



Contact geometry and quantum mechanics

Gabriel Herczeg^a, Andrew Waldron^{b,*}

^a Department of Physics, University of California, Davis, CA 95616, USA

^b Center for Quantum Mathematics and Physics (QMAP), Department of Mathematics, University of California, Davis, CA 95616, USA

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ABSTRACT

We present a generally covariant approach to quantum mechanics in which generalized positions, momenta and *time* variables are treated as coordinates on a fundamental “phase-spacetime”. We show that this covariant starting point makes quantization into a purely geometric flatness condition. This makes quantum mechanics purely geometric, and possibly even topological. Our approach is especially useful for time-dependent problems and systems subject to ambiguities in choices of clock or observer. As a byproduct, we give a derivation and generalization of the Wigner functions of standard quantum mechanics.

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1. Contact geometry

Mechanics is usually formulated in terms of an even $2n$ -dimensional phase-space (or symplectic manifold) with time treated as an external parameter and dynamics determined by a choice of Hamiltonian. Yet classical physics ought not depend on choices of clocks. However, Einstein’s principle of general covariance can be applied to this situation by introducing an odd $(2n + 1)$ -dimensional phase-spacetime manifold Z . Dynamics is now encoded by giving Z a (strict) contact structure—*i.e.*, a one-form α subject to a non-degeneracy condition on the (phase-spacetime) volume form:

$$\text{Vol}_\alpha := \alpha \wedge (d\alpha)^{\wedge n} \neq 0. \quad (1.1)$$

Physical phase-spacetime trajectories γ are determined by extremizing the action

$$S = \int_\gamma \alpha. \quad (1.2)$$

Since the integral of a one-form along a path γ is a coordinate invariant quantity, general covariance (both worldline and target space) is built in from the beginning [1]. The equations of motion are

$$\varphi(\dot{\gamma}, \cdot) = 0, \quad (1.3)$$

where the two-form $\varphi := d\alpha$ is maximal rank by virtue of Eq. (1.1) and $\dot{\gamma}$ is a tangent vector to the path γ in Z .

The structure (Z, α) is called a (strict) contact geometry and Eq. (1.3) determines its Reeb dynamics [2]. In addition to general covariance, this formulation of mechanics enjoys a Darboux theorem, which implies the existence of *local* coordinates (ψ, π_A, χ^A) such that $\alpha = \pi_A d\chi^A - d\psi$ (where $A = 1, \dots, n$) that trivialize the dynamics. Hence one might hope to treat classical *and* quantum mechanics as contact topology problems.

2. Goal

We aim to develop a generally phase-spacetime covariant formulation of quantum mechanics. We find a formulation of quantum mechanics in terms of intrinsic geometric structures on a contact manifold. Our approach is similar to Fedosov’s quantization of symplectic manifolds [3], and indeed we were partly inspired by that work and subsequent applications of Fedosov quantization to models of higher spins [4]. Quantization based on contact geometry has been studied before: For example, Rajeev [5] considers quantization beginning with (classical) Lagrange brackets (the contact analog of Poisson brackets). Fitzpatrick [6] has extended this work to a rigorous geometric quantization setting. There is also earlier work by Kashiwara [7] that studies sheaves of pseudodifferential operators over contact manifolds. Investigations motivated by quantum cosmology of the so-called “clock ambiguity” in the quantum dynamics of time reparameterization invariant theories may be found in [8]. Contact geometry has also been employed in studies of choices of quantum clocks in [9].

* Corresponding author.

E-mail addresses: Herczeg@ms.physics.ucdavis.edu (G. Herczeg), wally@math.ucdavis.edu (A. Waldron).

3. BRST analysis

Because it is worldline diffeomorphism invariant, the system with action (1.2) has one first class constraint. From the Darboux expression for the contact form α we see that there are also $2n$ second class constraints (the canonical momenta for the coordinates χ^A are constrained to equal the coordinates π_A). The quantization of constrained systems is well understood, thanks to the seminal work of Becchi, Rouet, Stora and Tyutin (BRST) [10]. We employ the Hamiltonian BRST technology of Batalin, Fradkin and Vilkovisky (BFV) [11] as well as its extension to systems with second class constraints [12]:

Let z^i be phase-spacetime coordinates and introduce canonical momenta p_i with Poisson brackets

$$\{z^i, p_j\}_{\text{PB}} = \delta^i_j.$$

The second class constraints are

$$C_i = p_i - \alpha_i,$$

where $\alpha = \alpha_i dz^i$, $\varphi = \frac{1}{2} \varphi_{ij} dz^i \wedge dz^j$, and

$$\{C_i, C_j\}_{\text{PB}} = \varphi_{ij}.$$

Second class constraints require Dirac brackets; alternatively one may introduce $2n$ new variables s^a with Poisson brackets

$$\{s^a, s^b\}_{\text{PB}} = J^{ab},$$

where J is a constant, maximal rank, $2n \times 2n$ matrix [12]. At least locally, we can introduce $2n$ linearly independent soldering forms e^a (analogous to the vielbeine/tetrads of general relativity) such that

$$\varphi = \frac{1}{2} J_{ab} e^a \wedge e^b,$$

and $J_{ab} J^{bc} = \delta_a^c$. In these terms our system is now described by an extended action functional subject only to $2n + 1$ first class constraints:

$$S_{\text{ext}}[z(\tau), s(\tau)] = \int \left[\frac{1}{2} s^a J_{ab} \dot{s}^b + \dot{z}^i (\alpha_i(z) + s^a J_{ab} e_i^b(z) + \omega_i(z, s)) \right] d\tau. \quad (3.1)$$

In the above, τ is an arbitrary choice of worldline parameter, and the s -dependent one-form $\omega(z, s)$ on Z must be chosen to obey¹

$$d\Omega + \frac{1}{2} \{\Omega \wedge \Omega\}_{\text{PB}} = 0,$$

where $\Omega = \alpha + s^a J_{ab} e^b + \omega$, in order that the extended constraints $C_i^{\text{ext}} = p_i - \Omega_i$ are first class. Locally, the Darboux theorem implies that a set of one-forms e^a with a flat connection exists.

The gauge invariances

$$\delta z^i = \varepsilon^i(\tau), \quad \delta s^a = \varepsilon^i(\tau) J^{ab} \frac{\partial \Omega_i}{\partial s^b},$$

ensure² that the equations of motion

$$J_{ab} \dot{s}^b + \dot{z}^i \frac{\partial \Omega_i}{\partial s^a} = 0 = \dot{z}^i (\partial_i \Omega_j - \partial_j \Omega_i) - \dot{s}^a \frac{\partial \Omega_j}{\partial s^a},$$

are equivalent to Reeb dynamics.

¹ Note that $\{\Omega \wedge \Omega\}_{\text{PB}} := dz^i \wedge dz^j \{\Omega_i, \Omega_j\}_{\text{PB}}$. In related work, the authors of [14] have constructed a flat Cartan Maurer connection from a central extension of the group of canonical transformations.

² Here we assume that the rectangular matrix $\frac{\partial \Omega_i}{\partial s^a}$ has maximal rank, which is guaranteed at least in a neighborhood of $s = 0$.

Now that we are dealing with a first class constrained system, the BFV quantum action follows directly

$$S_{\text{qu}} = \int [\Theta + \{Q, \Phi\}_{\text{PB}}].$$

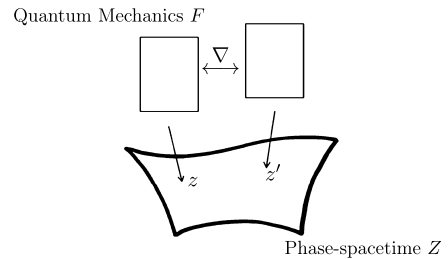
Here Φ is the gauge fixing fermion for some choice of gauge and Θ is the BRST-extended symplectic current

$$\Theta = p_i \dot{z}^i + \frac{1}{2} s^a J_{ab} \dot{s}^b + b_i \dot{c}^i,$$

where (b_i, c^i) are canonically conjugate Grassmann ghosts. The BRST charge $Q = c^i C_i^{\text{ext}}$ is determined by the first class constraints.

4. Quantization

We are now ready to quantize the contact formulation of classical mechanics. The physical picture underlying our method closely mimics general relativity: Spinors in curved space are described by gluing a copy of a flat space Clifford algebra and its spin representation to each point in spacetime using vielbeine and the spin connection to compare spinors at differing spacetime points. Mathematically, this is an example of a vector bundle in which context vielbeine are called soldering forms. Here we want to glue a copy of standard quantum mechanics to each point z in the phase-spacetime Z , which we view as the fibers of a suitable vector bundle, and then construct a connection ∇ to compare differing fibers, as depicted below:



In this picture, quantum mechanics along the fibers is described in terms of the variables s^a which are quantized in the standard way by choosing some polarization in which

$$\hat{s}^a = \left(S^A, \frac{\hbar}{i} \frac{\partial}{\partial S^A} \right).$$

