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Renormalization of Heavy-Light Bilinears and f_B for Wilson Fermions

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Abstract

The B meson decay constant can be measured on the lattice using the static effective field theory. We present the order α_S comparison of the matrix elements of heavy-light bilinears measured on the lattice using Wilson fermions to their counterparts in the continuum. The time component of the axial current determines f_B. A subtlety associated with a linear divergence in the heavy quark self-energy is discussed.

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It is presently not possible to do numerical simulations of QCD on lattices much larger than $24^3 \times 48$. Thus, requiring a spatial extent of two Fermis implies that the lattice spacing cannot be much less than $(2\mathrm{GeV})^{-1}$. Under these circumstances it is impossible to directly simulate arbitrary processes involving b quarks. However for several matrix elements, including that which determines f_B, the large rest energy of the b quark is not deposited into lighter hadrons. For these matrix elements an effective field theory has been proposed in which all the dependence on the large rest mass of the heavy quark has been removed analytically. The effective field theory is based on the zeroth order approximation in an expansion in the heavy quark's kinetic energy or spatial momentum over its rest energy, termed the $1/m$ expansion. The effective field theory action is presented in reference [1], which brings together lines developed by Eichten and Feinberg [2], Caswell and Lepage [3] and Politzer and Wise [4]. The remaining scales are then small enough that these matrix elements can be calculated reliably on the lattice.†

Since the quantity measured numerically is a matrix element of the lattice-regularized bare operator in the effective theory, the ratio of the matrix element of the lattice operator to its renormalized counterpart in the full theory must be calculated [7][8]. Since the origin of the difference in normalization is a difference in ultraviolet behavior and regulators, the comparison can be done reliably in perturbation theory. The comparison is most easily done in two steps. First the lattice operator is related to an operator in the effective field theory with a more convenient regularization and renormalization prescription. Then the operator in the effective theory is related to the operator in the full theory.

We perform this calculation for an arbitrary bilinear made out of a heavy quark treated in the static approximation and a light quark treated as a Wilson fermion. We begin by writing down a discretization of the Euclidean static effective field theory action and giving the Feynman rules for lattice perturbation theory. We then display one of the order α_S calculations in detail. The calculation uses the techniques and results developed in reference [1]. A result for the time component of the axial current, the bilinear used to determine f_B, has been obtained by a

† An alternative procedure for measuring heavy meson decay constants on the lattice [5] should be noted: Measure the meson decay constant as a function of meson mass up to meson masses for which corrections proportional to the heavy quark mass over the lattice cutoff become significant. Then extrapolate from these values using the scaling laws [4][6] for heavy-light systems to obtain the B meson decay constant.

different method by Boucaud, Lin and Pène [9]. Our results differ from theirs. We conclude by discussing a subtlety in taking the continuum limit associated with linear divergences in the self-energy of the heavy quark.

The Euclidean static effective field theory Lagrangian is [1]

$$\mathcal{L} = b^\dagger \left(i\partial_0 + gA_0 \right) b. \tag{1}$$

The two-component field b annihilates heavy quarks and $b^\dagger$ creates them. Note that a fixed momentum $(m, \mathbf{0})$ has been removed from the momentum of the heavy quark. Many discretizations of the action with the same naive continuum limit are possible. We make the following choice:

$$S = ia^3 \sum\nolimits_n \left[b^\dagger(n) \left(b(n) - U_0(n-\hat{0})^\dagger b(n-\hat{0}) \right) \right]. \tag{2}$$

This action reproduces the position space propagator previously suggested for use in lattice gauge theory calculations [10] which is being used in numerical simulations [11]. There is no doubling problem as would occur with a symmetric derivative.

The most general heavy-light bilinear in the full theory is $\bar{b}\Gamma q$. Γ is any Dirac matrix, and q is the light quark field. Parametrize Γ by two-by-two blocks:

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{3}$$

The corresponding operator in the effective theory is

$$b^\dagger (\alpha\, \beta) q. \tag{4}$$

The obvious choice of discretization of this operator is the zero distance bilinear

$$b^\dagger(n)(\alpha\, \beta) q(n). \tag{5}$$

The light quark will be treated as a Wilson fermion. In reference [12], the light quark is treated as a staggered fermion.

