ANALYSIS OF NUCLEON-NUCLEON SCATTERING EXPERIMENTS (*)

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The experiments reported here today show clearly that we are entering a new phase in the study of the two-nucleon interaction in the elastic scattering region. A theorist must be somewhat apologetic about past failures to present any reliable predictions to guide the course of these experiments. You may recall that the triple scattering experiments at 310 MeV were in part motivated by this failure, and were intended to provide a direct determination of the nucleon-nucleon scattering matrix at this energy, taking from theory only those basic elements which allow a phase shift analysis to be performed. This was only partially successful in that eight distinct phase shift solutions were found [1]. Theory was of modest help in showing that three of these were incompatible with the final state interaction in $\pi^+ + d \rightarrow p + p$. More recently, recognition of the fact that the one pion exchange contribution (OPEC) to the nuclear interaction, which can be predicted exactly, can be expected to dominate the scattering in higher angular momentum states [2], allowed the ambiguity to be further reduced to two physically distinct phase shift solutions [3]. Very similar solutions were found at 210 MeV [4], together with a new type which has recently been excluded by the measurement of D [5]. As Tinlot has just reported, two large angle measurements of A make it very probable that one of the two remaining solutions is spurious, leaving for the first time a unique phase shift solution for proton-proton scattering at high energy. The close similarity between the phase shift sets at 210 and 310 MeV make it very reasonable to assume that Solution 2 at 310 MeV is also spurious.

The situation at lower energies is not so clean. Until the conflict between the two sets of experiments at 145 MeV is decided experimentally, one cannot talk of an experimentally determined scattering matrix at that energy. At 95 MeV, there is certainly more than one way to fit the existing data. However, no model yet proposed for the two-nucleon interaction could be reconciled with a negative ${}^1D_2$ phase shift at that energy. If one accepts this theoretical restriction, both Perring [6] and our own group [7] are left with only one solution; this looks quite reasonable when compared to the (we believe) correct solutions at 210 and 310 MeV. At 68 MeV one must also add the restriction that ε_2 be negative to distinguish the analog of this solution. These solutions are given in Table I, but the reader is warned that the errors and correlations in error are large.

Below 68 MeV the situation is still worse, primarily because no triple scattering data exist. This is not surprising since, even if values of the S and D phases are assumed known, the differential cross section allows four different sets of P waves [8]. Polarizations are small, and have only been used to show that one must also include ${}^3P_2 - {}^3F_2$ coupling in the analysis at 40 MeV [9]. Higher partial waves are small, so including OPEC in the analysis does not help [10]. Recently Iwadare [11] has examined the effect of ${}^3P_2 - {}^3F_2$ coupling on the four-fold P-wave ambiguity. He comes to the somewhat encouraging conclusion that even a rough measurement of $C_{nn}(90°)$ and *either* R or A at the same angle would uniquely determine the scattering matrix at that energy. Therefore these difficult triple scattering measurements, which we trust the development of polarized beams and/or targets will soon make possible, need only be set up for single measurements; the added complication of measuring angular distributions is unnecessary.

We conclude from this survey that the experimental determination of a unique proton-proton scattering

(*) This work was performed under the auspices of the U.S. Atomic Energy Commission.

Table I. Unique proton-proton phase shift solutions
Nuclear bar phase shifts in degrees. Phases not listed are taken from OPEC.

Energy MeV	$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3P_2$	ε_2	$^3F_2$	$^3F_3$	$^3F_4$
68 (*)	30.45°	2.62°	18.59°	−10.49°	6.96°	−2.38°			
95 (*)	22.17°	3.87°	14.24°	−11.98°	11.17°	−2.78°			
210	6.27°	7.36°	−0.434°	−20.85°	16.92°	−2.09°	−0.15°	−2.24°	0.60°
310	−8.92°	11.87°	−11.27°	−27.49°	16.65°	−1.55°	1.21°	−3.53°	3.54

(*) Unique only if it is assumed that $^1D_2>0$ *and* that $^3F_2$ and higher phases are given by OPEC.

matrix has probably already succeeded at four energies, and is likely to succeed at reasonably well-spaced values of the energy up to 400 MeV in the next few years. Clearly, the next step should be to try to tie all these different experiments together to produce a unique model for the proton-proton interaction over this whole range. At first sight, it might be thought that simply combining the analyses already discussed would suffice. A minor objection to this is that the complicated triple scattering experiments can only be expected to be performed at a very few energies, while a much larger amount of very precise differential cross section measurements at other energies is already available. It is obvious that the precision should be improved if these data also can be utilized. The second reason for attempting to get the highest possible precision out of the proton-proton data is that it will give us an enormously powerful tool in analysing neutron-proton scattering. For the latter, no triple scattering experiments exist at present, and it is very doubtful if they can ever be performed to the requisite precision to allow unique analysis at a single energy. Therefore, it is only by knowing the proton-proton scattering matrix in detail, and using charge independence in the analysis, that we can reasonably hope to attain precision in *n-p* analyses. Since it is often inconvenient to arrange *n-p* experiments at precisely the same energy as the *p-p* experiments, and the data would almost certainly be less complete, the need for precise knowledge of the *p-p* scattering matrix at *all* energies is clear.

