Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 356–367

# Aspects of Quantum Groups and Integrable Systems

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Some algebro-geometric origins of $q$-differential equations are examined.

## 1 Introduction

Many $q$-differential operators arise in the study of $q$-special functions, Casimir operators, quantum groups, and representation theory. There are also natural origins via quantum integrable systems and the quantization of classical integrable systems. The latter is often expressed via a $q$-hierarchy picture akin to the standard Hirota–Lax–Sato formulation and this has many canonical aspects. On the other hand one can produce a great number of $q$-differential operators by more or less ad hoc manipulation of noncommutative differential calculi or by variations of classical Lie group methods applied to quantum groups. We examine first the hierarchy picture briefly and notice that although the standard methods generalize quite readily the resulting KP or KdV equations for example seem to have an infinite number of terms whereas many "ad hoc" derivations from differential calculi or e.g. Maurer–Cartan arguments have only a finite number of terms and there is no clear way to determine if in fact such equations have any intrinsic meaning. We show that the standard derivation of KdV via vector fields on the unit circle and the smooth dual space of Lax operators can be extended to a $q$-situation using a $q$-Virasoro algebra and we produce a corresponding qKdV equation with an infinite number of terms; this seems to be a fairly canonical derivation and we suggest that it could be equivalent to the hierarchy qKdV equation (not yet proved). This approach to KdV via the unit circle with its attendant UrKdV–mKdV equations, Schwarzian derivatives, projective geometry, etc. has another connection to quantum mechanics (QM) via the beautiful equivalence principle of Faraggi–Matone and we use this as a motivational background (cf. [1]). In this connection we remark that in fact KP for example can already be considered as a Moyal quantization of dKP (dispersionless KP) and it is not clear just what role the qKP or qKdV equations can play in quantum mechanics; however for completeness we also indicate a few approaches to Moyal type integrable equations.

## 2 The equivalence principle

The equivalence principle (EP) of Faraggi–Matone (cf. refs. [1, 2]) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton–Jacobi equation (QSHJE) which is a third-order nonlinear differential equation providing a trajectory representation of quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. ref. [3]) leading to **(A1)** $\dot{q} = p/m_Q \neq p/m$ where **(A2)** $m_Q = m(1-\partial_E Q)$ is the "quantum mass" and $Q$ the "quantum potential". Thus the EP is reminscient of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics should be invariant under general coordinate transformations (cf. ref. [4]). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. $c$, $\hbar$, $e$, $m$ and $\eta_{\mu\nu}$ = Minkowski tensor). Now for the EP, the classical picture with $S^{cl}(q, Q^0, t)$ the Hamilton principal function ($p = \partial S^{cl}/\partial q$) and $P^0$, $Q^0$ playing the role of initial conditions involves the classical HJ equation (CHJE) **(A3)** $H(q,p) = (\partial S^{cl}/\partial q), t) + (\partial S^{cl}/\partial t) = 0$. For time independent V one writes $S^{cl} = S_0^{cl}(q, Q^0) - Et$ and arrives at the classical stationary HJ equation (CSHJE) **(A4)** $(1/2m)(\partial S_0^{cl}/\partial q)^2 + \mathfrak{W} = 0$ where $\mathfrak{W} = V(q) - E$. In the Bohm theory one looked at Schrödinger equations **(A5)** $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi = \psi(q)\exp(-iEt/\hbar)$ and **(A6)** $\psi(q) = R(q\exp(i\hat{W}/\hbar)$ leading to

$$\left(\frac{1}{2m}\right)(\hat{W}')^2 + V - E - \frac{\hbar^2 R''}{2mR} = 0, \qquad (R^2\hat{W}')' = 0, \tag{1}$$

where **(A7)** $\hat{Q} = -\hbar^2 R''/2mR$ was called the quantum potential; this can be written in the Schwartzian form **(A8)** $\hat{Q} = (\hbar^2/4m)\{\hat{W}; q\}$ (via $R^2\hat{W}' = c$). Here **(A9)** $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$. Writing $\mathfrak{W} = V(q) - E$ as in **(A4)** we have the quantum stationary HJ equation (QSHJE) **(A10)** $(1/2m)(\partial\hat{W}'/\partial q)^2 + \mathfrak{W}(q) + \hat{Q}(q) = 0$ $(\equiv \mathfrak{W} = -(\hbar^2/4m)\{\exp(2iS_0/\hbar); q\})$. This was worked out in the Bohm school (without the Schwarzian connections) but **(A6)** is not appropriate for all situations; the Bohm theory is incomplete and can lead to incorrect predictions. The technique of Faraggi–Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzians, Legendre duality, and the geometry of a second-order differential operator $D_x^2 + V(x)$ (Möbius transformations play an important role here) to arrive at the QSHJE in the form

$$\frac{1}{2m}\left(\frac{\partial S_0^v(q^v)}{\partial q^v}\right)^2 + \mathfrak{W}(q^v) + \mathfrak{Q}^v(q^v) = 0, \tag{2}$$

where $v : q \to q^v$ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in $ker\,(D_x^2 + V(x))$ is twice $V(x)$. In particular given an arbitrary system with coordinate $q$ and reduced action $S_0(q)$ the system with coordinate $q^0$ corresponding to $V - E = 0$ involves **(A11)** $\mathfrak{W}(q) = (q^0; q)$ where $(q^0, q)$ is a cocycle term which has the form **(A12)** $(q^a; q^b) = -(\hbar^2/4m)\{q^a; q^b\}$. In fact it can be said that the essence of the EP is the cocycle condition **(A13)** $(q^a; q^c) = (\partial_{q^c} q^b)^2[(q^a; q^b) - (q^c; q^b)]$.

In addition FM developed a theory of $(x, \psi)$ duality (cf. ref. [1]) which related the space coordinate and the wave function via a prepotential (free energy) in the form $\mathfrak{F} = (1/2)\psi\bar{\psi} + iX/\epsilon$ for example. A number of interesting philosophical points arise (e.g. the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in refs. [5, 6] in a sort of extended WKB spirit. One should note here that although a form **(A6)** is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation $\psi$ and $\bar{\psi}$ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator $L^2 = D_x^2 + V(x)$ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on $S^1$ (see e.g. refs. [7–10]. Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of $D_x = d/dx$ and the circle $S^1$. In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM to a study of $S^1$ and $\partial_x$!

