

Holographic hydrodynamics with a chemical potential

Robert C. Myers,^a Miguel F. Paulos^b and Aninda Sinha^a

^a*Perimeter Institute for Theoretical Physics,*

Waterloo, Ontario N2L 2Y5, Canada

^b*Department of Applied Mathematics and Theoretical Physics,*

Cambridge CB3 0WA, U.K.

E-mail: rmyers@perimeterinstitute.ca, m.f.paulos@damtp.cam.ac.uk,

asinha@perimeterinstitute.ca

ABSTRACT: We consider five-dimensional gravity coupled to a negative cosmological constant and a single U(1) gauge field, including a general set of four-derivative interactions. In this framework, we construct charged planar AdS black hole solutions perturbatively and consider the thermal and hydrodynamic properties of the plasma in the dual CFT. In particular, we calculate the ratio of shear viscosity to entropy density and argue that the violation of the KSS bound is enhanced in the presence of a chemical potential. We also compute the electrical conductivity and comment on various conjectured bounds related to this coefficient.

KEYWORDS: Gauge-gravity correspondence, Black Holes in String Theory, AdS-CFT Correspondence

ARXIV EPRINT: [0903.2834](https://arxiv.org/abs/0903.2834)

Contents

1	Introduction	1
2	Charged black holes in higher derivative gravity	3
2.1	AdS/CFT dictionary	4
2.2	Charged black hole solutions	5
2.3	Black hole thermodynamics	8
3	Holographic hydrodynamics	11
3.1	Corrections to η/s	12
3.2	Conductivity and higher derivative terms	16
4	Discussion	20
A	Field Redefinitions and the four-derivative action	26

1 Introduction

Recent years have seen a remarkable confluence of string theory and nuclear physics, with efforts to gain new theoretical insights into the strongly coupled quark-gluon plasma (sQGP) using holographic techniques [1]. The AdS/CFT correspondence [2, 3] has proven to be a powerful tool to investigate the thermal and hydrodynamic properties for certain strongly coupled gauge theories [4]. Of course, the gauge theories which are amenable to such holographic study are somewhat exotic compared to QCD but one may suppose that certain properties of the corresponding plasmas may be universal. The latter suggestion was reinforced by the observation that the ratio of shear viscosity to entropy density seemed to be a universal property of the holographic theories yielding $\eta/s = 1/4\pi$ [5, 6]. Further, experimental data indicates that this ratio is also unusually small for the sQGP and even appears to yield roughly $\eta/s \sim 1/4\pi$ [7]. It is now well understood that the holographic result $\eta/s = 1/4\pi$ emerges for gauge theories described by Einstein gravity as the gravitational dual. Still this encompasses a remarkably wide class of theories and situations, e.g., with various gauge groups and matter content, with or without chemical potentials, with non-commutative spatial directions or in external background fields [5, 6]. It is also well understood that higher curvature corrections in the gravitational dual will modify this ratio [8]–[14]. In fact, it was shown that for certain theories these corrections produce even lower values [12–14] thus disproving¹ a longstanding conjecture that $\eta/s = 1/4\pi$ represented a strict lower bound for the viscosity of any physical system, i.e., the KSS

¹See also [15].

bound [16]. Still one may interpret these new holographic calculations with higher curvature interactions as broadening the universality class of conformal gauge theories under study [11, 14].

The focus of the present paper is to use the AdS/CFT correspondence to investigate how a nonvanishing chemical potential μ effects the hydrodynamics of strongly coupled gauge theories. In particular, we consider how η/s is modified at finite μ . As noted above, with an Einstein gravity dual, the result remains $\eta/s = 1/4\pi$, even though individually the viscosity and entropy density have a complicated dependence on μ [17, 18]. Hence, μ can only modify this ratio through the correction terms appearing from higher derivative interactions in the gravitational dual, as we will explicitly illustrate. Since the effects of the chemical potential are tied to the higher derivative interactions, it is interesting to examine violations of the KSS bound in this context. For example, one might find that the chemical potential limits any violations and that the bound is restored with sufficiently large μ , i.e., $\eta/s > 1/4\pi$ for $\mu > \mu_c$. However, we identify a broad class of theories where in fact the opposite result is found, i.e., increasing μ only enhances the violation of the KSS bound.

In principle, studying the effect of the chemical potential on hydrodynamic properties is also of phenomenological interest. The higher derivative modifications are associated with corrections emerging from finite N_c and λ in the QCD plasma and these may be significant for the sQGP [10, 11, 14]. So it is again of interest to determine whether finite μ enhances or suppresses these effects. Unfortunately the relevant chemical potential for baryon number is not expected to be large, i.e., $\mu_B \sim 30\text{MeV}$ or $\mu_B/T \lesssim 0.15$ for recent experiments at RHIC [19] and so any effects will be limited. However, they may still play a role as the determination of η/s becomes more precise in the coming years.

Turning to the holographic hydrodynamics described by the charged black holes more broadly, we also investigate the conductivity, σ . It was suggested that the ratio of the conductivity to the shear viscosity could obey a bound similar to η/s [20]. The heuristic reasoning behind this conjecture was as follows [20]: in any four-dimensional CFT, we expect $\eta \sim cT^3$ while $\sigma \sim kT$ where c and k are basically central charges of the CFT. The first of these is related to the total number of degrees of freedom while k is related to the charge degrees of freedom. Thus it is natural to expect an upper bound on $\sigma T^2/(\eta e^2) \propto k/c$, which in turn may be related to the weak gravity conjecture of [21]. While this ratio depends on the relative normalization of the current and the stress tensor in the CFT, it was also suggested in [20] that this relative normalization would not appear in the ratio of the conductivity to the susceptibility and so it may be more natural for this ratio to obey a universal bound. We extend this discussion to a framework of general four-derivative interactions, as described below.

An overview of the paper is as follows: In section 2, we present our action for five-dimensional gravity coupled to a negative cosmological constant and a single U(1) gauge field, including a general set of four-derivative interactions. Further we examine how the higher-derivative terms modify charged planar AdS black holes, both the solution and their thermodynamic properties. In section 3, we investigate the hydrodynamic properties of these black holes with the four-derivative corrections. In particular, we calculate both the viscosity and the conductivity of the dual CFT plasma. Finally, a concluding discussion

is presented in section 4. We also show in appendix A that one can use field redefinitions to reduce the most general four-derivative action to include only the five interactions explicitly studied in main text.

2 Charged black holes in higher derivative gravity

We begin with five-dimensional gravity coupled to a negative cosmological constant and a $U(1)$ gauge field in the following action:

$$I = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[\frac{12}{L^2} + R - \frac{1}{4}F^2 + \frac{\kappa}{3}\varepsilon^{abcde} A_a F_{bc} F_{de} + L^2 \left(c_1 R_{abcd} R^{abcd} \right. \right. \quad (2.1) \\ \left. \left. + c_2 R_{abcd} F^{ab} F^{cd} + c_3 (F^2)^2 + c_4 F^4 + c_5 \varepsilon^{abcde} A_a R_{bcfg} R_{de}{}^{fg} \right) \right],$$

where $F^2 = F_{ab}F^{ab}$ and $F^4 = F^a{}_b F^b{}_c F^c{}_d F^d{}_a$. As well as the conventional Einstein and Maxwell terms, our two-derivative action also includes the Chern-Simons term proportional to ε^{abcde} , which naturally arises in five-dimensional supergravity [22]. The above action (2.1) also contains a general set of four-derivative interactions. We will treat these terms in a perturbative framework where each of the dimensionless coefficients $c_i \ll 1$. As discussed in [14], it is natural to expect that each of these coefficients is suppressed by a factor of ℓ_p^2/L^2 , which we are assuming is very small. We demonstrate in appendix A that within this perturbative framework, one can use field redefinitions to reduce the most general four-derivative action to include only the five interactions appearing above in (2.1).

