Teleparallel Gravity with Non-Vanishing Curvature

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Abstract

Guided by the rules of Einstein's geometrization philosophy, a pure geometric field theory is constructed. The Lagrangian used to derive the field equations of the theory is a curvature scalar of a version of absolute parallelism (AP-)geometry known in the literature as the parameterized absolute parallelism (PAP-)geometry. The linear connection of this version has simultaneously non-vanishing curvature and torsion. Analysis of the theory obtained shows clearly that it is a pure gravity theory. The theory is a teleparallel one, since the building blocks of both PAP and AP geometries are the same. It is shown analytically that the theory has a trivial version in the AP-geometry, if gravity is attributed to curvature not to torsion. In the case of spherical symmetry, solutions of the field equations give rise to the Schwarzchild exterior field. The theory depends on two principles: covariance and unification. The weak equivalence principle (WEP) is satisfied under a certain condition. The work preserves Einstein's main idea that gravity is just space-time curvature, although it is not a metric theory. It is shown that the theory reduces to vacuum general relativity upon taking the parameter of the geometry $b = 0$.

keywords:Field theory; Geometric material distribution; General relativity; Geometric Poisson equation; PAP-Geometry

0− Motivation:

In the context of Einstein's principal ideas, gravity is just a geometric property of the space-time. He has constructed a successful theory for gravity, the general theory of relativity (GR), by attributing gravity to the curvature of Levi-Civita linear connection.

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GR has many advantages and successful applications. The theory still treated as the standard theory for gravity, so far, although it has difficulties in some applications.
On the other hand, the teleparallel equivalent of general relativity (TEGR) is an alternative theory of gravity. It is constructed in the context of teleparallel geometry, the absolute parallelism (AP) geometry. The Lagrangian function, used to derive the field equations of this theory, is a torsion scalar T. Torsion of the AP-geometry is the skew symmetric part of the Weitzenböck linear connection. It is well known (cf.[1],[2]) that the curvature of this connection vanishes identically. So, in the context of TEGR, gravity is attributed to torsion not to curvature. This represents one of the differences between GR and TEGR.

As it is well known the geometric structure used to construct GR has a non-vanishing curvature and a vanishing torsion. On the other hand TEGR is constructed in the AP-geometry having a vanishing curvature and a non-vanishing torsion. However, a version of the AP-geometry, known in the literature as the *parameterized absolute parallelism (PAP) geometry,* has simultaneously non-vanishing curvature and torsion. This motivates us to explore the consequence of writing a gravity theory using the curvature of the parameterized Weitzenböck linear connection. This may help in reattributing gravity to the curvature of a linear connection, preserving Einstein's principal ideas. In general, the resulting theory is not a metric one but a teleparallel theory. In other words the gravitational potential is defined in terms of the building blocks (BB) of the PAP-geometry, the teleparallel vector fields. It can be reduced to GR under certain condition.

1 Introduction

One of the big achievements of the twentieth century is the successful theory of gravity, the General Theory of Relativity(GR) suggested by A. Einstein in 1915 [3]. This theory has been applied successfully in the context of pseudo-Riemannian geometry. The field equations of the theory can be written in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -k T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is Ricci tensor and R is Ricci scalar, both defined in the above mentioned geometry, while $T_{\mu\nu}$ is a phenomenological 2^{nd} order symmetric tensor giving the material distribution. The equation of motion of a free massive (massless) particle in GR is the time like (null) geodesic equation. This geodesic is represented by the equation,

$$\frac{dU^\alpha}{dS} + \{^{\alpha}_{\mu\nu}\} U^\mu U^\nu = 0, \qquad (2)$$

where U^α is the unit tangent at each point along the geodesic curve, S is a parameter varying along this curve and $\{^{\alpha}_{\mu\nu}\}$ is Christoffel symbol giving the coefficients of the Levi-Civita linear connection. The theory has been constructed depending on two principles: the general covariance and the equivalence principles. It has many successful applications in the domain of Solar and stellar systems. Predictions of GR, giving rise to new phenomenae, are still considerable, so far. For example, gravitational waves (cf. [4]), which has been predicted through a linear version of the theory very early in the 20^{th} century [5], are detected recently in the present century [6].

At the end of the 20^{th} century, it seems that large scale (cosmological) observation contradict some of the predictions of orthodox GR. The most famous of these observation are the SN type Ia [7], [8] . Authors have tried to interpret these observation by modifying GR or using some exotic expressions like "Dark Energy", and "Dark Matter". Some authors started to modify GR by reinserting the cosmological constant in its structure (cf. [9]), or by assuming a material distribution with certain unusual properties (cf. [10],[11, 12]). Another group of authors attempted to construct other theories of gravity, alternative to GR.

To review these attempts from the point of view of constructing their field equations, let us recall first the methods used to construct the field equations of GR. The first method is the most famous one. It depends on an action principle with certain Lagrangian function, Hilbert method (cf.[13]), to construct the field equations of the theory. The Lagrangian function usually used, in the context of geometry, is the Ricci scalar R. The second method is less famous. It uses differential identities, as representing conservation, to write the field equation. Einstein [3] has used Bianchi identities, to write his field equations. Although the two methods seem to be equivalent, we prefer the use of the second one.

Nowadays, there are two main streams, for choosing the lagrangian, to derive the field equations of gravity theories, alternative to GR. The first one is known in the literature as $f(R)$ theories (cf.[14],[15],[16]). This stream is running in the context of the same geometry of GR. In place of using Ricci scalar R as a Lagrangian function, authors use a general function of R to construct alternative theories. The main problem with this stream is that the resulting differential equations, in application, are of 4^{th} order which are difficult to solve and have unstable solutions [15, 17]. The second stream is treated by using the torsion of the AP-geometry. Many authors thought that the (AP) geometry has vanishing curvature but non-vanishing torsion (cf. [1]). This property is attributed to the connection used not to the whole geometry. This point will be discussed in Section 2. GR can be constructed in the AP-geometry by using a certain torsion scalar T, as stated above, in place of the Ricci scalar R. The second stream is known in the literature as $f(T)$ theories (cf.[18], [19], [20]). Both streams use the first method, for deriving the field equations of alternative theories, the Hilbert method.

However, there is a third less known stream running in the context of the AP-geometry or one of its versions, e.g. PAP-geometry. This stream started early in the seventies of the past century [21, 22]. It can be shown that the AP-geometry and PAP-geometry each has more than one curvature and W-tensors [23, 24]. Using scalars of such tensors as Lagrangian functions, one can construct different theories, alternative to GR, (e.g. [22, 25, 26, 27, 28, 29]) with successful applications (e.g. [30, 31, 32, 33, 34, 35]).

As will be shown in Section **2**, the PAP-geometry has the following parameterized linear connection, $\nabla^{\alpha}{}_{\mu\nu}$:

$$\nabla^{\alpha}{}_{\mu\nu} \stackrel{\text{def}}{=} \{^{\alpha}_{\mu\nu}\} + b\,\gamma^{\alpha}{}_{\mu\nu}, \tag{3}$$

where b is a dimensionless parameter, $\gamma^{\alpha}{}_{\mu\nu}$ is the contortion. Taking $b = 0$, (3) covers the domain of Riemannian geometry with its Levi-Civita linear connection. Also, if we take $b = 1$, (3) will reduce to the Weitzenböck linear connection, the case of AP-geometry. As stated above, Weitzenböck connection has a vanishing curvature, but its parameterized connection (3) has a simultaneously non-vanishing curvature and torsion. Due to the importance of curvature in describing gravity, it would be of interest to explore the consequences of constructing a field theory depending on the curvature of (3). The resulting theory would have a trivial AP-limit, as will be shown in the following Sections.

In constructing the theory, we confine ourselves only, to geometrization philosophy. In particular we confine ourselves to two principles:

- (i) The general covariance principle.

- (ii) The unification principle, which is the main important part of this philosophy.