In any event KdV in its geometrical glory is important and we want to look at qKdV and qKP. A main theme here is to understand the relation of the qKdV hierarchy picture in terms of a geometrical origin of qKdV type equations. The hierarchy picture is described below. We also give various derivations of qKdv and qKP type equations from various geometrical or quantum group points of view (most of which are not equivalent to the hierarchy pictures). However we do find a qKdV equation from a $q$-Virasoro context which could agree with the hierarchy picture (see Section 6). Even so it is not entirely obvious that the hierarchy picture is the only legitimate object of study – although the analytic approach of ref. [11] and the algebraic (oper) approach of ref. [12] do point in this direction.

## 3 KP and KDV hierarchies

For classical KP and KdV one has (cf. refs. [5, 6, 13, 14]

$$L = \partial + \sum_1^\infty u_{n+1}(t)\partial^{-n}, \qquad W = 1 + \sum_1^\infty w_n \partial^{-n}, \qquad L = W\partial W^{-1} \tag{3}$$

with **(A14)** $\psi = We^{\xi} = W(\lambda)e^{\xi}$; $\xi = \sum_1^\infty t_n\lambda^n$; $L\psi = -\lambda\psi$; $\partial_m\psi = B_m\psi = L_+^m\psi$ and **(A15)** $\partial_m L = [B_m, L]$; $\partial_n B_m - \partial_m B_n = [B_n, B_m]$; $\partial_n W = -(L_-^n)W$. Recall **(A16)** $\psi(t,\lambda) = \frac{X(\lambda)\tau}{\tau} = \frac{e^{\xi}\tau\left(t_j - \frac{1}{j\lambda^j}\right)}{\tau(t)} = \frac{e^{\xi}\tau_-}{\tau}$ where $\tau$ is the famous tau function with **(A17)** $u = \partial_x^2\log(\tau)$ with $u \sim u_2$ in 3. Typical forms for KP and KdV are **(A18)** $u_t - u_{xxx} + 6uu_x - 3\partial^{-1}u_{yy} = 0$ and $u_t - u_{xxx} + 6uu_x = 0$. For KdV one has $L_+^2 = L^2$ and the hierarchy picture is much simplified; we discuss this below in the context of qKdV.

## 4 *Q*-hierarchies

For qKP we go to refs. [11, 12, 14–23]; the hierarchy picture is straightforward but the resulting equations are much more complicated. Thus one writes e.g. **(A19)** $\partial_q f(x) = [f(qx) - f(x)]/(q-1)x$ and $Df(x) = f(qx)$. Then in the hierarchy constructions $x$ and $t = t_1$ are both used in the first variable position with

$$\tau_q(x,t) = \tau(t_i + c(x)_i), \qquad c(x) = \left(\frac{(1-q)x}{1-q}, \ldots, \frac{(1-q)^n x^n}{n(1-q^n)}, \ldots\right), \tag{4}$$

where $\tau$ is an ordinary tau function for KP. One takes then ($\partial_1 \sim \partial/\partial t_1, \ldots, \partial_n \sim \partial/\partial t_n$)

$$L_q = \partial_q + \sum_0^\infty a_n(t)\partial_q^{-n}, \qquad \partial_n L_q = [(L_q^n)_+, L_q] = [B_n^q, L_q] \tag{5}$$

(note $a_0 \neq 0$ appears here and $u \sim a_1$). Further ($\xi = \sum_1^\infty t_k\lambda^k$)

$$\psi_q = \frac{\tau_q(x, t - [\lambda^{-1}])}{\tau_q(x,t)} e_q(x\lambda)\exp(\xi), \qquad (q;q)_k = (1-q)\cdots(1-q^k), \tag{6}$$
$$\exp_q(x) = \sum_1^\infty \frac{(1-q)^k x^k}{(q;q)_k} = \exp\left(\sum_1^\infty \frac{1-q)^k x^k}{k(1-q^k)}\right), \qquad \partial_q e_q(x\lambda) = \lambda e_q(x),$$

where $[\lambda^{-1}] = (1/n\lambda^n)$. Note also **(A20)** $\psi_q = W_q e_q(x\lambda)e^{\xi}$; $L_q\psi_q = \lambda\psi_q$ with

$$L_q = W_q\partial_q W_q^{-1}, \qquad W_q = 1 + \sum_1^\infty \tilde{w}_j\partial_q^{-j}, \qquad \partial_j W_q = -(L_q^j)_- W_q. \tag{7}$$

Next note $e_q(x\lambda)^{-1} = e_{1/q}(-x\lambda)$ and the adjoint wave function $\psi_q^*$ is determined via

$$\psi_q^* = (W^*)_{x/q}^{-1}\exp_{1/q}(-x\lambda)e^{\xi}, \qquad L_{x/q}^*\psi_q^* = \lambda\psi_q^*,$$
$$\psi_q^* = \frac{\tau_q(x,t+[\lambda^{-1}])}{\tau_q(x,t)}\exp_{1/q}(-x\lambda)e^{-\xi}, \tag{8}$$

where $V = \sum v_i\partial_q^i \sim V^* = \sum(\partial_q^*)^i v_i$ with $\partial_q^* = -q^{-1}\partial_{1/q}$ and $V_{x/q} \sim \sum v_i(x/q)q^i\partial_q^i$. Recall also the Schur polynomials

$$\sum p_n(t)\lambda^n = \exp(\xi(t,\lambda)), \qquad \sum \tilde{p}_n(x,t)\lambda^n = \exp_q(x\lambda)e^{\xi(t,\lambda)},$$
$$\tilde{p}_k(x,t) = p_k(t+c(x)). \tag{9}$$

Next the classical Hirota bilinear identity is

$$\oint \psi^*(t,\lambda)\psi(t',\lambda)d\lambda = 0 \quad \Rightarrow \quad \oint \tau(t+y+[\lambda^{-1}])\tau(t-y-[\lambda^{-1}])e^{-2\sum y_i\lambda^i} = 0 \tag{10}$$

and this leads to classical Hirota equations **(A21)** $\partial_1\partial_n\tau\cdot\tau = 2p_{n+1}(\tilde{\partial})\tau\cdot\tau$ (Hirota notation – also $\tilde{\partial} = (\partial_1, (1/2)\partial_2, \ldots)$). One can produce analogous formulas in $q$-theory but their usefulness is limited and not even clear (cf. ref. [14] for an extensive discussion). The problem lies in the fact that $u = \partial^2\log\tau$ does not generalize but instead

$$u = -(q-1)x\partial_q\left(\frac{p_2(\tilde{\partial})\tau_q}{\tau_q}\right) + \frac{\partial_q\tau_q}{\tau_q}(q-1)x\partial_q\left(\frac{\partial_1\tau_q}{\tau_1}\right) + \partial_q\left(\frac{\partial_q\tau_q}{\tau_q}\right). \tag{11}$$

The classical Hirota equations are compatible with and lead to formulas involving $\log\tau$ (and hence $u$) but in the $q$-situation (11) makes this impossible. In fact I have never seen a qKP equation analogous to **(A18)** for example.