In order to carry through this program, the model we use to fit the proton-proton scattering data would ideally have to meet certain specific requirements. (1) It should incorporate all that is reliably known theoretically about the two-nucleon interaction. (2) The parameters introduced to obtain detailed agreement with experiment should be capable of theoretical interpretation consistent with the theory. (3) Since these parameters are to be adjusted to give a least-squares fit to the data, they should be capable of being varied independently without losing any theoretical or experimental information already obtained. (4) The framework should be flexible enough to incorporate new theoretical or experimental information as it becomes available. Before discussing a specific proposal which I believe goes a long way toward satisfying these requirements, I will discuss briefly other approaches to this problem.

Nearly all previous attempts to predict or describe the energy dependence of the two-nucleon scattering matrix have been based on potential models [12]. The one-pion exchange potential (OPEP) is reliably given by almost any type of meson theoretic calculation, and is in good agreement with those parts of the low-energy nucleon-nucleon data which are sensitive to it [13]. Unfortunately, there is at present no agreement between theorists as to what meson theory predicts for the shorter range parts of the potential [14], and considerable question whether it is meaningful to take the static limit of the field theoretic interaction in the way this has usually been done [15]. All existing models are therefore to a considerable extent phenomenological.

Gammel, Christian and Thaler [16], found it impossible to reconcile even differential cross section and polarization data with a 14-parameter hard-core model of Yukawa type, but the inclusion of an $L\cdot S$

term by Gammel and Thaler [17] allowed semi-quantitative agreement with the triple scattering experiments at 310 MeV as well as with the data at lower energies. Signell and Marshak [18] found that, by adding an $L\cdot S$ term to the potential calculated by Gartenhaus [19] from the cutoff version of meson theory, they could fit experiment reasonably well below 200 MeV and not so well above that energy. In both cases, the range of the $L\cdot S$ term was incompatible with very general requirements of meson theory. By adjusting the shorter range parts of the potential predicted by one calculation from meson theory, it may be possible to fit all data below 150 MeV without using an $L\cdot S$ term [13, 20]. Recently Bryan [21] and Hamada [22] have succeeded in obtaining reasonable agreement with the data up to 310 MeV with models that have the OPEP tail and spin-orbit terms whose ranges at least are compatible with meson theory, but at the expense of considerable complication in the inner regions of the potential. Hamada also uses a quadratic $L\cdot S$ interaction to improve agreement with the singlet phases at 310 MeV.

The most recent models do reproduce in detail the observed features of proton-proton scattering. However, when the discrepancy between prediction and experiment is computed at any particular energy, they do not in general give a value of χ^2 which would be considered acceptable for a phase shift analysis. A much more ambitious attempt to achieve this type of precision has been made by Breit and his collaborators [23]. Starting from the phase shift values given by one or another of the potentials, they introduce parameters which allow departures from these values as a function of energy and attempt to fit the data at all energies simultaneously. By examining the centrifugal shielding of the inner regions of the potential in the higher angular momentum states, they determine beyond which l value the phases can be reliably calculated from OPEP, and include as many of these as make a significant contribution to the scattering amplitude. When they analyze the data, assuming charge independence, they obtain the very interesting result that the value of the pion-nucleon coupling constant (which determines the strength of the OPEP tail) that best fits the data is the same for n-p as for p-p scattering [24].

The alternative approach to this problem I am about to propose has been made possible by the new representation of the two-body scattering matrix conjectured by Mandelstam [25]. Eden [26] reported yesterday that Mandelstam's conjecture as to the analytic structure of the scattering matrix is valid to all orders in perturbation theory for the case of interest here (no anomalous thresholds). It could still turn out, of course, that it is not a rigorous consequence of the postulates of local field theory; but it can also be thought of as the most successful candidate up to now for replacing that somewhat dubious structure. In either case the uncertainties lie only in the behavior of the scattering matrix at very high energy, and therefore affect the present application only through the theoretical interpretation of the phenomenological parameters that we require to describe the interaction at very short distances. This gives me confidence that this framework will prove quantitatively reliable for a description of the nucleon-nucleon scattering up to at least 400 MeV. Since we work from the start with the Lorentz-invariant theory, we successfully bypass the ambiguities that have plagued field theoretical calculations of the potential. Further, Wong and I believe that we have successfully included electrostatic effects in this framework, and can give at least as adequate discussion of the question of charge independence as is possible in terms of potential models.

