# ASYMPTOTIC LIMITS IN POTENTIAL SCATTERING AND IN FIELD THEORY

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## INTRODUCTION

A great deal of interest, both theoretical and experimental, has been devoted recently to the study of high-energy interactions. This interest has been greatly stimulated by the valuable application of the results obtained by Regge in potential scattering to high-energy elastic scattering. This paper will discuss an approach to high-energy physics which is to some extent complementary to the one based on Regge poles and has the following features:

(1) It is based on a model for high-energy interactions obtained as the natural generalization of the peripheral model which has been successful in understanding many features of interactions between $1 \sim 3$ GeV. This model describes the high-energy collisions as the result of the combination of a large number of low-energy interactions.

(2) The techniques used to evaluate the asymptotic limits are based on the direct study of the linear integral equations of the model and do not involve the use of analytic continuation in the angular momentum explicitly.

(3) The model allows an estimate of the main features both of elastic diffraction scattering and of multiple production. The results concerning elastic scattering are closely analogous to the ones obtained on the basis of Regge poles.

In order to explain the mathematical techniques in the simplest possible manner we have discussed in detail (Section 1) their application to potential scattering which is a very useful "laboratory" for theoretical physicists. Section 2 deals with the prediction of the model for elastic diffraction scattering and with the relativistic two-body equation. Finally in section 3 the different predictions concerning multiple production are discussed.

The paper by Stroffolini* is complementary to the present one since the mathematical techniques which can be used in order to evaluate the different "trajectories" both in potential scattering and in field theory are discussed there.

## 1. POTENTIAL SCATTERING

We shall consider the case of potential scattering first. Consider the transition matrix element $\Phi(\vec{k}, \vec{k}_0)$ satisfying the Lippmann-Schwinger equation

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* These proceedings.

$$\Phi(\vec{k},\vec{k}_0) = V(\vec{k},\vec{k}_0) + \frac{1}{(2\pi)^3}\int V(\vec{k},\vec{k}')\frac{1}{k'^2-k_0^2-i\eta}\Phi(\vec{k}',\vec{k}_0)d^3k' \quad (1.1)$$

where $V(\vec{k},\vec{k}_0)$ represents the Fourier transform of the static potential. We recall that

$$\Phi(\vec{k},\vec{k}_0) = \langle \vec{k}|V|\vec{k}_0\rangle$$

where $|\vec{k}\rangle$ is an eigenstate of the free Hamiltonian and $|\vec{k}_0\rangle$ an eigenstate of the total Hamiltonian corresponding to momenta $\vec{k}$ and $\vec{k}_0$ respectively. We shall define:

$$u = k^2,\ s = k_0^2,\ t = -(\vec{k}-\vec{k}_0)^2. \quad (1.3)$$

On the mass shell, i.e. for $|\vec{k}|^2 = |\vec{k}_0|^2$, we obtain the usual scattering amplitude

$$f(k^2\cos\theta) = \Phi_{|\vec{k}|^2=|\vec{k}_0|^2}(\vec{k}_0,\vec{k}). \quad (1.4)$$

Assume now that the potential V is given by a superposition of Yukawa potentials:

$$V(k,k') = \int_{\mu^2}^{\infty}\frac{v(t_0)dt_0}{t_0+(\vec{k}-\vec{k}')^2}. \quad (1.5)$$

We insert in the right-hand side of Eq. (1.1) the following "Ansatz":

$$\Phi(\vec{k},\vec{k}_0) = \frac{1}{\pi}\int\frac{\varphi(k^2,k_0^2,t')dt'}{t'+(\vec{k}-\vec{k}_0)^2} = \frac{1}{\pi}\int\frac{\varphi(u,s,t')dt'}{t'-t}. \quad (1.6)$$

We insert Eqs. (1.5) and (1.6) into Eq. (1.1). The integral on $d^3k'$ can be separated into a radial and an angular part giving

$$\iiint dt_0 v(t_0)dt'du'\varphi(u',t',s)\int\frac{d^3k'\,\delta(k'^2-u)}{[t_0+(\vec{k}-\vec{k}')^2][t'+(\vec{k}'-\vec{k}_0)^2]}.$$