The metric equation of motion arising from (2.1) is

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{1}{2}F_{ac}F_b{}^c - \frac{1}{8}F^2g_{ab} + \frac{6}{L^2}g_{ab} \quad (2.2) \\ + L^2 c_1 \left(\frac{1}{2}R_{cdef}R^{cdef}g_{ab} - 2R_{(a|cde}R_{|b)}{}^{cde} + 4\nabla^c \nabla^d R_{c(ab)d} \right) \\ + L^2 c_2 \left(\frac{1}{2}R_{cdef}F^{cd}F^{ef}g_{ab} + 3R^{cde}{}_{(a}F_{b)e}F_{cd} + 2\nabla^c \nabla^d (F_{c(a}F_{b)d}) \right) \\ + L^2 c_3 \left(\frac{1}{2}(F^2)^2g_{ab} - 4F^2F_{ac}F_b{}^c \right) + L^2 c_4 \left(\frac{1}{2}F^4g_{ab} - 4F_{ac}F^c{}_d F^d{}_e F^e{}_b \right) \\ + 2L^2 c_5 \varepsilon^{cdef}{}_{(a} (R^g{}_{|b)ef} \nabla_g F_{cd} + 2F_{cd} \nabla_e R_{f|b})$$

while the vector equation of motion is given by

$$\nabla_b F^{ba} + \kappa \varepsilon^{abcde} F_{bc} F_{de} = -4L^2 c_2 \nabla_b (R^{abcd} F_{cd}) \quad (2.3) \\ + 8L^2 c_3 \nabla_b (F^2 F^{ba}) + 8L^2 c_4 \nabla^d (F^a{}_b F^b{}_c F^c{}_d) \\ - L^2 c_5 \varepsilon^{abcde} R_{bcfg} R_{de}{}^{fg}.$$

As discussed above, we will solve these equations of motion perturbatively in the coefficients c_i of the interactions in the four-derivative action.

2.1 AdS/CFT dictionary

With a negative cosmological constant, the gravitational theory described by (2.1) naturally has an AdS_5 vacuum and is dual to a four-dimensional CFT. In this holographic context, the bulk vector field will be dual to the current generating a global $U(1)$ symmetry in the CFT. In all, the action (2.1) is characterized by seven dimensionless parameters: L^3/ℓ_p^3 , κ and the five coefficients c_i . The AdS/CFT correspondence then relates each of these gravitational couplings to various parameters that characterize the dual field theory. For example, the holographic framework relates the two central charges, a and c , of four-dimensional CFT to [23, 24]

$$\frac{L^3}{\ell_p^3} \simeq \frac{c}{\pi^2} \left(1 - \frac{3}{8} \frac{c-a}{c} \right), \quad c_1 \simeq \frac{1}{8} \frac{c-a}{c}. \quad (2.4)$$

As described in [14], a key assumption in working with the effective action (2.1) is that the five-dimensional gravity theory is described by a sensible derivative expansion. That is, we are implicitly assuming that couplings of the four- and higher-derivative interactions are systematically suppressed by powers of the Planck length over the (bare) AdS scale, ℓ_p/L . In particular then, we expect that $c_1 \propto \ell_p^2/L^2 \ll 1$. From the perspective of the AdS/CFT correspondence then, we are restricted by (2.4) to consider CFT's for which

$$c \sim a \gg 1 \quad \text{and} \quad |c-a|/c \ll 1. \quad (2.5)$$

Further, our assumption about the derivative expansion in the gravity action then restricts the size of the field theory parameters related to the four-derivative couplings, i.e., the CFT's of interest should have the corresponding parameters being suppressed by inverse powers of the central charge c .

Turning to the other couplings, it is natural to consider κ and c_5 together since they both appear in interactions proportional to ε^{abcde} . Both of these Chern-Simons-like terms are not invariant under ‘large’ gauge transformations and as a result, in the present holographic context, they play the distinguished role of determining anomalies for the global $U(1)$ symmetry in the dual CFT [25, 26]. In our present analysis, we leave these coefficients to be arbitrary constants but we should also note that their precise values are irrelevant here since these terms play no role below in determining the geometry or thermal properties of our background solutions.

In contrast, the interactions parameterized by c_2 , c_3 and c_4 all play a role in our perturbative analysis of the charged black holes, as well as c_1 . From the point of view of the dual CFT, c_2 characterizes the form of the three-point function of two currents with the stress tensor [27]. Similarly, c_3 and c_4 provide two independent couplings in the four-point function of four currents. Again, we leave all of these coefficients arbitrary in our general analysis. However, in the context of a supersymmetric theory, a special case arises for the R -symmetry current which is in the same supermultiplet as the stress tensor. In this case, all of the corresponding CFT couplings will be proportional to the difference of the central charges [27], as found for c_1 in (2.4). The holographic dual of such a supersymmetric CFT is an $N = 2$ supergravity theory. While the latter may be gauged

or ungauged depending on the details of the CFT, the dual of the R -symmetry current is the particular $U(1)$ vector appearing in the five-dimensional graviton supermultiplet. In this framework, supersymmetry dictates the form of the four-derivative corrections to the leading supergravity action and so all of the relevant couplings c_i are again related [26, 28]. This supersymmetric setting will be of particular interest in our discussion in section 4.

We should note that there is one other (dimensionless) parameter implicit in our analysis, which can be described as the relative normalization of the gauge and gravity kinetic terms or alternatively as the ratio of the five-dimensional gauge coupling to, say, the AdS scale. One can see there is an issue here since in the conventions used in (2.1), the gauge field is dimensionless while a conventional gauge connection should have units of energy or inverse length. Hence, we should scale the gauge field by some appropriate scale, $A_\mu = L_* \tilde{A}_\mu$. With this choice, the Maxwell term in (2.1) becomes $-\frac{1}{4g_5^2} \int d^5x \sqrt{-g} \tilde{F}^2$ where the five-dimensional gauge coupling is given by $g_5^2 = 2\ell_p^3/L_*^2$. In any particular setting, one is typically guided by the details of the AdS/CFT correspondence or the string theory construction to give the proper normalization of the gauge field, i.e., choosing the scale L_* — for example, see [17, 29]. For simplicity, in the following we make a particular convenient choice for L_* , but of course it is a straightforward exercise to reinstate a general L_* in our results.

To close this subsection, we observe that typically in supergravity actions, the gauge kinetic terms will couple to various scalars. From the dual CFT perspective, such coupling would indicate a nontrivial three-point function mixing two currents with some scalar operator. Hence, from this point of view, the action (2.1) is not the most general since we are making a special choice for the form of the vector kinetic term in the two-derivative action. Beyond this choice, we note that while we are also dropping any possible scalar couplings in the four-derivative interactions, such couplings would only contribute at the next order in our perturbative expansion [14].

2.2 Charged black hole solutions

We consider charged planar black hole solutions with the following ansatz:²

$$ds^2 = -\frac{r^2 f(r)}{L^2} dt^2 + \frac{L^2}{r^2 g(r)} dr^2 + \frac{r^2}{L^2} (dx^2 + dy^2 + dz^2), \quad (2.6)$$

$$A_t = h(r).$$

The leading order solution (of the two-derivative equations of motion) may be determined to be:

$$f_0 = g_0 = \left(1 - \frac{r_0^2}{r^2}\right) \left(1 + \frac{r_0^2}{r^2} - \frac{q^2}{r_0^2 r^4}\right), \quad (2.7)$$

$$h_0 = \frac{1}{2} Q L^3 \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) \quad \text{where} \quad Q = \frac{2\sqrt{3}q}{L^4}. \quad (2.8)$$

²Charged black hole solutions with spherical horizons were constructed for a general four-derivative action in [30].

Here r_0 denotes the position of the (outer) event horizon. There is also an inner horizon at

$$r_-^2 = \frac{1}{2}r_0^2 \left(\sqrt{1 + 4\frac{q^2}{r_0^6}} - 1 \right). \quad (2.9)$$

The solution is characterized by the charge density, which is given by $(*F)_{xyz} = Q$, and the mass density, which is proportional to $M = r_0^4 + q^2/r_0^2$. Implicitly, we have fixed the integration constant in h_0 such that the gauge field vanishes at the horizon, as required by regularity.³ This leading order solution is extremal with $q^2/r_0^6 = 2$ for which (2.9) shows the two horizons coincide, i.e., $r_-^2 = r_0^2$. With $q^2/r_0^6 > 2$, $r_-^2 > r_0^2$ and the solutions actually describe the same set of nonextremal black holes as with $q^2/r_0^6 < 2$ but with r_0 and r_- exchanging roles.⁴ Solutions where ratio of charge to mass densities exceeds that in the extremal black hole (i.e., $Q^2/M^{3/2} > 8/\sqrt{3}L^8$) are found by allowing r_0^2 to become negative but, of course, such solutions all contain a naked singularity at $r = 0$.