In order to see clearly the relationship between potential models and a discussion based on the analytic properties of the scattering matrix, I will start by discussing scattering by a Yukawa potential $-f^2e^{-\mu r}/r$ as predicted by the Schroedinger equation, but in language that can be easily adapted to the field theoretic case. The radial wave function is given by

$$u_l = j_l + f^2 M\int_0^\infty \frac{e^{-\mu r'}}{r'} G(r,r') u_{l'}(qr')dr' \tag{1}$$

where
$$G(r,r') = \frac{1}{q} u_l^R(qr) u_l^I(qr') \tag{2}$$

Hence the tangent of the phase shift is

$$\frac{1}{q}\tan\delta_l = f^2 M\int_0^\infty \frac{e^{-\mu r'}}{r'} u_l^R(qr')\psi_l(q,r')dr' \tag{3}$$

$$= f^2 M\int_0^\infty \frac{e^{-\mu r'}}{r'} [u_l^R(qr')]^2 dr' + O(f^4M^2)$$

Looking at this expression as a function of q^2, considered as a complex variable, we see that $(\tan\delta)/q$ is real for real $q^2 > -\frac{1}{4}\mu^2$ but develops an imaginary part for q^2 more negative. Further, $(\tan\delta)/q$ is a real analytic function [i.e., $f^*(q^{2*}) = f(q^2)$] of q^2 in the entire complex q^2 plane, except for a cut running along the negative real axis up to this point. If we attempt to compute this cut as a power series in f^2M by computing the Born series from Eq. (3) this cut contains all powers of f^2 (cf. Fig. 1 (*a*)). It turns out to be more useful to consider instead the partial wave scattering amplitude

$$t_l(q^2) \equiv \frac{\tan\delta_l}{q(1-i\tan\delta_l)} \equiv \frac{1}{q\tan\delta_l - iq} \equiv \frac{e^{i\delta_l}\sin\delta_l}{q}. \quad (4)$$

We see that t has a second cut starting at $q^2 = 0$ and running along the positive real axis. We will refer to this as the unitarity cut, since it arises from the unitarity requirement that t be given by the right-hand expression in Eq. (4) with δ_l real. The cut for $q^2 \leq -\frac{1}{4}\mu^2$ will be called the interaction cut, since it disappears as f^2 goes to zero. If we now consider the Born series for t generated by substituting the Born series for $(\tan\delta_l)/q$ into Eq. (4), we find that for $-\mu^2 \leq q^2 \leq -\mu^2/4$ all the higher powers of f^2M in the cut from $\tan\delta_l$ are identically cancelled by the expansion of $(1-i\tan\delta_l)^{-1}$. For $-9\mu^2/4 < q^2 \leq -\mu^2$ we find that only the f^2M term already given by the first Born approximation and an $(f^2M)^2$ term survive, and so on (cf. Fig. 1 (*b*)).

Turning to field theory, we find that the Mandelstam representation implies the same analytic structure for t, but gives a different prescription for the imaginary part of t along the interaction cut. Briefly, this is given by the imaginary part of the transition amplitude for $N+\bar{N}\rightarrow I$, where I is some intermediate state of mass m_I which has the same quantum numbers as the nucleon-antinucleon state we are considering. This imaginary part is zero for $q^2 > -\frac{1}{4}m_I^2$, which determines the starting point of the cut for that particular intermediate state. For the nucleon-nucleon system, the lowest mass state is the single pion, and it turns out that the discontinuity is given *exactly* by the imaginary part of the first Born approximation, if we use in it the physical value for the pion mass and coupling constant. The next lowest mass state is two pions, so this contributes a discontinuity starting at $-\mu^2$. We can compute this discontinuity if we know the transition amplitude from the nucleon-antinucleon system to two *interacting* pions. For

