{x}_1 \vec{x}_2 \vec{x}_3 \vec{x}_4)\, f_2(\vec{x}_3, \vec{x}_4) \geqslant 0 \qquad (1.11)$$

So far our discussion has followed ref. 2.

Glaser has pointed out [3] that locality and the edge of the wedge theorem can be used to further extend the domain of definition and analyticity of the state vectors $\psi(z_1 \dots z_n)$ and, therefore, $\psi^E(x_1 \dots x_2)$:

Let π a permutation of 1...n and consider the vector

$$\psi_\pi(z_1 \dots z_n) = \psi(z_{\pi 1} \dots z_{\pi n}) = \text{analyt. cont. of } \phi(x_{\pi 1}) \dots \phi(x_{\pi n})\Omega \quad (1.12)$$

This is holomorphic for $y_{\pi 1} \in \overline{V}_+,\ y_{\pi\, j+1} - y_{\pi\, j} \in \overline{V}_+$
Locality tells us that on Minkowski space

$$\psi_\pi(x_1 \dots x_n) = \psi(x_1 \dots x_n) \quad \text{for all } \pi \text{ if all } (x_i - x_j)^2 < 0 \quad (1.13)$$

Thus, the boundary values of the holomorphic functions $\psi(z_1 \dots z_n)$ agree on a real neighborhood (they are known to be vector-valued distributions). The edge of the wedge theorem asserts that then the functions ψ_π are in fact analytic continuations of one and the same holomorphic function ψ $(z_1 \dots z_n)$, this function is thus defined on the union of the original domains of definition of the original ψ_π and is symmetric in its arguments there by (1.13). Specializing to Euclidean arguments this contains the union over π of the sets of arguments with $0 < x^4_{\pi 1} < \dots < x^4_{\pi n}$, i.e. n-tuples of Euclidean arguments with non-coinciding positive times $x^4_i > 0$, $x^4_i \neq x^4_j$ $(i \neq j)$. It can be shown [5] that the restriction to noncoinciding times can be weakened to non-coinciding arguments. Summing up:

> state vectors $\psi^E(\vec{x}_1 \dots \vec{x}_n)$ in $\mathcal{H}$ are defined and (real) analytic for Euclidean arguments $\vec{x}_1 \dots \vec{x}_n$ such that $\vec{x}_i \neq \vec{x}_j$ for $i \neq j$ and all $x^4_i > 0$. They are symmetric in their arguments, viz. $\psi^E(\vec{x}_{\pi 1} \dots \vec{x}_{\pi n}) = \psi^E(\vec{x}_1 \dots \vec{x}_n)$ for all permutations π.

Using this and the fact that the Green functions depend only on differences of their arguments, we see that the Euclidean Green functions $G_n(\vec{x}_1 \dots \vec{x}_n)$ are defined by Eq. (1.7) for arbitrary noncoinciding arguments $\vec{x}_1 \dots \vec{x}_n$. By symmetry of ψ^E,

$$G_n(\vec{x}_{\pi 1} \dots \vec{x}_{\pi n}) = G_n(\vec{x}_1 \dots \vec{x}_n) \quad \text{for all permutations } \pi \qquad (E.3)$$

Finally it is known that the Euclidean Green functions so defined are invariant under the Euclidean Poincaré-group, viz

$$G_n(m\vec{x}_1 + \vec{a} \dots m\vec{x}_n + \vec{a}) = G_n(\vec{x}_1 \dots \vec{x}_n) \qquad (E.1)$$

for arbitrary 4-rotations $m \in SO(4)$.

Osterwalder and Schrader have shown [2] that postulates (E.1), (E.2) and (E.3) as stated above together with a standard cluster property and a rather involved distribution theoretic axiom are sufficient conditions to guarantee that the Euclidean Green functions G_n (n=0,1...) determine a local quantum field theory satisfying the Wightman axioms.

Lastly it may be worthwhile pointing out that positivity (E.2) will actually hold for more general finite sequences of Schwartz test functions than stated there; it suffices that they vanish with all their derivatives when $\vec{x}_i = \vec{x}_j$ for some $i \neq j$, or some $x_i^4 \leq 0$. This follows immediately from assertion (1.14).

2. Conformal Invariance

From now on we shall restrict our attention to quantum field theories whose Euclidean Green functions are exactly conformal invariant. The Euclidean conformal group is $\mathcal{G} \simeq SO_\ell(5,1)$, it is compounded from [1]

$$\begin{aligned}
&\text{4-rotations } \vec{x} \to m\,\vec{x},\ m \in SO(4)\\
&\text{translations } \vec{x} \to t_c\vec{x} = \vec{x} + \vec{c}\\
&\text{dilations } \vec{x} \to \rho\vec{x},\ \rho > 0\\
&\text{special conformal transformations } \vec{x} \to R t_c R^{-1}\\
&\text{where } R\vec{x} = \frac{\vec{x}}{x^2}\ .
\end{aligned}$$

We will assume that our theory is parity invariant, it follows then that the Euclidean Green functions are invariant under Euclidean time reversal Θ . It is known that ΘR is an element of the identity component of the conformal group $\mathcal{G}$, thus parity invariance plus conformal invariance implies R-invariance.
Conversely R-invariance alone implies full conformal invariance in a Poincaré invariant theory, because special conformal transformations, 4-rotations and translations generate the whole group $\mathcal{G}$.

Let d the dimension of the fundamental field in the theory (the dimension is a new quantum number) then the requirement of R-invariance for the Euclidean Green functions is

$$G_n(\vec{x}_1 \dots \vec{x}_n) = (x_1^2 \cdots x_n^2)^{-d}\, G_n(R\vec{x}_1 \dots R\vec{x}_n)$$

One knows from Wilson's work [6] that the dimension d can in general be noninteger and is dynamically determined. Positivity of the 2-point function requires however that $d \geqslant 1$ resp. $d \geqslant \frac{1}{2}D-1$ in a world with D space time dimensions. In order not to have to distinguish between several cases we will assume that in fact $\frac{1}{2}D-1 \leq d \leq \frac{1}{2}D$. It is interesting to consider theories in an arbitrary even number D of space time dimensions, the considerations of Sec. 1 generalize immediately to this case.