The problems are more easily seen with qKdV and there one has (recall $Df(x) = f(qx)$)

$$L_q^2 = \partial_q^2 + (q-1)xu\partial_q + u, \qquad u = \partial_q\partial_1\log[\tau_q(x,t)D\tau_q(x,t)],$$
$$L_q = \partial_q + s_0 + s_1\partial_q^{-1} + \cdots. \tag{12}$$

Considerable calculation leads to ($u_1 = (q-1)xu$)

$$s_0 = (q-1)x\partial_1\partial_q\log\tau_q, \qquad \partial_t u = (\partial_q^3 u) + w_2(\partial_q^2 u) + w_1(\partial_q u) - [(\partial_q^2 w_0) + u_1(\partial_q w_0)], \tag{13}$$
$$w_2 = D^2 s_0 + u_1 = D^2 s_0 + Ds_0 + s_0, \tag{14}$$
$$w_1 = (q+1)(D\partial_q s_0) + \tau^2 s_1 + [(Ds_0) + s_0](Ds_0) + u,$$
$$w_0 = \partial_q^2 s_0 + (q+1)(D\partial_q s_1) + u_1\partial_q s_0 + u_1(Ds_1) + us_0 + D^2 s_2$$

and the determination of the $s_i$ requires infinite series calculations based on formulas like

$$s_1 + Ds_1 = u - \partial_q s_0 - s_0^2 = f \quad \Rightarrow \quad s_1(x) = \sum_0^\infty (-1)^n f(q^n x). \tag{15}$$

Similarly **(A22)** $Ds_2 + s_2 = -\partial_q s_1 - s_0 s_1 - s_1 D^{-1}s_0$, etc. This seems to suggest that an explicit form of qKdV (as in **(A18)**) arising in the hierarchy picture with coefficients specified in terms of $u$ (or $s_0$) will involve an infinite number of terms.

# 5 Origins of $Q$-equations

## 5.1 Linear equations

In refs. [14–16] we sketched a number of sources for linear $q$-differential equations in various contexts (see the references there for details and cf. also refs. [24, 25]). This includes many relations involving $q$-special functions, Casimir operators, quantum groups, and representation theory with linear $q$-differential equations arising naturally. A good source of general information with Lie theoretic background is refs. [26, 27]. There is also a development involving special functions and $q$-special functions where tau functions arise as generating functions for matrix elements in general group representations (cf. ref. [15] and references there).

**Remark 1.** The development in ref. [26] for example emphasizes the ideas of intertwining and group invariant differential operators in a general Lie theoretic context with representations involving Verma modules (cf. also ref. [27] and references there). One arrives e.g. at a Maxwell hierarchy and its quantum group version in the context of a particular $q$-Minkowski space-time. The spirit is that of QG symmetries described via invariant $q$-differential operators and here the equations are also $q$-conformally invariant. Other work involving Dirac equations is described in ref. [15] for example and the differential operators involved seem to be primarily linear. However the treatment in ref. [27] provides interesting nonlinear differential operators in the context of multilinear intertwining differential operators. Bilinear and trilinear formulas for example are written out for $\mathfrak{sl}(2)$ and there is an interesting connection to KdV. Thus let $G$ be a Lie group with representations $T$, $T'$ in spaces $C$, $C'$. An intertwining operator $\mathfrak{I}$ is a continuous linear map **(A23)** $\mathfrak{I} : C \to C' : f \to j$ such that **(A24)** $\mathfrak{I} \circ T(g) = T'(g) \circ \mathfrak{I}$. The equation **(A25)** $\mathfrak{I}f = j$ is then a $G$-invariant equation. Omitting details one defines multilinear intertwining differential operators via

$$ {}_k\mathfrak{I} : f \otimes \cdots \otimes f \to j;\ {}_k\mathfrak{I} \circ T(g) \otimes \cdots \otimes T(g) = T'(g) \circ {}_k\mathfrak{I} \tag{16} $$

and arrives at generalized Schwarzians **(A26)** $\widetilde{Sch}_n(\phi) = [1/(\phi')^2]_2\mathfrak{I}^0_{n\alpha}(\phi)$. Recalling that the KdV equation can be written in the Krichever–Novikov form **(A27)** $\partial_t f + \widetilde{Sch}_4(f)f' = 0$ (note $\widetilde{Sch}_4(f) = S(f)$ is the Schwarzian derivative) one can pass to the standard KdV form **(A28)** $\partial_t u + u''' - 6uu' = 0$ where the substitution $u = -(1/2)\widetilde{Sch}_4(f)$ is used. It is conjectured in ref. [27] that the equations **(A29)** $\partial_t f + \widetilde{Sch}_n(f)f' = 0$ ($n \in 2\mathbb{N}+2$) are integrable, and if so they should coincide with the KdV hierarchy. One expects that a similar analysis in the $q$-context would be of great interest in understanding qKdV.

## 5.2 Differential calculi

When we come to nonlinear equations such as KP or KdV there are a number of classical derivations of intrinsic geometrical or algebraic interest which should morally have a $q$-version. Moreover one would expect the $q$-versions to bear some natural or canonical relation to the hierarchy version described earlier. In refs. [14–16] we examined some such derivations but without being able to establish a clear connection to the hierarchy picture. The constructions however seem interesting enough and well modeled on meaningful classical situations so some further study is indicated. First in a somewhat experimental manner consider a differential calculus approach following refs. [14–16, 28]. Thus

**Example 1.** Consider a calculus based on **(A)** $dt^2 = dx^2 = dxdt + dtdx = 0$ **(B)** $[dt, t] = [dx, t] = [dt, x] = 0$ and $[dx, x] = \eta dt$. Assuming the Leibnitz rule $d(fg) = (df)g + f(dg)$ for functions and $d^2 = 0$ one obtains **(A30)** $df = f_x dx + (f_t + (1/2)\eta f_{xx})dt$. For a connection $A = wdt + udx$ the zero curvature condition $F = dA + A^2 = 0$ leads to **(A31)** $(u_t - w_x + (\eta/2)u_{xx} + \eta u u_x = 0$ which for $w_x = 0$ is a form of Burger's equation.