The free propagator in momentum space derived from the discretized Euclidean action is

$$\frac{1}{-\frac{i}{a}(e^{ip_0 a} - 1) + i\epsilon}. \tag{6}$$

Even in Euclidean space the pole prescription is necessary [1]. The interactions are obtained by writing

$$U_0(n-\hat{0}) = e^{igaA_0(n-\hat{0})}, \tag{7}$$

and expanding the exponential in a power series. Calculations to order α_S only require the first two terms in the expansion. The order g term produces the gluon-heavy quark vertices in the diagrams depicted in figures 1 and 2. The order g^2 term produces the vertex depicted in figure 3.

When fixing the relative normalization of an operator defined in two different regularizations, it is important to calculate the same unambiguous quantity in the two schemes. We will take a matrix element of the heavy-light bilinear between an incoming light quark and an outgoing heavy quark. For calculational convenience, we will set the light quark mass to zero, and take our mass shell point to be where the incoming and outgoing momenta are zero. The infrared divergences which appear at zero external momenta are eliminated by giving the gluon a mass, λ. No problem with a gluon mass arises in one loop, because the required diagrams are QED-like. For a nice review of this procedure, including a comparison of matrix elements measured on the lattice with the continuum in a case without heavy quarks where the result is known, see reference [7].

To calculate the matrix elements, we need the vertex correction graphs in the lattice effective theory, the continuum effective theory and the continuum full theory, and the self-energy graphs for the heavy quark and the light quark in the three theories. Light quark wave function renormalization is standard [13], and all the continuum computations were done in [1]. So it remains only to calculate the vertex correction graph and the heavy quark self-energy graphs on the lattice.

We simply quote the result for the vertex correction graph, fig. 1. The result is

$$\frac{g^2}{12\pi^2}\left(-\ln\lambda^2 a^2 + d\right) \tag{8}$$

times the zeroth order result. For a general bilinear, we write $d = d_1 + d_2 G$, where G is defined by $G\Gamma = \gamma_0\Gamma\gamma_0$. We used the method of isolating $a \to 0$ divergences introduced by Bernard, Soni and Draper [14], and evaluated the residual integrals for several values of the Wilson mass coefficient r with VEGAS [15]. The results for d_1 and d_2 are tabulated below.

We will evaluate the heavy quark self energy graph of figure 2 in sufficient detail to expose our method. The integral for this graph is

$$\frac{4}{3}g^2\frac{1}{a}\int\frac{d^4l}{(2\pi)^4}e^{i(p_0 a + l_0)}\frac{1}{\Delta_g(l-pa)+\lambda^2a^2}\,\frac{1}{-i(e^{il_0}-1)+i\epsilon}, \tag{9}$$

where $\Delta_g(l) = \sum_\mu 4\sin^2(l_\mu/2)$. The phases in the numerator are present because the action contains interactions of fields with one site difference in their arguments.

The integration region is $[-\pi, \pi]^4$. The external momentum is routed through the gluon propagator to avoid differentiating the non-covariant pole when obtaining wave function renormalization. The $\mathbf{p}$ dependence is illusory.

To obtain the contribution to wave function renormalization from the graph in figure 2, take the derivative of the self-energy correction, equation (9), with respect to p_0 at zero momentum. The result from differentiating $e^{ip_0 a}$ is

$$\frac{4}{3} i g^2 \int \frac{d^4 l}{(2\pi)^4} e^{il_0} \frac{1}{\Delta_g(l) + \lambda^2 a^2} \frac{1}{-i(e^{il_0} - 1) + i\epsilon}. \tag{10}$$

Now replace the gluon propagator in this result by

$$\frac{1}{\Delta_g(l) + \lambda^2 a^2} - \frac{1}{\Delta_g(0, \mathbf{l}) + \lambda^2 a^2} + \frac{1}{\Delta_g(0, \mathbf{l}) + \lambda^2 a^2}. \tag{11}$$

The term we have subtracted and added has been chosen to isolate the noncovariant pole and for ease of integration. In the difference, we don't have to worry about the $i\epsilon$ prescription. We simplify by symmetrizing under $l_0 \to -l_0$. No singularities are encountered as $\lambda \to 0$, and we obtain integrals easy to evaluate numerically[15]. The term added back in factors into an l_0 integration and an $\mathbf{l}$ integration. The l_0 integration is shown to vanish by contour integration. There is still a contribution to wave function renormalization obtained from differentiating the gluon propagator with respect to p_0. After differentiation, it is seen that this term has a factor of $\sin l_0/2$ in the numerator, so the $i\epsilon$ prescription is unnecessary. One can extract the dependence on a as $a \to 0$ easily, again leaving integrals to evaluate numerically.