3. Conformal Partial Wave Expansion

Given a conformal invariant Euclidean n-point Green function, graphically represented by a bubble with n legs, we can select two if its arguments and decompose it into terms corresponding to the exchange of definite conformal quantum numbers flowing between the selected pair of legs and the remaining ones. This is called the (Euclidean) conformal partial wave expansion [1].

Its virtues are first of all these: The connected Green functions in a Lagrangian quantum field theory are known to satisfy dynamical equations which amount to a coupled set of infinitely many nonlinear integral equations. All these integral equations are "solved" simultaneously by the conformal partial wave expansion, i.e. they are converted from integral equations to algebraic constraints. These algebraic constraints amount to demanding presence of certain simple poles in the partial waves qua analytic functions of the continuous conformal quantum number ("dimension") δ, with factorizing residues.

All this has been derived and discussed in detail in earlier lectures by the author which are already published [1]. We will therefore only review here very briefly the formulae which we will need later on.

The conformal quantum numbers (Casimir invariants of SO(5,1) resp. SO(D+1,1) D = # of space time dimensions) are $\chi = [\ell, \delta]$, with ℓ an SO(4)-spin (resp. SO(D)-spin) and δ a complex number, the "dimension". We will only be interested here in completely symmetric traceless tensor representations of SO(4), they can be characterized by their rank ℓ. Thus ℓ will be a nonnegative integer from now on.

There are several series of unitary representations of SO(D+1,1). If the space time dimension D is even, they are [7]

<u>identity representation</u> (1-dimensional)
<u>principal series</u>: ℓ arbitrary, $\delta = \frac{1}{2} D + i\sigma$, $-\infty < \sigma < \infty$
<u>supplementary series</u>: includes in particular $\ell = 0$, $0 < \delta < D$ real
<u>exceptional series:</u> associated with certain "integer points" (integer δ)

The exceptional series does never appear in the decomposition of the Euclidean Green functions for $D > 2$, and for $D=2$ it is equivalent to principal series representations [9] .

More generally, there exist (Banach-space) representations of the Euclidean conformal group $\mathcal{G}$ for arbitrary $\chi = [\ell, \delta]$, δ complex. They contain the unitary ones as special cases (resp. irreducible parts thereof for the exceptional series) and are constructed as induced representations as follows [1,8] .

The representation space $\mathcal{E}^{\chi}$ (or a dense subspace thereof) consists of functions $\varphi_{\alpha}(x)$ on Euclidean space $\{\vec{x}\}$, $\alpha = (\alpha_1 \dots \alpha_\ell)$ being tensor indices, viz $\alpha_i = 1 \dots 4$ resp. D. The little group H of $\vec{x} = 0$ consists of Euclidean 4-rotations $m \in M$, dilations $a \in A$ and special conformal transformations $n \in N$, viz $H = MAN$. The transformation law of functions $\varphi \in \mathcal{E}^{\chi}$ under $\Lambda \in \mathcal{G}$ is

$$\left(T(\Lambda)\varphi\right)_{\alpha}(\vec{x}) = D^{\chi}_{\alpha\beta}(h)\varphi_{\beta}(\Lambda^{-1}\vec{x}) \quad \text{(sum over } \beta \text{)} \tag{3.1}$$

Herein the little group element $h \in H$ depends on $\vec{x}$ and Λ and is given by

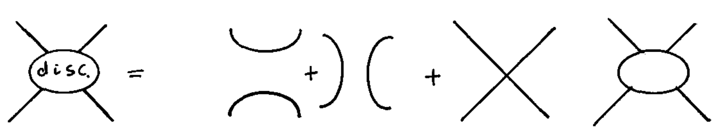
$$h = t_x^{-1} \Lambda t_{x'} \; , \quad \vec{x}' = \Lambda^{-1}\vec{x} \, ,$$

t_x the translation taking 0 to $\vec{x}$. The inducing representation D^{χ} is given by

$$D^{\chi}_{\alpha\beta}(m\,a\,n) = |a|^{-\delta} D^{\ell}_{\alpha\beta}(m)$$

if a is a dilation by $|a|$. D^{ℓ} is the completely symmetric ℓ-th rank tensor representation of $M \simeq SO(4)$, note that all $D^{\ell}_{\alpha\beta}(m)$ are therefore <u>real</u>. This representation can be extended to a representation of the group O(4) which is obtained by adjoining to M the reflection by Θ , so that also $D^{\ell}(\Theta)$ is defined. Representations $\chi = [\ell, \delta]$ and $-\chi = [\ell, D-\delta]$ are equivalent except at integer points. In addition there are further "partial equivalences" at integer points which are very important but cannot be discussed here (see refs. 9 and 16).

Since we will be interested in discussing positivity, we will need to consider the full disconnected Green functions as we did throughout Sec. 1. In terms of the connected Green functions

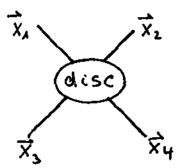


etc. (3.2)

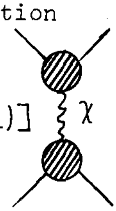
We will for simplicity consider a theory of one hermitian scalar field ϕ , e.g. a ϕ^3-theory. [The considerations can be extended to ϕ^4-theory with only small changes by introducing also the field $\phi^2(x)$ right from the start, i.e. a theory of two hermitian scalar fields having different dimensions.] The 2-point function is specified up to normalization by the dimension d of the field ϕ ; we will fix its normalization by [1]

$$\text{———} = G(\vec{x}_1\vec{x}_2) = (2\pi)^{-\frac{1}{2}D}\,\Gamma(d)\,(\tfrac{1}{2}|\vec{x}_1-\vec{x}_2|^2)^{-d}/\,\Gamma(\tfrac{1}{2}D-d)$$

We will now write down the conformal partial wave expansion for the full disconnected 4-point Green function



$$= \sum\!\!\!\!\!\!\int d\chi\,[1+g(\chi)]$$



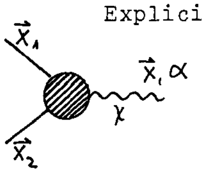
$$+ \text{ Id.} \qquad (3.3)$$

$$= \sum\!\!\!\!\!\!\int d\chi\,[1+g(\chi)]\int dx\;\Gamma^{\chi}(\vec{x}_1\vec{x}_2|\vec{x})\,\Gamma^{-\chi}(\vec{x}_3\vec{x}_4|\vec{x}) + \text{Id.}$$

Id. stands for a contribution belonging to χ = identity representation: it is given explicitly by the very first term on the right hand side (r.h.s) of Eq. (3.2), viz. Id. = $G_2(x_1x_2)\ G_2(x_3x_4)$.