Now we wish to construct perturbative solutions to first order in the c_i . We maintain the ansatz (2.6) and parameterize the perturbative solution as

$$\begin{aligned} f(r) &= f_0(r)(1 + F(r)), \\ g(r) &= f_0(r)(1 + F(r) + G(r)), \\ h(r) &= h_0(r) + H(r), \end{aligned} \quad (2.10)$$

where $F(r)$, $G(r)$ and $H(r)$ are $O(c_i)$ corrections. It is then straightforward to solve the equations of motion (2.2) and (2.3) to first order:⁵

$$\begin{aligned} 6f_0F(r) &= 2(2c_1 - 3g_1) + \frac{r_0^4}{r^4}f_1 + 12\frac{q^2}{r^6}(h_2 - 26c_1 - 12c_2) + 12\frac{r_0^8}{r^8}\left(1 + \frac{q^2}{r_0^6}\right)^2 c_1 \\ &\quad + 8\frac{q^2r_0^4}{r^{10}}\left(1 + \frac{q^2}{r_0^6}\right)(5c_1 + 6c_2) + \frac{q^4}{r^{12}}(17c_1 - 24(c_2 + 6c_3 + 3c_4)), \end{aligned} \quad (2.11)$$

$$G(r) = g_1 - \frac{8}{3}\frac{q^2}{r^6}(13c_1 + 12c_2), \quad (2.12)$$

$$\begin{aligned} H(r) &= h_1 - \sqrt{3}\frac{q}{Lr^2}h_2 - 8\sqrt{3}\frac{qr_0^4}{Lr^6}\left(1 + \frac{q^2}{r_0^6}\right)c_2 \\ &\quad + \frac{1}{\sqrt{3}}\frac{q^3}{Lr^8}(48c_2 + 144c_3 + 72c_4 - 13c_1), \end{aligned} \quad (2.13)$$

where f_1 , g_1 , h_1 and h_2 are (dimensionless) integration constants.

³To see this, note that the event horizon in (2.6) is the Killing horizon where $|\partial_t|^2 = 0$. However, as a Killing horizon, it also contains the bifurcation surface which is a fixed point of the Killing flow, i.e., $\partial_t = 0$ on the bifurcation surface, as opposed to the previous null condition [31]. Hence if the gauge field A is to be a well defined one-form, then A_t must vanish there. This is, of course, the Lorentzian analog of the topological constraint which arises more intuitively for the corresponding Euclidean black hole.

⁴This symmetry between r_0 and r_- is readily seen by noting that (2.9) comes from demanding the vanishing of the second factor in f_0 , i.e., $r_0^2 r_-^4 + r_0^4 r_-^2 - q^2 = 0$.

⁵Our approach was as follows: Examining the linear combination of the metric equations (2.2) proportional to $G^t_t - f/g G^r_r$ (where G^a_b is the Einstein tensor), one finds a first-order linear ODE for $G(r)$ which is readily soluble. Given $G(r)$, the t component of the vector equations (2.3) is easily solved for $H(r)$. Finally with the solution for these two perturbations, $F(r)$ can be determined by solving the first-order linear ODE coming from the G^r_r equation.

We fix the integration constants as follows:

- The background metric for the dual CFT can be extracted from the asymptotic behaviour in the black hole metric (2.6) as

$$ds_{\text{CFT}}^2 = -f_\infty dt^2 + dx^2 + dy^2 + dz^2 \quad (2.14)$$

where we defined $f_\infty \equiv f(r \rightarrow \infty)$. Hence to fix the speed of light to be one in the dual gauge theory, we require that $f_\infty = 1$. From (2.11), we find

$$g_1 = \frac{2}{3}c_1. \quad (2.15)$$

Note that this fixes the asymptotic behaviour $g(r \rightarrow \infty) \rightarrow 1 + 2c_1/3$ which reflects the fact that, as noted in [14], the AdS scale of the background geometry is perturbed to be

$$\frac{1}{\hat{L}^2} = \frac{1}{L^2} \left(1 + \frac{2}{3}c_1 \right). \quad (2.16)$$

when c_1 is nonvanishing.

- For simplicity, we require a regular $F(r)$ and fix the position of the event horizon to remain at $r = r_0$. That is, we require that $f_0 F(r = r_0) = 0$. Again from (2.11), this fixes f_1 to be

$$\begin{aligned} f_1 = & -2(8c_1 - 3g_1) + 4\frac{q^2}{r_0^6}(62c_1 + 24c_2 - 3h_2) \\ & - \left(\frac{q^2}{r_0^6} \right)^2 (69c_1 + 24(c_2 - 6c_3 - 3c_4)). \end{aligned} \quad (2.17)$$

- We require that A_t vanishes on the horizon — as described in footnote 3. Setting $H(r = r_0) = 0$ in (2.13), we fix h_1 to be

$$h_1 = \sqrt{3}\frac{q}{Lr_0^2}(h_2 + 8c_2) + \frac{1}{\sqrt{3}}\frac{q^3}{Lr_0^8}(13c_1 - 24(c_2 + 6c_3 + 3c_4)). \quad (2.18)$$

- Finally we may use the remaining freedom to require that the charge density is fixed as in the leading order solution, i.e., $(*F)_{xyz} = Q$. The perturbed vector equation of motion (2.3) can be written in the form $\nabla_b X^{ba} = 0$ for the appropriate antisymmetric tensor X_{ab} . In the perturbed solution, $(*X)_{xyz}$ is a constant independent of radius and it is natural to define the charge density to be $(*X)_{xyz} = Q$. This allows us to fix h_2 , which is most simply done by examining this constraint for asymptotic r where

$$\begin{aligned} \lim_{r \rightarrow \infty} (*X)_{xyz} &= \lim_{r \rightarrow \infty} \left[\frac{r^3}{L^3} \sqrt{g/f} (F_{rt} - 8L^2 c_2 R_{rt}{}^{rt} F_{rt}) \right] \\ &\simeq \left(1 + \frac{1}{2}g_1 + h_2 + 8c_2 \right) Q. \end{aligned} \quad (2.19)$$

Hence we fix

$$h_2 = -\frac{1}{2}g_1 - 8c_2 = -\frac{1}{3}c_1 - 8c_2, \quad (2.20)$$

where in the last expression, we substituted for g_1 as in (2.15).

2.3 Black hole thermodynamics

For the thermodynamics of the above charged black holes, let us begin by first reviewing the results for the leading order solution, (2.7) and (2.8).⁶ The temperature of the dual CFT is precisely the Hawking temperature calculated as the inverse of the periodicity of time in the corresponding Euclidean solution:

$$T = \frac{r_0}{\pi L^2} \left(1 - \frac{q^2}{2r_0^6} \right). \quad (2.21)$$

Note that the temperature vanishes for the extremal black hole with $q^2/r_0^6 = 2$.

Next we would like to consider the chemical potential of the system which is related to the asymptotic value of the potential A_t . However, as discussed in section 2.1, we must scale the gauge field by some appropriate scale, $A_\mu = L_* \tilde{A}_\mu$, to produce a chemical potential with the appropriate units of energy. The chemical potential then becomes

$$\mu = \lim_{r \rightarrow \infty} \tilde{A}_t = \frac{\sqrt{3} q}{L_* L r_0^2}. \quad (2.22)$$

In any particular setting, one would be guided by the details of the AdS/CFT construction or the string theory construction to give the proper normalization of the chemical potential. However, for simplicity in our general analysis, we will make the convenient choice

$$L_* = \pi L \quad (2.23)$$

in the following. Of course, it is a straightforward exercise to reinstate a general L_* in the following calculations. Note that with the preceding choice, we may write

$$r_0 = \pi L^2 \frac{T}{2} \left(1 + \sqrt{1 + \frac{2}{3} \frac{\mu^2}{T^2}} \right). \quad (2.24)$$

Further in the extremal limit $T = 0$, the horizon radius remains finite with $r_0 = \pi L^2 \mu / \sqrt{6}$. It will also be convenient to denote the ratio of the chemical potential to the temperature as

$$\bar{\mu} \equiv \frac{L_*}{\pi L} \frac{\mu}{T} = \frac{\sqrt{3} \frac{q}{r_0^3}}{1 - \frac{q^2}{2r_0^6}}. \quad (2.25)$$

This formula may be inverted to yield

$$\frac{q}{r_0^3} = \frac{2}{\sqrt{3}} \bar{\mu} \left(1 + \sqrt{1 + \frac{2}{3} \bar{\mu}^2} \right)^{-1} \simeq \frac{\bar{\mu}}{\sqrt{3}} \left(1 - \frac{1}{6} \bar{\mu}^2 + \frac{1}{18} \bar{\mu}^4 + \dots \right), \quad (2.26)$$

where the last expression is a Taylor series for small $\bar{\mu}$.⁷

⁶The thermodynamics of charged AdS black holes has been well studied [32, 33], of course, but the focus was on solutions with spherical horizons.

⁷We provide such a Taylor series expansion for all of our results with an eye towards the fact that μ_B/T is small at RHIC.