The integral on $d^3k'$ can be performed by standard techniques giving

$$\int\frac{d^3k'\,\delta(k'^2-u)}{[t_0+(\vec{k}-\vec{k}')^2][t'+(\vec{k}'-\vec{k}_0)^2]} = \pi\int\frac{K(t,t',t_0,u,u',s)}{t'-t}dt'$$

where

$$K = \frac{\theta(\sqrt{t} - \sqrt{t'} - \sqrt{t_0})\theta(\Delta)}{\sqrt{\Delta}} \tag{1.7}$$

and

$$\Delta = -\begin{vmatrix} u & \frac{t_0 + u + u'}{2} & \frac{t + s + u}{2} \\ \frac{t_0 + u + u'}{2} & u' & \frac{t' + s + u'}{2} \\ \frac{t + s + u}{2} & \frac{t' + s' + u'}{2} & s \end{vmatrix} \tag{1.8}$$

so that we finally get the following integral equation for the spectral function $\varphi$:

$$\varphi(u, t) = v(t) + \iint Q(u, t; u', t') \frac{\varphi(u't')\, du' dt'}{u' - s - i\eta} \tag{1.9}$$

where

$$Q(u, t; u', t') = \frac{1}{8\pi^2}\int K_{t_0}(u, t; u't'; t_0)\, v(t_0) dt_0. \tag{1.10}$$

Let us now discuss the properties of Eq. (1.9)(which has been obtained from (1.1) through the transformation (1.6)).

We see that the role of the kernel K is essentially to fix, (through the $\theta$ functions) the boundaries of the phase space in which the integration variables t , u can vary, for given values of t and u. In particular, the equation $\Delta = 0$ is a quadratic equation in u, whose solutions are the minimum and maximum value which $u'$ can attain for fixed t, u and $t'$.

The limitation $\sqrt{t} > \sqrt{t_0} + \sqrt{t'}$ has a very important effect on the structure of the equation. If we solve Eq. (1.9) by means of an iteration procedure:

$$\varphi(u, t) = \sum_n \varphi_n(u, t), \tag{1.11}$$

$$\varphi_0(t) = v(t), \tag{1.12}$$

$$\varphi_{n+1}(t) = \iint Q_s(u, t; u't')\, \varphi_n(u', t') du' dt'. \tag{1.12.a}$$

This limitation has the consequence that <u>for each finite value of t</u> only a finite number $\varphi_n$ will be different from zero. So for each finite t the perturbation expansion for $\varphi(t)$ not only converges but also stops after a finite number of terms. This might at first look rather paradoxical, since we know that the perturbation series for the transition matrix element

$$\Phi(t) = \int_{\mu^2}^{\infty} \frac{\varphi(t')dt'}{t'-t} \tag{1.13}$$

is indeed divergent in all cases in which bound states are present. The reason of this paradox is easy to understand: in the spectral integral (1.13) the integration goes until infinity and the number of $\varphi_n$ also becomes infinite. This means that for finite values of t the perturbation series for $\Phi(t)$ contains an infinite number of terms.

These arguments suggest that the behaviour of $\varphi(t)$ when $t\to\infty$ must be very interesting and is in some way connected with the presence of bound states or resonances, since it is the only possible cause for the divergence of the perturbation series of the S matrix. We shall therefore concentrate our attention on the problem of finding the limit of

$$\varphi(t) = \sum_n \varphi_n(t) \tag{1.14}$$

when $t\to\infty$ and the number of terms of the series likewise goes to infinity.

It will be seen in the next section that the relativistic analogues of the $\varphi_n(t)$ have a very important physical meaning. In order to obtain this asymptotic limit we consider the form taken from Eqs. (1.7), (1.8), (1.9) and (1.10) for large values of t. First of all we note that, for convergent forms of the potential (1.5), the integration on $u'$ is dominated by values of $u'$ of the order of $\mu^2$; this can be checked directly on each term of the iteration series. So, for very large values of t we can disregard $u, u'$ and s as compared to t in the determinant $\Delta$. We thus get for $\Delta$ the simplified form:

$$\Delta \approx - \begin{vmatrix} u & \frac{t_0+u+u'}{2} & \frac{t}{2} \\ \frac{t_0+u+u'}{2} & u' & \frac{t'}{2} \\ \frac{t}{2} & \frac{t'}{2} & 0 \end{vmatrix} .$$

Note that we have not disregarded $t'$ as compared to t since the ratio $x = t'/t$ can indeed be of the order of 1. Moreover, we have assumed the spectral function $v(t)$ to tend to zero for $t\to\infty$ (in order to give a convergent integral (1.5)) so that for large t the contribution of this term to the r.h.s of Eq. (1.9) will be negligible. Thus we are led to the following asymptotic equation:

$$\varphi_{as}(u,t) = \frac{1}{t}\iint Q_{as}(x,u,u')\,\frac{\varphi_{as}(u',t')}{u'-s-i\eta}dt' \tag{1.15}$$

where

$$Q_{as}(x,u,u') = \frac{1}{8\pi^2}\int \frac{\theta[u'-ux-t_0x/(1-x)]r(t_0)dt_0}{(1-x)^{1/2}[u'-ux-t_0x/(1-x)]^{1/2}} \tag{1.16}$$

and where $x = t'/t$. The asymptotic equation [Eq. (1.15)] satisfies a very important property; it is invariant under the dilatation

$$\begin{aligned} t &\to \text{Const. } t, \\ t' &\to \text{Const. } t'. \end{aligned} \tag{1.17}$$