The integration $\sum\!\!\!\!\!\int d\chi$ runs over a certain subset of the unitary representations of $\mathcal{G}$, we will come back to this below. All the dynamical information on the disconnected Greenfunction is in the partial wave amplitudes $1+g(\chi)$. It is written in this way to agree with the notation of ref.[1]. All the rest are kinematical factors determined by group theory.

Explicitly, the Clebsch Gordan kernels



$$= \Gamma^{\chi}_{\alpha_1\ldots\alpha_\ell}(\vec{x}_1\vec{x}_2/\vec{x})$$

$$= N(\chi)\,(\tfrac{1}{2}\,x_{12}^2)^{-d+\frac{1}{2}\delta-\frac{1}{2}\ell}\,(\tfrac{1}{2}\,x_{13}^2\;\tfrac{1}{2}\,x_{23}^2)^{-\frac{1}{2}(\delta-\ell)}\,(\hat{x}_{\alpha_1}\ldots\hat{x}_{\alpha_\ell}-\text{traces})$$

(3.4)

with $x_{ij} = \vec{x}_i - \vec{x}_j$, $\hat{x}_\alpha = \frac{(x_{13})_\alpha}{\frac{1}{2} x_{13}^2} - \frac{(x_{23})_\alpha}{\frac{1}{2} x_{23}^2} = \nabla_{3\alpha} \ln (x_{23}^2 / x_{13}^2)$

and for D = 2h

$$N(\chi) = (2\pi)^{-h} \left\{ \frac{\Gamma(-h+d+\frac{1}{2}\delta+\frac{1}{2}\ell)\Gamma(d-\frac{1}{2}\delta+\frac{1}{2}\ell)\Gamma(\frac{1}{2}\delta+\frac{1}{2}\ell)^2}{\Gamma(2h-d-\frac{1}{2}\delta+\frac{1}{2}\ell)\Gamma(h-d+\frac{1}{2}\delta+\frac{1}{2}\ell)\Gamma(h-\frac{1}{2}\delta+\frac{1}{2}\ell)^2} \right\}^{1/2}$$

Because of the equivalence of representations χ and $-\chi$, the partial waves can, and will be required to satisfy a symmetry relation

$$g(\chi) = g(-\chi) \tag{3.5}$$

Our choice of overall factors is such that [11] the Clebsch Gordan kernels for representations χ and $-\chi$ are related by

$$\Gamma_\beta^\chi(\vec{x}_1, \vec{x}_2 | \vec{x}) = \int dx' \, \Gamma_\alpha^{-\chi}(\vec{x}_1, \vec{x}_2 | \vec{x}') \Delta_{\alpha\beta}^\chi(\vec{x}', \vec{x}) \tag{3.6}$$

with intertwining kernel

$$- \Delta_{\alpha\beta}^\chi(\vec{x}, \vec{x}') = n(\chi)(\tfrac{1}{2} x^2)^{-\delta} \{ g_{\alpha_1\beta_1}(\vec{x}) \ldots g_{\alpha_\ell\beta_\ell}(\vec{x}) - \text{traces} \}$$

with

$$g_{\alpha\beta}(\vec{x}) = -\delta_{\alpha\beta} + 2 x_\alpha x_\beta / \vec{x}^2$$

$$n(\chi) = (2\pi)^{-h} \frac{\Gamma(\delta+\ell)\Gamma(2h-\delta-1)}{\Gamma(h-\delta)\Gamma(2h-\delta+\ell-1)} \tag{3.7}$$

Since (3.5) is true for all χ, hence also for $-\chi$, it follows that $\Delta^{-\chi}$ is the inverse of Δ^χ in the convolution sense. The graphical notation used e. g. in (3.3) takes all this into account if we picture $\Gamma^{-\chi}$ as a bubble with a short wiggly line. Inserting expression (3.6) for Γ^χ into the rhs. of the second equality (3.3), a more symmetrical expression results, which involves however two x-integrations.

We will not now write expansions for the higher n-point functions, connected or otherwise, they were given in ref. 1. The expansion for the connected 4-point Green function is obtained from (3.3) by substituting

$g(\chi)$ for $1+g(\chi)$ and omitting the contribution from the identity representation.

The dynamical integral equation [12] for Green functions mentioned before imply [1] for $g(\chi)$ that it should have a simple pole in δ for $\ell = 0$ at $\delta = d$, shortly: a pole at $\chi = [0,d]$. Its residue must be positive and fixes the square of the coupling constant. As a result, the integration over representations in (3.3) can be deformed to path integrals which run as follows:

$$\int\!\!\!\!\!\!\sum d\chi = \sum_{\ell=0,2..} \frac{1}{2\pi i} \int c(\chi)\, d\delta ;$$

$$c(\chi) = \frac{1}{\ell!}\, \Gamma(\tfrac{1}{2}D+\ell)\, \tfrac{1}{2}\, (2\pi)^{\frac{1}{2}D}\, n(\chi)\, n(-\chi)$$

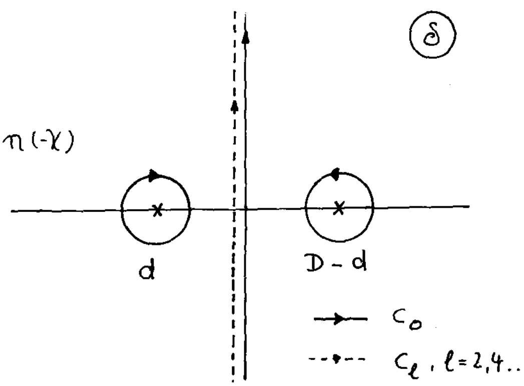


Fig. 1 paths of δ -integration

The Plancherel measure $c(\chi)$ is a polynomial. Note that the pole of $g(\chi)$ at $\chi = \chi_0 = [0,d]$ is accompanied by a brother at $-\chi_0$ by symmetry (3.5).