Quantum wavefunctions $\Psi(S^A)$ depend on half the s -variables S^A spanning \mathbb{R}^n , and the inner product is the usual one: $\langle \Psi, \Psi' \rangle = \int_{\mathbb{R}^n} \Psi^* \Psi'$. The ‘‘Schrödinger equation’’ along each fiber as well as the parallel transport of quantum mechanics from fiber to fiber is controlled by the connection ∇ given by the quantum BRST charge \hat{Q} . To compute this connection, we quantize the contact coordinates z^i and their momenta using the polarization

$$\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial z^i}.$$

In addition, we identify the Grassmann ghosts c^i with a basis of one-forms dz^i along Z . Hence BRST wavefunctions now depend on (z^i, dz^i, S^A) and may be viewed as differential forms on the contact manifold Z taking values in the Hilbert space $L^2(\mathbb{R}^n)$. The quantum BRST charge $\hat{Q} = \frac{\hbar}{i} \nabla$ where ∇ is the operator-valued connection³

³ In fact, exactly such a connection over a symplectic manifold has been introduced in [13].

$$\nabla = d - \frac{i}{\hbar} \widehat{\Omega}(z, \hat{s}).$$

Here the operator-valued one-form

$$\widehat{\Omega} = \alpha + e_a \hat{s}^a + \widehat{\omega}(z, \hat{s}),$$

where $\widehat{\omega}$ is an expansion in two and higher powers of \hat{s} . By nilpotence of \widehat{Q} , the operator $\widehat{\Omega}$ is solved for by requiring that the connection ∇ is flat:

$$\nabla^2 = 0.$$

Again, existence of a solution on a local patch of Z is guaranteed by the Darboux theorem.

Physical quantum states are given by the BRST cohomology at ghost number zero, *i.e.*, zero-forms $\Psi(z^i, S^A)$ subject to the parallel transport condition

$$\nabla \Psi(z^i, S^A) = 0.$$

We then search for solutions labeled by a set of quantum numbers \mathcal{E} such that at each point $z \in Z$, $\Psi_{\mathcal{E}}(z, S)$ are complete and orthonormal:

$$\int \Psi_{\mathcal{E}'}^*(z, S) \Psi_{\mathcal{E}}(z, S) d^n S = \delta_{\mathcal{E}, \mathcal{E}'}. \tag{4.1}$$

Explicit solutions to this condition for a broad class of models are given at the end of this letter.

Finally we are ready to build a set of generalized (contact covariant, non-diagonal) Wigner distributions encoding the quantum mechanical system: In a ket notation, we may think of the wavefunction $\Psi_{\mathcal{E}}(z, S)$ as a state $|\mathcal{E}; z\rangle$ labeled by quantum numbers \mathcal{E} and indexed by control parameters z given by points in the contact manifold Z . In this notation, the display (4.1) reads $\langle \mathcal{E}'; z | \mathcal{E}; z \rangle = \delta_{\mathcal{E}, \mathcal{E}'}$. Then the generalized Wigner distribution is given by

$$\mathcal{W}_{\mathcal{E}, \mathcal{E}'}(z, z') := \langle \mathcal{E}'; z' | \mathcal{E}; z \rangle.$$

The above expression can also be viewed as a two-point correlator for a field theory on the contact manifold.

The Wigner distribution can be interpreted physically as follows. The control parameters z^i correspond to values of dials, knobs, meters *and* clocks in the classical laboratory, while \mathcal{E} labels a quantum state prepared by the experimenter. Note that the parameters z are mutually commuting variables consistent with the phase space dependence of quantum mechanical Wigner functions. The quantity $|\mathcal{W}_{\mathcal{E}, \mathcal{E}'}(z, z')|^2$ then measures the probability of observing the state \mathcal{E}' given the state \mathcal{E} was initially prepared and the classical laboratory has “evolved” in control parameter space from z to z' .

5. Example: Hamiltonian mechanics

Consider a phase-spacetime with coordinates (q^A, p_A, t) and contact form

$$\alpha = p_A dq^A - H(p, q, t) dt.$$

Choosing the worldline parameterization $t(\tau) = \tau$, the action principle of Eq. (1.2) then gives Hamilton’s equations for (q^A, p_A) with Hamiltonian $H(p, q, t)$. Here

$$\varphi = d\alpha = e_A \wedge f^A$$

with

$$e_A = dp_A + \frac{\partial H}{\partial q^A} dt, \quad f^A = dq^A - \frac{\partial H}{\partial p_A} dt.$$

It is not difficult to formally solve order by order in the fiber operators $\hat{s}^a = (S^A, \frac{\hbar}{i} \frac{\partial}{\partial S^A})$ for the operator-valued one-form $\widehat{\Omega}$ and find a flat connection

$$\nabla = d - \frac{i}{\hbar} \alpha + \frac{i}{\hbar} e_A S^A - f^A \frac{\partial}{\partial S^A} + \frac{i}{\hbar} dt \sum_{\sigma \geq 2} \frac{1}{\sigma!} \frac{\partial^\sigma H(Z, t)}{\partial Z^{a_1} \dots \partial Z^{a_\sigma}} \hat{s}^{a_1} \dots \hat{s}^{a_\sigma},$$

where $Z^a := (q^A, p_A)$.