This property will be common to all the asymptotic equations we shall be considering and enables us to obtain a solution of (1.15) in the form:

$$\varphi(u,t) = f_\alpha(u)t^\alpha \tag{1.18}$$

where $f_\alpha(u)$ satisfies the equation

$$f_\alpha(u) = \int R_\alpha(u,u')\,\frac{f_\alpha(u')du'}{u'-s-i\eta} \tag{1.19}$$

where

$$R_\alpha(u,u') = \frac{1}{8\pi^2}\int v(t_0)dt_0\int_0^1 \frac{dx\,x^\alpha}{(1-x)^{1/2}}\,\frac{\theta[u'-ux-t_0x/(1-x)]}{[u'-ux-t_0x/(1-x)]^{1/2}}. \tag{1.20}$$

Equation (1.19) is an homogeneous linear integral equation of the Fredholm kind giving rise to an eigenvalue problem. For a fixed value of the total energy s, the equation is satisfied only in correspondence with well defined values of $\alpha$. For $s > 0$, i.e. in the scattering region, the presence of the $u' - s - i\eta$ denominator will lead to complex values of $\alpha$. For $s < 0$ the denominator $u - s$ cannot vanish and so the eigenvalues will be real. The eigenvalue of $\alpha$ having the largest real part is of particular interest since this gives rise to the dominating term as $t \to \infty$.

Let us summarize the results obtained. We have started from the usual Lippman-Schwinger equation [Eq. (1.1)] and we have transformed it into Eq. (1.9) for the spectral function $\varphi(t)$. We have then considered the "reduced" [Eq. (1.15)] obtained by taking the large t limit on Eq. (1.9). Finally the solution of Eq. (1.15) leads to the asymptotic form (1.18) where the fundamental exponent $\alpha$ is determined by the homogeneous equation (1.20).

We wish to emphasize the heuristic character of the derivation of Eq. (1.18) since the procedure of taking the asymptotic form of an equation in order to obtain the asymptotic solution, although frequently used by physicists, is not a rigorous one. We shall, however, show that the use of this procedure is indeed justified in our case and that Eqs. (1.18) and (1. 19) lead to the correct asymptotic limit of $\varphi(t)$. We shall now turn to the problem of determining the asymptotic limit of the scattering amplitude $\Phi$ itself, related to $\varphi$ by the dispersion integral (1. 6). For convergence reasons this dispersion relation has actually to be written down with m subtractions, where m is the minimum integer greater than Re $\alpha(s)$

$$\Phi(t) = P_{m-1} + \frac{t^m}{\pi}\int_{\mu^2}^{\infty} \frac{\varphi(t')}{t^m(t'-t)}\,dt' \tag{1.21}$$

where $P_{m-1}(t)$ is a polynomial in t with maximum power (t-1).

From Eq. (1. 21) we obtain the asymptotic form for $\Phi(t)$ by making the following approximations: first of all we substitute for $\varphi(t)$ its asymptotic form (1.18) arguing that terms whose asymptotic form is smaller than (1.18) cannot contribute to the asymptotic form of $\Phi(t)$. We then extend the integration range between 0 and $\infty$ since the contribution between 0 and $\mu$ is negligible. Finally we neglect in Eq. (1.21) the subtraction polynomial (whose maximum power is m-1) and get:

$$\Phi_{as}(t) = f_\alpha(u)\,\frac{t^m}{\pi}\int_0^{\infty}\frac{dt'}{(t')^{m-\alpha}(t'-t)} = f_\alpha(u)t^\alpha\int_0^{\infty}\frac{dz}{(z)^{m-\alpha}(z-1)}\,.$$

The integral in z is a well-known one (see the theory of the $\Gamma$ function) and we finally obtain:

$$\Phi_{as}(s,t,u) = f_\alpha(u)\,\frac{e^{i\pi\alpha(s)}}{\sin\pi\alpha(s)}\,t^{\alpha(s)}\,. \tag{1.22}$$

The asymptotic form (1.22) coincides completely with result of Regge, based on theory of continuation in the angular momentum variable.

Equation (1.22) now clearly shows the relation between the asymptotic behaviour of $\varphi(t)$ and the bound state problem. Indeed we see that the amplitude $\Phi_{as}$ has poles in s in correspondence to values of s for which

$$\alpha(s) = \ell,$$

$\ell$ being any positive integer. Those poles correspond to bound states or resonances (depending on whether they correspond to real or complex s) in states of angular momentum $\ell$. (this is because the coefficient $t^\ell$ represents the asymptotic limit of $P_\ell(\cos\theta)$). This means that Eq. (1. 19), which determines $\alpha$, also leads through Eq. (1.23) to a determination of the bound states and resonances of the problem. Stroffolini has shown that for entire values of $\alpha$, Eq. (1.19) is just a different form of the Schroedinger equation for bound states and particularly suited for continuation in the complex angular momentum.