**Example 2.** Next consider **(A)** $[dt,t]=[dx,t]=[dt,x]=[dy,t]=[dt,y]=[dy,y]=0$ with **(B)** $[dx,x]=2bdy$ and $[dx,y]=[dy,x]=3adt$. Further **(C)** $dt^2=dy^2=dtdx+dxdt=dydt+dtdy=dydx+dxdy=0$. Then **(A32)** $df=f_xdx+(f_y+bf_{xx})dy+(f_t+3af_{xy}+abf_{xxx}dt)$. For $A=vdx+wdt+udy$ one finds that $dA+A^2=F=0$ implies

$$u_x=v_y+bv_{xx}+2bvv_x,\qquad w_x=3av_{xy}+abv_{xxx}+3auv_x+3av(v_y+bv_{xx}),\tag{17}$$
$$w_x+bw_{xx}=u_t+3au_{xy}+abu_{xxx}+3auu_x-v[2bw_x-3a(u_y+bu_{xx})].$$

Taking e.g. $w_x=(3a/2b)u_y+(3a/2)u_{xx}$ in the last equation to decouple one arrives at **(A33)** $\partial_x(u_t-(ab/2)u_{xxx}+3auu_x)=(3a/2b)u_{yy}$; for suitable $a$, $b$ this is KP. If the equation is independent of $y$ we obtain a version of KdV.

It is surprisingly difficult to convert these examples into meaningful $q$-calculus equations and in that spirit for guidance we were motivated to develop many formulas concerning qKP, qKdV, etc. One has e.g. a first-order differential calculus (FODC) $\Gamma_+$ from ref. [25] on a quantum plane or Manin plane (cf. also ref. [15]); this is based on $xp=qpx$ with standard formulas as in ref. [25]. In this FODC the partial derivatives $\partial_i$ of $\Gamma_+$ act on $\mathfrak{A}$ = formal power series with $x$, $p$ ordering, via ($\partial_p x^n=q^nx^n\partial_p$ and $\partial_x p^n=q^np^n\partial_x$)

$$\partial_x(f(x)h(p))=(D^x_{q^2}f(x))h(p),\qquad \partial_p(f(x)h(p))=(T_qf(x))(D^p_{q^2}h(p)),\tag{18}$$

$$\partial_x(x^n)=D_{q^2}x^n=[[n]]_{q^2}x^{n-1},\qquad [[n]]_{q^2}=\frac{q^{2n}-1}{q^2-1},\qquad \partial_pp^n=[[n]]_{q^2}p^{n-1}.\tag{19}$$

**Example 3.** The $q$-plane itself doesn't seem immediately fruitful here so consider the generalized $q$-plane with an algebra generated by $x$, $y$, $x^{-1}$, $y^{-1}$ where $xy=qyx$, $xdx=qdxx$, $ydx=q^{-1}dxy$, $xdy=qdyx$, and $ydy=q^{-1}dyy$. Also from $qdyx=dxy$ we have $qdydx=dxdy$ and a little calculation yields **(A34)** $dx^n=[(1-q^{-n})/(1-q^{-1})]x^{n-1}dx$ with $dy^m=[(1-q^m)/(1-q)]y^{m-1}dy$. Working from $f=\sum a_{nm}x^ny^m$ one obtains then (note $dxy^m=q^my^mdx$)

$$df=D_yD^x_{q^{-1}}fdx+D^y_qfdy.\tag{20}$$

Set then $A=wdy+udx$ with $dA=D_yD^x_{q^{-1}}wdxdy+D^y_qudydx$ and, noting that $dyx^n=q^{-n}x^ndy$, $dyy^m=q^my^mdy$, and $dxy^m=q^my^mdx$ with $dxx^n=q^{-n}x^ndx$ one gets $dyw=D_x^{-1}D_ywdy$ and $dxu=D_x^{-1}D_yudx$ leading to $A^2=wD_x^{-1}D_yudydx+uD_x^{-1}D_ywdxdy$; and

$$dA+A^2=0=qD_yD^x_{q^{-1}}w+D^y_qu+wD_x^{-1}D_yu+quD_x^{-1}D_yw.\tag{21}$$

Setting then e.g. $qw=D_y^{-1}D^x_{q^{-1}}u$ one gets

$$D^y_qu+(D^x_{q^{-1}})^2u+q^{-1}(D_x^{-1}D_yu)(D_y^{-1}D^x_{q^{-1}}u)+uD_x^{-1}D^x_{q^{-1}}u=0.\tag{22}$$

For $q\to1$ we have **(A35)** $u_y+u_{xx}+2uu_x=0$ so this appears to be an exact $q$-form of Burger's equation.

**Example 4.** We will try now a somewhat different approach. First we take $q$-derivatives only in $x$, as in the case of qKP for example and we know from Example 2 that **(A32)** leads to interesting consequences so begin with an assumption ($f_y=\partial f/\partial y$, etc.)

$$df=D^x_qfdx+\left(f_y+b(D^x_q)^2f\right)dy+\left(f_t+3a\partial_yD^x_qf+ab(D^x_q)^3f\right)dt.\tag{23}$$

Then we can determine what elementary commutation relations between the variables are consistent with (23). This is rather ad hoc but we stipulate $x$, $y$, $t$ ordering and then there are relations

$$dxx=qxdx+b[2]_qdy,\qquad dyx=q^2xdy+a[3]_qdt,$$
$$dtx=q^3xdt,\qquad dxy=ydx+3adt\tag{24}$$

along with

$$[dt, y] = [dy, y] = [dy, t] = [dx, t] = [dt, t] = [dx, y] = 0 \tag{25}$$

which are determined by (23). The underlying structure for $x, y, t$ is not visible from (23) but (24)–(25) do lead to (23) and whatever zero curvature equations subsequently arise (see ref. [16] for details). Returning now to (23) it remains to check now the zero curvature equation arising in the spirit of Example 2 (some extra factors and terms will arise via noncommutativity). Thus assume first $dx^2 = dy^2 = dt^2 = 0$ and take $A = vdx + wdt + udy$; after some computation modeled on Example 2 one arrives at a rather clumsy $q$-version of KP. For $y$ independent $u$ we obtain

$$\begin{aligned} &u_t + ab(D_q^x)^3 u - \frac{[3]_q ab}{[2]_q} D_q^x D_x (D_q^x)^2 u + [3]_q au D_x^2 D_q^x u \\ &\quad = \frac{[3]_q a}{[2]_q b} \left( D_x D_q^x u D_x^3 u - u D_x^3 D_q^x u \right) \end{aligned} \tag{26}$$

which is a quantum KdV type equation.

## 5.3 Geometry

We indicate two "geometrical" contexts in a classical vein and subsequently give "quantum" versions of these.