One finds in addition that $g(\chi)$ should also have a pole at $\chi = [2,D]$, this is required by the existence of a stress energy tensor.

Throughout this paper we will restrict our attention to theories in which the partial waves $g(\chi)$ are meromorphic function of δ for all ℓ with only simple poles. There is a good physical reason, for this assumption is necessary to have validity of operator product expansions with only a discrete number of (composite) fields appearing in them. This has also been discussed ref. 1. Further light on this analyticity assumption will be shed by our later considerations of positivity, where we will need the hypothesis that $g(\chi)$ are holomorphic in a cut δ -plane.

4. A Semigroup and Its Contractive Representations

We will now introduce a maximal noncommutative semigroup S contained in the Euclidean conformal group $\mathcal{G}$. It is defined to consist of those conformal transformations Λ in $\mathcal{G}$ which leave invariant the halfspace with positive Euclidean time x^4 resp. x^D:

$$\Lambda \in S \subset \mathcal{G} \text{ iff } 0 < x^4 < \infty \quad \text{implies} \quad 0 < (\Lambda x)^4 < \infty \tag{4.1}$$

and similarly for D space time dimensions. Henceforth such generalization will be left to the reader. In the 6-dimensional language, S consists of pseudo-orthogonal transformations of positive lightlike 6-vectors ξ which leave invariant the halfspace $\xi^4 > 0$.

Evidently S contains a subgroup $U \simeq SO_e(4.1)$ which consists of those pseudo-rotations which leave invariant ξ^4. It also contains the 1-parameter semigroup of pseudorotations b_τ ($\tau \geq 0$) in the 4-6 direction. This is seen as follows: Introduce hyperbolic coordinates, viz

$$\xi^k = r \cdot e^k \; (k = 1235) \, ; \; \xi^4 = r \sinh \sigma \; ; \; \xi^6 = r \cosh \sigma \tag{4.2}$$

Evidently $\xi^4 > 0$ if and only if $\sigma > 0$. Thus the halfspace $\xi^4 > 0$ is left invariant by the pseudorotations b_τ with $\tau \geq 0$ which translate the hyperpolic coordinate σ to $\sigma + \tau$. The generator H of this 1-parameter semigroup will be of special interest, it is defined by

$$b_\tau = e^{-H\tau} \tag{4.3}$$

The interior S^o of S is also a semigroup and so is $S^\sim = S^o \cup U$. It follows from the work of Toller and collaborators [13] that

$$\Lambda \in S^\sim \text{ iff } \Lambda = u_1 b_\tau u_2 \quad \text{with } u_i \in U \; (i=1,2), \; \tau \geq 0 \tag{4.4}$$

We observe that if Λ is in S resp. $S^\sim$ then so is $\bar{\Lambda} \equiv \Theta \Lambda^{-1} \Theta$. Consider now any of the (possibly nonunitary) induced representations of the Euclidean conformal group $\mathcal{G}$ which were described in the last section, with representation space $\mathcal{E}^\chi$ consisting of functions $\varphi_\alpha(x)$ We restrict this representation to a representation of the semigroup $S^\sim \subset \mathcal{G}$.

One notes then that there exists a subspace $\mathcal{E}_+^\chi$ of $\mathcal{E}^\chi$ which consists of those functions which vanish in a halfspace

$$\mathcal{E}_+^\chi : \quad \varphi_\alpha(x) = 0 \quad \text{if} \quad x^4 < 0 \tag{4.5}$$

It is evidently invariant under the action of the semigroup S.

It was discovered by Toller et al. that there also exists an invariant complement (or almost $\sim$). That is another invariant subspace $\mathcal{E}_-^\chi$ such that the direct sum $\mathcal{E}_+^\chi \oplus \mathcal{E}_-^\chi = \mathcal{E}^{\circ\chi}$ is dense in the (Banach) space $\mathcal{E}^\chi$.

This decomposition corresponds to a split of functions $\varphi_\alpha(x) \in \mathcal{E}^{\circ\chi}$ as follows

$$\varphi_\alpha(\vec{x}) = \varphi_\alpha^+(\vec{x}) + \int dx' \, \Delta^\chi_{\alpha\beta}(\vec{x},\vec{x}') \, \varphi_\beta^-(\vec{x}') \tag{4.6}$$
$$\text{with} \quad \varphi_\alpha^\pm(\vec{x}) = 0 \quad \text{if} \quad x^4 < 0$$

Note that the intertwining operator Δ^χ is involved, thus the possibility of the split must be related to the equivalence of representations χ and $-\chi$ of $\mathcal{G}$. We will only sketch an argument for the possibility of the split: Suppose that φ^- is already known, then obviously it is trivial to find also φ^+. Now φ^- is determined entirely by the values of $\varphi_\alpha(x)$ for $x^4 < 0$ and can be determined by solving the integral equation obtained from (4.6) by setting $x^4 < 0$ so that the first term on the rhs. is absent. This integral equation can be solved by partial wave expansion over the group $U \simeq SO(4.1)$. This follows from the fact that U acts transitively on the halfplane $x^4 > 0$. Restrictions on the functions φ that can be split come from the requirement that the U-partial waves of φ^- are well behaved at infinity qua functions of the continuous Casimir-invariant, this restricts the behaviour of φ for $x^4 < 0$ (in particular φ must be real analytic there).

We will next introduce a bilinear form on the representation space $\mathcal{E}_+^{-\chi}$ of $\tilde{S}$ by

$$(\varphi,\psi)_\chi = \Gamma(\tfrac{1}{2}D-\delta)\int dx\, dx' \, \bar{\varphi}_\alpha(\vec{x}) \, D^\ell_{\alpha\beta}(\Theta) \, \Delta^\chi_{\beta\gamma}(\Theta\vec{x},\vec{x}') \, \psi_\gamma(\vec{x}') \tag{4.7}$$

We ask ourselves for which χ the bilinear form (4.7) defines a positive semi-definite scalar product. For suitable choice of phase in $D^\ell(\Theta)$ the answer is given by the following

Lemma: If $\chi = [\ell, \delta]$ with either $\ell = 0$, $\delta > \frac{1}{2}D-1$ or $\ell > 0$, $\delta > D-2+\ell$, then $(\varphi, \varphi)_\chi \geqslant 0$, and $\| T(\Lambda) \| \leq 1$ for all $\Lambda \in S^\sim$ in the operator norm induced by the scalar product $(,)_\chi$.