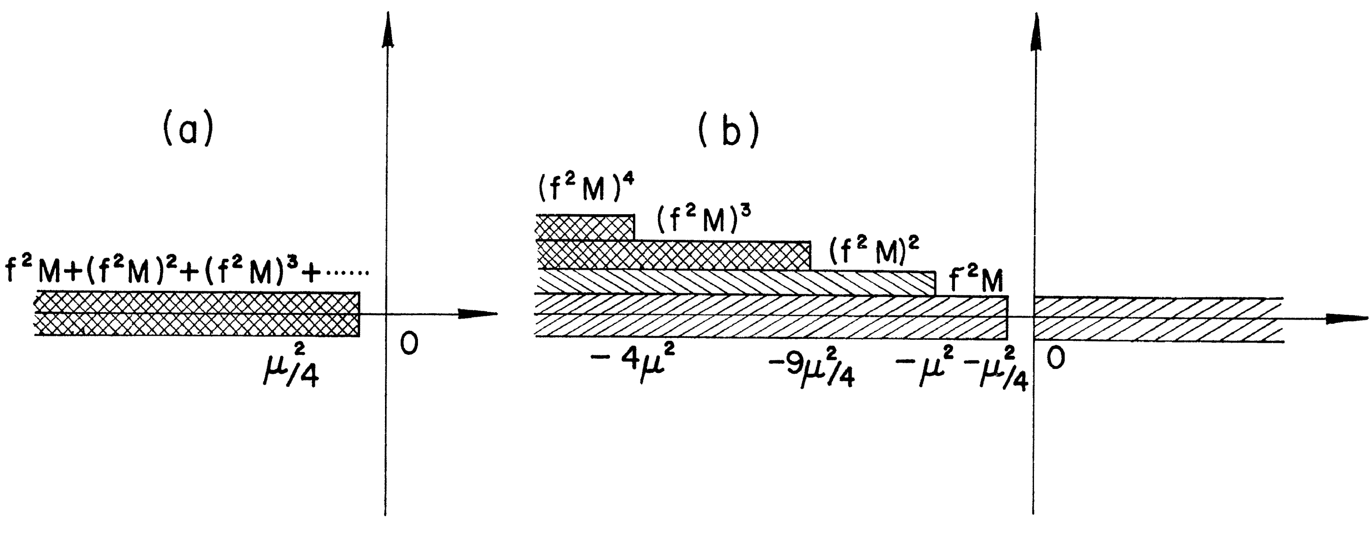

Fig. 1 Analytic structure predicted by the Schroedinger equation for scattering by a Yukawa potential $-f^2Me^{-\mu r}/r$. *(a)* Singularities of $(\tan\delta)/q$ in the complex q^2 plane. *(b)* Singularities of $e^{i\delta}(\sin\delta)/q$ in the complex q^2 plane.

more negative values of q^2 we must consider still more complicated multiparticle states.

We can now discuss the difference between a " meson potential " and field theory. The customary procedure has been to interpret the first Born approximation as the Fourier transform of a static potential, which as we have seen generates an interaction cut containing successively higher powers of f^2M as q^2 becomes more negative. From the above discussion we see that these in some sense represent the discontinuity due to *non-interacting* multipion states, and also from the expansion of $(1-i\tan\delta)^{-1}$ a contribution from satisfying the unitarity requirement for $q^2>0$. If we now include in addition the two-meson " potential ", it is a very delicate matter to insure that we do not count twice $(f^2M)^n$ terms already included in the one-pion " potential ". Further, if we stop at this point, we have no guarantee that the discontinuity beyond $-9\mu^2/4$ correctly represents multiparticle exchanges, and there is no unambiguous way of taking out these unwanted terms if we represent multiparticle exchanges phenomenologically by assuming some radial form at short distances. The approximation suggested by the Mandelstam approach is superior, in that it allows us to include exactly the entire effect of those interacting states whose transition amplitudes we know how to compute. Further, even though we are unable to compute the multiparticle exchanges, we know that their transition amplitudes are limited by unitarity, so we can make realistic estimates of the magnitude of the discontinuity for larger negative values of q^2. As can be seen from the formalism about to be developed, the structure of the singularities beyond $q^2=-9\mu^2/4$ will seriously affect the behavior of the phase shifts at 95 MeV and above, so I do not believe it is possible to make realistic calculations from meson " potentials " in this region. This is borne out by the extreme sensit.vity of such calculations to the precise behavior of the potential near the hard core, which there is no way to calculate. I therefore think it will be much safer to base our phenomenological treatment of the short range interactions firmly on field theory.

We will now show how this discussion of the analytic structure of partial wave amplitudes can be made to give us a practical tool for the analysis of nucleon-nucleon scattering experiments. Using for simplicity non-relativistic kinematics (which restriction we remove below), it has been shown [27] that the phase shift is given by

$$\frac{1}{q}\tan\delta=\frac{\displaystyle\int_0^1 dy\frac{R(y)E(y)}{1+4yq^2}}{\displaystyle 1+2q^2\int_0^1 dy\frac{\sqrt{y}R(y)E(y)}{1+4yq^2}} \tag{5}$$

where $E(y)$ is the solution of the integral equation

$$E(y)=1+\frac{1}{2}\int_0^1 dz\frac{R(z)E(z)}{\sqrt{y}+\sqrt{z}} \tag{6}$$

Here we have assumed that the imaginary part of t for $q^2<-\frac{1}{4}\mu^2$ (which is to be computed from field theory) is given by $\pi\, r(q^2)$ and have defined the quantity $R(y)$ which specifies this interaction by

$$-\frac{4q^2}{\mu^2}r(q^2)\equiv R(y),\ y=-\mu^2/4q^2.$$

Returning for the moment to our discussion of the Yukawa potential, we see that $R(y)$ for $0.25<y\leq 1$ can be roughly pictured as the magnitude and sign of the interaction due to quanta of pionic mass. Similarly, the region $0.11<y\leq 0.25$ corresponds to twice pionic mass, and so on. Thus, using the usual uncertainty principle relationship between quantum mass and range, we can still form an intuitive picture of the strength of the interaction as a function of the separation of the particles, even though we have abandoned potential models.

Since there is no hope at present of making more than rough estimates for $R(y)$ for $y<0.11$ (three-pion and other more massive multiparticle exchanges), we will introduce phenomenological parameters to describe this region. Note that a numerical solution of Eq. (6) in which the integral is replaced by a finite sum is identical to assuming that $R(y)$ is replaced by a finite number of delta functions, or that the continuous-interaction cut in t is replaced by a finite number of poles. We therefore use this approximation for the phenomenological region, and obtain for the phase shift

$$\frac{1}{q}\tan\delta = \frac{\int_0^1 dy \frac{R_N(y)f(y)}{1+4yq^2} + \sum_i \alpha_i \left(\frac{1}{1+4y_i q^2} + \int_0^1 dy \frac{R_N(y)g_i(y)}{1+4yq^2} \right)}{1+2q^2 \left[\int_0^1 dy \frac{\sqrt{y}R_N(y)f(y)}{1+4yq^2} + \sum_i \alpha_i \left(\frac{\sqrt{y_i}}{1+4y_i q^2} + \int_0^1 dy \frac{\sqrt{y}R_N(y)g_i(y)}{1+4yq^2} \right) \right]} \tag{7}$$