To proceed further, we apply the standard path integral techniques [34] in which we identify the Euclidean action $I_E = W/T$, where $W(T, \mu)$ is the Gibbs free energy, i.e., the thermodynamic potential in the grand canonical ensemble.⁸ To calculate the Euclidean action, as well as the bulk action (2.1), one includes the Gibbons-Hawking boundary term [34] and the appropriate boundary ‘counter-term’ action [23, 35]. Alternatively, one can use background subtraction and consider the difference in the (bulk) action for the charged black hole and for AdS₅ with a constant gauge potential. Further since we are considering planar black holes (2.6), the spatial volume in the dual CFT is infinite and so we divide our result for any extensive quantities by a regulator volume V_x and work with the corresponding density. We do not go into the details of the calculations here but only present the final result for the free energy density:

$$\begin{aligned} w &= -\frac{r_0^4}{2\ell_p^3 L^5} \left(1 + \frac{q^2}{r_0^6} \right) \\ &= -\frac{1}{2\ell_p^3 L^5} \left(r_0^4 + \frac{\pi^2 L^4}{3} \mu^2 r_0^2 \right). \end{aligned} \quad (2.27)$$

While in principle we could use (2.24) to express the free energy density entirely in terms of T and μ , the last expression above with $w(r_0(T, \mu), \mu)$ is sufficient for most calculations. In particular, the standard thermodynamic identities yield the entropy density and the charge density:

$$s = - \left. \frac{\partial w}{\partial T} \right|_{\mu} = \frac{2\pi r_0^3}{\ell_p^3 L^3}, \quad (2.28)$$

$$n_q = - \left. \frac{\partial w}{\partial \mu} \right|_T = \frac{\sqrt{3}\pi}{\ell_p^3 L^3} q. \quad (2.29)$$

Note that the result for the entropy density matches the expected result for the Bekenstein-Hawking entropy of the black hole horizon. Using the previous expressions, we can also express these quantities in terms of the temperature and chemical potential:

$$\begin{aligned} s &= \frac{\pi^4 L^3}{4\ell_p^3} T^3 \left(1 + \sqrt{1 + \frac{2}{3} \bar{\mu}^2} \right)^3 \\ &\simeq \frac{2\pi^4 L^3}{\ell_p^3} T^3 \left(1 + \frac{1}{2} \bar{\mu}^2 + \frac{1}{216} \bar{\mu}^6 + \dots \right), \end{aligned} \quad (2.30)$$

$$\begin{aligned} n_q &= \frac{\pi^4 L^3}{4\ell_p^3} \mu T^2 \left(1 + \sqrt{1 + \frac{2}{3} \bar{\mu}^2} \right)^2 \\ &\simeq \frac{\pi^4 L^3}{\ell_p^3} \mu T^2 \left(1 + \frac{1}{3} \bar{\mu}^2 - \frac{1}{36} \bar{\mu}^4 + \frac{1}{108} \bar{\mu}^6 + \dots \right), \end{aligned} \quad (2.31)$$

⁸One could also consider the microcanonical ensemble with a fixed charge density n_q by making the standard Legendre transform to the Helmholtz free energy: $F(T, n_q) = W(T, \mu) + \int d^3x n_q \mu$. In the AdS₅ language, this corresponds to adding an additional boundary term to the action which ensures that the appropriate boundary condition corresponds to fixing the (radial) electric field, rather than the gauge potential, at asymptotic infinity — for example, see [29, 32].

where in both cases, we have also given a Taylor series for small $\bar{\mu}$. Hence for large T (or small μ/T) our holographic model yields $s \propto T^3$ and $n_q \propto \mu T^2$, as may have been anticipated. We can also combine the above expressions, (2.28) and (2.29), to calculate the energy density:

$$\begin{aligned}\rho_E &= w + Ts + \mu n_q \\ &= \frac{3}{2} \frac{r_0^4}{\ell_p^3 L^5} \left(1 + \frac{q^2}{r_0^6} \right) = \frac{3}{2} \frac{M}{\ell_p^3 L^5}.\end{aligned}\quad (2.32)$$

We now turn to the first order solution (2.10) which takes into account $O(c_i)$ terms in the equations of motion. The temperature of the perturbed black hole becomes

$$\begin{aligned}T &= \frac{r_0}{\pi L^2} \left(1 - \frac{q^2}{2r_0^6} \right) \left(1 + F(r_0) + \frac{1}{2} G(r_0) \right) \\ &= \frac{r_0}{\pi L^2} \left[1 - \frac{5}{3} c_1 - \frac{q^2}{2r_0^6} \left(1 + \frac{31}{3} c_1 + 16c_2 \right) - \left(\frac{q^2}{r_0^6} \right)^2 (9c_1 - 4c_2 - 24(2c_3 + c_4)) \right].\end{aligned}\quad (2.33)$$

Note that the corrections shift the condition for the extremal limit $T = 0$ to be

$$\frac{q^2}{r_0^6} = 2 [1 - 48(c_1 - 2(2c_3 + c_4))]. \quad (2.34)$$

The asymptotic value of A_t determines the chemical potential, as in (2.22). The modified result is given by

$$\begin{aligned}\mu &= \frac{\sqrt{3} q}{\pi L^2 r_0^2} + \frac{h_1}{\pi L} \\ &= \frac{\sqrt{3} q}{\pi L^2 r_0^2} \left[1 - \frac{1}{3} c_1 + \frac{q^2}{r_0^6} \left(\frac{13}{3} c_1 - 8c_2 - 24(2c_3 + c_4) \right) \right],\end{aligned}\quad (2.35)$$

where we have chosen L_* as in (2.23) and h_1 is fixed as in (2.18).

The free energy density is most easily calculated using background subtraction, as described above, with the final result:

$$\begin{aligned}w &= -\frac{r_0^4}{2\ell_p^3 L^5} \left[1 + \frac{19}{3} c_1 + \frac{q^2}{r_0^6} \left(1 - \frac{113}{3} c_1 - 32c_2 \right) + \frac{q^4}{r_0^{12}} \left(\frac{23}{2} c_1 + 4c_2 - 12(2c_3 + c_4) \right) \right] \\ &= -\frac{\pi^4 L^3}{2\ell_p^3} T^4 \left[1 + 13c_1 + \left(1 + \frac{11}{3} c_1 \right) \bar{\mu}^2 \right. \\ &\quad \left. + \left(1 + \frac{26}{3} c_1 + 24(c_2 + 2c_3 + c_4) \right) \frac{\bar{\mu}^4}{6} - \frac{1}{108} (1 - 15c_1) \bar{\mu}^6 + \dots \right].\end{aligned}\quad (2.36)$$

Now using the same thermodynamic identities as above, we arrive at the following expres-

sions for the entropy and charge densities:

$$\begin{aligned}
s &= \frac{2\pi r_0^3}{\ell_p^3 L^3} \left(1 + 8c_1 - 4(7c_1 + 6c_2) \frac{q^2}{r_0^6} \right), \\
&= \frac{\pi^4 L^3}{4\ell_p^3} T^3 \left[\left(1 + \sqrt{1 + \frac{2}{3}\bar{\mu}^2} \right)^3 + \frac{c_1}{3} \left(1 + \sqrt{1 + \frac{2}{3}\bar{\mu}^2} \right) \left(78 - 2\bar{\mu}^2 + 6 \frac{13 + 4\bar{\mu}^2}{\sqrt{1 + \frac{2}{3}\bar{\mu}^2}} \right) \right] \\
&\simeq \frac{2\pi^4 L^3}{\ell_p^3} T^3 \left[1 + 13c_1 + \frac{\bar{\mu}^2}{2} \left(1 + \frac{11}{3}c_1 \right) + \frac{\bar{\mu}^6}{216} (1 - 15c_1) + \dots \right], \tag{2.37}
\end{aligned}$$

$$\begin{aligned}
n_q &= \frac{\pi^4 L^3}{4\ell_p^3} \mu T^2 \left[\left(1 + \sqrt{1 + \frac{2}{3}\bar{\mu}^2} \right)^2 + \frac{c_1}{3} \left(33 + 46\bar{\mu}^2 + \frac{33 - 4\bar{\mu}^2}{\sqrt{1 + \frac{2}{3}\bar{\mu}^2}} \right) + 32(c_2 + 2c_3 + c_4) \bar{\mu}^2 \right] \\
&\simeq \frac{\pi^4 L^3}{\ell_p^3} \mu T^2 \left[1 + \frac{11}{3}c_1 + \frac{\bar{\mu}^2}{3} \left(1 + \frac{26}{3}c_1 + 24(c_2 + 2c_3 + c_4) \right) \right. \\
&\quad \left. - \frac{\bar{\mu}^4}{36} (1 - 15c_1) + \frac{\bar{\mu}^6}{108} \left(1 - \frac{73}{3}c_1 \right) + \dots \right], \tag{2.38}
\end{aligned}$$

Note that the first expression for the entropy density matches precisely the result found using Wald's formula for higher curvature theories [36]. It is interesting to observe that when the entropy is expressed in terms of T and μ , it becomes independent of c_2 , which appears in the original 'geometric' expression for the entropy. In contrast, all of the coefficients, c_1 , c_2 , c_3 and c_4 , appear in the charge density. We also note here that just as for the leading order results, one finds $\rho_E = -3w$ when the first order corrections are included.