Moreover, one checks by a straightforward computation that for real δ

$$(\varphi, T(\Lambda)\psi) = (T(\bar{\Lambda})\varphi, \psi) \quad \text{for } \Lambda \in S, \ \bar{\Lambda} = \Theta \Lambda^{-1} \Theta \tag{4.8}$$

It follows from this that the subspace of zero-norm vectors of $\mathcal{E}_+^{\cdot\chi}$ is invariant under $S^\sim$.

Thus, for χ such as are specified in the lemma, we may complete the representation space $\mathcal{E}_+^{-\chi}$ to a Hilbert space $\mathcal{H}_+^{-\chi}$ after dividing out the invariant subspace of zero norm vectors, and we are supplied by a contractive representation of $S^\sim$ acting in this Hilbert space $\mathcal{H}_+^{-\chi}$, and satisfying a pseudo-hermiticity condition (4.8) which implies in particular that U is represented unitarily and b_τ is represented by selfadjoint contraction operators so that $H \geqslant 0$, selfadjoint. The last assertion holds because $\Theta H \Theta = -H$ and $\Theta u \Theta = u$ for $u \in U$. The operator H is called the conformal Hamiltonian for reasons explained elsewhere [5] .

We will not prove the lemma but make it understandable by mentioning the following theorem which was stated and proven by Lüscher and the author in Appendix C of ref. [5] .

<u>Theorem</u>: Let T a continuous representation of $S^\sim$ by contraction operators in a Hilbert space, satisfying the condition (4.8), viz $T(\Lambda) = T(\bar{\Lambda})^*$. Then T can be analytically continued to a unitary representation of the universal (∞-sheeted) covering group $\mathcal{G}^*$ of the Minkowskian conformal group $SO(4,2)/Z_2$ resp. $SO(D,2)/Z_2$.

Continuity is satisfied if $\| T(\Lambda)-1 \| \to 0$ when $\Lambda \to 1$ through values in $S^\sim$. It can be shown to be satisfied for the representations considered so far. Taking lemma and theorem together we see that we end up with a class of unitary representations of $\mathcal{G}^*$. They ought to be equivalent to the known analytic representations of $\mathcal{G}^*$ studied by Rühl [14] .

5. Positivity of the 4-Point Function

Let us start by rewriting the conformal partial wave expansion (3.3) of the 4-point function. We consider the Clebsch Gordan kernel $\Gamma_\alpha^{-\chi}(\vec{x}_3\vec{x}_4 \mid \vec{x})$ as a function of $\vec{x}$ and split in the manner of Eq. (4.6), viz.

$$\Gamma_\alpha^{\chi}(\vec{x}_3\vec{x}_4 \mid \vec{x}) = Q_\alpha^{\chi}(\vec{x}_3\vec{x}_4 \mid \vec{x}) + \int dx' \, \Delta^{\chi}_{\alpha\beta}(\vec{x},\vec{x}') \, Q_\beta^{-\chi}(\vec{x}_3\vec{x}_4 \mid \vec{x}') \tag{5.1a}$$

$$\text{for } x_3^4, x_4^4 > 0 \quad \text{with } Q_\alpha^{\pm\chi}(\ldots \mid \vec{x}) = 0 \text{ if } x^4 < 0 .$$

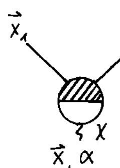
The notation takes into account the symmetry property (3.6) of Γ^χ under $\chi \to -\chi$. Because of Θ -invariance of Δ^χ and Γ^χ it follows that also

$$\Gamma_\alpha^{\chi}(\vec{x}_1\vec{x}_2 \mid \vec{x}) = Q^{\chi}_{\Theta\alpha}(\Theta\vec{x}_1, \Theta\vec{x}_2 \mid \Theta\vec{x}) + \int dx' \, \Delta^{\chi}_{\alpha\beta}(\vec{x},\vec{x}') \, Q^{-\chi}_{\Theta\beta}(\Theta\vec{x}_1 \Theta\vec{x}_2 \mid \Theta\vec{x}') \tag{5.1b}$$

$$\text{for } x_1^4, x_2^4 > 0 \quad \text{with } Q^{\pm\chi}_{\Theta\alpha} \equiv D^{\ell}_{\alpha\beta}(\Theta)\, Q_\beta^{\pm\chi}$$

The split (5.1a) may be performed in the manner sketched in Sec. 4. The Q^χ unfortunately turn out not to be good functions of $\vec{x}$, even though their U-partial waves are well defined, $\Gamma_\alpha^{\chi}(\vec{x}_3\vec{x}_4 \mid \vec{x})$ being a smooth function of $\vec{x}$ for $x^4 < 0$, $x_3^{\,4}$, $x_4^{\,4} > 0$. In the present note we will for simplicity ignore this complication, and proceed heuristically.*

We will use a graphical notation

$Q_\alpha^{-\chi}(\vec{x}_1\vec{x}_2 \mid \vec{x}) =$ 

$= Q^{-\chi}_{\Theta\alpha}(\Theta\vec{x}_1, \Theta\vec{x}_2 \mid \Theta\vec{x})$ (5.2)

if $x_1^4, x_2^4 > 0$ resp. if $x_1^4, x_2^4 < 0$

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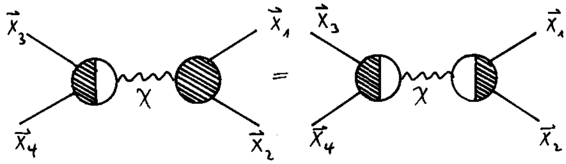
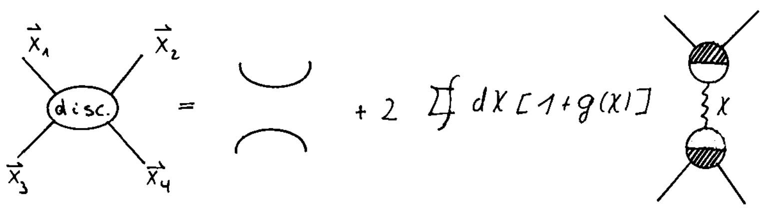
* A related problem is that the Hilbert spaces $\mathcal{H}_+^\chi$ of Sec. 4 do not anymore consist of equivalence classes of functions after completion in the norm, because Cauchy sequences need not converge in any function space topology. Both problems can be overcome by working with U-partial waves throughout.