The later principle obliges one to write every object in the theory using only the building blocks (BB) of the geometry used, and preventing the use of any other object from outside. Such theories are termed **" Pure Geometric"** field theories. For example GR with field equations (1) is not a pure geometric theory because of the use of a phenomenological $T_{\mu\nu}$ from outside the geometry used. For the same reason, TEGR, $f(T)$ and $f(R)$ theories are not pure geometric. On the other hand, GR in free space (from which most of the advantages of the theory are obtained), is a pure geometric field theory. Physical contents of such pure geometric theories are not known before analyzing the theory. In the present article we give three different methods for analyzing the theory. In such theories we are not going to impose any restrictions from old theories or laboratory physics on a pure geometric theory unless when dealing with special cases. Also we do not assume that terrestrial laboratory physics is the physics running, especially in stellar interiors or generally in the cosmos. Our goal is to explore physics resulting from geometry, providing that it reduces to laboratory physics under certain conditions.

The article is arranged as follows, a brief account on the PAP-geometry is given in Section **2** . In Section **3** we select a Lagrangian density using a commutation relation and apply a variational method to construct the field equation of the theory. The physical meanings of the geometric objects, emerged from the equations, are extracted in Section **4**, using three different methods. Solutions with spherical symmetry are given in Section **5**. The work is discussed and some concluding remarks are given in Section **6**.

2 A Brief Review of PAP-Geometry

A PAP-space $(M, \underset{i}{\lambda})$ is an n dimensional smooth manifold M, and at each point we define n globaley independent vectors $\underset{i}{\lambda}$ $(i = 0, 1, ..., n-1)$ [1]. Since these vectors are linearly independent, the n^2 matrix $\underset{i}{\lambda}^{\mu}$ is non-degenerate

[1] in the present article we use Greek indices to characterize coordinate components and Latin indices for vector numbers. For more details about PAP-geometry the reader is refered to [36] and [37].

i.e $det(\underset{i}{\lambda}^{\mu}) = \lambda \neq 0$. Consequently, the covariant components of the above mentioned vectors, $\underset{i}{\lambda}_{\alpha}$, can be defined such that:

$$\underset{i}{\lambda}^{\mu}\,\underset{i}{\lambda}_{\nu} = \delta^{\mu}_{\nu} \qquad , \qquad \underset{i}{\lambda}^{\mu}\,\underset{j}{\lambda}_{\mu} = \delta_{ij}. \tag{4}$$

Note that Einstein summation convention is carried out over repeated latin indices wherever they appear in each term.

In order to facilitate comparison with metric theories of gravity, specially GR, using the above vectors, one can define the following symmetric tensors:

$$g_{\mu\nu} = \eta_{ij}\,\underset{i}{\lambda}_{\mu}\,\underset{j}{\lambda}_{\nu} \qquad , \qquad g^{\mu\nu} = \eta_{ij}\,\underset{i}{\lambda}^{\mu}\,\underset{j}{\lambda}^{\nu}, \tag{5}$$

where $\eta_{ij}(= diag(+1, -1, -1, -1))$. Due to the above mentioned properties of the vector fields $\underset{i}{\lambda}$, the tensors defined by (5) satisfy the relation

$$g^{\alpha\mu} g_{\nu\alpha} = \delta^{\mu}_{\nu}. \tag{6}$$

It can be shown that these tensors possess the same properties of a metric tensor and can be used to define a pseudo-Riemannian structure (M, g) with its Levi-Civita linear connection preserving metricity, given by,

$$\left\{{}^{\alpha}_{\mu\nu}\right\} \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \tag{7}$$

In addition to the Levi-Civita linear connection (7), one can define another linear connection, the Weitzenböck connection, by

$$\Gamma^{\alpha}{}_{\mu\nu} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu,\nu}, \tag{8}$$

$$= \left\{{}^{\alpha}_{\mu\nu}\right\} + \gamma^{\alpha}{}_{\mu\nu}$$

where comma(,) is used for ordinary partial differentiation, $\gamma^{\alpha}{}_{\mu\nu}(\stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha}\,\underset{i}{\lambda}_{\mu;\nu})$ is the contortion and the semi-colon is an infix covariant differential operator carried out using (7). It has been shown that the connection (8) preserves metricity ($g_{\alpha\beta\underset{+}{|}\gamma} = 0$) and parallelism ($\underset{i}{\lambda}^{\alpha}{}_{\underset{+}{|}\beta} = 0 = \underset{i}{\lambda}_{\alpha\underset{+}{|}\beta}$) (cf.[2]). Since the Weitzenböck connection (8) is non-symmetric, its dual $\tilde{\Gamma}^{\alpha}{}_{\mu\nu}(\stackrel{\text{def}}{=} \Gamma^{\alpha}{}_{\nu\mu})$ and its symmetric part,

$$\Gamma^{\alpha}{}_{(\mu\nu)} \stackrel{\text{def}}{=} \frac{1}{2}(\Gamma^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\nu\mu}), \tag{9}$$

are both linear connections.

So far, we have four natural [2] linear connections, $\{^{\alpha}_{\mu\nu}\}$, $\Gamma^{\alpha}{}_{\mu\nu}$, $\tilde{\Gamma}^{\alpha}{}_{\mu\nu}$ and $\Gamma^{\alpha}{}_{(\mu\nu)}$. Combining these four connections linearly and imposing some conditions, one can get the result [36]

$$\nabla^{\alpha}{}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\} + b\,\gamma^{\alpha}{}_{\mu\nu}, \tag{10}$$

where b is a dimensionless parameter. It is easy to show that the object given by (10) is a metric linear connection [37]. This connection will be called parameterized canonical connection or parameterized Weitzenböck connection. It is clearly non-symmetric, so its dual $\tilde{\nabla}^{\alpha}{}_{\mu\nu}(\stackrel{\text{def}}{=} \nabla^{\alpha}{}_{\nu\mu})$ and its symmetric part

$$\nabla^{\alpha}{}_{(\mu\nu)} \stackrel{\text{def}}{=} \frac{1}{2}(\nabla^{\alpha}{}_{\mu\nu} + \nabla^{\alpha}{}_{\nu\mu}), \tag{11}$$

are linear connections. This adds three linear connections, $\nabla^{\alpha}{}_{\mu\nu}$, $\tilde{\nabla}^{\alpha}{}_{\mu\nu}$ and $\nabla^{\alpha}{}_{(\mu\nu)}$, to the above mentioned four connections.

Now we have the following important notes.

Note 1: The parameterized linear connection (10), characterizing the PAP-geometry, has the following properties:

- If $b = 0$, (10) reduces to (7) covering the domain of pseudo-Riemannian geometry.

- While $b = 1$, (10) reduces to the Weitzenböck connection (8) characterizing the (AP)geometry.

So, the pseudo-Riemannian and the AP-structures are just special cases of the PAP-geometry.

Note 2: It is worth of mention that the PAP-geometry has the same BB as the AP-geometry.

In what follows, we are going to use the semicolon (;), stroke (|) and double stroke (||) as infix operators for tensor derivatives using the connections (7), (8) and (10), respectively. As the connections (8), (10) are non-symmetric, each will be associated with three tensor derivatives. We are going to characterize the last two infix operators by a(+) sign, a (-) sign and without signs to distinguish between tensor derivatives using the connection (8), (or (10))

[2]**natural** here is used to characterize objects constructed only from the building blocks(BB) of the geometry. In the present case the BB are the vector fields $\underset{i}{\lambda}$.

its dual and its symmetric part, respectively.
For example, if A_μ is an arbitrary covariant vector field defined in the PAP-space, one can write

$$A_{\underset{-}{\mu}||\nu} \stackrel{\text{def}}{=} A_{\mu,\nu} - A_\alpha \tilde{\nabla}^\alpha{}_{\mu\nu}. \tag{12}$$

Again since (8) is non-symmetric, torsion $\Lambda^\alpha{}_{\mu\nu}$, contortion $\gamma^\alpha{}_{\mu\nu}$ and the basic vector C_μ of the AP-space can be written, in the AP-space as (cf. [1]),

$$\left.\begin{array}{l} \Lambda^\alpha{}_{\mu\nu} \stackrel{\text{def}}{=} \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} = \gamma^\alpha{}_{\mu\nu} - \gamma^\alpha{}_{\nu\mu}, \\ \gamma^\alpha{}_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}(\Lambda^\alpha{}_{\mu\nu} - \Lambda_\mu{}^\alpha{}_\nu - \Lambda_\nu{}^\alpha{}_\mu). \\ C_\mu \stackrel{\text{def}}{=} \gamma^\alpha{}_{\mu\alpha} = \Lambda^\alpha{}_{\mu\alpha}. \end{array}\right\} \tag{13}$$

It can be easily shown that [3]

$$\left.\begin{array}{l} \overset{*}{\Lambda}{}^\alpha{}_{\mu\nu} \stackrel{\text{def}}{=} b\ \Lambda^\alpha{}_{\mu\nu}, \\ \overset{*}{\gamma}{}^\alpha{}_{\nu\mu} \stackrel{\text{def}}{=} b\ \gamma^\alpha{}_{\mu\nu}, \\ \overset{*}{C}_\mu \stackrel{\text{def}}{=} b\ C_\mu, \end{array}\right\} \tag{14}$$

which are the torsion, contortion and the basic vector of the PAP-space, respectively. It is obvious that(14) reduces to (13) upon taking $b = 1$.