3 Holographic hydrodynamics

We now turn to computing the shear viscosity and the conductivity of the holographic plasma represented by the charged black holes in the previous section. We follow the approach of expressing the transport coefficients in terms of field theory correlators using the Kubo formula and then calculating these correlators with holographic techniques [37]. However, as we are working with the higher curvature interactions in (2.1), we must generalize these standard calculations. The analogous calculations for a particular four-curvature interaction first appeared in [8] and the latter are readily adapted to the present case. Our presentation also builds on the recent work of [6] which focussed on the hydrodynamic limit of low frequency and momenta, showing that there is a simple relation between quantities computed in the membrane paradigm approach and those calculated using AdS/CFT. In particular, the shear viscosity of a field theory with a gravity dual may be related in a simple fashion to the membrane coupling constant of a certain minimally coupled scalar in the dual gravitational background. Our calculation closely parallels the discussion of [38] which presented a general framework to calculate the shear viscosity for higher curvature theories.

Before proceeding, we make a change of coordinates $u = r_0^2/r^2$ which is more readily adapted to the hydrodynamic calculations. With this coordinate choice, the background

solution (2.6) becomes

$$ds^2 = -\frac{r_0^2}{L^2} \frac{f(u)}{u} dt^2 + \frac{L^2}{4u^2 g(u)} du^2 + \frac{r_0^2}{L^2} \frac{1}{u} (dx^2 + dy^2 + dz^2), \quad (3.1)$$

$$A_t = h(u),$$

where the leading order solution (2.7) and (2.8) now takes the form:

$$f_0 = g_0 = (1 - u) \left(1 + u - \frac{q^2}{r_0^6} u^2 \right), \quad (3.2)$$

$$h_0 = \frac{\sqrt{3}q}{Lr_0^2} (1 - u). \quad (3.3)$$

Of course, one must also make the appropriate substitution in the perturbative solution given by (2.10)–(2.13). A simplifying feature, however, is that with our choice of the integration constants described above, in particular for f_1 , the event horizon remains fixed at $r = r_0$ or $u = u_0 = 1$ in the perturbative solution. Of course, in terms of the new radial coordinate, the asymptotic boundary corresponds to $u = 0$.

3.1 Corrections to η/s

Kubo's formula relates the shear viscosity to the low frequency and zero momentum limit of the retarded Green's function of the stress tensor in the CFT

$$G_{xy,xy}^R(\omega, \mathbf{k} = 0) = -i \int dt d\mathbf{x} e^{i\omega t} \theta(t) \langle [T_{xy}(x), T_{xy}(0)] \rangle. \quad (3.4)$$

Concretely one has

$$\eta = -\lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_{xy,xy}^R(\omega, \mathbf{k} = 0). \quad (3.5)$$

The retarded Green's function may be computed using the prescription first set out in [37]. Translating the calculation of the correlator to a holographic one, one first finds the effective action for the metric perturbation $h_x{}^y(t, u) = \int \frac{d^4 k}{(2\pi)^4} \phi_k(u) e^{-i\omega t + i k z}$. Evaluating the action (2.1) to quadratic order in the fluctuations $\phi_k(u)$ yields

$$I_\phi^{(2)} = \frac{1}{2\ell_p^3} \int \frac{d^4 k}{(2\pi)^4} du \left(A(u) \phi_k'' \phi_{-k} + B(u) \phi_k' \phi_{-k}' + C(u) \phi_k' \phi_{-k} \right. \\ \left. + D(u) \phi_k \phi_{-k} + E(u) \phi_k'' \phi_{-k}'' + F \phi_k'' \phi_{-k}' \right) + \mathcal{K}. \quad (3.6)$$

This form of the effective action originally appeared in [8], where the effect of certain R^4 terms were considered, but this general form will arise for any action involving any powers of the curvature tensor (but not derivatives of the curvature⁹) and so appears again in the present context with the action (2.1). If we consider an action where the higher derivative terms come coupled through some parameter γ , then the two functions E and F are $O(\gamma)$. Following [8], we have also added a generalized Gibbons-Hawking boundary term \mathcal{K} in (3.6),

$$\mathcal{K} = \frac{1}{2\ell_p^3} \int \frac{d^4 k}{(2\pi)^4} (K_1 + K_2 + K_3)|_{u=0}^{u=1}. \quad (3.7)$$

⁹A generalization to include derivatives of the curvature appears in [38].

with

$$\begin{aligned} K_1 &= -A\phi'_k\phi_{-k} & K_2 &= -\frac{F}{2}\phi'_k\phi'_{-k} \\ K_3 &= E(p_1\phi'_k + 2p_0\phi_k)\phi'_{-k} \end{aligned} \quad (3.8)$$

The first term K_1 is essentially the contribution of original Gibbons-Hawking term while K_2 and K_3 are new $\mathcal{O}(\gamma)$ contributions. In the term K_3 , the coefficient functions p_0, p_1 are defined in terms of the linearized equation of motion for ϕ :

$$\phi'' + p_1\phi' + p_0\phi = \mathcal{O}(\gamma). \quad (3.9)$$

With this boundary term (3.7), the variational principle is valid up to $\mathcal{O}(\gamma^2)$ [8].

Given the effective action, let us proceed in trying to compute the shear viscosity. First, we integrate by parts to rewrite the action in a more symmetric fashion:

$$\begin{aligned} \tilde{I}_\phi^{(2)} &= \frac{1}{2\ell_p^3} \int \frac{d^4k}{(2\pi)^4} du \left((B - A - F'/2)\phi'_k\phi'_{-k} + E\phi''_k\phi''_{-k} \right. \\ &\quad \left. + (D - (C - A')'/2)\phi_k\phi_{-k} \right) + \tilde{\mathcal{K}}. \end{aligned} \quad (3.10)$$

The integration by parts eliminates K_1 and K_2 in the boundary term and $\tilde{\mathcal{K}}$ is given by:

$$\tilde{\mathcal{K}} = \frac{1}{2\ell_p^3} \int \frac{d^4k}{(2\pi)^4} \left(K_3 + \frac{1}{2}(C - A')\phi_k\phi_{-k} \right) \Big|_{u=0}^{u=1} \quad (3.11)$$

At this point, it is convenient to define the (radial) canonical momentum for our effective scalar as:

$$\Pi_k(u) \equiv \frac{\delta \tilde{I}_\phi^{(2)}}{\delta \phi'_{-k}} = \frac{1}{\ell_p^3} \left((B - A - F'/2)\phi'_k(u) - (E\phi''_k(u))' \right), \quad (3.12)$$

where in the variation we regard ϕ'' as $(\phi')'$. The scalar equation of motion then takes the simple form

$$\partial_u \Pi_k(u) = M(u)\phi_k(u), \quad M(u) \equiv \frac{1}{\ell_p^3} (D - (C - A')'/2). \quad (3.13)$$

together with the definition of the canonical momentum (3.12).

To compute the retarded Green's function, we now evaluate the effective action on-shell, which reduces to a boundary term using the linearized equation of motion (3.13)

$$I_{\text{on-shell}} = \int \frac{d^4k}{(2\pi)^4} \mathcal{F}_k \Big|_{u=0}^{u=1}. \quad (3.14)$$

The retarded Green's function is then given by the flux factor [37] evaluated at the asymptotic boundary

$$G_{xy,xy}^R(\omega, \mathbf{k}) = - \lim_{u \rightarrow 0} \frac{2\mathcal{F}_k}{\phi_k(u)\phi_{-k}(u)} \quad (3.15)$$

where the factors $\phi_k(u)\phi_{-k}(u)$ ensure the appropriate normalization for the Green's function. Implicitly, this expression is also evaluated on $\phi(u)$ with infalling boundary conditions at the horizon. In the present case, the flux factor reduces to

$$\begin{aligned} 2\mathcal{F}_k &= \Pi_k \phi_{-k} + (C - A') \phi_k \phi_{-k} + E \phi_k'' \phi_{-k}' + K_3 \\ &= \Pi_k \phi_{-k} + (C - A') \phi_k \phi_{-k} + E p_0 \phi_k' \phi_{-k}, \end{aligned} \quad (3.16)$$

where on the second line we have used the lowest order equation of motion (3.9) for ϕ_k . We see that the flux is given almost entirely by the canonical momentum term. However, at this point we note that, since according to (3.5) the shear viscosity is given by the imaginary part of the Green's function, then the second term will not contribute as $\phi_k \phi_{-k}$ is real. It turns out that we can discard the third term as well since it is of $O(\omega^2)$, as we now explain.