Let us now consider the full disconnected Euclidean Green function $G_4(\vec{x}_1\vec{x}_2\vec{x}_3\vec{x}_4)$ for x_1^4, $x_2^4 < 0$; x_3^4, $x_4^4 > 0$. After inserting the split (5.1a) for one of the Γ^χ-kernels in the expansion (3.3), one has two terms. They can however be grouped together again by a change of χ -integration variable, using the symmetry property $g(\chi) = g(-\chi)$. Next one can use the identity



for $x_1^4, x_2^4 < 0$; $x_3^4, x_4^4 > 0$ (5.3)

This follows from the split (5.1b) and the support properties of $Q^{-\chi}$. As a result we get the new expansion



$$= G_2(\vec{x}_1 \vec{x}_2)\, G_2(\vec{x}_3 \vec{x}_4) + 2 \oint d\chi\, [1+g(\chi)] \int dx\, dx'\, Q_\alpha^{-\chi}(\theta\vec{x}_1\, \theta\vec{x}_2 \,|\, \vec{x})\, D^{\ell}_{\alpha\beta}(\theta)\, \Delta^{\chi}_{\beta\gamma}(\vec{x}, \vec{x}')\, Q_\gamma^{-\chi}(\vec{x}_3 \vec{x}_4 \,|\, \vec{x}')$$

Given a function $f_2(\vec{x}_1,\vec{x}_2)$ satisfying support properties stated after (E.2) of Sec. 1, let us define

$$\varphi_\alpha(\vec{x}) = \{N(\chi) N(-\chi)\}^{-\frac{1}{2}} \int dx_1\, dx_2\, f_2(\vec{x}_1 \vec{x}_2)\, Q_\alpha^{-\chi}(\vec{x}_1 \vec{x}_2 \,|\, \vec{x})$$

$\varphi'_\alpha(\vec{x})$ = same with $\bar{f}_2$ in place of f_2.

Of course these functions depend on χ ($\varphi_\alpha \in \mathcal{E}_+^{-\chi}$ if $Q^{-\chi}$ were a good function).

We may now inspect the expression (1.11) which ought to be positive. Comparing with the definition (4.7) of the bilinear form $(,)_\chi$ and recalling that G_4 is symmetric in its arguments we find

$$G_4(\theta f_2^* \times f_2) = 2 \sum\!\!\!\!\!\!\int d\chi\, M(\chi)[1+g(\chi)](\bar{\varphi}', \varphi)_\chi + \mathrm{Id}. \tag{5.4a}$$

$$\text{where} \quad M(\chi) = N(\chi)N(-\chi)\Gamma(\tfrac{1}{2}D-\delta)^{-1} \tag{5.4b}$$

and Id. stands for a contribution from the identity representation which is automatically positive by itself. The normalization factors N(.) where given in Eq. (3.4) and f.

We will now try to deform the path of the χ -integration in such a way that the assertion of the lemma becomes applicable.

Let us assume that the partial waves $g(\chi)$ satisfy growth conditions for $|\delta| \to \infty$ such that the path of the δ-integration can be closed to the right in Fig. 1. We also assume temporarily that there are no poles of the integrand in (5.4a) inside this closed path which come from the factors $M(\chi)(\bar{\varphi}', \varphi)_\chi$. Lastly, we observe that $\{N(\chi)N(-\chi)\}^{-1/2}\Gamma^\chi$ and Δ^χ are real for real δ, and so is therefore $\{N(\chi)N(-\chi)\}^{-1/2}Q^{-\chi}$ by its definition; it follows that $\bar{\varphi}' = \varphi$ for real δ. Suppose that $g(\chi)$ has a pole at $\chi = \chi_a = [\ell_a, \delta_a]$, (i.e. a pole in δ at δ_a for $\ell = \ell_a$). We define*

$$\operatorname*{res_*}_{\chi_a}[1+g(\chi)] = -c(\chi_a)M(\chi_a)\operatorname*{res}_{\delta=\delta_a}[1+g(\ell_a,\delta_a)] \tag{5.5}$$

Note that the definition depends on the dimension d of the fundamental field through $M(\chi)$.

By Cauchy's theorem we have then

$$G_4(\theta f_2^* \times f_2) = 2\sum_{\text{poles}} \mathrm{res}_*[1+g(\chi)](\varphi,\varphi)_{\chi_a} + \mathrm{Id}' \tag{5.6}$$

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* In the second paper of ref. 1 a factor $M(\chi)$ is missing in the statement of positivity constraints.

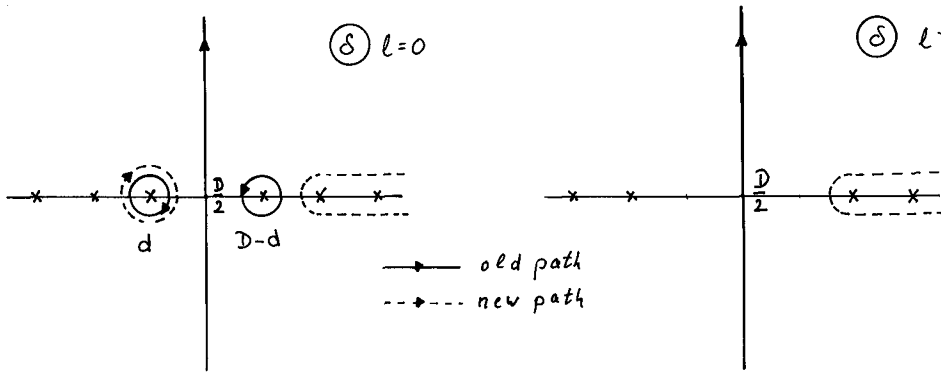


Fig. 2 Deformation of paths of δ-integrations.

with summation running over the pole of g(χ) at $\chi_0 = [0,d]$ plus all poles with $\delta > \frac{1}{2}$ D except the pole at $-\chi_0$. We see that the expression is manifestly positive if all the residues of the aforementioned poles are positive and these poles are positioned at real $\delta_2 > D-2+\ell_a$ if $\ell_a > 0$, for then the hypotheses of the lemma in Sec. 4 are met.