**Example 5.** One can devise a procedure directly from ref. [29]. Thus look at $SL(2, \mathbb{R})$ with matrices **(A36)** $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$. The right invariant Maurer–Cartan (MC) form is **(A37)** $\omega = dXX^{-1} = (w^i_j)$ $(i, j = 1, 2)$ where $\omega^1_1 + \omega^2_2 = 0$. The structure equation of $SL(2, \mathbb{R})$ or MC equation is **(A38)** $d\omega = \omega \wedge \omega$ or explicitly **(A39)** $d\omega^1_1 = \omega^2_1 \wedge \omega^1_2$; $d\omega^2_1 = 2\omega^1_1 \wedge \omega^2_1$; $d\omega^1_2 = 2\omega^1_2 \wedge \omega^1_1$. Now let U be a neighborhood in the $(x, t)$ plane and consider a smooth map $f : U \to SL(2, \mathbb{R})$. The pullback of the MC form can be written as **(A40)** $\omega^1_1 \sim \eta dx + Adt$; $\omega^2_1 \sim Qdx + Bdt$; $\omega^1_2 \sim rdx + Cdt$ with coefficient functions of $x$, $t$. The equations **(A39)** become **(A41)** (i) $-\eta_t + A_x - QC + rB = 0$, (ii) $-Q_t + B_x - 2\eta B + 2QA = 0$, and (iii) $-r_t + C_x - 2rA + 2\eta C = 0$. Take $r = 1$ with $\eta$ independent of $(x, t)$ and set $Q = u(x, t)$. Then from (i) and (iii) one gets **(A42)** $A = \eta C + \frac{1}{2} C_x$; $B = uC - \eta C_x - \frac{1}{2} C_{xx}$. Putting this in the (ii) above yields $u_t = K(u)$ where **(A43)** $K(u) = u_x C + 2uC_x + 2\eta^2 C_x - \frac{1}{2} C_{xxx}$. In the special case $C = \eta^2 - (1/2)u$ one gets the KdV equation **(A44)** $u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} uu_x$.

**Example 6.** Following ref. [30], let Vec $(S^1)$ denote the Lie algebra of smooth vector fields on $S^1$ and then the Virasoro algebra is Vir $= \mathrm{Vec}(S^1) \oplus \mathbb{R} = \mathfrak{W} \oplus \mathbb{R}$ with (note the minus sign convention involving $f'g - fg'$)

$$[(f(x)\partial_x, a), (g(x)\partial_x, b)] = \left( (f'g - fg')\partial_x, \int_{S^1} f'g'' dx \right) \tag{27}$$

($\mathfrak{W} \sim$ Witt algebra). Here $\int_{S^1} f'g''dx$ is called the Gelfand–Fuks cocycle, where a cocycle on a Lie algebra $\mathfrak{g}$ is a bilinear skew symmetric form $c(\cdot, \cdot)$ satisfying **(A45)** $\sum c([f, g], h) = 0$ over cyclic permutations of $f$, $g$, $h$. This means that $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ (central extension) with commutator $[(f, a), (g, b)] = ([f, g], c(f, g))$ satisfies the Jacobi identity of a Lie algebra. Now the Euler equation corresponding to geodesic flow is a 1-parameter family of KdV equations. To see how this arises consider **(A46)** $\mathrm{Vir}^* = \{(u(x)dx^2, c);\ u \text{ smooth on } S^1 \text{ and } c \in \mathbb{R}\}$. Then **(A47)** $\langle (v(x)\partial_x, a), (u(x)dx^2, c) \rangle = \int_{S^1} v(x)u(x)dx + ac$. The coadjoint action of $(f\partial_x, a) \in$ Vir on $(udx^2, c) \in \mathrm{Vir}^*$ is $(\mathrm{ad}^*_v : \mathfrak{g}^* \to \mathfrak{g}^*,\ \mathrm{ad}^*_v w(u) = w(ad_v u))$ **(A48)** $\mathrm{ad}^*_{(f\partial_x, a)}(udx^2, c) =$

$(2f'u + fu' + cf''')dx^2, 0)$ which arises from the identity **(A49)** $\langle[(f\partial_x, a), (g\partial_x, b)], (udx^2, c)\rangle = \langle(g\partial_x, b), \mathrm{ad}^*_{(f\partial_x,a)}(udx^2, c)\rangle$. Now for $S^1$ there are no boundary terms in integration (for single valued functions) so, integrating by parts,

$$\int_{S^1} g(2f'u + fu' + cf''')dx = \int_{S^1} [u(gf' - fg') + cf'g'']dx. \tag{28}$$

Now a function $H$ on $\mathfrak{g} = \mathrm{Vir}$ determines a tautological inertia operator $A : \mathrm{Vir} \to \mathrm{Vir}^* : (u\partial_x, c) \to (udx^2, c)$ and hence a quadratic Hamiltonian on $\mathrm{Vir}^*$ via

$$H(udx^2, c) = \frac{1}{2}\langle(u\partial_x, c), (udx^2, c)\rangle = \frac{1}{2}\langle(u\partial_x, c), A(u\partial_x, c)\rangle. \tag{29}$$

Following ref. [30] the corresponding Euler equation is $\dot{m} = -\mathrm{ad}^*_{A^{-1}m} m$ $(m \in \hat{\mathfrak{g}})$ which here takes the form **(A50)** $\partial_t(udx^2, c) = -\mathrm{ad}^*_{A^{-1}(udx^2,c)}(udx^2, c)$. Using **(A51)** $(f\partial_x, a) = A^{-1}(udx^2, c) = (u\partial_x, c)$ this becomes an equation **(A52)** $\partial_t u = -2u'u - uu' - cu''' = -3uu' - cu'''$ where $c$ is independent of time.

# 6 *Q*-versions

## 6.1 *Q*-Virasoro

We recall first from refs. [16, 31] the following information regarding $q$-Virasoro constructions. Thus work on $S^1$ with $(q \neq 0, \pm 1)$

$$\partial_q z = \frac{q^m z^m - q^{-m} z^m}{(q - q^{-1})z} = z^{m-1}[m], \qquad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{30}$$