The other graph contributing to heavy quark wave function renormalization, figure 3, is easy enough that we present it's evaluation completely. The integral is

$$-\frac{2}{3} i g^2 \frac{1}{a} e^{ip_0 a} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{\Delta_g(l)}. \tag{12}$$

Take the derivative with respect to p_0 at zero momentum to isolate the contribution to wave function renormalization. We obtain

$$\frac{2}{3} g^2 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{\Delta_g(l)}. \tag{13}$$

The result of the numerical evaluation of this integral is $g^2/12\pi^2$ times 12.23. The spherical symmetry of continuum integrals makes one expect results of $g^2/12\pi^2$ times a number of order one. However in lattice perturbation theory there is no spherical symmetry; integrands are order one right to the edges of the Brillouin zone, and tadpole graphs frequently give such large coefficients.

We summarize the heavy quark self energy results by writing the wave function renormalization as

$$\frac{g^2}{12\pi^2}\Big(-2\ln\lambda^2a^2+e\Big). \tag{14}$$

The full result from both graphs is $e = 24.48$.

The light quark self energy graphs which are required for the matrix element are standard [13]. We parametrize them as

$$\frac{g^2}{12\pi^2}\Big(\ln\lambda^2a^2+f\Big). \tag{15}$$

The numerically evaluated constant f is also tabulated for several values of r. Errors for d_1, d_2, e and f are at most $\mathcal{O}(1)$ in the last decimal place.

r	d_1	d_2	f
1.00	5.46	−7.22	13.35
0.75	5.76	−7.23	11.96
0.50	6.30	−7.00	10.22
0.25	7.37	−5.72	8.07
0.00	8.79	0.00	6.54

The corresponding graphs in the continuum effective field theory with dimensional regularization and $\overline{\mathrm{MS}}$ with a scale μ have the same form as expressions (8), (14), and (15). The only changes are that $1/\mu$ replaces a and the constant term differs. The constant corresponding to d we call D, and the constant corresponding to e is E. In reference [1], $D = 1$ and $E = 0$. We find the constant corresponding to f, F, is $1/2$.

To order α_S the ratio of a matrix element of a bilinear in the lattice regulated static effective field theory to that in the dimensionally regularized effective theory is then

$$1+\frac{g^2}{12\pi^2}\left[d+\frac{1}{2}e+\frac{1}{2}f-\Big(D+\frac{1}{2}E+\frac{1}{2}F\Big)\right]. \tag{16}$$

The ratio of the operators could not depend on λ since the lambda dependence must be the same for the two different regulators. Similarly, if we hadn't set the light quark mass to zero, dependence on it would have had to drop out of the ratio. Thus the result only depends on the scales μ and a, it is dimensionless, and the dependence on μa has been eliminated by setting $\mu = 1/a$.

To use (16) to get the ratio of the matrix element of the the bilinear in the lattice regulated effective theory to the $\overline{\mathrm{MS}}$ subtracted bilinear in the full theory, we need the ratio of the $\overline{\mathrm{MS}}$ subtracted bilinear in the continuum effective theory to the $\overline{\mathrm{MS}}$ subtracted bilinear in the full theory calculated in [1]. It is

$$1-\frac{g^2}{12\pi^2}\left(C_1\ln\frac{m^2}{\mu^2}+C_2\right), \tag{17}$$

where $C_1=5/2-H^2/4$ and $C_2=-4+3H^2/4-HH'-GH/2$. H is defined by $H\Gamma=\gamma_\mu\Gamma\gamma_\mu$, and H' is the derivative with respect to d of H in d dimensions. For the case of the time component of the axial current, the bilinear used to determine f_B, $G=-1$ and $H=2$. If we use the extension of the gamma matrix algebra that γ_5 anticommutes with all the γ_μ, $1\le\mu\le d$, then $H'=1$. So in this case, we have $C_1=3/2$ and $C_2=-2$. These two numbers can be extracted from equations (2.10) and (2.30) of Boucaud, Lin and Pène [9], and we agree.