Now consider the following commutation relations

$$A_{\underset{+}{\mu}|\nu\sigma} - A_{\underset{+}{\mu}|\sigma\nu} = A_\alpha M^\alpha{}_{\mu\nu\sigma} - A_{\underset{+}{\mu}|\alpha}\ \Lambda^\alpha{}_{\nu\sigma}, \tag{15}$$

$$A_{\underset{+}{\mu}||\nu\sigma} - A_{\underset{+}{\mu}||\sigma\nu} = A_\alpha \overset{*}{M}{}^\alpha{}_{\mu\nu\sigma} - A_{\underset{+}{\mu}||\alpha}\ \overset{*}{\Lambda}{}^\alpha{}_{\nu\sigma}. \tag{16}$$

The tensors M, $\overset{*}{M}$ represent the curvature tensors of the AP and PAP linear connections (8),(10), respectively.

Note 3: It can be easily shown that $\overset{*}{M} \to M$ as $b \to 1$. Also it is well known that $M^\alpha{}_{\mu\nu\sigma}$ is identically vanishing because of (8) (cf. [1]). So, a field theory constructed using $\overset{*}{M}{}^\alpha{}_{\mu\nu\sigma}$ would have a trivial limit in the AP-geometry. This is the cornerstone of the present work, and will be discussed in Section 6 .

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[3] From now on, we are going to decorate tensors by a star if the parameter b generally appears in its structure.

3 A Suggested field theory

In what follows, we are going to construct a field theory in the context of the PAP-geometry which is briefly reviewed in Section 2. In general, we are going to relay on two important principles:

1. General covariance principle,

2. Unification principle [29] which means that all physical entities related to the theory are to be defined from the BB of the PAP-geometry, $\underset{i}{\lambda}$. This will be discussed in Section 6.

This principle can be considered as an explicit statement of geometrization philosophy [38] which can be summarized as follows [2] *"To understand Nature, one should start with Geometry, end with Physics"*.

In the context of the above mentioned philosophy, we have three important rules to be followed:
rule 1: Physical quantities are represented by geometric objects.
rule 2: Physical conservation laws are just differential identities in the geometry chosen.
rule 3: Physical trajectories of test particles are geodesics of connections.

3.1 The Field Equations

In the present Subsection, we construct the field equations of the theory using the second method mentioned in the introduction, the differential identity method. So, it is important to find such identities in the PAP-geometry. Fortunately, Dolan and McCrea in 1963 [39] have suggested a variational method to look for identities in Riemannian geometry. This method has been modified to suite the AP-geometry [21], [22]. The result is the general differential identity of the AP-geometry,

$$E^{\mu}{}_{\nu|\underline{\mu}} \equiv 0, \tag{17}$$

where $E^{\mu}{}_{\nu}$ is a non-symmetric tensor defined in terms of the BB of the AP-geometry as

$$E^{\mu}{}_{\nu} \stackrel{\text{def}}{=} \frac{1}{\lambda} \frac{\delta \pounds}{\delta \underset{j}{\lambda}_{\mu}} \underset{j}{\lambda}_{\nu}, \tag{18}$$

and $\frac{\delta \pounds}{\delta \underset{j}{\lambda}_\mu}$ is the Hamiltonian derivative of the Lagrangian density $\pounds$ defined by

$$\frac{\delta \pounds}{\delta \underset{j}{\lambda}_\mu} \stackrel{\text{def}}{=} \frac{\partial \pounds}{\partial \underset{j}{\lambda}_\mu} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \pounds}{\partial \underset{j}{\lambda}_{\mu,\alpha}}\right). \tag{19}$$

The Lagrangian density $\pounds$ is defined using a Lagrangian function L as

$$\pounds \stackrel{\text{def}}{=} \lambda L, \tag{20}$$

$$L = L(\underset{i}{\lambda}_\mu, \underset{i}{\lambda}_{\mu,\nu}). \tag{21}$$

The differential identity (17) is a general identity. It can be considered as a generalization of Bianchi differential identity. In the case of the PAP-geometry, it can be easily shown [40] that(17) holds. The proof is easy if we consider note 2, which states that the BB of the AP and the PAP geometries are the same [4]. The difference is that we decorate the objects $E^\mu{}_\nu$, $\pounds$, L appearing in expressions (18), (19), (20), (21) using a star. Then the identity (17) can be written in the PAP-geometry as

$$\overset{*}{E}{}^\mu{}_{\nu|\underline{\mu}} \equiv 0. \tag{22}$$

Following **rule 2** mentioned above, we consider the differential identity (22) as conservation of the entities represented by $\overset{*}{E}{}^\mu{}_\nu$. Immediately, the field equations of the theory can be written, in general, as

$$\overset{*}{E}{}^\mu{}_\nu = 0. \tag{23}$$

Now, for the present work, we are going to use the curvature, of the parameterized linear connection (10), defined from the commutation relation (16). This can be defined in terms of (10) as,

$$\overset{*}{M}{}^\alpha{}_{\mu\nu\sigma} \stackrel{\text{def}}{=} \nabla^\alpha{}_{\mu\sigma,\nu} - \nabla^\alpha{}_{\mu\nu,\sigma} + \nabla^\epsilon{}_{\mu\sigma}\nabla^\alpha{}_{\epsilon\nu} - \nabla^\epsilon{}_{\mu\nu}\nabla^\alpha{}_{\epsilon\sigma}\,, \tag{24}$$

which we use to construct our Lagrangian function. Substituting from (10) into (24) and contracting this tensor twice, we get after some reductions

$$\overset{*}{M} = R + (\overset{*}{C}{}^\epsilon \overset{*}{C}_\epsilon - \overset{*}{\gamma}{}^{\alpha\nu}{}_\epsilon\, \overset{*}{\gamma}{}^\epsilon{}_{\nu\alpha}) + 2bC^\alpha{}_{;\alpha}, \tag{25}$$

[4] This has also been proved using theorem-1 of [40].

where R is Ricci scalar defined completely from Levi-Civita connection (7). The scalar curvature (25) can be considered as the Lagrangian function for the present work. The corresponding Lagrangian density can be written as

$$\left.\begin{aligned} \overset{*}{\mathcal{M}} &= \lambda \overset{*}{M}, \\ &= \lambda\ R + \lambda\ (\overset{*}{C}{}^{\epsilon}\overset{*}{C}_{\epsilon} - \overset{*}{\gamma}{}^{\alpha\nu}{}_{\epsilon}\ \overset{*}{\gamma}{}^{\epsilon}{}_{\nu\alpha}) \\ &\quad + 2b\,\lambda\ C^{\alpha}{}_{;\alpha}. \end{aligned}\right\} \quad (26)$$

As it is well known, the last term gives no contribution to the variation results. Then (26) can be written in the equivalent form,

$$\overset{*}{\mathcal{M}} = \lambda\ R + \lambda\ (\overset{*}{C}{}^{\epsilon}\overset{*}{C}_{\epsilon} - \overset{*}{\gamma}{}^{\alpha\nu}{}_{\epsilon}\ \overset{*}{\gamma}{}^{\epsilon}{}_{\nu\alpha}). \quad (27)$$