It is an important point that in general the effective mass $M(u)$ in (3.13) is $O(\omega^2)$ and therefore can be set to zero in the low frequency approximation. Consider setting $\phi(u)$ to a constant, in which case the corresponding radial momentum (3.12) automatically vanishes. The equation of motion (3.13) must be satisfied, since a constant $\phi(u)$ simply corresponds to a rotation and rescaling of the x, y coordinates. Therefore $M(u)$ must be $O(\omega)$, but time reversal invariance demands that it must be proportional to ω^2 . We conclude that on general grounds we must have $M(u) = O(\omega^2)$, and therefore can be set to zero in the low frequency limit, which is taken in calculating the shear viscosity via (3.5). In particular, with regard to the flux in (3.16), the third term is proportional to the mass term of the lowest order equation of motion and so is $O(\omega^2)$. Hence this term is also irrelevant in calculating η in the low frequency limit. We conclude that the only relevant piece in the flux is the canonical momentum term, and so

$$\eta = - \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G_{xy,xy}^R(\omega, \mathbf{k} = 0) = \lim_{u, \omega \rightarrow 0} \frac{\Pi(u)}{i\omega \phi(u)}, \quad (3.17)$$

Here $\Pi(u) \equiv \Pi_{\{\omega, \mathbf{k}=0\}}(u)$ and $\phi(u) \equiv \phi_{\{\omega, \mathbf{k}=0\}}(u)$. Further, in this limit, the equation of motion (3.13) for the canonical momentum is simply

$$\partial_u \Pi_k(u) = 0, \quad (3.18)$$

i.e., $\Pi_k(u)$ is independent of the radius. Therefore we are free to evaluate the value of $\Pi(u)$ in (3.17) at any radius and in particular, it can be evaluated at the horizon.

Hence, the final ingredient to evaluate (3.17) is to determine $\omega \phi(u)$. Now to fix the fluctuations, we must impose infalling boundary conditions on $\phi_k(u)$ at the horizon $u = u_0$. These together with regularity at the horizon imply [6]:

$$\begin{aligned} \partial_u \phi(u_0, t) &= -i\omega \left(\sqrt{\frac{g_{uu}}{-g_{tt}}} \right) \Big|_{u_0} \phi(u_0) + \mathcal{O}(\omega^2) \\ \partial_u^2 \phi(u_0, t) &= -i\omega \partial_u \left(\sqrt{\frac{g_{uu}}{-g_{tt}}} \right) \Big|_{u_0} \phi(u_0) + \mathcal{O}(\omega^2) \\ \partial_u^3 \phi(u_0, t) &= -i\omega \partial_u^2 \left(\sqrt{\frac{g_{uu}}{-g_{tt}}} \right) \Big|_{u_0} \phi(u_0) + \mathcal{O}(\omega^2). \end{aligned}$$

Following [6], keeping $\omega\phi(u)$ and $\Pi(u)$ constant in the low frequency limit, the definition of Π implies

$$\omega\phi'(u) = \gamma(C_1(u)\phi''(u) + C_2(u)\phi'''(u)) \quad (3.19)$$

with some functions C_1, C_2 whose detailed form is irrelevant to our purposes. Working perturbatively in γ ,¹⁰ one performs a split $\phi(u) = \phi_0(u) + \gamma\phi_1(u)$. Then, to lowest order in γ , (3.19) yields the solution: $\omega\phi_0(u) = \omega\phi_0(0)$, i.e., $\omega\phi_0(u)$ is also constant in the low frequency limit. At the next order in γ , the equation of motion for $\omega\phi_1(u)$ then also reduces to $\omega\phi_1'(u) = 0$ and again with the solution: $\omega\phi_1(u) = \omega\phi_1(0)$. We conclude that $\omega\phi(u) = \omega\phi(0)$ to leading order in the low frequency limit.

Hence we arrive at the result

$$\eta = \lim_{\omega \rightarrow 0} \frac{\Pi(u)}{i\omega\phi(u)} = \frac{1}{\ell_p^3} (\kappa_2(u_0) + \kappa_4(u_0)) . \quad (3.20)$$

where in the second expression, we have evaluated the ratio at the horizon $u = u_0$ and defined the quantities

$$\kappa_2(u) = \sqrt{-\frac{g_{uu}(u)}{g_{tt}(u)}} \left(A(u) - B(u) + \frac{F'(u)}{2} \right) , \quad \kappa_4(u) = \left(E(u) \left(\sqrt{-\frac{g_{uu}(u)}{g_{tt}(u)}} \right)' \right)' . \quad (3.21)$$

The indices on κ_i indicate the number of derivatives appearing in the corresponding terms in (3.10). Note that up to now our discussion of calculating η has been completely general and this approach applies to any higher derivative action involving powers of the curvature tensor (but not derivatives of the curvature). We have verified that this approach reproduces the known results in the literature for theories containing four-curvature [8, 9] and two-curvature [12, 13] interactions.

Now we specialize the discussion to considering the action (2.1). Using the action given in equation (2.1), the functions A, B, E, F turn out to be:

$$\begin{aligned} \frac{A}{4\sqrt{-g}g^{uu}} &= 1 + 4c_1(f'_0(u) - f_0(u)) \\ \frac{B}{3\sqrt{-g}g^{uu}} &= 1 - \frac{4}{3}c_1 \frac{f_0(u)^2 - 2u^2 f'_0(u)^2}{f_0(u)} \\ \frac{E}{\sqrt{-g}(g^{uu})^2} &= 4c_1 \\ \frac{F}{\sqrt{-g}g^{uu}} &= 16c_1 f'_0(u) \end{aligned} \quad (3.22)$$

where we have used that to lowest order in c_i , $f(u) = g(u) = f_0(u)$. With these expressions

¹⁰For certain higher curvature actions, e.g., Gauss-Bonnet gravity, the equations of motion for $\phi(u)$ are still second order in derivatives, in which case $C_1 = C_2 = 0$ and our conclusion follows immediately.

in hand, it is straightforward to obtain the shear viscosity:¹¹

$$\eta = \frac{r_0^3}{2L^3\ell_P^3} (1 - 8c_1(f_0'' - f_0')) . \quad (3.23)$$

Combining this with our previous result for the entropy density (2.37), we arrive at

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 - 8c_1 + 4(c_1 + 6c_2) \frac{q^2}{r_0^6} \right) . \quad (3.24)$$

This agrees with the well established result when $q = 0$ [12, 13]. Note that c_3 and c_4 do not appear in this expression. We can combine the above with (2.26) to express the ratio in terms of $\bar{\mu} = \mu/T$:

$$\begin{aligned} \frac{\eta}{s} &= \frac{1}{4\pi} \left[1 - 8c_1 + \frac{16\bar{\mu}^2 (c_1 + 6c_2)}{3(1 + \sqrt{1 + 2\bar{\mu}^2/3})^2} \right] \\ &= \frac{1}{4\pi} \left[1 - 8c_1 + \frac{4}{3}(c_1 + 6c_2)\bar{\mu}^2 \left(1 - \frac{1}{3}\bar{\mu}^2 + O(\bar{\mu}^4) \right) \right] . \end{aligned} \quad (3.25)$$

We note that the infalling boundary conditions are modified for the extremal black holes and so in principle, our computation would need to be modified in this case. Nevertheless we can consider our result (3.24) in the extremal limit,¹² where i.e., $q^2/r_0^6 \rightarrow 2 + O(c_i)$. This limit yields at $T = 0$:

$$\frac{\eta}{s} = \frac{1}{4\pi} (1 + 48c_2) . \quad (3.26)$$

Notice that the leading correction is now independent of c_1 unlike the $T \neq 0$ case.

3.2 Conductivity and higher derivative terms

We now turn to computing the DC conductivity in the perturbative background corrected by the four-derivative interactions. This can be obtained by using a Kubo formula similar to the one for the shear viscosity. Let us define

$$G_{x,x}^R(\omega, \mathbf{k} = 0) = -i \int dt d\mathbf{x} e^{i\omega t} \theta(t) \langle [J_x(x), J_x(0)] \rangle , \quad (3.27)$$

where J_μ is the CFT current dual to the bulk gauge field A_μ . Then the DC conductivity is given by:

$$\sigma = - \lim_{\omega \rightarrow 0} \frac{e^2 L_*^2}{\omega} \text{Im} G_{x,x}^R(\omega, \mathbf{k} = 0) . \quad (3.28)$$

Here the factor L_*^2 appears in the prefactor so that σ corresponds to the conductivity of the current dual to the properly normalized potential \tilde{A}_μ . Further, in order to interpret the

¹¹We should point out that to obtain this result the corrections to the lowest order background play no role other than defining the temperature and the chemical potential. This could have been anticipated since the corrected background only comes in at the two derivative level, but there on general grounds the shear viscosity has the universal form $\eta = V_3/(2\ell_P^3)$ where $V_3 = (g_{xx})^{3/2}$ [37].