This result fully confirms the validity of the whole procedure which has led to the asymptotic limit (1.22). Indeed a scattering amplitude has the same poles in s independently on the value of t and hence also in the limit $t \to \infty$. So the fact that for $\alpha$ entire, Eq. (1.19) coincides with the exact Schroedinger equation for bound states confirms that Eq. (1.22) gives the correct asymptotic limit of $\Phi(s, t, u)$.

## 2. RELATIVISTIC TWO-BODY PROBLEM

The simplest relativistic generalization of the potential model discussed in the previous section is the Bethe-Salpeter equation in the ladder approximation. This equation is summing the series of graphs shown in Fig. 1 which represents elastic scattering

$$A_1 + A_2 \to A_1 + A_2 \tag{2.1}$$

where $q_1, q_2, n_1, n_2$ are the initial and final momenta.

We define

$$\begin{aligned} (q_1 + q_2)/2 &= (n_1 + n_2)/2 = \Delta, \\ (q_1 - q_2)/2 &= Q, \\ (n_1 - n_2)/2 &= N. \end{aligned} \tag{2.2}$$

where $2\Delta$ is the total momentum of the system and Q and N are the relative momenta in the initial and final states respectively. The Bethe-Salpeter equation has the form

$$\Phi(Q, \Delta, N) = V(Q, N) + \frac{2}{(2\pi)^4} \int \frac{k \cdot V(Q, \varphi')\Phi(\varphi', \Delta, N)d^4Q'}{[(\Delta + \varphi)^2 - \mu^2][(\Delta - \varphi')^2 - \mu^2]} \tag{2.3}$$

where the "potential" V is the propagator of the systems exchanged between $A_1$ and $A_2$. If what is exchanged is a single particle:

$$V = g^2/[(Q - Q')^2 + m^2].$$

If a system of particles is exchanged, then V is represented by a weighted sum of propagators.

$$V(Q, Q') = \int \frac{v_0(s_0)ds_0}{(Q - Q')^2 - s_0}. \tag{2.4}$$

Finally $\mu$ represents the masses of $A_1$ $A_2$ (which for simplicity are assumed equal). The analogy between the relativistic four-dimensional scattering equation, [Eq. (2. 3)], and the non relativistic one discussed in the previous section is quite evident. So the method we shall use to treat both equations will be closely analogous.

There is, however, a very important difference because of the field theoretical nature of Eq. (2. 3). The same amplitude $\Phi$ describes at the same time the reactions

$$(I); \quad A_1 + A_2 \rightarrow A_1 + A_2$$

and

$$(II); \quad A_1 + \bar{A}_1 \rightarrow A_2 + \bar{A}_2.$$

The initial momenta for this second reaction are $q_1$, $-n_1$ and the final momenta $-q_2$, $n_2$ so that in this new channel $2\Delta$ now represents the momentum transfer and $Q + N = q_1 + n_1 = q_2 + n_2$ the total momentum.

If we define

$$4\Delta^2 = -t \text{ and } (Q+N)^2 = s \qquad (2.5)$$

we have:

In channel I: t is the square of the CM energy, s the momentum transfer;

In channel II: s is the square of the CM energy, t the momentum transfer. (The notation here is adapted to channel II). The existence of the substitution rule is of the utmost importance for the physical interpretation of the asymptotic limit of the B. S. equation. Indeed scattering in channel I has a strong analogy with potential scattering so that we shall find the asymptotic limit for small energy t and for momentum transfer $s \rightarrow \infty$. On the other hand, s plays the role of energy in channel II and therefore the asymptotic result can be interpreted as limit of the $A_1 + \bar{A}_1 \rightarrow A_2 + \bar{A}_2$ amplitude for large values of energy s and small momentum transfer t.

Let us now discuss the asymptotic solution of Eq. (2. 3). We shall only sketch the main points, since it is very analogous to potential scattering. We define the virtual "masses" of $q_1$, $q_2$ as:

$$q_1^2 = (Q+\Delta)^2 = -u_1,$$
$$q_2^2 = (Q-\Delta)^2 = -u_2 \qquad (2.5a)$$

and we introduce for $\Phi$ the ansatz:

$$\Phi(s, u_1, u_2, t) = \frac{1}{\pi}\int \frac{\varphi(s', u_1, u_2, t)}{s' - s} ds'. \qquad (2.6)$$