It can be easily shown that the Lagrangian density (27) vanishes identically in the AP-geometry. This will be discussed in Section 6. Considering (27) as the Lagrangian density for the theory and using the following derivatives necessary to evaluate the Hamiltonian derivative (19), we have

$$\frac{\partial \overset{*}{\mathcal{M}}}{\partial \underset{j}{\lambda}_{\beta}} = \frac{\partial(\lambda R)}{\partial \underset{j}{\lambda}_{\beta}} + \lambda\ b^2[\underset{j}{\lambda}{}^{\beta}C^{\mu}C_{\mu} - 2\,\underset{j}{\lambda}{}^{\alpha}C^{\beta}C_{\alpha} + 2\,\underset{j}{\lambda}{}^{\alpha}C^{\mu}\Lambda^{\beta}{}_{\alpha\mu} + \underset{j}{\lambda}{}^{\beta}\gamma^{\epsilon\mu}{}_{\alpha}\gamma^{\alpha}{}_{\epsilon\mu}$$

$$+2\,\underset{j}{\lambda}{}^{\epsilon}g^{\mu\nu}\gamma^{\beta}{}_{\mu\alpha}\gamma^{\alpha}{}_{\nu\epsilon} - 2\,\underset{j}{\lambda}{}^{\epsilon}g^{\mu\nu}\gamma^{\beta}{}_{\mu\alpha}\gamma^{\alpha}{}_{\epsilon\nu}], \quad (28)$$

$$\frac{\partial \overset{*}{\mathcal{M}}}{\partial \underset{j}{\lambda}_{\beta,\alpha}} = \frac{\partial(\lambda R)}{\partial \underset{j}{\lambda}_{\beta,\alpha}} + 2\,\lambda\, b^2[\underset{j}{\lambda}{}^{\alpha}g^{\nu\beta}C_{\nu} - \underset{j}{\lambda}{}^{\beta}g^{\nu\alpha}C_{\nu} - \underset{j}{\lambda}{}^{\epsilon}g^{\nu\beta}\gamma^{\alpha}{}_{\nu\epsilon}]. \quad (29)$$

Then substituting from (28), (29) into (19), we get

$$\frac{\delta \overset{*}{\mathcal{M}}}{\delta \underset{j}{\lambda}_{\beta}} = (\frac{\partial(\lambda R)}{\partial \underset{j}{\lambda}_{\beta}} - \frac{\partial}{\partial x^{\alpha}}\frac{\partial(\lambda R)}{\partial \underset{j}{\lambda}_{\beta,\alpha}}) + \lambda\, b^2[\underset{j}{\lambda}{}^{\beta}C^{\mu}C_{\mu} - 2\,\underset{j}{\lambda}{}^{\alpha}C^{\beta}C_{\alpha} + 2\,\underset{j}{\lambda}{}^{\alpha}C^{\mu}\Lambda^{\beta}{}_{\alpha\mu} + \underset{j}{\lambda}{}^{\beta}\gamma^{\epsilon\mu}{}_{\alpha}\gamma^{\alpha}{}_{\epsilon\mu}$$

$$+2\,\underset{j}{\lambda}{}^{\epsilon}g^{\mu\nu}\gamma^{\beta}{}_{\mu\alpha}\gamma^{\alpha}{}_{\nu\epsilon} - 2\,\underset{j}{\lambda}{}^{\epsilon}g^{\mu\nu}\gamma^{\beta}{}_{\mu\alpha}\gamma^{\alpha}{}_{\epsilon\nu}] - \frac{\partial}{\partial x^{\alpha}}(2\,\lambda b^2[\underset{j}{\lambda}{}^{\alpha}g^{\nu\beta}C_{\nu} - \underset{j}{\lambda}{}^{\beta}g^{\nu\alpha}C_{\nu} - \underset{j}{\lambda}{}^{\epsilon}g^{\nu\beta}\gamma^{\alpha}{}_{\nu\epsilon}]),$$

where the first term in the above relation represent a Hamiltonian derivative of Ricci tensor R with respect to $\underset{j}{\lambda}_\nu$. For the Ricci scalar R, it is easy to show that,

$$\underset{j}{\lambda}^\mu \frac{\delta(\lambda R)}{\delta \underset{j}{\lambda}_\nu} = 2\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}},$$

where λ is replaced by $\sqrt{-g}$ according to (5) and it is well known that (cf. [4])

$$\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} = -G^{\mu\nu}.$$

Then we can write (23), after some rearrangements, as [5]

$$\overset{*}{M}_{\nu\sigma} \overset{\text{def}}{=} -2G_{\nu\sigma} + b^2[g_{\nu\sigma}((1-2b)C^\mu C_\mu + \gamma^{\epsilon\mu}{}_\alpha \gamma^\alpha{}_{\epsilon\mu} + 2C^\mu{}_{||\underset{+}{\mu}}) + 2(b-1)\gamma^\alpha{}_{\epsilon\sigma}\gamma^\epsilon{}_{\nu\alpha}$$
$$+2(1-2b)C_\epsilon \gamma^\epsilon{}_{\nu\sigma} - 2C_{\nu||\underset{+}{\sigma}} + 2b\gamma^\epsilon{}_{\nu\alpha}\gamma^\alpha{}_{\sigma\epsilon} + 2\gamma^\alpha{}_{\nu\sigma||\underset{+}{\alpha}}]. \qquad (30)$$

It is to be noted that the tensor $\overset{*}{M}_{\nu\sigma}$ is subject to the differential identity (22) so, according to the scheme outlined above we can write the field equations (23) of the present theory as,

$$\overset{*}{M}_{\nu\sigma} = 0. \qquad (31)$$

It is worth of mention that the tensor $\overset{*}{M}_{\nu\sigma}$, given by (30), is in general non-symmetric and is formed completely from the BB of the PAP-geometry. So the field equations of the theory (31) satisfy the two principles mentioned at the beginning of the present Section.

3.2 The Equations of Motion

According to **rule 3**, modified geodesic resulting from the parameterized connection (10) can be used to study trajectories of test particles moving in the field given by the field equations. These curves are governed by the equation [36, 37]

$$\frac{d^2x^\alpha}{d\tau^2} + \left\{{\alpha \atop \mu\nu}\right\}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -b\ \Lambda_{(\mu\nu)}{}^{\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \qquad (32)$$

[5] $\overset{*}{M}_{\nu\sigma}$ is used in the place of $\overset{*}{E}_{\nu\sigma}$ for the present work.

where τ is a parameter varying along the curve. The R.H.S of this equation represents torsion effect on the curve and b is the same parameter of (10). Also, equation (32) satisfies the above mentioned two principles. Note that the curve (32) reduces to the geodesic (2) upon taking $b = 0$, as expected.

4 Extraction of Physics from Geometry

So far, all objects appeared in the treatment are pure geometric objects constructed from the BB of the PAP-geometry. In the present section, we are going to consider **rule 1** given in Section 3. It is of importance to attribute physical meaning to the geometric quantities admitted in the theory. In order to do so, we are going to use three different schemes usually used in such theories (e.g. [27, 29]).

In particular, the aim of the present section is threefold. First, is to get physical meaning for the pure geometric objects of the suggested theory, which is done in subsection 4.1. The second is to get the relation between the values of any geometric quantity when written in geometric units and when written in physical units. This is done in subsection 4.2 which facilitates conversion between units. The third (subsection 4.3) is a covariant scheme suggested to classify geometric structures, physically, in the context of the theory.