We adapt the formalism of ref. [31] as follows. Let $D_n = -z^{n+1}\partial$ with $\partial : z^m \to q^m[m]z^{m-1}$ so $\partial \sim \partial_q \tau$ where $\tau f(z) = f(qz)$. Generally we will think of $z = e^{i\theta} \in S^1$ so $(1/2\pi i)\int_{S^1} z^n dz = (1/2\pi)\int z^{n+1} d\theta = \delta_{(-1,0)}$ which will be written as **(A52)** $\int z^n = \delta_{(-1,0)}$. Write also **(A53)** $\ell_n \sim D_n = -z^{n+1}\partial_q \tau$ and it is known that $q$-brackets are needed now where **(A54)** $[\ell_m, \ell_n]_q = q^{m-n}\ell_m\ell_n - q^{n-m}\ell_n\ell_m = [m-n]\ell_{m+n}$. For a central term in a putative $\mathrm{Vir}_q$ one wants (cf. refs. [16, 31]) a formula **(A55)** $c[m+1][m][m-1]\delta_{m+m,0}$ (see below for an optimal term). First we want to formulate the $q$-bracket in terms of vector fields as follows (the central term will be added later). This can be done as a direct calculation using the basic definition of $\partial$ above (cf. also (37) below). Thus

$$[z^n\partial, z^m\partial]_q \sim q^{n-m} z^n \partial(z^m\partial) - q^{m-n} z^m \partial(z^n\partial) = [n-m](-z^{m+n-1}\partial). \tag{31}$$

Let now $v \sim \sum a_n z^n$ and $w \sim \sum b_m z^m$; then we define a bracket in $Vec(S^1)$ via **(A56)** $[v\partial, w\partial]_q = -\sum a_n b_m [n-m] z^{m+n-1}\partial$. We defined a bracket of vector fields in ref. [16] so that from **(A56)** there resulted a correspondence **(A57)** $v'w - vw' \sim -[v\partial_x, w\partial_x] \sim -[v\partial, w\partial]_q = -\{(\tau v)(\partial_q w) - (\tau w)(\partial_q v)\}\tau$. This dangling $\tau$ creates some complications and is removed below.

**Remark 2.** In ref. [31] one defines the $q$-analogue of the enveloping algebra of the Witt algebra $\mathfrak{W}$ as the associative algebra $\mathfrak{U}_q(\mathfrak{W})$ having generators $\ell_m$ $(m \in \mathbb{Z})$ and relations **(A54)**. The $q$-deformed Virasoro algebra is defined as the associative algebra $\mathfrak{U}_q(\mathrm{Vir})$ having generators $\ell_m$ $(m \in \mathbb{Z})$ and relations ($q \neq$ root of unity)

$$q^{m-n}\ell_m\ell_n - q^{n-m}\ell_n\ell_m = [m-n]\ell_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3]\langle m\rangle}\hat{c}, \tag{32}$$

where $\langle m\rangle = q^m + q^{-m}$ and $\hat{c}\ell_m = q^{2m}\ell_m\hat{c}$ (thus $\hat{c}$ is an operator which we examine below and we refer to refs. [16, 31] for the central term). Then $\mathfrak{U}_q(\mathrm{Vir}) \sim \mathrm{Vir}_q$ is a $\mathbb{Z}$ graded algebra with deg $(\ell_m) = m$ and deg $(\hat{c}) = 0$. One also introduces in ref. [31] a larger algebra $\mathfrak{U}(V_q) =$ associative algebra generated by $J^{\pm 1}$, $\hat{c}$, $d_m$ $(m \in \mathbb{Z})$ with relations

$$JJ^{-1} = J^{-1}J = 1, \qquad Jd_mJ^{-1} = q^m d_m, \qquad \hat{c}J = J\hat{c}, \qquad \hat{c}d_m = q^m d_m\hat{c}, \tag{33}$$
$$q^m d_m d_n J - q^n d_n d_m J = [m-n]d_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3]\langle m\rangle}\hat{c}.$$

The subalgebra of $\mathfrak{U}(V_q)$ generated by $\ell'_m = d_mJ$ and $\hat{c}' = \hat{c}J$ $(m \in \mathbb{Z})$ is the same as $\mathfrak{U}_q(\mathrm{Vir})$. It is stated in ref. [31] that $V_q$ is the universal quantum central extension of $\mathfrak{W}_q$ and thus (33) is better adapted for optimal algebraic and geometric meaning; it is this aspect which we emphasize below.

Now we mimic the framework of Example 6 and it is interesting to note that an ordinary integral $\int_{S^1}$ will suffice. One does not need a Jackson type integral in order to deal with integration by parts. Thus we observe that **(A58)** $\int_{S^1} f = \int \sum f_n z^n = f_{-1}$; $\int \partial_q f = (q - q^{-1})^{-1}(f_0 - f_0) = 0$. Since $\partial_q(fh) = (\tau f)(\partial_q h) + (\partial_q f)(\tau^{-1}h)$ we have an integration by parts formula

$$\int (\tau f)(\partial_q h) = -\int (\partial_q f)(\tau^{-1}h) \Rightarrow \int f\partial_q(\tau h) = -\int \partial_q(\tau^{-1}f)h. \tag{34}$$

This can be written as (recall $\partial \sim \partial_q\tau$) **(A59)** $\int f\partial h = -\int h\hat{\partial} f$ for $\hat{\partial} = \partial_q\tau^{-1}$. Now we think of $\mathfrak{U}_q(\mathrm{Vir})$ with elements $(f\partial, a)$ satisfying **(A60)** $[(f\partial, a), (g\partial, b)] = (-[f\partial, g\partial]_q\partial, \psi(f\partial, g\partial))$. The central term could be defined tentatively via e.g. **(A61)** $\int(\tau\partial^3 f)(\tau g)\hat{c} = \psi(f\partial, g\partial)$ (where one notes that **(A62)** $\psi(f\partial, g\partial) = q^{-1}\int g\partial^3 f\hat{c}$). We will want to put the central operator $\hat{c}$ into the integral, acting on $f$, and will see below that $\hat{c} \sim \tau^2$ for example and $\tau^2 F(z) = F(q^2 z)\tau^2$ so it eventually automatically passes to the right in our qKdV type equations. Hence for the moment think of $\hat{c} = \tau^2$ put into **(A61)** via e.g. $\tau\partial^3\tau^{-2}\hat{c}f \equiv \tau\partial^3 f$ and ignored at the end except when exhibiting formulas like (32) on generators (see also remarks below).

We recall now from ref. [31] (second paper)

$$[d_m, d_n] = Jd_mJ^{-1}d_nJ - Jd_nJ^{-1}d_mJ = q^{-n}Jd_md_n - q^{-m}Jd_nd_m. \tag{35}$$

Further $\hat{c}d_m = q^m d_m\hat{c}$ suggests $\hat{c} = \tau$ here and $d_m$ is being used as $\ell_m\tau^{-1}$. Dropping the minus sign momentarily, from $\ell_m = z^{m+1}\partial_q\tau$ we get then **(A63)** $d_m = z^{m+1}\partial_q$. Then, using **(A64)** $\partial_q\tau = q\tau\partial_q$, one obtains $\tau d_m = \tau(z^{m+1}\partial_q) = q^m z^{m+1}\partial_q\tau$ and $\tau^{-1}d_m = q^{-m}d_m\tau^{-1}$. In addition $Jd_mJ^{-1} = q^m d_m$ corresponds to $Jd_m = q^m d_m J$ so we identify $J = \tau$. Writing $\ell_m = d_mJ = d_m\tau$ we can also easily see that the brackets $[d_m, d_n]$ above are exactly the $q$-brackets **(A65)** $[\ell_m, \ell_n]_q = q^m\ell_m\ell_n - q^n\ell_n\ell_m$.