Continuing the comparison for this special case, we find from the table above for $r=1$ that $d=12.68$, $e=24.48$ and $f=13.35$. Boucaud Lin and Péne [9] found $2(d-D)/3=2.97$, $(e-E)/3=0.17$ and $(f-F)/3=4.29$. Our results for these combinations are 7.79, 8.16 and 4.28 respectively. Only the result for f agrees. Evaluating expression (16) for $\beta\equiv 6/g^2=6.0$ using our results gives 1.26. Using the continuum value of α_S with $\Lambda_{QCD}=250\,\mathrm{MeV}$ for four active quarks, $\mu=2\,\mathrm{GeV}$, and $m=5\,\mathrm{GeV}$ we find that expression (17) is 0.97. Taking the product of the two ratios, we find that the value of f_B measured on a $2\,\mathrm{GeV}$ lattice using the static approximation should be reduced by a factor of 1.22. The largest source of uncertainty in this result is the choice of the value of α_S to use in the continuum effective theory to lattice effective theory matching. At $\mu=2\,\mathrm{GeV}$, the lattice value of α_S is smaller than the continuum value by a factor of 2.7 [7]. Using the continuum value of α_S in this matching would thus result in a substantially larger reduction factor for f_B.

One part of the discrepancy in e between this work and the previous work brings up an interesting subtlety which we will explore and resolve in the rest of this paper. Define $G_B(n_0)$ by

$$\frac{1}{a^3}\delta_{\mathbf{n},0}G_B(n_0)=\langle J_B^\dagger(n)J_B(0)\rangle, \tag{18}$$

where J_B denotes the discretization, (5), of the time component of the axial current. In their position space calculation, Boucaud, Lin and Pène [9] found that the tadpole graph gives a contribution to (18) proportional to n_0 times the $O(g^0)$ result, which leads them to conclude that the tadpole graph only contributes to mass renormalization and not to wave function renormalization (see the text below equation (C22) of their paper). However, if the action is modified to include mass and wave function renormalization

$$S_R = ia^3 \sum_n \left[Z\, b^\dagger(n) \left(b(n) - U_0(n-\hat{0})^\dagger b(n-\hat{0}) \right) + a\, \delta m\, b^\dagger(n) b(n) \right] \qquad (19)$$

one finds that the correction of order δm is proportional to n_0+1, so the tadpole graph cannot be completely absorbed by mass renormalization. While the difference between n_0 and n_0+1 is naively irrelevant in the continuum limit, the tadpole graph is linearly divergent, and there is a finite contribution to wave function renormalization, as we found in equation (13).

This linear divergence requires the reconsideration of other usual assumptions. In particular, Boucaud and Pène [16] raise the issue of whether f_B should be extracted from numerical simulations by fitting $G_B(n_0)$ to

$$A\mathrm{e}^{-Bn_0 a}, \qquad (20)$$

or to

$$A\mathrm{e}^{-B(n_0+1)a}, \qquad (21)$$

when using the usual relationship (27) between f_B and A. Again, the difference is naively irrelevant, but since B contains linear divergences, there is a finite difference between the two definitions of A. Now that the issue has been exposed, we demonstrate how to resolve it.

What is measured on the lattice is the correlator of bare operators, $G_B(n_0)$ defined in equation (18), using the action (2), which has not been tuned to give cutoff independent Green's functions as the cutoff goes to infinity. Let the renormalized current $J(n)$ be defined by $Z_J J(n) = J_B(n)$, and let the correlator of it calculated using the renormalized action (19) be $G(n_0)$. Z_J is adjusted so that matrix elements of $J(n)$ agree with matrix elements of the current in the full theory in the continuum renormalized using $\overline{\mathrm{MS}}$. To order α_S, Z_J is the product of (16) and (17). It is $G(n_0)$ for which we can show by inserting a complete set of intermediate states that in the limit $a \to 0$, na fixed,

$$G(n_0) = \frac{(f_B m_B)^2}{2m_B} \mathrm{e}^{-E_B n_0 a}. \qquad (22)$$

E_B is the energy (not including the rest energy of the heavy quark) of the B meson. Here, whether n_0 or n_0+1 appears in the exponential is irrelevant since E_B is finite as $a \to 0$. In fact, this could be taken to be the renormalization condition determining δm.

We can obtain the relation between $G(n_0)$ and $G_B(n_0)$ exactly. First, we give a functional integral expression for $G(n_0)$, suppressing the gauge and light fermion integrations:

$$\begin{aligned}\frac{1}{a^3}\delta_{\mathbf{n},0}G(n_0) &= \frac{\int (db)(db^\dagger)J^\dagger(n)J(0)\mathrm{e}^{-S_R}}{\int (db)(db^\dagger)\mathrm{e}^{-S_R}} \\ &= Z_J^{-2}\frac{\int (db)(db^\dagger)J_B^\dagger(n)J_B(0)\mathrm{e}^{-S_B}}{\int (db)(db^\dagger)\mathrm{e}^{-S_B}},\end{aligned} \tag{23}$$

where we have introduced

$$S_B = ia^3\sum\nolimits_n \Big[b^\dagger(n)\,\Big(b(n)-U_0(n-\hat{0})^\dagger b(n-\hat{0})\Big)+(a\,\delta m/Z)\,b^\dagger(n)b(n)\Big]. \tag{24}$$

S_B is the renormalized action rewritten in terms of the bare fields.