4.1 Comparison with Non-linear Theories

The symmetric part of the field equations (31) can be written as

$$\overset{*}{M}_{(\nu\sigma)} \overset{\text{def}}{=} \frac{1}{2}(\overset{*}{M}_{\nu\sigma} + \overset{*}{M}_{\sigma\nu}) = 0. \tag{33}$$

Using the definition of $\overset{*}{M}_{\nu\sigma}$ given by (30) and the second order tensors defined in Table **A**.1, equation (33) can be written in the form

$$R_{\nu\sigma} - \frac{1}{2}g_{\nu\sigma}R = \overset{*}{T}_{\nu\sigma}, \tag{34}$$

where,

$$\overset{*}{T}_{\nu\sigma} \overset{\text{def}}{=} (\frac{1}{2})b^2\{g_{\nu\sigma}[-\alpha - \omega + \theta] + \psi_{\nu\sigma} - \theta_{\nu\sigma} - \phi_{\nu\sigma} + 2\omega_{\nu\sigma}\}. \tag{35}$$

Note that the L.H.S. of equation (34) can be obtained using Christoffel symbols alone. This is not the situation with the R.H.S. of (34) given by (35). This is an important point which will be discussed in Section 6. The symmetric part of the field equations is written in the form (34) in order to facilitate comparison with the corresponding equation of GR (1). From this comparison we can extract the following physical meanings:

- (i) The tensor $g_{\nu\sigma}$ is the gravitational potential.
- (ii) The tensor $\overset{*}{T}_{\nu\sigma}$ is a geometric energy-momentum tensor for the present theory, defined from the BB of the geometry used.
- (iii) The tensor $\overset{*}{T}_{\nu\sigma}$ is the source of the gravitational field described by Einstein tensor, the L.H.S of (34).
- (iv) Due to (22), the tensor $\overset{*}{T}_{\nu\sigma}$ satisfies a differential identity and consequently conservation. This can be shown as follows. As a consequence of theorem-2 of [40], the identity for the symmetric part of the field equations takes the form,

$$\overset{*}{M}{}^{(\nu\sigma)}{}_{;\nu} \equiv 0.$$

From the symmetric field equations (34) we can write,

$$\overset{*}{M}{}^{(\nu\sigma)} \overset{\text{def}}{=} G^{\nu\sigma} - \overset{*}{T}{}^{\nu\sigma} = 0.$$

Using theorem-2 of [40] we can rewrite the identity for the above equation as

$$G^{\nu\sigma}{}_{;\nu} - \overset{*}{T}{}^{\nu\sigma}{}_{;\nu} \equiv 0.$$

Consequently,

$$\overset{*}{T}{}^{\nu\sigma}{}_{;\nu} \equiv 0, \qquad (35.a)$$

which has been proved in detailed in [40], and shows that the $\overset{*}{T}{}^{\nu\sigma}$ satisfies conservation.

Similarly, the skew part of the field equations (31) can be defined by

$$\overset{*}{M}_{[\nu\sigma]} \overset{\text{def}}{=} \frac{1}{2}(\overset{*}{M}_{\nu\sigma} - \overset{*}{M}_{\sigma\nu}) = 0. \qquad (36)$$

Again, using the skew tensors defined in Table **A**.1, we can write (36), after some reductions as

$$\overset{*}{M}_{[\nu\sigma]} \overset{\text{def}}{=} (b^2 - 1)(\varepsilon_{\nu\sigma} + \eta_{\nu\sigma} - \chi_{\nu\sigma}) = 0. \tag{37}$$

But, we have the algebraic identity (cf.[1])

$$\varepsilon_{\nu\sigma} + \eta_{\nu\sigma} - \chi_{\nu\sigma} \equiv 0,$$

then the skew part of the field equation is satisfied identically and the suggested theory represents pure gravity. This point will be discussed in Section 6.

4.2 Linearization Scheme

In order to gain more clear physical information about the present theory, one has to compare it with linear field theories. Since the present theory is highly non-linear in the general potential $\underset{i}{\lambda}_{\mu}$, one has to linearize its equations. This can be done using the following linearization scheme [21], [41]. Let the generalized potential be written as

$$\underset{i}{\lambda}_{\mu} = \delta_{i\mu} + \epsilon\, \underset{i}{h}_{\mu}, \tag{38}$$

where 'ϵ' is a small parameter compared to unity and $\underset{i}{h}_{\mu}$ are functions of the coordinates. Using the definitions of the geometric objects given in Section 2 and the form (38), we can expand all such objects in terms of the parameter 'ϵ' as follows

$$Q = \epsilon^0 \overset{(0)}{Q} + \epsilon^1 \overset{(1)}{Q} + \epsilon^2 \overset{(2)}{Q} + \cdots\cdots\cdots\,, \tag{39}$$

where Q is any geometric object defined from the BB of geometry used. A list of geometric objects expanded as (39), is tabulated in Table **A**.2 in the appendix. Consequently, the metric tensor $g_{\mu\nu}$ (cf.[41]) will take the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon y_{\mu\nu} + \epsilon^2 \eta_{ij}\, \underset{i}{h}_{\mu}\, \underset{j}{h}_{\nu}, \tag{40}$$

where

$$y_{\mu\nu} \overset{\text{def}}{=} \eta_{i\nu}\, \underset{i}{h}_{\mu} + \eta_{\mu j}\, \underset{j}{h}_{\nu}. \tag{41}$$

Also, the linearized form of the determinant λ (cf.[29]) can be written as:

$$\lambda = 1 + \epsilon(\underset{0}{h_0} + \underset{1}{h_1} + \underset{2}{h_2} + \underset{3}{h_3}). \tag{42}$$

The linearized form of the symmetric part (34) may be written as:

$$\overset{(1)}{R}_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\overset{(1)}{R} = \overset{(1)}{\overset{*}{T}}_{\nu\sigma}, \tag{43}$$

the superscript (1) above tensors is used to indicate linearity.
Equation (43) may be written in another equivalent form as:

$$\overset{(1)}{R}_{\nu\sigma} = \overset{(1)}{\overset{*}{T}}_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\overset{(1)}{\overset{*}{T}}, \tag{44}$$

where,

$$\overset{(1)}{\overset{*}{T}} \overset{\text{def}}{=} g^{\nu\sigma}\overset{(1)}{\overset{*}{T}}_{\nu\sigma}. \tag{45}$$

For the material energy tensor which is defined in (35), we can write it, to the first order (using Table **A**.2), as

$$\overset{(1)}{\overset{*}{T}}_{\nu\sigma} = \frac{b^2}{2}(\eta_{\nu\sigma}\overset{(1)}{\theta} - \overset{(1)}{\theta}_{\nu\sigma} + \overset{(1)}{\psi}_{\nu\sigma}). \tag{46}$$

Consequently the scalar $\overset{*}{T}$ (45) has the form

$$\overset{(1)}{\overset{*}{T}} = \frac{b^2}{2}(3\overset{(1)}{\theta} + \overset{(1)}{\psi}). \tag{47}$$

Note that $\psi = -\theta$ (cf.[29]), then (47) can be written as,

$$\overset{(1)}{\overset{*}{T}} = -b^2\overset{(1)}{\psi} = b^2\overset{(1)}{\theta}. \tag{48}$$

Substitute from (46), (48) in the R.H.S. of (44) we get,

$$\overset{(1)}{\overset{*}{T}}_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\overset{(1)}{\overset{*}{T}} = \frac{b^2}{2} \quad (\overset{(1)}{\psi}_{\nu\sigma} - \overset{(1)}{\theta}_{\nu\sigma}). \tag{49}$$

The L.H.S. of (44) can be written as (cf.[4]) ;

$$\overset{(1)}{R}_{\nu\sigma} = \frac{1}{2}(y_{\nu\sigma,\alpha\alpha} + y_{\alpha\alpha,\nu\sigma} - y_{\nu\alpha,\sigma\alpha} - y_{\sigma\alpha,\nu\alpha}). \tag{50}$$

Substitute from (50), (49) into (44), we obtain

$$y_{\nu\sigma,\alpha\alpha} + y_{\alpha\alpha,\nu\sigma} - y_{\nu\alpha,\sigma\alpha} - y_{\sigma\alpha,\nu\alpha} = b^2(\overset{(1)}{\psi}_{\nu\sigma} - \overset{(1)}{\theta}_{\nu\sigma}). \tag{51}$$

For $\nu = \sigma = 0$, the above equation reduces to

$$y_{00,\alpha\alpha} = b^2(\overset{(1)}{\psi}_{00} - \overset{(1)}{\theta}_{00}),$$

i.e.