Here $R_N(y)$ is that part of the discontinuity in t which we believe we have computed reliably from field theory, and the positions of our phenomenological poles y_i are to lie between 0 and 0.11. The functions $f(y)$ and $g_i(y)$ are solutions of the integral equations

$$f(y) = 1 + \frac{1}{2}\int_0^1 dz \frac{R_N(z)f(z)}{\sqrt{y}+\sqrt{z}}, \tag{8}$$

and

$$g_i(y) = \frac{1}{2(\sqrt{y}+\sqrt{y_i})} + \frac{1}{2}\int_0^1 dz \frac{R_N(z)g_i(z)}{\sqrt{y}+\sqrt{z}}. \tag{9}$$

Thus if we pick some reasonable set of y_i's, the phase shift is given in terms of known functions of the energy $q^2 = T_{\text{lab}}/41.65$ MeV, and the arbitrary parameters α_i. The simple dependence on our phenomenological parameters α_i is of considerable practical importance. It is easy to show [28] that we can determine some of them by requiring the phase shift to have particular values at particular energies, and define new functions of the energy and a new set of parameters which can be varied independently. Thus we have achieved our goal of deriving an explicit formula for the energy dependence of the nucleon-nucleon phase shifts which (1) incorporates all that can be computed from theory $[R_N(y)]$, (2) can include specific experimental information, and (3) can be fitted to arbitrarily extensive nucleon-nucleon scattering data by the least squares adjustment of independent parameters of theoretical significance.

Note that thus far we have no guarantee that the phase shift will go to zero as q^{2l+1} at zero energy. This would be true only if $R(y)$ were exact. We can therefore always pick l of the α_i to insure this behavior. This has the expected effect of making the phase shifts for large l depend less and less on the short range part of the interaction as l increases.

In order to obtain the relativistic generalization of this approach, it is necessary to go to the formulation of the scattering amplitude in terms of the five relativistic invariants (for each isotopic spin state) and investigate in detail what combination of these in fact has only the Mandelstam singularities. This is a very complicated calculation [29, 30], but the result is simple for the uncoupled states: we must consider $(M^2+q^2)^{\frac{1}{2}}e^{i\delta}$ $(\sin\delta)/q$ rather than $t(q^2)$. This introduces a somewhat more complicated kernel in the integral equation for $E(y)$ and corresponding complications in the formula for $q \operatorname{ctn} \delta$, but does not alter anything else I have said so far. A more serious change is that the S-wave equation is no longer convergent unless we introduce at least one phenomenological parameter. If one goes into details it turns out that this constant is not arbitrary, but is in principle determined by a relationship between the different amplitudes at $q^2 = -M^2$. Since this falls in a region where we cannot make a reliable calculation, we know in advance that we will have to use instead the empirical values for the S-wave scattering lengths (or in the isotopic singlet case, the binding energy of the deuteron). The relative success or failure of the theory will be measured by how few additional parameters will be required.

So far we have ignored the correction due to inelastic scattering which makes the imaginary part of t^{-1} increase slowly from $-q$ to $-2q$ above 280 MeV. D. Wong [31] has made an approximate calculation of this effect and the effect of using relativistic kinematics. While the inelastic correction is small in the 200-300 MeV region, using relativistic kinematics changes the phase shift considerably from the value given by the nonrelativistic calculation. Therefore the more elaborate integral equation which includes this correction [28] may well be required for quantitative work.

Before we can proceed to a detailed comparison of this field theoretic approach with experiments of the existing precision, and in particular before we can hope to invoke charge independence in going from *p-p* to *n-p* data, it is necessary to include the the Coulomb interaction between two protons in our treatment. D. Wong and I[32] have done this by using the fact that at low energy the wave function must approach the usual combination of regular and irregular hypergeometric functions at large distances, and by exploiting the close connection between the one meson exchange interaction and the Yukawa potential discussed above. The details are rather complicated, but the results are reasonably simple. Briefly, we are led to consider the amplitude

$$\frac{e^{i\delta}\sin\delta}{e^2 q} = \frac{1}{S(q^2)-i(C^2 q - iQ)} = \frac{1}{S(x)-\dfrac{f(x)}{2\sqrt{x}}} \qquad (10)$$

$$(x = -\mu^2/4q^2).$$

Here S is the usual effective range function first introduced by Breit[33]

$$S(q^2) = C^2 q \operatorname{ctn}\delta + Q, \qquad (11)$$

$$C^2 = \frac{2\pi\eta}{e^{2\pi\eta}-1} \qquad \eta = \frac{Me^2}{2q}$$

$$Q = Me^2\left[\tfrac{1}{2}\psi(i\eta)+\tfrac{1}{2}\psi(-i\eta)-\ln\eta\right]. \qquad (12)$$