Because this functional integral is essentially one-dimensional, one can relate it to the functional integral with the action S, defined in equation (2), which reproduces the propagator being used in numerical simulations. One finds

$$\begin{aligned}\frac{1}{a^3}\delta_{\mathbf{n},0}G(n_0) &= Z_J^{-2}\left(\frac{1}{1+a\,\delta m/Z}\right)^{n_0+1}\frac{\int (db)(db^\dagger)J_B^\dagger(n)J_B(0)\mathrm{e}^{-S}}{\int (db)(db^\dagger)\mathrm{e}^{-S}} \\ &= Z_J^{-2}\left(\frac{1}{1+a\,\delta m/Z}\right)^{n_0+1}\frac{1}{a^3}\delta_{\mathbf{n},0}G_B(n,0).\end{aligned} \tag{25}$$

Now substitute expression (21) for $G_B(n_0)$ and expression (22) for $G(n_0)$ to get

$$\frac{(f_Bm_B)^2}{2m_B}\mathrm{e}^{-E_Bn_0a} = Z_J^{-2}\left(\frac{1}{1+a\,\delta m/Z}\right)^{n_0+1}A\mathrm{e}^{-B(n_0+1)}. \tag{26}$$

The constants Z and δm are chosen so that the right hand side has a cutoff independent limit as $n\to\infty$, na fixed. This implies $B+\ln(1+a\,\delta m/Z)$ is order a. Call the coefficient of the order a term h. Then what we find in the limit is $E_B = h$ and

$$\frac{(f_Bm_B)^2}{2m_B} = Z_J^{-2}A. \tag{27}$$

This is the equation usually used to report f_B [11] and concludes our argument for using (21).

It could be argued that omissions of the form coming from the tadpole graph are compensated by the way f_B is normally extracted from numerical simulations.

That is, equation (20) has been used to fit $G_B(n_0)$. So we note that in that case instead of (27), the result would have been

$$\frac{(f_B m_B)^2}{2m_B} = Z_J^{-2} \frac{1}{1 + a\,\delta m/Z} A. \tag{28}$$

There is a residue from the cancellation of the fraction and the exponential, and because δm contains linear divergences this results in a different value of f_B.

If one makes the error in analysis of using (20) with (27), it is clear from (28) that it can be compensated for in perturbation theory by reducing the value of e by an amount taken from the linearly divergent part of mass renormalization. Because contributions of the type coming from the tadpole graph contribute the same way to Z_J^{-2} and $a\,\delta m$, part of the effect is to eliminate the contribution of the tadpole graph. We find that the sum of the one-loop contributions to the self-energy which must be cancelled by δm is

$$\frac{g^2}{12\pi^2} \frac{1}{a} 19.95. \tag{29}$$

12.23 of the constant came from the tadpole graph, and 7.72 from the graph of figure 2. The reduced value of e is 4.53, giving $(e-E)/3 = 1.51$, which comes considerably closer to Boucaud, Lin and Pène's result. If we use the reduced value for e, and the same sample values for μ and Λ_{QCD} as we did previously, we find that the value of f_B measured on the lattice should be divided by 1.14. Boucaud, Lin and Pène's values of d, e and f give 1.06. As noted previously, use of the continuum value of α_S would substantially increase this.

The clear formulation of the static approximation we have used has allowed for a simple derivation of the normalization of an arbitrary heavy-light bilinear measured on the lattice. The discrepancy with previous work has uncovered a subtlety in the extraction of f_B from lattice calculations arising from linear divergences in the self-energy of the heavy quark. However, even accounting for this effect we are unable to reproduce the results of Boucaud, Lin, and Pène [9]. The formulation employed here will simplify calculations of the renormalization of other operators and make the calculation of $1/m$ and $O(a)$ corrections tractable.

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Figure Captions

Fig. 1: Vertex Correction Graph

Fig. 2: Self-Energy Correction

Fig. 3: Tadpole Self-Energy Correction