$$\left.\begin{aligned} &\Box^2 y_{00} = b^2(\overset{(1)}{\psi}_{00} - \overset{(1)}{\theta}_{00}), \\ \textit{where}, \quad &\Box^2 \overset{\text{def}}{=} \frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial(x^1)^2} - \frac{\partial^2}{\partial(x^2)^2} - \frac{\partial^2}{\partial(x^3)^2}. \end{aligned}\right\} \tag{52}$$

Taking into consideration the assumption of a static field

$$\frac{\partial y_{00}}{\partial x^0} = 0,$$

and recalling that [1],

$$\overset{(1)}{\theta}_{\nu\sigma} \overset{\text{def}}{=} \overset{(1)}{C}_{\nu|\underset{+}{\sigma}} + \overset{(1)}{C}_{\sigma|\underset{+}{\nu}} \longrightarrow \overset{(1)}{\theta}_{00} = 0,$$

then equation (52) reduces to

$$\nabla^2 y_{00} = b^2 \overset{(1)}{\psi}_{00}, \tag{53}$$

where,

$$\nabla^2 \overset{\text{def}}{=} \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2}.$$

But the classical Poisson equation (cf.[4]) for gravitational potential φ within a fluid of density ρ is given by :

$$\nabla^2 \varphi = 4\pi\gamma\rho, \tag{54}$$

where γ is Newton's gravitational constant. By comparing the two equations (53), (54) we can conclude that: y_{00} is the classical gravitational potential and ψ_{00} represents the fluid density in certain units. Consequently, we can assume that ψ_{00} is a geometric representation of fluid density in the linearized scheme. In case of $\psi_{00} = 0$ or $b = 0$ equation (53) will reduce to;

$$\nabla^2 y_{00} = 0, \tag{55}$$

which is similar to Laplace equation (cf.[4]) describing the gravitational field in empty space, (again, with y_{00} representing Newtonian potential φ).

Equation (53) shows that the component ψ_{00} has the correct geometric units of matter (energy) density. On one hand the L.H.S. of this equation has the geometric units cm^{-2} (note that the potential y_{00} is dimensionless). So, ψ_{00} would have the same geometric units. On the other hand the density $(\frac{mass}{volume})$ has physical units $gm\ cm^{-3}$, so its geometric units would be cm^{-2}, as mass is measured geometrically in cm units. Therefore, ψ_{00} has the correct units of density. Furthermore, the matching between the pure geometric relation (53) and the physical relation (54) gives a conversion between the two systems of units.

4.3 Type Analysis

This method was first suggested in 1981 [41] in order to physically classify AP-geometric structures. It has been reformulated for different geometric field theories (cf. [27, 28, 29]). Now, we are going to reformulate this method to suite the present field theory. Let us define the tensor,

$$\overset{*}{S}_{\upsilon\sigma} \overset{\text{def}}{=} 2b^2\omega_{\upsilon\sigma} - b^2\phi_{\upsilon\sigma} - b^2 g_{\nu\sigma}[\alpha + \omega]. \tag{56}$$

The expansion formula for this tensor starts with 2^{nd} order terms in 'ϵ'. In other words ,(56) has neither zero nor 1^{st} order terms in 'ϵ' (see Table A.2). So, we can rewrite (35) in the form,

$$\overset{*}{T}_{\nu\sigma} \overset{\text{def}}{=} (\frac{1}{2})\{b^2 g_{\nu\sigma}\theta + b^2\psi_{\nu\sigma} - b^2\theta_{\nu\sigma} + \overset{*}{S}_{\upsilon\sigma}\}. \tag{57}$$

Then from (30), (56) and (57) we can write the type analysis of (34) in the following table (G is a code for gravitation). The importance of this method appears clearly in applications (e.g. see Section 5).

Table 1: Type Analysis

Tensors	Physical Meaning	Code
$\overset{*}{M}{}^{\alpha}{}_{\mu\nu\sigma} = 0$	No gravitational field	$G0$
$\overset{*}{M}{}^{\alpha}{}_{\mu\nu\sigma} \neq 0, \overset{*}{T}_{\nu\sigma} = 0$	Gravitational field in empty space	GI
$\overset{*}{M}{}^{\alpha}{}_{\mu\nu\sigma} \neq 0, \overset{*}{T}_{\nu\sigma} \neq 0,$ $\overset{*}{S}_{\upsilon\sigma} = 0$	Gravitational field within a material distribution	GII
$\overset{*}{M}{}^{\alpha}{}_{\mu\nu\sigma} \neq 0, \overset{*}{T}_{\nu\sigma} \neq 0,$ $\overset{*}{S}_{\upsilon\sigma} \neq 0$	Strong gravitational field within a material distribution	$GIII$

The procedure of type analysis depends on Table **A**.2 in the Appendix.

5 Spherical Symmetric Solution

In the case of $n = 4$, the standard tetrad (BB of AP or PAP) having spherical symmetry has been constructed by Robertson [42]. It can be written in the matrix [6],

$$\underset{i}{\lambda}{}^{\mu} = \begin{pmatrix} A & Dr & 0 & 0 \\ 0 & B\sin\theta\cos\varphi & \frac{B}{r}\cos\theta\cos\varphi & \frac{-B\sin\varphi}{r\sin\theta} \\ 0 & B\sin\theta\sin\varphi & \frac{B}{r}\cos\theta\sin\varphi & \frac{B\cos\varphi}{r\sin\theta} \\ 0 & B\cos\theta & \frac{-B}{r}\sin\theta & 0 \end{pmatrix}, \tag{58}$$

(the coordinate system used: $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$).
The unknowns A, B, and D are functions of the coordinate (r) only.

[6]Rows in the matrix are characterized by $i = 0, 1, 2, 3$.

Using (5) and (58), the metric tensor of the pseudo-Riemannian space associated with (58) can written in the form

$$g_{\mu\nu} = \begin{pmatrix} \frac{(B^2-D^2r^2)}{A^2B^2} & \frac{Dr}{AB^2} & 0 & 0 \\ \frac{Dr}{AB^2} & -\frac{1}{B^2} & 0 & 0 \\ 0 & 0 & -\frac{r^2}{B^2} & 0 \\ 0 & 0 & 0 & -\frac{r^2\sin^2\theta}{B^2} \end{pmatrix}. \tag{59}$$

Using the tetrad components (58), to evaluate the tensor defined in Table **A**.1 and write the field equation (31), we get the following set of differential equations:

$$\left.\begin{array}{l} \overset{*}{\mathbf{M}}_{\mathbf{00}}: \ 2(b^2-1)\frac{(B^2-D^2r^2)}{A^2B^4r}[-4BB'(B^2-2D^2r^2)+2BB'DD'r^3-3B^2D^2r \\ -2B^2DD'r^2-B'^2r(-3B^2+5D^2r^2)+2BB''r(-B^2+D^2r^2)]=0, \\ \overset{*}{\mathbf{M}}_{\mathbf{01}}: \ 2(b^2-1)\frac{-D}{AB^4}[4BB'(B^2-2D^2r^2)-2BB'DD'r^3+3B^2D^2r \\ +2B^2DD'r^2-B'^2r(3B^2-5D^2r^2)+2BB''r(B^2-D^2r^2)]=0, \\ \overset{*}{\mathbf{M}}_{\mathbf{11}}: \ 2(b^2-1)\frac{1}{AB^4r}[2AB^3B'+3AB^2D^2r+2AB^2DD'r^2-8ABB'D^2r^2 \\ +2A'B^4-2ABB'DD'r^3+5AB'^2D^2r^3-AB^2B'^2r \\ -2ABB''D^2r^3-2A'B'B^3r]=0, \\ \overset{*}{\mathbf{M}}_{\mathbf{22}}: \ 2(b^2-1)\frac{r^2}{A^2B^4}[\frac{A^2B^3B'}{r}+A^2B^3B''+A^2B^2(3D^2+6DD'r+D'^2r^2) \\ -8A^2BB'D^2r-A^2B^2B'^2-5A^2BB'DD'r^2-2A^2BB''D^2r^2+5A^2B'^2D^2r^2 \\ +\frac{AA'B^4}{r}-4AA'B^2D^2r+3A'B'ABD^2r^2-3A'AB^2DD'r^2 \\ +(B^2-D^2r^2)(A''AB^2-2A'^2B^2)+A^2B^2DD''r^2]=0 \\ \overset{*}{\mathbf{M}}_{\mathbf{33}}=\sin^2\theta\ \overset{*}{\mathbf{M}}_{\mathbf{22}}. \end{array}\right\} \tag{60}$$