If we let e^2 go to zero, $S \to q\operatorname{ctn}\delta$ and $f(x)\to 1$, so this reduces to the partial wave amplitude already considered. (If $l\neq 0$, we must also include a factor $1/q^{2l}\prod_{\lambda=1}^{l}(1+\eta^2/\lambda^2)$. Although this amplitude has an essential singularity at $q^2 = 0$, there are no more singularities before the start of the one-pion cut at $-\frac{1}{4}\mu^2$. We find that we can compute this discontinuity exactly for a Coulomb plus a Yukawa potential down to $q^2 = -\mu^2$. To better than 1% it is the same as for the *n-p* case except for a factor

$$h(x) = \left(\frac{1}{\sqrt{x}}-1\right)^{-Me^2\sqrt{x}}. \qquad (13)$$

Beyond this point, this factor is everywhere less than 3%, so in our language the charge independence hypothesis consists of the statement that $R(y)$ is the same for *n-p* and *p-p* scattering, except for this factor, and possible further modifications of the order of a few % for $y<0.25$. In this multiparticle exchange region there can also be electrodynamic corrections (e.g., the charged-neutral pion mass differences $\sim 3.5\%$) of the same order, which we do not know how to compute. But we feel justified in stating that we have included exactly the electrostatic correction to single pion exchange.

Further, we can show that $S(q^2)$ has no singularities for q^2 greater than zero or in the neighborhood of $|q^2| = 0$. Therefore we can approximate S by a sequence of poles along the interaction cut, and require their residues to give the discontinuity in the partial wave amplitude we have just computed. Passing to the limit of a continuous distribution of poles, this leads to the integral equation

$$E_{pp}^{(x)} = 1+\frac{1}{2}\int_0^1 dy\frac{\sqrt{x}f(x)-\sqrt{y}f(y)}{x-y}R(y)h(y)E_{pp}(y) \qquad (14)$$

and the formula for the phase shift

$$C^2 q \operatorname{ctn}\delta + Q = \frac{1+2q^2\displaystyle\int_0^1 dx\frac{\sqrt{x}f(x)R(x)h(x)E_{pp}(x)}{1+4xq^2}}{\displaystyle\int_0^1 dx\frac{R(x)h(x)E_{pp}(x)}{1+4xq^2}}. \qquad (15)$$

It is interesting to see what happens if we make the shape-independent approximation $S(q^2) = A + Rq^2$, which is equivalent to replacing the interaction cut by a single pole, and require this pole to have the same position and strength for the *n-p* and *p-p* $^1S_0$ state. This leads to the simple result

$$A_{pp}-A_{np} = \frac{1}{4\sqrt{z}}(1+g(z)-2f(z)),$$

$$R_{pp}-R_{np} = 2\sqrt{z}(g(z)-f(z)). \qquad (16)$$

The function $g(z)$ is defined by $\dfrac{d}{dz}\displaystyle\int\frac{f(z)}{\sqrt{z}} = -\frac{1}{2z^{3/2}}g(z)$, and, like f, goes to 1 if we let the electric charge go to zero. z is just the position of the pole, so it measures the average range of the nuclear force. If we fit two low-energy *p-p* phase shifts, $z = 0.226$, corresponding to an average range of about $0.7\,f$ and a scattering length of -7.8 f. Using these numbers and an *n-p* scattering length of -23.74 f, we find that Eq. (16) predicts $A_{pp}-A_{np}$ with only about 10% error. It also states that the *n-p* effective range should be about 10% smaller than the *p-p* effective range. They are probably closer to each other than this; the failure of the simple formula comes from replacing the long range part of the interaction by a single pole.

We now improve the calculation by taking $R(y) = f^2 M$ (single pion exchange) and a single pole to represent multimeson exchange, which we again adjust to fit two *p-p* phases. Since the *n-p* effective range is not well known, we take the position of the pole to be the same as in the *p-p* calculation, and adjust its strength to fit the *n-p* scattering length. Our statement of the charge independence hypothesis is that this strength should turn out to be within $\sim 3\%$ of strength of the pole which fits the *p-p* data.

There is an additional complication in calculating *n-p* scattering, since both charged and neutral pions are exchanged. Because of their mass difference, there are two single pion cuts of different length, and

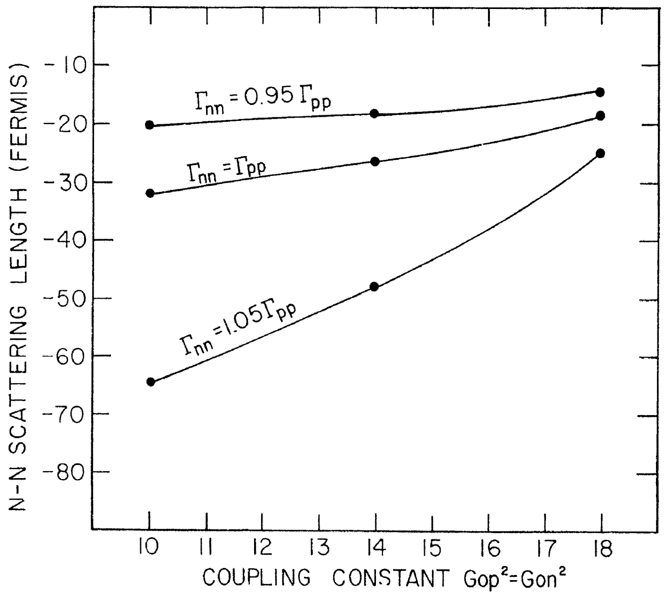

Fig. 3 Prediction of the $n-n$ scattering length, assuming exact charge symmetry or a 5% departure from it for multiparticle exchange.