Now in ref. [31] (second paper) a Jacobi type identity is used involving an operator $\sigma(x) = (1/2)(\tau + \tau^{-1})(x)$ for $x \in \oplus\mathbb{C}d_n$. This seems to be better phrased in terms of an operator **(A66)** $\Gamma(d_p) = \langle p\rangle d_p$ which avoids the need to carry $\tau$ around to other terms. Then we can check that the rule in (33) rewritten as **(A67)** $[d_m, d_n] = [m-n]d_{m+n} + \gamma_m\delta_{m+n,0}\hat{c}$ will yield

$$[[d_m, d_n], \Gamma(d_p)] + [[d_n, d_p], \Gamma(d_m)] + [[d_p, d_m], \Gamma(d_n)] = \Xi_{m,n,p} = 0. \tag{36}$$

This is based on two identities; one is **(A68)** $[m-n][m+n-p]\langle p\rangle + [n-p][n+p-m]\langle m\rangle + [p-m][p+m-n]\langle n\rangle = 0$ and the second is

$$[p+1][p][p-1][m-n] + [m+1][m][m-1][n-p]$$
$$+ [n+1][n][n-1][p-m] = 0 \qquad (m+n+p=0). \tag{37}$$

The proofs are essentially straightforward (cf. ref. [32] for details). Thus $V_q$ will be a genuine central extension of $\mathfrak{W}_q$, with a reasonable Jacobi identity (36).

**Remark 3.** Now in 31–**(A57)** we recall $\ell_m \sim -z^{m+1}\partial = -z^{m+1}\partial_q\tau$ and a $d_m$ formulation would drop the $\tau$. Thus work with $\partial_q$ instead of $\partial = \partial_q\tau$ with **(A69)** $\partial_q z^{p+1} = q^{p+1}z^{p+1}\partial_q + [p+1]z^p\tau^{-1}$ based on $\partial_q f = (\tau f)\partial_q + (\partial_q f)\tau^{-1}$. Then

$$[d_m, d_n] = [z^{m+1}\partial_q, z^{n+1}\partial_q]$$
$$= q^m d_m d_n \tau - q^n d_n d_m \tau = [n-m]z^{n+m+1}\partial_q = [m-n]d_{m+n}. \tag{38}$$

In this context the $\delta_{m+n,0}$ term does not arise. Note here $\tau^{-1}\partial_q = q\partial_q\tau^{-1}$ and **(A70)** $q^{m+1}[n+1] - q^{n+1}[m+1] = [n-m]$. Let now $v \sim \sum v_{n+1}z^{n+1}$ and $w = \sum w_{m+1}z^{m+1}$; then

$$[v\partial_q, w\partial_q] = [\sum v_{n+1}d_n, \sum w_{m+1}d_m] = \sum v_{n+1}w_{m+1}[n-m]z^{n+m+1}\partial_q. \tag{39}$$

Going back to (38) this corresponds then to **(A71)** $[v\partial_q, w\partial_q] = [(\tau v)(\partial_q w) - (\tau w)(\partial_q v)]\partial_q$ and this replaces **(A57)**.

Now we build in a cocycle term by using (33) in (38); a term **(A72)** $[z^{m+1}\partial_q, z^{n+1}\partial_q] = [d_m, d_n] = [m-n]d_{m+n} + \gamma_m\delta_{m+n,0}\tau$ arises. For $[v\partial_q, w\partial_q]$ we get then an additional term

$$\sum v_{n+1}w_{m+1}\gamma_n\delta_{m+n,0}\tau = \sum v_{n+1}w_{-n+1}\frac{[n+1][n][n-1]\tau}{\langle n\rangle[2][3]}. \tag{40}$$

Then for $a = 1/[2][3]$ and $(\tau + \tau^{-1})z^n = (q^n + q^{-n})z^n = \langle n\rangle z^n$ define

$$a\int w(\partial_q^2(\tau+\tau^{-1})^{-1}(\partial_q v)) = \sum v_{n+1}w_{-n+1}\gamma_n = \phi(v\partial_q, w\partial_q). \tag{41}$$

One checks that $\phi$ is antisymmetric and satisfies the $q$-cocycle condition

$$\phi([v\partial_q, w\partial_q], \Gamma(u\partial_q)) + \phi([w\partial_q, u\partial_q], \Gamma(v\partial_q)) + \phi([u\partial_q, v\partial_q], \Gamma(w\partial_q)) = 0. \tag{42}$$

Consequently (cf. ref. [32] for details).

**Theorem 1.** *The term $\phi(v\partial_a, w\partial_q)$ in* (41) *is a q-cocycle and following the constructions in ref.* [16] *one has a possibly canonical qKdV equation in the form* ($a = 1/[2][3]$)

$$u_t + c'\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q u + \partial_q(u\tau u) + \tau^{-1}u\partial_q\tau^{-1}u. \tag{43}$$

**Proof.** We modify slightly the constructions in ref. [16] and write

$$q\langle[f\partial_q, a), (g\partial_q, b)], (u, c)\rangle$$
$$= -q\int[(\tau f)(\partial_q g) - (\tau f)(\partial_q f)]u + caq\int g\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q^2 f. \tag{44}$$

We note from ref. [16] that **(A73)** $q\int fg = \int \tau^{-1}f\tau^{-1}g$ and $\int \partial_q f = 0$ while via **(A74)** $\partial_q(gfu) = \partial_q f\tau^{-1}(fu) + (\tau g)\partial_q(fu)$ (via $\partial_q(ab) = (\tau a)\partial_q b + (\partial_a)\tau^{-1}b$). Then **(A75)** $\int g[\tau^{-1}\times \partial_q(\tau u\tau^2 f) + \tau^{-1}(u\partial_q f) + aqc\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q f]$ follows from (44). Putting $f = u$ we obtain the Euler equation as in ref. [16], namely **(A76)** $qu_t = -qca\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q u - \tau^{-1}\partial_q(\tau u\tau^2 u) - \tau^{-1}(u\partial_q u)$. Using also $\tau^{-1}\partial_q = q\partial_q\tau^{-1}$ we obtain (43) with $c' = ac$. ■

**Remark 4.** In view of the expression **(A77)** $(\tau+\tau^{-1})^{-1} \sim \tau\sum(-1)^n\tau^{2n}$ the equation (43) involves an infinite number of terms (much as are indicated for qKdV in the hierarchy picture in ref. [16]. Since we now have a derivation with all of the classical algebraic and geometrical structure duplicated it seems that (43) could be a good candidate for a canonical form.