The integral equation obtained by substituting Eq. (2. 6) into Eq. (2. 3) is:

$$\varphi(s, u_1, u_2, t) = v(s) + \int Q(s, u_1, u_2; s', u_1', u_2', t)\varphi(s', u_1', u_2', t) ds' du_1' du_2', \tag{2.7}$$

$$Q(s, u_1, u_2; s', u_1', u_2', t) = \frac{2}{(2\pi)^4}\int ds_0 v(s_0) K(s, u_1, u_2; s', u_1', u_2', t, s_0), \tag{2.8}$$

$$K = \int d^4Q\, \delta[(Q - Q')^2 - s_0]\delta[(Q' + \Delta)^2 + u_1']\delta[(Q' - \Delta)^2 + u_2']$$

$$\cdot\, \delta[(Q' + N)^2 - s'] = (1/8)\ (\Delta)/\sqrt{\Delta}, \tag{2.9}$$

$$\Delta = \begin{vmatrix} -\frac{t}{2} & \frac{u_1' - u_2'}{2} & \frac{u_1 - u_2}{2} & 0 \\ \frac{u_1' - u_2'}{2} & u_1' + u_2' + \frac{t}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & s' - \mu^2 + \frac{u_1' + u_2' + t}{2} \\ \frac{u_1 - u_2}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & u_1 + u_2 + \frac{t}{2} & s - \mu^2 + \frac{u_1 + u_2 + t}{2} \\ 0 & s' - \mu^2 + \frac{u_1' + u_2' + t}{2} & s - \mu^2 + \frac{u_1 + u_2 + t}{2} & 2\mu^2 - \frac{t}{2} \end{vmatrix}. \tag{2.10}$$

The analogy with the corresponding potential Eqs. (2.8), (2.9) and (2.10) is quite striking. The greater complication of the new equations is naturally as result of the four-dimensional nature of the relativistic problem.

The kernel K vanishes for $\sqrt{s} < \sqrt{s'} + \sqrt{s_0}$, ensuring that for a finite value of s the series obtained by iterating Eq. (2.7) stops after a finite number of terms. The terms of the iteration series can be represented by the graphs of Fig. 1 in which each exchanged particle propagators $1/[(\Phi - \Phi')^2 - s_0]$ are substituted by $2\pi\delta[(\Phi - \Phi')^2 - s_0]$. In other words, in the iteration series of $\varphi$ all exchanged particles are taken on the mass shell. The physical meaning of this important fact will be discussed in the next section.

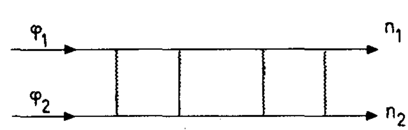


Fig. 1

We turn now to the problem of obtaining the asymptotic limit of $\varphi(s, t, u_1, u_2)$. Here there will also be nothing new, as compared with the preceding section.

We can disregard, in the determinant $\Delta, u_1, u_2, u_1', u_2', t, s_0$ as compared to s. We then get:

$$\frac{\Delta}{s^2} = \begin{vmatrix} -\frac{t}{2} & \frac{u_1'-u_2'}{2} & \frac{u_1-u_2}{2} & 0 \\ \frac{u_1'-u_2'}{2} & u_1'+u_2'+\frac{t}{2} & \frac{u_1+u_2+u_1'+u_2'+t}{2}+s_0 & x \\ \frac{u_1-u_2}{2} & \frac{u_1+u_2+u_1'+u_2'+t}{2}+s_0 & u_1+u_2+\frac{t}{2} & 1 \\ 0 & x & 1 & 0 \end{vmatrix} \qquad (2.11)$$

where $x = s'/s$. Moreover we can neglect the contribution of $v(t_0)$ for large s so that we get the asymptotic form of the equation

$$\varphi_{as}(s, u_1, u_2) = \frac{1}{s}\iiint Q_{as}(x, u_1, u_2; u_1' u_2') \cdot \frac{\varphi_{as}(u_1', u_2')\, du_1'\, du_2'}{(u_1'+\mu^2)(u_2'+\mu^2)} \qquad (2.12)$$

where

$$Q_{as} = \frac{1}{2(2\pi)^4}\int v(s_0)\, ds_0\, \frac{\theta[H(z, z_1, z_2)]}{[H(z, z_1, z_2)]^{1/2}} \qquad (2.13)$$

where

$$H(z, z_1, z_2) = -(1/4)[z^2 + z_1^2 + z_2^2 - 2zz_1 - 2zz_2 - 2z_1 z_2] \qquad (2.14)$$

and

$$\begin{aligned} z &= -t(1-x), \\ z_1 &= u_1' - u_1 x - s_0 x/(1-x), \\ z_2 &= u_2' - u_2 x - s_0 x/(1-x). \end{aligned} \qquad (2.15)$$