We have so far disregarded possible singularities of the factors $M(\chi)(\tilde{\varphi}', \varphi)_\chi$. Preliminary computations indicate that one must anticipate poles of two types: i) the poles of $N(\chi)N(-\chi)$ at $\delta = 2d+\ell+2n$, n = 0, 1,... ii) poles at certain integer points (δ integer). In order that such poles do not ruin positivity, partial waves g(χ) must satisfy additional kinematical constraints. Todorov and collaborators have recently shown [9] that in addition to (3.5) g(χ) must in any case satisfy further equalities relating its values at partially equivalent integer points, viz

$$g(\chi) = g(\chi') \quad \text{if } \chi = [\ell, D+\ell+n-1]\ ;\ \chi' = [\ell+n, D+\ell-1]\ , n = 2,4,\ldots$$

It is hoped that these constraints will lead to cancellation of the contributions from integer point poles. Concerning poles i) it seems attractive, though not really necessary, to demand that they are cancelled by zeroes of 1 + g(χ), so that expression (5.6) is valid without extra terms.

Summing up, positivity of the 4-point function will hold, if the partial waves g(χ) fulfill conditions of the following type as a function of $\chi = [\ell, \delta]$; $\ell = 0, 2 \ldots$

1. g(χ) is a meromorphic function of δ for each ℓ, with poles

only at real δ satisfying $|\delta - \frac{1}{2}D| > |\frac{1}{2}D-2+\ell|$ if $\ell > 0$. The residues of the pole at $\chi_0 = [0,d]$ and of all poles with $\delta > \frac{1}{2}D$ apart from $-\chi_0$ must be positive.

2. $g(\chi)$ satisfies growth conditions as $|\delta| \to \infty$ ($\mathrm{Re}\,\delta > \frac{1}{2}D$) so that the path of integration in (5.4a) can be closed to the right.

3. $g(\chi)$ must satisfy certain kinematical constraints related to the existence of kinematical poles of $M(\chi)(.,.)_\chi$, cp. text.

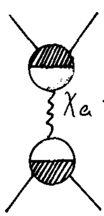
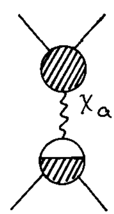
One can also look at the result in another way. The above conditions are imposed in order that the sum in (5.6) converges and consists of a sum of positive terms. That is

$$G_4(\Theta f_2^* \times f_2) = \sum_a r_a (\varphi,\varphi)_{\chi_a} + |d'| \quad , \qquad \text{with } r_a > 0 \text{ and}$$

$$\chi_a = [\ell_a, \delta_a]\,, \quad \delta_a > \tfrac{1}{2}D - 1 \qquad \text{for } \ell = 0 \text{ and } \delta_a > D-2+\ell \text{ otherwise.}$$

We assume here and in the following that $1+g(\chi) = 0$ at the poles of $N(\chi)N(-\chi)$.

Because of the Cauchy-Schwartz inequality, an expansion of this type must then also be convergent if we smear with arbitrary test function $\Theta f_2^* \times g_2$ instead of $\Theta f_2^* \times f_2$, and thus in the distribution theoretic sense

$G_4(\vec{x}_1 \ldots \vec{x}_4) = \sum_a r_a$  $\chi_a + |d'| = \sum_a r_a$  $\chi_a + |d'|$ (5.7)

The second equality follows from (5.3). We have thus arrived at an expansion of the type suggested in ref. [15] and Eq. (9.3) of ref. [1] except that it has not been shown that the Q^χ used here are the same (in some sense) as those used in ref. [1].

The results of Osterwalder and Schrader imply that the expansion

(5.7) remains valid, i.e. convergent when we analytically continue in the external arguments $\vec{x}_1 \ldots \vec{x}_4$ to Minkowski space through values $x_1^4 < x_2^4 < 0 < x_3^4 < x_4^4$ (x^4 = imaginary part of time) and we will in this way obtain an expansion of the Wightman function $W(x_1x_2x_3x_4)$ We expect, in view of the theorem cited in Sec. 4, that if amounts to a partial wave expansion on the simply connected covering $\mathcal{G}^*$ of the Minkowskian conformal group $SO_e(4,2)/Z_2$.

$$W_4(x_1x_2x_3x_4) = \sum_a r_a \int_{\substack{\text{Eucl. space} \\ x^4 > 0}} d^4\vec{x}\, \Gamma^{\chi_a}(\vec{z}_1\vec{z}_2 | \vec{x})\, Q^{-\chi_a}(\vec{z}_3\vec{z}_4 | \vec{x}) \tag{5.8}$$

$$\vec{z}_i = (\underline{z}_i, z_i^4) = (\underline{x}_i, -ix_i^0 + \varepsilon_i), \quad \varepsilon_1 < \varepsilon_2 < 0 < \varepsilon_3 < \varepsilon_4 \quad \text{infinitesimal}$$

Note that there is still an integration over half of Euclidean space involved as the formula stands now.

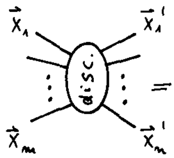
If the above interpretation is correct, one may even be able to show that the conditions for positivity of the 4-point function mentioned above are not only sufficient but also necessary ones. Absence of cuts in $g(\chi)$ needs a separate argument though, cp. Sec. 3, and also the precise form of the kinematical constraints is open to further study.

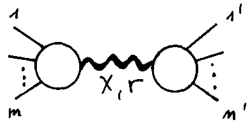
6. Generalization to Arbitrary n-point Functions

So far we have only investigated a special consequence of the positivity condition which involves the 4-point function alone. We now want to come to the general case.

Since we are presently interested in sufficient conditions for positivity, we will proceed by Ansatz.