**Remark 5.** The Maurer–Cartan (MC) formulas were indicated in Example 5 and a version of this in a $q$-plane context appears in ref. [25]. A version of this can also be developed directly from the discussion of duality (cf. ref. [16] for details). A more interesting version comes from a quantum line $\mathbb{R}^1_q$ coupled with a time variable (e.g. $A_q = C(\mathbb{R}) \otimes \mathbb{R}^1_q$) with e.g. $x\Lambda = q\Lambda x$; $xdx = qdxx$; $dx\Lambda = q\Lambda dx$; $xd\Lambda = qd\Lambda x$; $e_1 x = q\Lambda x$ and $e_1\Lambda = 0$; $df(e_1) = e_1 f$; $e_2\Lambda = q\Lambda x$; $e_2 x = 0$; $df(e_2) = e_2 f$ ($\Lambda$ is introduced for technical reasons (cf. ref. [16]). The MC equations are **(A78)** $d\omega_0 = \mathfrak{Q}(\omega_0 \wedge \omega_1)$; $d\omega_1 = -\omega_0 \wedge \omega_2$; $d\omega_2 = \mathfrak{Q}(\omega_1 \wedge \omega_2)$ ($\mathfrak{Q} = q^2 + q^4$) and we must find expressions $df$ and $d\omega$ for functions and 1-forms. After considerable calculation based on zero curvature ideas (and some assumptions on $\Lambda$, $d\Lambda$) one can say that if $u = u(x,t)$ does not depend on $\Lambda$ a qKdV type equation based on the quantum line arises in the form **(A79)** $u_t = \frac{1}{\mathfrak{Q}^2}\partial^x_{q^{-1}}(\partial^x_q)^2 u + \frac{\eta}{\mathfrak{Q}}\left((\partial^x_{q^{-1}}D_x)^2 - D_x^{-1}(\partial^x_q)^2\right)u - \frac{1}{\mathfrak{Q}}[q(\partial^x_q)D_x u + (1+q)u\partial^x_q u] + \eta u(1-D_x)u$.

# 7 Moyal approach

One knows that Moyal quantization plays an important role in mathematical physics (for discussion cf. ref. [5, 15]). In fact KP can be identified with a Moyal quantization of dKP (cf. ref. [15] for details). Generally speaking one writes

$$f * g = f \exp\left[\kappa(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right] g = f\left(x + \kappa\overrightarrow{\partial}_p, p - \kappa\overrightarrow{\partial}_x\right) g(x,p) \tag{45}$$

and $\{f,g\}_M = (f*g - g*f)/2\kappa$ is the Moyal bracket. This leads to **(A80)** $\{f,g\}_M = \frac{1}{\kappa}\{f \sin[\kappa(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)]g\}$. One can also write (using $x$, $t$ variables and $\kappa \sim \theta/2$) **(A81)** $f * g = m \circ \exp(\theta P/2)(f \otimes g)$; $m(f \otimes g) = fg$; $P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t$. Now go to ref. [28] as developed in ref. [15] where one deals with bicomplexes (BC) $M = \oplus_{r\geq 0} M^r$ with linear maps $d, \delta : M^r \to M^r$ satisfying **(A82)** $d^2 = \delta^2 = d\delta + \delta d = 0$ (note $\delta$ is not the standard metric adjoint). In the spirit of Examples 1, 2 one develops zero curvature equations to produce integrable equations such as KdV and KP.

**Example 7.** Let $M = C^\infty(\mathbb{R}^3) \otimes \Lambda_2$ ($\Lambda_2 \sim \oplus_0^2 \Lambda^j$) with **(A83)** $df = (f_t - f_{xxx})\tau + (1/2)(f_y - f_{xx})\xi$ and $\delta f = (3/2)(f_y + f_{xx})\tau + f_x\xi$ where $\tau$, $\xi \in \Lambda^1$. Deform now (or "dress") d to $Df = df + \delta(vf) - v\delta(f)$ so

$$Df = [f_t - f_{xxx} + (3/2)(v_y + v_{xx})f + 3v_x f_x]\tau + (1/2)(f_y - f_{xx} + 2v_x f)\xi. \tag{46}$$

Then the required BC condition $D^2 = 0$ becomes **(A84)** $v_{xt} - (1/4)v_{xxxx} + 3v_x v_{xx} - (3/4)v_{yy} = 0$ which is equivalent to KP for $u \sim -v_x$. There are also other forms of dressing of one BC to another while preserving the BC conditions. The underlying ideas here are zero curvature, cohomology (to get hierarchies), and gauge transformations (involving Seiberg–Witten (SW) maps – cf. ref. [5, 15]). In particular SW maps preserve zero curvature and e.g. solutions of the KdV equation determine solutions of a noncommutative KdV (NCKdV) equation in a manner similar to what happens with the SW map between commutative and noncommutative gauge theories (cf. ref. [15, 28] for details). Thus take $M = C^\infty(\mathbb{R}^2) \otimes \Lambda_2$ with $\tau$, $\xi \in \Lambda^1$ satisfying $\tau^2 = \xi^2 = \tau\xi + \xi\tau = 0$. One can then define $d$, $\delta$ on $M^0 = C^\infty(\mathbb{R}^2)$ and extend by linearity via $d(f\tau + h\xi) = (df)\tau + (dh)\xi$ for example. Start with **(A85)** $df = -f_{xx}\xi + (f_t + 4f_{xxx})\tau$ and $\delta f = f_x\xi - 3f_{xx}\tau$ (similar to Example 1). Apply a dressing to d in the form ($u = \phi_x$)

$$Df = df + \delta(\phi * f) - \phi * \delta f = -(f_{xx} + u * f)\xi + (f_t + 4f_{xxx} - 6u * f_x - 3u_x * f)\tau \tag{47}$$

(note $\partial_x$ and $\partial_t$ are derivations for $*$ which is a product as in (45) based on $x$, $t$). The only nontrivial BC equation is now $D^2 = 0$ and this is equivalent to the NCKdV equation **(A86)** $u_t + u_{xxx} - 3(u * u_x + u_x * u) = 0$. There are many more developments but we stop here.

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