We recall from elementary geometry that $(1/2)\sqrt{H}$ represents the area of the plane triangle whose sides are $\sqrt{z}, \sqrt{z_1}, \sqrt{z_2}$. Also here we find the remarkable invariance of the equation under the transformation:

$$s \to e\,s \quad \text{and} \quad s' \to e\,s', \tag{2.16}$$

which allows to factorize $\varphi$ in the form

$$\varphi_{as}(s, u_1, u_2, t) = s^{\alpha(t)} f_\alpha(u_1, u_2, t) \tag{2.17}$$

where f satisfies the homogeneous equation

$$f_\alpha(u_1, u_2) = \int R_\alpha(u_1, u_2, u_1', u_2') \frac{f_\alpha(u_1', u_2')\, du_1'\, du_2'}{(u_1' + \mu^2)(u_2' - \mu^2)}, \tag{2.18}$$

$$R_\alpha(u_1, u_2, u_1', u_2') = \frac{1}{2(2\pi)^4} \int v(s_0)\, ds_0 \int_0^1 dx\, x^\alpha \frac{\theta[H(z, z_1, z_2)]}{[H(z, z_1, z_2)]^{1/2}}. \tag{2.19}$$

The eigenvalue Eq. (2.13) determines the exponent $\alpha$ as a function of t. For fixed values of t such equations have a discrete spectrum of eigenvalues. Eq. (2.13) is identical with the corresponding result in potential scattering and coincides with that obtained by extending the Regge results in relativistic theory. The use of the optical theorem gives for the total cross-section:

$$\sigma = f(-\mu^2 - \mu_0^2)\, S^{\alpha(0) - 1}. \tag{2.20}$$

The experimental evidence for the high energy total cross-section indicates that the actual value of $\alpha(0)$ is not very different from 1.

## 3. THE MULTIPERIPHERAL MODEL

We shall discuss in more detail the physical meaning of the ladder graphs treated in the last section. This will enable us to gain a deeper understanding of the significance of the formulae for elastic scattering obtained and at the same time to derive a general model for the inelastic processes taking place at high energy.

We shall consider the ladder graphs of Fig. 1 and we explicitly refer to channel II appropriate to high energy low momentum transfer scattering. Consider the amplitude $\varphi$ in the forward direction whose structure is shown in Fig. 2. In channel II, the amplitude $\varphi$ is just the absorptive part of the full amplitude $\Phi$ so that the forward elastic amplitude

$$\varphi(s, 0, u, u) = A(s, u) \tag{3.1}$$

is related to the total cross-section by the well known optical relation:

$$A(s_1 - \mu^2) = \sigma_T / 2q\sqrt{s} = \sigma_T / s \tag{3.2}$$

where $q = (1/2)\sqrt{s} - u\mu^2$ is the CM momentum of the incoming particle.

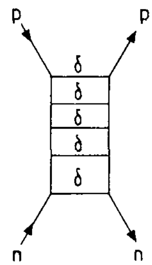


Fig. 2

Eq. (3.2) shows that if one makes a model for high-energy elastic scattering, one also implicitly constructs a model for the different production processes whose usm gives rise to the total cross-section appearing in Eq. (3.2). This is quite clear physically, since we know that at high energy, elastic scattering is essentially shadow scattering so that the form of the diffraction peak depends essentially on the multiple production processes responsible for the absorption. If one looks at the graph in Fig. 2 one sees that the production graphs giving rise to the diffraction pattern discussed in the last section are the ones shown in Fig. 3. We are therefore led to a model for multiple production which is the generalization to very high-energy of the peripheral model of Chew-Low, Drell and Saltzman. The external outgoing lines represent groups of particles whose mass distribution is given by the spectral function $v(s_0)$ which can be related to low energy cross-sections.

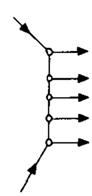

Fig. 3

The absorptive amplitude $A(s, u)$ is obtained from $(s, u_1, u_2, t)$ by making $u_1 = u_2$ and $t = 0$. So that in the high-energy limit the energy for $A(s, u)$ is particularly simple (see Fig. 4):

$$A(s, u) \equiv \frac{1}{16\pi^3 s} \int v(s_0)\, ds_0 \int_0^s ds' \int_{ux + \frac{s_0 x}{1-x}} du' \frac{A(s', u')}{(u' + \mu^2)^2}. \qquad (3.3)$$

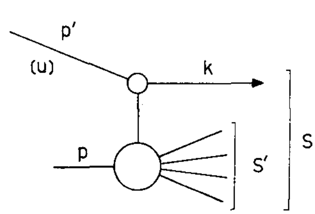


Fig. 4

So, by applying the usual factorization:

$$A(s,u) = s^{\alpha(0)} f_\alpha(u) \tag{3.4}$$

one obtains:

$$f_\alpha(u) = \frac{1}{16\pi^3}\int v(s_0)\,ds_0 \int_0^1 x^\alpha dx \int_{ux+\frac{s_0 x}{1-x}}^{\infty} du' \frac{f_\alpha(u')\,du'}{(u'+\mu^2)^2}\,. \tag{3.4 a}$$

Estimates of the exponent $\alpha$ using Eq. (3.4) with a "potential" suggested by low-energy cross-sections lead to values which are not inconsistent with the experimental value $\alpha \cong 1$.