The above set of equations has many solutions, let us review and study some of these solutions:

(1) Equation $\overset{*}{M}_{01}=0$ of the set (60) is satisfied if $D(r)=0$. Consequently, substituting $D(r)=0$, in the set (60), then it will be reduced to,

$$\overset{*}{M}_{00}: \quad 4\frac{B'}{Br}-3\frac{B'^2}{B^2}+2\frac{B''}{B}=0, \tag{61.a}$$

$$\overset{*}{M}_{11}: \quad 2\frac{B'}{Br} + 2\frac{A'}{Ar} - \frac{B'^2}{B^2} - 2\frac{A'B'}{AB} = 0, \qquad (61.b)$$

$$\overset{*}{M}_{22}: \quad \frac{B'r}{B} + \frac{B''r^2}{B} - \frac{B'^2r^2}{B^2} + \frac{A'r}{A} + \frac{A''r^2}{A} - 2\frac{A'^2r^2}{A^2} = 0, \qquad (61.c)$$

$$\overset{*}{M}_{33} = \overset{*}{M}_{22} sin^2\theta. \qquad (61.d)$$

Note that the above set holds for any value of b except $b = 1$. Equation (61.a) can be written in the form,

$$\frac{d(B'B^{-\frac{3}{2}}r^2)}{dr} = 0. \qquad (62)$$

After double integration, we get,

$$B = (\frac{c_1}{2r} + c_2)^{-2}, \qquad (63)$$

where c_1 and c_2 are arbitrary constants of integration. Substituting from (63) into (61.b) and integrating we get,

$$A = c_3\left(\frac{c_2 + \frac{c_1}{2r}}{c_2 - \frac{c_1}{2r}}\right), \qquad (64)$$

where c_3 is an arbitrary constant of integration.

In order to compare the results obtained with those of GR, under similar conditions, we have to write the metric of the pseudo-Riemannian space associated with the structure (58). Using (59), (63), (64) and (5), we can write the metric of pseudo- Riemannian space, associated with the structure (58), as [7]

$$d\tau^2 = (c_3(\frac{c_2 - \frac{c_1}{2r}}{c_2 - \frac{c_1}{2r}}))^{-2}\ dt^2 - (\frac{c_1}{2r} + c_2)^4(dr^2 + r^2d\theta^2 + r^2\sin^2\theta\ d\phi^2). \quad (65)$$

Choosing $c_1 = -m,\ c_2 = 1, c_3 = 1$ then (65) will reduces to

$$d\tau^2 = (\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}})^2\ dt^2 - (1 + \frac{m}{2r})^4(dr^2 + r^2d\theta^2 + r^2\sin^2\theta\ d\phi^2), \qquad (66)$$

where τ is the proper time. The expression (66) is identical with the metric of Schwarzchild exterior field, in isotropic coordinates, in the case of GR.

[7]Note that Lorentz signature (+2) corresponds to the affine parameter (s), while (-2) corresponds to the proper time (τ). The relation between them is $ds = icd\tau$.

(2) If we assume that $A = \alpha_1, B = \alpha_2, D \neq 0$, where α_1, α_2 are non-vanishing constants, then the set (60) will reduce to

$$\left.\begin{array}{l} 2(b^2-1)(\frac{{\alpha_2}^2+D^2r^2}{{\alpha_1}^2{\alpha_2}^2 r})(-3D^2r-2DD'r^2)=0,\\ 2(b^2-1)(\frac{-D}{\alpha_1{\alpha_2}^2})(-3D^2r-2DD'r^2)=0,\\ 2(b^2-1)(\frac{1}{{\alpha_2}^2 r})(-3D^2r-2DD'r^2)=0,\\ 2(b^2-1)(\frac{r^2}{{\alpha_2}^2})(-3D^2-6DD'r-D'^2r^2-DD''r^2)=0. \end{array}\right\} \quad (67)$$

As $(b^2-1) \neq 0$,$D \neq 0$ and $r \neq 0$, then the second equation of set of equations (67) reduced to

$$3D + 2D'r = 0,$$

which has a solution $D^2 = \frac{c_4}{r^3}$, that satisfy all equations in the set (67), where c_4 is a constant of integration.
Again using (59) and $D^2 = \frac{c_4}{r^3}$ we get,

$$d\tau^2 = (\frac{1}{{\alpha_1}^2} - \frac{c_4}{{\alpha_1}^2\alpha_2^2 r})dt^2 + \frac{2}{\alpha_1\alpha_2^2}\sqrt{\frac{c_4}{r}}dtdr - \frac{1}{\alpha_2^2}dr^2 - \frac{r^2}{\alpha_2^2}d\theta^2 - \frac{r^2\sin^2\theta}{\alpha_2^2}\, d\phi^2. \quad (68)$$

To remove the cross term, we use the coordinate transformation (cf. [43])

$\hat{t} = t + h(r)$, $\quad R = \frac{r}{\alpha_2}$, $\quad$ where $\quad h(r) \stackrel{\text{def}}{=} \int \frac{\sqrt{\frac{c_4}{r}}\alpha_1}{(\alpha_2^2 - \frac{c_4}{r})}dr$,

then the metric(68) will reduce to

$$d\tau^2 = (\frac{1}{{\alpha_1}^2} - \frac{c_4}{{\alpha_1}^2\alpha_2^3 R})d\hat{t}^2 - \frac{1}{(1-\frac{c_4}{\alpha_2^3 R})}dR^2 - R^2d\theta^2 - R^2\sin^2\theta\, d\phi^2. \quad (69)$$

By substituting $\alpha_1 = 1$, $\alpha_2 = 1$ and $c_4 = 2m$ in (69), we get

$$d\tau^2 = (1 - \frac{2m}{r})d\hat{t}^2 - \frac{1}{(1-\frac{2m}{r})}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2, \quad (70)$$

by omitting the hat ($\hat{}$) in (70) then the above metric will become,

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{(1-\frac{2m}{r})}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2, \quad (71)$$

which gives rise, again, to the Schwarzchild exterior field of GR, in its standard form.

The above results imply that the present theory is reduced to GR in the special cases given above. This point is further discussed in Section 6.

6 Discussion and Concluding Remarks

1- The theory suggested in this article has the following properties:

- (i) It is a teleparallel theory since the BB of the geometry used (see note 2 of Section 2) are the vectors $\underset{i}{\lambda}$, subject to the condition

$$\underset{i}{\lambda}_{\underset{+}{\mu|\nu}} = 0,$$

giving rise to the linear connection(8), which implies teleparallelism.

- (ii) It is a pure gravity theory as shown from the analysis carried out in Section 4. This point is further supported by the solutions, obtained so far, for its field equations in the case of spherical symmetry (see the metrics (66), (71)).

- (iii) The theory re-attributes gravity to curvature, which in the present case the non-vanishing curvature (24) of the parameterized Weitzenböck connection (10). This preserves Einstein's main idea that gravity is just space-time curvature.

2- The theory is valid for any value of the parameter 'b' except for $b = 1$ (the case of conventional AP-geometry). For example, as mentioned in **Note 3** Section 2 the curvature (24) reduces to zero as $b = 1$, and consequently the Lagrangian (27) vanishes identically. As a conclusion, the theory is trivial in absolute parallelism geometry, although it is teleparallel as shown in point 1, above.

3- For $b = 0$ we get the following results:

- (i) The parameterized Weitzenböck connection (10) reduces to Levi-Civita connection.

- (ii) The identity (22) reduces to Bianchi identity, (theorem 2 in [40]).

- (iii) The Lagrangian function of the theory (25), reduces to Ricci scalar.

- (iv) The equations of motion (32) reduce to the geodesic equation (2), preserving the WEP.