¹²We can use the leading order result found from (2.21) here since q only appears in the correction term in (3.24).

result as the ‘electrical’ conductivity of the plasma, we imagine coupling the CFT current to an external or auxiliary vector field, following [20, 39]. This auxiliary vector gauges the corresponding global U(1) symmetry in the CFT with a (small) coupling e . Then to leading order in e , the effects of the auxiliary vector are negligible and the conductivity can be determined from the original CFT alone. The same result can also be related to the thermal conductivity [17] which determines the response of the heat flow to temperature gradients, i.e., $T^t{}_i = -\kappa_T \partial_i T$ in the hydrodynamic limit.¹³ The full expression for the thermal conductivity can be written as [17]

$$\kappa_T = \left(\frac{s}{n_q} + \frac{\mu}{T} \right)^2 \frac{T}{e^2} \sigma. \quad (3.29)$$

The computations are most conveniently performed within an effective action approach, as in the previous subsection. Since the A_t component of the bulk vector is nonvanishing in the background (2.6), the perturbations A_x can couple to the shear mode graviton, i.e., metric perturbations of the form h_{xi} . However, gauge invariance imposes a relation between the two sets of perturbations which we use to integrate out the h_{xi} and obtain an action that involves only the A_x fluctuation.

Starting with the action (2.1), we compute the quadratic action for the perturbations

$$\begin{aligned} h_t^x &= \int \frac{d^4 k}{(2\pi)^4} t_k(u) e^{-i\omega t + ikz}, \\ h_u^x &= \int \frac{d^4 k}{(2\pi)^4} h_k(u) e^{-i\omega t + ikz}, \\ A_x &= \int \frac{d^4 k}{(2\pi)^4} a_k(u) e^{-i\omega t + ikz}. \end{aligned} \quad (3.30)$$

To begin, we consider only the leading order background, (2.7) and (2.8), setting $c_i = 0$ in the action. We would like to set the perturbation h_u^x to zero as a gauge choice. The corresponding component of Einstein’s equations then becomes a constraint which yields:

$$g_{xx} t'_k = -A'_t a_k. \quad (3.31)$$

Plugging this constraint back into the effective action along with $h_k(u) = 0$, the quadratic action takes the simple form:

$$\tilde{I}_a^{(2)} = \frac{1}{2\ell_p^3} \int \frac{d^4 k}{(2\pi)^4} du \left(N(u) a'_k a'_{-k} + M(u) a_k a_{-k} \right), \quad (3.32)$$

where

$$N(u) = -\frac{r_0^2}{L^3} f_0(u), \quad M(u) = \frac{L \omega^2}{4u f_0(u)} - \frac{u}{L} A'_t(u)^2. \quad (3.33)$$

The equation of motion is solved near the horizon with the ansatz

$$a_k(u) = C f_0(u)^\alpha \quad (3.34)$$

¹³The full relativistic expression for the heat flow also includes a contribution proportional to the pressure gradient [17].

with $\alpha = \pm i \frac{\omega}{4\pi T}$ as usual. The infalling boundary condition corresponds to choosing the minus sign. The equation of motion for a_k can be re-expressed as

$$\partial_u j_k(u) = \frac{1}{\ell_p^3} M(u) a_k(u) \quad (3.35)$$

where, as in the previous section, we have defined the radial momentum for the effective scalar

$$j_k(u) \equiv \frac{\delta \tilde{I}_a^{(2)}}{\delta a_{-k}'} = \frac{1}{\ell_p^3} N(u) a_{-k}'(u). \quad (3.36)$$

The condition of regularity at the horizon $u = u_0$ corresponds to setting [6]

$$j_k(u_0) = -i\omega \lim_{u \rightarrow u_0} \frac{N(u)}{\ell_p^3} \sqrt{\frac{g_{uu}}{g_{tt}}} a_k(u_0) + \mathcal{O}(\omega^2), \quad (3.37)$$

where we are expanding in small ω with the zero-frequency limit of (3.28) in mind. Next one evaluates the on-shell action to identify the flux factor, which can be written as simply

$$2 \mathcal{F}_k = j_k(u) a_{-k}(u). \quad (3.38)$$

The Green's function (3.27) is given by evaluating the flux with the appropriate normalization at the asymptotic boundary. The DC conductivity (3.28) is then given by a formula analogous to (3.17) for the shear viscosity,

$$\sigma = \lim_{u, \omega \rightarrow 0} \frac{e^2 L_*^2}{\omega} \text{Im} \left[\frac{2 \mathcal{F}_k}{a_k(u) a_{-k}(u)} \right]_{\mathbf{k}=0} = e^2 L_*^2 \lim_{u, \omega \rightarrow 0} \frac{\text{Im}[j_k(u) a_{-k}(u)]}{\omega a_k(u) a_{-k}(u)} \Big|_{\mathbf{k}=0}, \quad (3.39)$$

where it is convenient not to cancel the factors of $a_{-k}(u)$ in the final expression, as will become apparent below. The key difference between the present case and the computation of the shear viscosity is that neither j_k nor ωa_k is independent of the radial position, even in the low frequency limit. As is evident from (3.33), the effective mass $M(u)$ no longer vanishes in this limit and so the equation of motion (3.35) still produces a nontrivial flow in the radial direction. However, if we apply (3.35) in examining the radial evolution of the numerator in (3.39), we find

$$\frac{d}{du} \text{Im}[j_k(u) a_{-k}(u)] = \text{Im}(f_1(u) a_k a_{-k} + f_2(u) j_k j_{-k}) = 0. \quad (3.40)$$

Notice that this result does not rely on the fact that we are taking a low frequency limit. Therefore we are free to evaluate $\text{Im}[j_k(u) a_{-k}(u)]$ at any radius, e.g., at the horizon. However, at the horizon, j_k is constrained by the regularity condition (3.37) and so we may write

$$\sigma = \frac{e^2 L_*^2}{\ell_p^3} \kappa_2^A(u_0) \frac{\mathcal{N}(u_0)}{\mathcal{N}(0)} \Big|_{\mathbf{k}=0}, \quad (3.41)$$

where we have defined

$$\kappa_2^A(u) = -N(u) \sqrt{\frac{g_{uu}}{-g_{tt}}} \quad \text{and} \quad \mathcal{N}_k(u) = a_k(u) a_{-k}(u). \quad (3.42)$$

The quantity $\mathcal{N}(u)$ is real and so independent of ω up to $\mathcal{O}(\omega^2)$. This also means that to this order, $\mathcal{N}(u)$ is completely regular at the horizon, since the logarithmic divergence of $a_k(u)$ there is always accompanied by a factor of $i\omega/T$. Therefore, in computing \mathcal{N} , we are free to solve $a_k(u)$ imposing regularity at the horizon and setting ω to zero, which simplifies the calculation considerably. In the leading order background (3.1)–(3.3), the solution is easy to obtain:

$$a_k(u) = a_k(0) \frac{1 + \frac{q^2}{2r_0^6}(2 - 3u)}{1 + \frac{q^2}{r_0^6}}. \quad (3.43)$$

The leading order conductivity then follows:

$$\begin{aligned} \sigma &= \frac{e^2 L_*^2 r_0}{2 \ell_p^3 L} \left(\frac{1 - \frac{q^2}{2r_0^6}}{1 + \frac{q^2}{r_0^6}} \right)^2 \\ &= \frac{\pi^3 e^2 L^3}{4 \ell_p^3} T \frac{\left(1 + \sqrt{1 + \frac{2}{3} \bar{\mu}^2} \right)^3}{\left(1 + \bar{\mu}^2 + \sqrt{1 + \frac{2}{3} \bar{\mu}^2} \right)^2}, \end{aligned} \quad (3.44)$$

using our previous formulae (2.23)–(2.26).

Extending these calculations to work at first order in the four-derivative couplings c_i is straightforward. One follows the same steps as above. That is, one first computes the quadratic effective action for a_k, t_k, h_k and obtains a constraint upon setting $h_k = 0$. Substituting the constraint back into the action and keeping terms linear in c_i one still gets an action of the form (3.32). In the subsequent steps to calculate σ , the equation of motion for a_k is technically harder to solve and so we only present the results to leading order for small $\bar{\mu}$, i.e., to order $O(\bar{\mu}^2)$ or $O(q^2)$. The conductivity then turns out to be

$$\begin{aligned} \sigma &= \frac{e^2 L_*^2 r_0}{2 \ell_p^3 L} \left(1 + 16c_2 + \frac{q^2}{r_0^6}(-3 + 68c_1 + 40c_2 + 96(2c_3 + c_4)) \right) \\ &= \frac{\pi^3 e^2 L^3 T}{2 \ell_p^3} \left(1 + \frac{5}{3}c_1 + 16c_2 - \frac{5\bar{\mu}^2}{6} \left[1 - \frac{1}{5}(153c_1 + 112c_2 + 192(2c_3 + c_4)) \right] + O(\bar{\mu}^4) \right). \end{aligned} \quad (3.45)$$

Now following [20], we examine the ratio

$$\frac{\sigma T^2}{\eta e^2} = 1 - \frac{4}{3}\bar{\mu}^2 - \frac{10}{3}c_1 + 16c_2 + \frac{8}{3}\bar{\mu}^2[13c_1 + c_2 + 12(2c_3 + c_4)] + O(\bar{\mu}^4). \quad (3.46)$$

As described in the introduction, [20] suggested that the simplicity of the leading result at $\bar{\mu} = 0$, i.e., 1, may indicate that this result is universal. Further they argued that this leading behaviour may then represent a universal upper bound for this ratio. The above result certainly indicates that $\sigma T^2/(\eta e^2) \leq 1$ even for $\mu \neq 0$. However, the leading order result is also modified by the four-derivative couplings and so the upper bound conjectured here may also be violated depending on the precise values of these couplings, similar to what was found for the KSS bound in [12, 13].