Let us now discuss some of the main trends of high-energy collisions which can be predicted on the basis of the multiperipheral model. First of all, let us consider the multiplicity of secondaries. This will be proportional to the average number of "blubs" in the graph of Fig. 3 since the number of particles coming from each blub is, in our model, constant. The evaluation of the behaviour of such "blub multiplicity" is very easy. We write the "potential" in the form

$$v(s_0) = \lambda\, u(s_0) \tag{3.5}$$

where $u(s_0)$ is normalized to $+\int u(s_0)\,ds_0 = 1$. Then we write the multiperipheral series exhibiting explicitly the $\lambda$ dependence

$$A(s) = \sum \lambda^n a_n(s) \tag{3.6}$$

where $\lambda^n a_n/A$ is the probability for production of n blubs. Thus the multiplicity can be written as

$$< N > = \lambda\left(\frac{\delta A}{\delta \lambda}\right)/A = \frac{\Sigma\, n\lambda^n a_n(s)}{\Sigma\, \lambda^n a_n(s)}\,. \tag{3.7}$$

But now the forward on-mass shell amplitude is (see Eq. (3.4))

$$A(s_1 - \mu^2) = C(\lambda)\, s^{\alpha(\lambda)} \tag{3.8}$$

where the parameter $\lambda$ enters through the functions $C(\lambda), \alpha(\lambda)$, so that using Eq. (3.7) we get:

$$< N > = \lambda\frac{d\alpha}{d\lambda}\log s + \frac{\lambda}{c}\left(\frac{dC}{d\lambda}\right) = \lambda\left(\frac{d\alpha}{d\lambda}\right)\log\frac{s}{s}\,. \tag{3.9}$$

We have obtained the important result that the multiplicity of secondaries grows with the logarithm of the incoming energy. This is not inconsistent with experiment, though the present data are still too rough to distinguish between a logarithmic or a slow power ($\sim s^{1/4}$) behaviour.

Let us now consider the average spectra of the produced particles. Let us first look at the spectrum of that final line which is directly connected with the incident particle (first line in the multiperipheral chain). The laboratory energy of this system (see Fig. 1) is given (in the high-energy limit) by:

$$E' = (s - s')/2\mu . \tag{3.10}$$

The energy distribution of this particle is simply obtained by adding in the integrand on the r, u, s of Eq. (3.3) an extra $\delta(E' - (s - s')/2\mu)$:

$$\frac{d\sigma}{dt'} \approx s\,\frac{dA(s-\mu^2)}{dE}$$

$$= \frac{1}{16\pi^3}\int v(s_0)\,ds_0 \int_0^s ds' \int_{-\mu^2 x + \frac{s_0 x}{1-x}}^{\infty} \frac{du'\,\delta(E' - [s - s']/2\mu)\,A(s', u')}{(u' + \mu^2)^2} \tag{3.11}$$

and using the asymptotic form (3.4) for A:

$$\frac{d\sigma}{d\epsilon} = \frac{s^{a-1}}{16\pi^3}\int v(s_0)\,ds_0 \int_0^1 x^a dx \int_{\mu^2 + \frac{s_0 x}{1-x}}^{\infty} \frac{du' f(u')}{(u' + \mu)^2}\,\delta(\epsilon - 1 - x) \tag{3.12}$$

where

$$\epsilon = \frac{E}{E}$$

is the ratio between the secondary energy E and the primary energy $E = s/2\mu$. So we have the very simple result:

$$\left(\frac{d\sigma}{d\epsilon}\right)/\sigma_T = F(\epsilon). \tag{3.13}$$

The shape of the energy specturm of the first secondary is completely independent of the value of the primary energy. In particular the inelasticity, i.e. the fraction of energy taken by the first secondary, is independent on the primary energy. The spectrum we are studying is particularly easy to measure when the incident particle is distinguishable from the secondary pions. This is the case for nucleon collisions, which have been extensively studied either with accelerators or cosmic rays. The analysis of high-energy jets yields that the average energy carried away by the nucleon in the lab system is nearly a constant fraction of the incident energy, which is just the prediction of our model.

It is also easy to study the spectrum of secondaries, regardless of their position in the multiperipheral chain. We shall not report here the calculation which does not offer any new difficulty and only give the results.

Let us call $k_L$ and $k_L$ the longitudinal and transverse momenta of the secondary. If $k_L \ll E$, the spectrum can be written in the form

$$F(k_T^2)\, dk_L / k_L \tag{3.14}$$

where $F(k_T^2)$ is a universal function independent both of E and of $k_L$ and strongly peaked for small values of $k_T$. These results, especially the separability of the transverse and longitudinal spectra, are not inconsistent with present experimental data.