- (v) The field equations (34) reduces to the Einstein field equations in free space
$$R_{\mu\nu} = 0.$$

4- The field equations of any theory written in the AP-geometry or one of its versions in four dimensions, are solved for sixteen field variable, $\underset{i}{\lambda}_{\mu}$. The field equations of the present theory (31) are sixteen in number. Six of these equations (37), are satisfied identically. So, we are left with the ten equations, (34), but with a geometric energy-momentum tensor (35).
5- The absence of any interaction other than gravity, in the present theory, is due to the identical vanishing of the skew part (37) of the field equations. This is a direct consequence of the algebraic identity between some skew tensors of Table A.1.
6- The tensors on the R.H.S. of (35), and consequently $\overset{*}{T}_{\nu\sigma}$, can not be defined using Levi-Civita connection only. They are defined using torsion (or contortion) as clear from their definitions (see Table A.1). For this reason they are attributed to physical meaning different from these of the L.H.S. of the field equations (34). In other words, the tensors in (35) represent the material distribution responsible for the gravitational field on the L.H.S. of (34).
7- Considering the " Unification Principle" mentioned at the end of Section 2, it is necessary but not sufficient principle for the goal of unifying fundamental interactions. This principle means simply that all physical entities are to be constructed using BB of the geometry used. In particular, in the present work, this principle gives the possibility of writing the material energy tensor (35) in terms of the BB of the PAP-geometry. As an important result, a geometric form of Poisson's equation (53) is derived in the Linearized regime of the theory.
8- The following table gives a brief comparison between GR, TEGR (cf. [44], [45]) and the theory suggested in the present work.

Table 2: Comparison between GR, TEGR and the present work.

Criteria	GR	TEGR	Present work
Geometry	pseudo-Riemannian	AP	PAP
BB	$g_{\mu\nu}$	$\underset{i}{\lambda}_{\mu}$	$\underset{i}{\lambda}_{\mu}$
Linear connection	Levi-Civita	Weitzenböck	parameterized Weitzenböck
Curvature	✓	✗	✓
Torsion	✗	✓	✓
Momentum-energy	phenomenological	phenomenological	geometric
Effect of torsion on motion	no effect	force	spin-torsion interaction

✓:nonvanishing, ✗:vanishing

9- In view of the structure of the theory suggested in this article, it is easy to conclude that flat space tetrad represents neither fields nor matter (energy). This is clear from equation (3 .8). The second term, on its R.H.S. represents deviation of the tetrad from its flat space values (ϵ is a small parameter). Taking $\epsilon = 0$ we get the components of the flat space tetrad (diagonal case). Substituting these components into the linear connection (10) leads to the identical vanishing of all components of this connection. Consequently, all components of its curvature and torsion identically vanish. It is easy to show, in this case, that all components of tensors listed in Table A.1 will identically vanish as they all depend on torsion (or contortion). This gives the above mentioned conclusion.

Finally, let us stress on two important points, that are usually taken into account in pure geometric field theories. The first point is that, in general, geometry has no physical meanings without a field theory. Physics is attributed to geometric objects, through the theory, satisfying certain relations or conditions. In the situation of seeking solutions for some problems using a theory, we are not obliged to get theoretical relations identical with those obtained from local laboratory physics. All what we need, in general, is that such relations reduce to local laboratory relations under certain conditions. We do not assume, a priori, that laws of nature are identical with laws of laboratory physics, laws that emerged in a narrow region of space-time. We consider the geometrization philosophy as a possible avenue for physics to be more near to nature. The second important point is that, for pure geometric field theories, field equations are just mathematical restrictions on the field variables (the BB of the geometry used). In addition we consider physical quantities to be just definitions in terms of the field variables.

Appendix A

Table A.1: Second Order World Tensors [1]

Skew-Symmetric Tensors	Symmetric Tensors
$\xi_{\mu\nu} \stackrel{\text{def}}{=} \gamma_{\mu\nu}{}^{\alpha}{}_{\stackrel{\alpha}{\vert +}}$	
$\zeta_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\ \gamma_{\mu\nu}{}^{\alpha}$	
$\eta_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\ \Lambda^{\alpha}{}_{\mu\nu}$	$\phi_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\ \Delta^{\alpha}{}_{\mu\nu}$
$\chi_{\mu\nu} \stackrel{\text{def}}{=} \Lambda^{\alpha}{}_{\mu\nu \stackrel{\alpha}{\vert +}}$	$\psi_{\mu\nu} \stackrel{\text{def}}{=} \Delta^{\alpha}{}_{\mu\nu \stackrel{\alpha}{\vert +}}$
$\varepsilon_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu \stackrel{\nu}{\vert +}} - C_{\nu \stackrel{\mu}{\vert +}}$	$\theta_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu \stackrel{\nu}{\vert +}} + C_{\nu \stackrel{\mu}{\vert +}}$
$\kappa_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}{}_{\mu\epsilon}\gamma^{\epsilon}{}_{\alpha\nu} - \gamma^{\alpha}{}_{\nu\epsilon}\gamma^{\epsilon}{}_{\alpha\mu}$	$\varpi_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}{}_{\mu\epsilon}\gamma^{\epsilon}{}_{\alpha\nu} + \gamma^{\alpha}{}_{\nu\epsilon}\gamma^{\epsilon}{}_{\alpha\mu}$
	$\omega_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\epsilon}{}_{\mu\alpha}\gamma^{\alpha}{}_{\nu\epsilon}$
	$\sigma_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\epsilon}{}_{\alpha\mu}\gamma^{\alpha}{}_{\epsilon\nu}$
	$\alpha_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu}C_{\nu}$

where,

$$\Delta^{\alpha}{}_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}{}_{\mu\nu} + \gamma^{\alpha}{}_{\nu\mu}.$$

This Table has been extracted from reference [1]. It contains all 2^{nd} order tensors associated with any AP structure or PAP structure apart from the parameter b. These tensors are of special importance in applications.

Table A.2: Expansion of the geometric objects [21].

Geometric Objects	Terms of 0-order	Terms of first order	Terms of second order	Terms of third and higher orders
$\underset{i}{\lambda}_{\mu}$	✓	✓	✗	✗
$g_{\mu\nu}$	✓	✓	✓	✗
$\underset{i}{\lambda}^{\mu}$	✓	✓	✓	✓
$g^{\mu\nu}$	✓	✓	✓	✓
$\lambda \stackrel{\text{def}}{=} \lvert \underset{i}{\lambda}_{\mu} \rvert$	✓	✓	✓	✓
$\Gamma^{\alpha}{}_{\mu\nu}$	✗	✓	✓	✓
$\{^{\alpha}_{\mu\nu}\}$	✗	✓	✓	✓
$\gamma^{\alpha}{}_{\mu\nu}$	✗	✓	✓	✓
$\nabla^{\alpha}{}_{\mu\nu}$	✗	✓	✓	✓
$\Lambda^{\alpha}{}_{\mu\nu}$	✗	✓	✓	✓
$\Delta^{\alpha}\mu\nu$	✗	✓	✓	✓
C_{α}	✗	✓	✓	✓
$\theta_{\mu\nu}$	✗	✓	✓	✓
$\psi_{\mu\nu}$	✗	✓	✓	✓
$\phi_{\mu\nu}$	✗	✗	✓	✓
$\varpi_{\mu\nu}$	✗	✗	✓	✓
$\omega_{\mu\nu}$	✗	✗	✓	✓
$\sigma_{\mu\nu}$	✗	✗	✓	✓
$\alpha_{\mu\nu}$	✗	✗	✓	✓
$\xi_{\mu\nu}$	✗	✓	✓	✓
$\chi_{\mu\nu}$	✗	✓	✓	✓
$\varepsilon_{\mu\nu}$	✗	✓	✓	✓
$\zeta_{\mu\nu}$	✗	✗	✓	✓
$\eta_{\mu\nu}$	✗	✗	✓	✓
$\kappa_{\mu\nu}$	✗	✗	✓	✓

This Table has been extracted from reference [21]. It lists the results of using the expansion formula (39) for different geometric objects. The Table is theory independent.

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