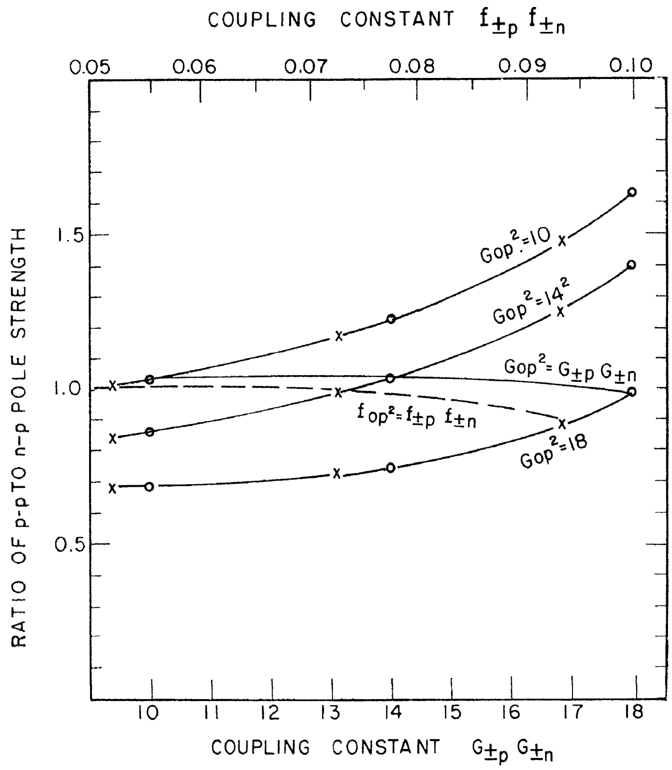

Fig. 2 Ratio of the multiparticle exchange pole strength required to fit low energy $p-p$ scattering to that needed to fit the $n-p$ scattering length as a function of the coupling constant for charged pions to neutron and proton, for fixed coupling constant for neutral pions to protons.

the coupling constants could also differ. We therefore present our result for the ratio of multiparticle pole strengths as a function of the charged coupling constant (see Fig. 2.) Looking at this figure, we see that if the coupling constants are equal, then the pole strengths for multiparticle exchange must also be very nearly the same, or vice versa. In view of Breit's result [24] that the coupling constants are nearly the same, we believe this is convincing evidence for the charge independence of the $^1S_0$ state. Of course this is not a new result, since comparable results have been obtained with potential models. However, we think it important because it makes clear the fact that the residual failure of charge independence in the multiparticle exchange interaction (if any) falls well within our *a priori* estimate. Thus our calculation gives more precision than that of Riaz-ud-din [34], who arrived at a large uncertainty for this effect.

A second advantage to our approach is that we believe it allows a reliable calculation of the neutron-neutron scattering length, the results of which are given in Fig. 3. We see that for a coupling constant of 12 ± 2 (ref. [3]) exact charge symmetry would require an *n-n* scattering length of -29 ± 3 f. Failure of charge symmetry by as little as 5% would allow it to lie anywhere between -20 f and -60 f. Therefore the experiment proposed by McVoy [35] to determine this quantity gives a very sensitive test of charge symmetry, and we hope it will soon be attempted.

The calculations just presented can be improved by adding a second phenomenological pole adjusted to fit two more *p-p* phase shifts at higher energy, and thus reproducing the effects explained by a " hard core " in potential models. Unfortunately, these calculations had not been completed when this report was written, so I cannot discuss the " shape sensitivity " of our prediction, or the expected " shape parameter " in *p-p* scattering.

Experimentally, the departure of the effective range function from the shape-independent approximation for *p-p* scattering at low energy is still unknown. Heller [36] obtains a shape parameter of opposite sign to that expected from hard-core potential models, after a careful consideration of vacuum polarization effects, from phase shifts at 0.3839, 2.425, and 4.203 MeV. If instead of the 4.203 MeV phase shift, one uses 1.397 MeV, the sign is reversed. Unfortunately, the slit corrections to the 4.203 MeV experiment are now known to be in error, and Knecht [37] has recently informed me that the published values of the small-angle cross sections at 1.397 and 2.425 MeV are also wrong.