## 4. CONCLUSIONS

We wish now to summarize briefly the different results and their physical meaning. We have discussed the predictions for the different high-energy processes obtained on the basis of the multiperpheral model. It has been possible to sum the whole series of multiperipheral graphs by means of a linear integral equation for the off-mass shell absorptive amplitude $\varphi(s, u_1, u_2, t)$. The kernel of this integral equation depends on the low-energy amplitude $v(s_0)$. The knowledge of this amplitude is sufficient to allow the computation both of the elastic scattering and of multiple production. The on-mass shell amplitude $\varphi(s, -\mu^2, -\mu^2, t)$ leads to the elastic diffraction cross-sections, while we can evaluate the average distributions of particles in multiple production on the basis of the forward off-mass shell amplitude $\varphi(s, u_1, u_2, 0)$.

The asymptotic behaviour of the amplitude is obtained by considering the high-energy limit of the integral equation. In this limit, the integral equation shows a very remarkable feature which is independent of the specific form of the amplitude $A^R$. The kernel depends only on the ratio $s'/s$, so that the equation is invariant under the transformation $s \to cs$, $s' \to cs'$ This allows us to factorize the s dependence of the amplitude in the simple form:

$$\varphi(s, u_1, u_2, t) = s^{\alpha(t)} f(u_1, u_2, t). \tag{4.1}$$

The problem is then reduced to the solution of an homogeneous integral equation for $f(u_1, u_2, t)$, whose solution determines both the exponent $\alpha(t)$ and the eigenfunction $f(u_1, u_2, t)$. As already pointed out, both eigenvalues and eigenfunctions have a physical meaning: the eigenvalue gives the well-known shrinking of the diffraction peak, whereas the eigenfunction is connected with the average properties of multiple production. A form of the scattering amplitude analogous to Eq. (4.1) has been obtained by many people by adapting the results of Regge in potential theory to high-energy scattering. This analogy can be understood by considering that the multiperipheral graphs observed in the crossed channel are the relativistic analogues of the different iterations of the potential model used by Regge.

The predictions obtained by means of the model can be divided into two categories:

(a) Many general trends of high-energy collisions depend only on the transformation property of the integral equation which is a consequence of the topology of the multiperipheral graphs. These general trends do not in fact depend on any special choice of the low-energy amplitude $v(s_0)$:

(b) The specific numerical answers (as, for example, the value of the total cross-sections) do, of course, depend on the choice of $v(s_0)$ and on the manner in which $v(s_0)$ is continued off the mass shell.

We shall now summarize the different conclusions obtained on the basis of the multiperipheral model including those which have not been discussed in this paper.

(1) Elastic amplitude

The high-energy behaviour of the scattering amplitude $\Phi(s,t)$ is

$$\Phi_j(s,t) = s^{\alpha_j(t)}\, C_j(t)\,[\,-\operatorname{cotg}(\pi\alpha_j(t)/2) + i\,] \tag{4.2}$$

for symmetric amplitudes under crossing $s \leftrightarrow \bar{s}$, as, for instance, absolute elastic scattering, and

$$\Phi_j(s,t) = s^{\alpha_j(t)}\, C_j(t)\,[\,\operatorname{tg}(\pi\alpha_j(t)/2) + i\,] \tag{4.3}$$

for antisymmetric amplitudes under the crossing. We obtain $d\alpha/dt > 0$. The exponent for the charge exchange amplitude is always smaller than the one for the purely elastic one. Eqs. (4.2) and (4.3) turn out to be independent of the scattering particles, apart from the value of $C(t)$. The $C(t)$ can be factorized in such a manner that the relation between different amplitudes (dominated by the same pole) is the following:

$$\Phi_{xy}(s,t)/\Phi_{zy}(s,t) = \Phi_{xw}(s,t)/\Phi_{zw}(s,t)$$

where x, y, z and w represent any kind of particles.

(2) Inelastic scattering

The average properties of inelastic scattering are also very simple and depend only on the general form of Eq. (4.4) The multiplicity grows with the logarithm of the energy, and the inelasticity is energy independent. The spectra of the secondary particles are given by

$$N(k)\,d^4k = F(k^2, \vec{k}_T^2)\, d\vec{k}_T dk^2 (dk_L/k_L)$$

for $k_L \ll$ the initial energy. $k_L$ ($k_T$ and $k_L$ being the transverse and longitudinal momenta), where $F(k^2, k_T^2)$ is a universal function independent both of s and $k_L$, and strongly peaked for $k_T^2 \lesssim \mu^2$.

These results, especially the separability of the transverse and longitudinal spectra, are not inconsistent with present experimental data.

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