Considering an m+n - point Green function, we try a decomposition into terms which correspond to exchange of definite conformal quantum numbers as follows



$= \sum_r \int\!\!\!\!\!\sum d\chi$


$$\equiv \sum_r \int dx\, dx'\, G_\alpha^{-\chi,r}(\vec{x}\,|\,\vec{x}_1 \ldots \vec{x}_m)\, \tilde{g}_r(\chi)\, \Delta^\chi_{\alpha\beta}(\vec{x},\vec{x}')\, G_\beta^{-\chi,r}(\vec{x}'\,|\,\vec{x}_1' \ldots \vec{x}_n') \tag{6.1}$$

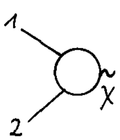
We have introduced here a new notation: the χ -integration is now supposed to include also a contribution from the identity representation (We leave it to the reader to work out the necessary convections for such χ), and a fat wiggly propagator

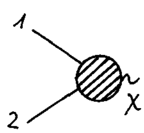
x  x' $= \tilde{g}_r(\chi)\, \Delta^\chi_{\alpha\beta}(x,x')$

r are some sort of internal quantum numbers. The $G^{-\chi,r}$ are conformal invariant, their transformation law as functions of the first argument is specified by $-\chi$ and given by Eq. (3.1), similarly for the other arguments ($\ell = 0$, $\delta = d$ there).

Our Ansatz consists in demanding that an expansion of the form (6.1) is valid with an integrand that factorizes. That is, for arbitrary number of arguments $x_1' \ldots x_n'$, the factor $G_\alpha^{-\chi,r}(\vec{x}\,|\,\vec{x}_1 \ldots \vec{x}_m)$ is always the same, and similarly for the second factor.

In case no summation over internal quantum numbers is necessary, we put



$= G_\alpha^{-\chi}(\vec{x}\,|\,\vec{x}_1,\vec{x}_2) = \bar{\Gamma}_\alpha^{-\chi}(\vec{x}_1,\vec{x}_2\,|\,\vec{x}) =$


(6.2)

so that

$$\tilde{g}(\chi) = 1 + g(\chi)$$

For m=2, partial wave expansions of the form (6.1) were already considered in ref. 1, and we know from the discussion given there that

the partial waves for arbitrary n = 2,3,... should share the poles of $g(\chi)$. For these poles are in correspondence with the local fields (including composite ones) in the theory, and their positions indicate tensor character and dimension of such fields.
For this reason we have pulled out a factor $1+g(\chi)$ resp $\hat{g}_r(\chi)$ in Eq. (6.1), it is supposed to contain all the "dynamical" poles of the integrand, while the factors $G^{-\chi,r}$ will be assumed to be holomorphic (apart from the possibility of certain kinematical singularities. They need a separate discussion much as in the case of the 4-point function).

Let there be given a finite sequence of test functions f_o, $f_1(\vec{x}_1),\dots f_N(\vec{x}_1\dots\vec{x}_v)$ with support properties as stated after (E.2) in Sec. 2. We define

$$\psi_\alpha^r(\vec{x}) = \sum_k \int dx_1 \dots dx_k\, f_k(\vec{x}_1\dots\vec{x}_k)\, G_\alpha^{-\chi,r}(\vec{x}|\vec{x}_1\dots\vec{x}_k) \tag{6.3}$$

and

$$\psi_\alpha'^r(\vec{x}) = \quad \text{same with } \bar{f}_k \text{ in place of } f_k.$$

Of course they depend on χ . With this notation,

$$\sum_{k,l} G_{k+l}(\Theta f_k^* \times f_l) = \sum_r \not\!\!\int d\chi\, \hat{g}_r(\chi) \int dx\,dx'\, \psi_\alpha'^r(\Theta\vec{x})\, \Delta_{\alpha\beta}^\chi(\vec{x},\vec{x}')\, \psi_\beta^r(\vec{x}') \tag{6.4}$$

assuming Θ -invariance of $G^{-\chi,r}$, i.e. parity invariance of the theory.

We will now split the functions ψ, ψ' in the manner described in Eq. (4.6), viz.

$$\psi_\alpha^r(\vec{x}) = \varphi_\alpha^{+,r}(x) + \int dx'\, \Delta_{\alpha\beta}^\chi(x,x')\, \varphi_\beta^{-,r}(x') \tag{6.5}$$

and similarly for ψ'. Because of equivalence of representations χ and $-\chi$ of the Euclidean conformal group, partial waves $G^{\chi,r}$ will (or can be required to) share a symmetry property analog to (3.6). As a consequence the same is true for ψ , ψ' and so $\varphi^{+,r}$ at χ is the same as $\varphi^{-,r}$ at $-\chi$.

Inserting the split (6.5) into (6.4) and simplifying the result with the help of symmetry and support properties as in Sec. 5, we end up with

$$\sum_{\kappa,l} G_{\kappa+l}(\Theta f_\kappa^* \times f_l) = \sum_r 2 \sum\!\!\!\!\!\!\int d\chi\, \hat{g}_r(\chi)\, M(\chi)\, (\bar{\varphi}'^r, \varphi^r)_\chi \qquad (6.6)$$

where $\varphi^r \equiv \varphi^{r,+}$ and $\varphi'^r \equiv \varphi'^{r,+}$

This equation is identical in appearance to Eq. (5.4a) of Sec. 5.

The further analysis then proceeds as in the case of the 4-point function. This results in some extra conditions in addition to those already satisfied by Ansatz or assumption stated above. They are of the same types as were stated in Sec. 5, i.e. the poles of $\hat{g}_r(\chi)$ should be in permissible parts of the real δ-axis and have positive residues if $\delta > \frac{1}{2}$ D, $\chi \neq -\chi_0$. In addition there are growth conditions and kinematical constraints. However, in contrast to the Γ^χ, the partial waves $G^{\chi,r}$ are not in general completely determined by kinematics. Thus, the growth conditions will be conditions not only on $\hat{g}_r(\chi)$ but they also involve the partial waves $G^{-\chi,r}$. Also the kinematical constraints relating $G^{-\chi}$ at partially equivalent integer points will look more complicated; we will not give the explicit expressions here they are however implicit in ref. 9. A detailed study of the growth conditions has not yet been carried out at the time of this writing. Also the connection between the expansion (5.7) resp. (5.8) and operator product expansion [17] deserves further study.

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