Previous attempts to determine the *n-p* singlet effective range parameters from *n-p* total cross sections were confined to experiments below 4.5 MeV, to avoid contributions from other than S waves, and could only be carried out by assuming some value for the shape parameter in both singlet and triplet states. However, we now have good reason to trust the OPEC predictions for $l \geq 2$ to much higher energy than this, and know that the isotopic triplet P waves are smaller than those predicted by OPEC at low energy [10, 11]. The isotopic singlet P wave is small even at 90 MeV, as is shown by the near symmetry of the *n-p* differential cross section about 90° c.m., so we can again assume OPEC gives an upper limit. We also know that the ${}^3S_1 - {}^3D_1$ coupling is well represented by OPEC [38]. I therefore assume that the total cross-section for $l > 0$ is at most equal to OPEC up to at least 14 MeV, and see whether or not subtracting this term from the total cross section affects the analysis.

A second objection to the analysis I am about to present might be that it should be sensitive to the triplet cross section. However, $q \operatorname{ctn} \delta_t$ is zero near 14 MeV, so the triplet cross section $\sigma_t = 3\pi/(q^2 \operatorname{ctn}^2 \delta_t + + q^2)$ is very insensitive to the triplet effective range parameters in this region. I have checked this by performing the analysis for extreme values of the triplet shape parameter, and verified that it does not change the results appreciably. The binding energy of the deuteron and the singlet scattering length introduce negligible error compared to the errors in the cross sections. The uncertainity in the triplet scattering length, taken to be 5.39 ± 0.03 f, is significant. I therefore adjust it at the same time as I fit the singlet effective range. From the above argument, it is sensitive to a different energy region than the singlet effective range; indeed I find that the correlation in error is small. E. M. Hafner has made a selection of seven total cross section measurements between 1 and 14 MeV using criteria of experimental reliability. The results of my least-squares fit to these values are given in Table II.

We see that, independent of the amount of scattering in the higher partial waves, the best value for the *n-p* effective range is 2.67 or 2.68 ± 0.03 f, and the shape parameter is in better agreement with the value predicted by one meson exchange plus a single attractive pole than that predicted by hard core potential models. The value predicted by the calculation just reported, assuming $G^2 = 12 \pm 2$ and an *n-p* pole fitted to the observed scattering length is 2.68 ± 0.07 f, in excellent agreement with this result. Until we have completed the two pole calculation which is fitted to a negative S phase at high energy, we cannot be sure that the predicted shape parameter will not change sign, and (as would then happen) the predicted effective range become smaller.

Finally, I have a few theoretical remarks to make about some results of an energy dependent phase shift analysis that has been carried out by H. P. Stapp, M. J. Moravcsik and myself (see below). Functional forms not exhibiting all of the required analyticity properties are being used for early exploratory work. For the S-wave the " *SF* " form is being used and this is equivalent to the replacement of the interaction cut by a finite number of poles.

If we calculate the positions of these poles from the values of the parameters obtained from experiment, we find for the ${}^1S_0$ state that only 2 of the 8 poles fall on the negative real q^2 axis. The remaining 6 come in the second sheet of the Riemann surface.

Table II. Analysis of seven $n-p$ total cross sections between 1 and 14 MeV to obtain S-wave effective range parameters: $q \operatorname{ctn} \delta = -1/a + \frac{1}{2} rq^2 - Pr^3 q^4$. (*)

Shape	P_s	r_s(fermi)	a_t(fermi)	2
If everything except S-waves is calculated from OPEC, the best fit is given by:				
Raphael (hard core)	−0.070	2.44±0.02	5.371±0.009	2.73
Shape independent .	0	2.61±0.02	5.385±0.009	1.47
Single pion+pole .	0.033	2.70±0.03	5.392±0.009	1.36
Best fit	0.017±0.028	2.67±0.03	5.390±0.009	1.36
If only S-waves are present, the best fit is given by:				
Raphael	−0.076	2.33±0.02	5.363±0.009	4.16
Shape independent .	0	2.51±0.02	5.377±0.009	2.03
Single pion+pole .	0.040	2.61±0.03	5.384±0.009	1.52
Best fit	0.046±0.028	2.68±0.03	5.390±0.009	1.36

(*) The following assumptions introduce negligible uncertainty into the analysis: $a_s = -23.74$ f, $\frac{1}{2} r_t = \hbar(ME_b)^{-1/2} - \hbar^2/(ME_b a_t)$ with $E_b = 2.226$ MeV. The value $a_t = 5.39 \pm 0.03$ is fitted in addition to the cross sections. $P_s r^3$ is either assigned a value or fitted simultaneously with a_t and r_s.

This means we cannot consistently interpret these two poles as representing the interaction. If we let their strength go to zero, 4 poles would still remain on the second sheet, and the scattering would not go to zero. Technically, this is called the " Castillejo-Dalitz-Dyson " ambiguity [39] and is interpreted as our having found a solution to the analyticity requirements we have imposed, other than the physically correct one.

Fortunately, the two pole calculations have progressed far enough to see what has gone wrong. If the one meson cut is included and only the multiparticle exchanges represented by poles, then it *is* possible to obtain at least approximate agreement with the observed behavior of the $^1S_0$ phase. We conclude that replacing cuts by poles is not a sufficiently accurate approximation for quantitative work, at least for the longest range part of the interaction, and consequently that the more elaborate formulae developed above are indeed required if we wish to give a consistent theoretical interpretation to our fits to empirical data.

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