The “contact Schrödinger equation” $\frac{\hbar}{i} \nabla \Psi = 0$ now gives a triplet of equations

$$\begin{cases} \frac{\hbar}{i} \left[\frac{\partial}{\partial q^A} - \frac{\partial}{\partial S^A} \right] \Psi - p_A \Psi = 0, \\ \frac{\hbar}{i} \frac{\partial \Psi}{\partial p_A} + S^A \Psi = 0, \\ \frac{\hbar}{i} \frac{\partial \Psi}{\partial t} + i \sum_{\sigma \geq 0} \frac{1}{\sigma!} \frac{\partial H}{\partial Z^{a_1} \dots \partial Z^{a_\sigma}} \hat{s}^{a_1} \dots \hat{s}^{a_\sigma} \Psi = 0. \end{cases}$$

The first two of these equations are solved via

$$\Psi(q, p, t, s) = e^{-\frac{i}{\hbar} p_A S^A} \psi(Q^A, t),$$

where $Q^A = q^A + S^A$. The summation in the third equation can be performed and gives the operator $\widehat{H} = (H(Q, p + \frac{\hbar}{i} \frac{\partial}{\partial S}))_{\text{Weyl}}$, where the symbols Q^A and $\partial/\partial S^A$ are Weyl ordered. Thus $\psi(Q^A, t)$ obeys the time dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(Q, t)}{\partial t} = \left[H\left(Q, \frac{\hbar}{i} \frac{\partial}{\partial Q}\right) \right]_{\text{Weyl}} \psi(Q, t).$$

Focusing on the case where \widehat{H} is time-independent, the analysis is now standard: A complete orthonormal set of states $\{\psi_E(Q, t) = e^{-\frac{i}{\hbar} E t} \psi_E(Q) | E \in \text{spec}(\widehat{H})\}$ is labeled by energies E (up to degeneracies). At equal values of $z = (q, p, t)$, the inner product of solutions yields

$$\mathcal{W}_{E, E'}(z, z) = \delta_{E, E'},$$

in agreement with Eq. (4.1), while at unequal values of the control parameters $z \neq z'$ but equal quantum numbers $E = E'$,

$$\mathcal{W}_{E, E}(z', z) = e^{\frac{i}{\hbar} [E \delta t - \delta p_A \bar{q}^A]} \int \frac{d^n S d^n P}{(2\pi \hbar^2)^n} e^{\frac{i}{\hbar} P_A \delta q^A} W_E(S, P),$$

with $\bar{q} := (q + q')/2$, $\delta p = p' - p$ and $\delta t = t' - t$. Here W_E is the Wigner function

$$W_E(S, P) = \int d^n y e^{\frac{i}{\hbar} P_A y^A} \psi_E^*(S - \frac{y}{2}) \psi_E(S + \frac{y}{2}).$$

This is the fundamental building block of the phase space formulation of quantum mechanics [15].

6. Summary and discussion

By BRST quantizing classical mechanics described in terms of contact geometry, we have reformulated quantum mechanics as the parallel transport equation of a flat connection on a vector bundle over phase-spacetime. This implies that we have turned quantum mechanics into cohomology. Our approach has a simple geometric interpretation: (i) We maintain general covariance with

respect to phase-space and time coordinates at all junctures, and (ii) we compare standard quantization at fixed phase-spacetime points using a connection, just as the Levi-Civita spin connection compares vectors, spinors *etc.*, from spacetime point to spacetime point. From a bundle viewpoint, this means that we quantize along fibers and compute inner products fiber-wise. Correlators are co-variantly labeled by pairs of phase-spacetime points. This provides a derivation and generalization of the Wigner functions which are usually postulated in standard quantum mechanics.

Our approach is intrinsic to the data of a strict contact manifold, which is necessarily locally trivial. This raises the tantalizing possibility that both classical and quantum dynamics can be completely described in terms of the topological data of vector bundles over contact manifolds.

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