In [20], it was also noted that the ratio above depends on the relative normalization of the current and the stress tensor in the CFT but they further observed that this relative normalization does not appear in the ratio of the conductivity to the susceptibility. Hence [20] suggested that it may be more natural for the latter ratio to obey a universal bound. To examine how the higher derivative couplings effect this ratio, we must first calculate the susceptibility Ξ :

$$\Xi(T) \equiv \left. \frac{\partial n_q}{\partial \mu} \right|_T. \quad (3.47)$$

where n_q is defined in (2.29). By using the result (3.45) for σ , we arrive at the following expression for σ/Ξ :

$$\frac{\sigma}{\Xi} = \frac{e^2}{2\pi T} \left(1 - \frac{11}{6} \bar{\mu}^2 - 2c_1 + 16c_2 + \frac{2}{3} \bar{\mu}^2 \left[\frac{115}{3} c_1 - 32c_2 + 12(2c_3 + c_4) \right] + O(\bar{\mu}^4) \right). \quad (3.48)$$

It was conjectured that $e^2/(2\pi T)$ would be the lower bound for σ/Ξ in [20], at least when $\bar{\mu} = 0$. However, this possibility must again be questioned in light of the fact that the universal behaviour observed to leading order is again affected by the four-derivative couplings.

Finally, let us turn to the thermal conductivity (3.29). The following interesting ratio was constructed in [17]:¹⁴

$$\frac{\kappa_T \mu^2}{\eta T} = 4\pi^2 \left(1 + \frac{2}{3} (23c_1 + 24c_2) + \frac{2}{9} (37c_1 + 36(c_2 + 4c_3 + 2c_4)) \bar{\mu}^2 + O(\bar{\mu}^4) \right). \quad (3.49)$$

Here we note that again the four-derivative interactions modify the leading behaviour but, in particular, also introduce a dependence on $\bar{\mu}$, similar to what was found for the ratio η/s in (3.25).

4 Discussion

In this paper, we calculated the thermal and hydrodynamic properties of the CFT plasma dual to a charged planar AdS black hole. These calculations were made within the perturbative framework where the leading Einstein-Maxwell action was extended to include a general set of four-derivative interactions (2.1). We should say that this analysis partially overlaps with previous results and so let us briefly summarize what was already known in the literature: In [40], the authors considered the transport properties of the charged planar AdS black hole solution with the standard two-derivative action, including the electrical conductivity σ . Our results agree with theirs. Born-Infeld black holes were considered in [41], i.e., the two-derivative action was extended to include the combination of four- F terms arising from the expansion of the DBI action. They found that $\eta/s = 1/4\pi$ which is obtained straightforwardly from (3.25) by observing that the ratio is independent of c_3 and c_4 . The effect of adding a Gauss-Bonnet term to the gravitational action was considered in [42] and η/s was calculated in the charged planar AdS black hole background.

¹⁴Note that the leading order result given in [17] was $8\pi^2$ because their normalization for the gauge kinetic term differs by a factor of 2 from that used here.

In [43], this was extended to include both the Gauss-Bonnet term and the two independent four- F terms. In both cases, our results are in agreement with theirs. The work was also extended in [44] to include a dilaton coupling to the Gauss-Bonnet term. However, as discussed in [14], this extra coupling does not effect our results with the present perturbative approach. The authors of [45] had previously considered the thermodynamics with Gauss-Bonnet gravity and the four- F terms for charged AdS black holes with flat, spherical and hyperbolic horizons. Our results for the thermodynamic behaviour agrees with theirs for the flat case. Finally in [46], the effect of the $R_{abcd}F^{ab}F^{cd}$ interaction was considered on σ when $\mu = 0$. Their result is reproduced by setting $c_1 = 0 = \bar{\mu}$ in (3.45). Hence our comprehensive analysis agrees with all previous results where it should.

As noted in subsection 2.3, our result for the entropy density (2.37) agrees with the result found using Wald's formula for higher curvature theories [36]. As expected then, the 'geometric' expression for the entropy only involves c_1 and c_2 since it is only for these couplings that the corresponding four-derivative interactions in (2.1) involve the curvature tensor. It is a nontrivial result that the couplings c_3 and c_4 still do not enter when this result is re-expressed in terms of the temperature and chemical potential. We might note that these two coefficients appear in (2.33) and (2.35) and so a nontrivial cancelation is required for the final result in (2.37) to be independent of these parameters. It is interesting to observe that a similar cancelation occurs for c_2 when the entropy is expressed in terms of T and μ , so that s is also independent of this parameter. Referring back to (2.4), this means that the entropy density is the sum of two contributions proportional to each of the central charges a, c appearing in the CFT

$$s = \frac{9\pi^2}{2}cT^3g(\bar{\mu}) - \frac{5\pi^2}{2}aT^3h(\bar{\mu}), \quad (4.1)$$

where $g(\bar{\mu} = 0) = 1 = h(\bar{\mu} = 0)$. The shear viscosity also depends only on c_1 and c_2 , as can be inferred from (3.24) or (3.25). In this case, c_2 still appears in the result when it is expressed in terms of T and μ but again c_3 and c_4 do not appear. That is, the shear viscosity in the CFT depends on the central charges a, c but also the coupling of the stress tensor to the U(1) current parameterized by c_2 .

As noted above, only c_1 and c_2 appear in corrections to the shear viscosity. This result is in keeping with the spirit of the recent work in [47]. This work considers generalized higher curvature theories of gravity and attempts to relate the shear viscosity to a 'gravitational coupling' evaluated at the horizon with a Wald-like formula [36]. While the conjectured formulae reproduce known results for certain higher curvature actions, it is known that these expressions fail to reproduce the correct shear viscosity in complete generality [38, 48].

All four of the couplings, c_1, c_2, c_3 and c_4 , appear in our expressions for the charge density (2.38) and conductivity (3.45), as well as the free energy and energy densities. Since c_3 and c_4 parameterize couplings in the four-point function of the U(1) currents, it is natural that they play a role in correcting the properties of the CFT plasma directly related to the corresponding charge. It is interesting to note that these two couplings only appear in (2.38) and (3.45) in the combination $2c_3 + c_4$. In fact, examining all of our expressions in sections 2 and 3, one finds that it is only this particular combination of c_3

and c_4 that appears everywhere. Hence if we organized the four- F interactions in the action in terms of $(F^2)^2 - 2F^4$ and, say, F^4 , then the first term would completely decouple from the present analysis. This observation was previously noted in [45]. We should add that this combination seems only to be distinguished by the particular black holes that we are considering here. For example, this combination does not appear as the four- F interaction in the low-energy expansion of the Dirac-Born-Infeld action [49] or in the four-derivative extension of the $N = 2$ supergravity action [26].

In the extremal limit (2.34), the entropy density (2.37) becomes

$$\begin{aligned} s &= \frac{2\pi r_0^3}{\ell_p^3 L^3} (1 - 48(c_1 + c_2)) , \\ &= \frac{\pi^4 L^3}{3\sqrt{6}\ell_p^3} \mu^3 (1 - c_1 - 24c_2) \end{aligned} \quad (4.2)$$

This result reflects the fact that the extremal black hole still has a finite size horizon. The interpretation in terms of the dual gauge theory is that even at zero temperature, a finite chemical potential will produce a deconfined ‘plasma’ in the dual CFT. It is interesting to note that the ratio η/s for this extremal plasma, given in (3.26), only depends on c_2 . We should say that we expect that this ‘exotic’ behaviour of the CFT at zero temperature reflects our restriction of including only the metric and a single vector in the gravitational theory. That is, in many supergravity scenarios, the gauge kinetic terms will couple to various scalars and from the dual CFT perspective, such couplings reflect a nontrivial three-point function mixing two currents with some scalar operator. In such a scenario, the area of the horizon typically shrinks to zero size in the extremal limit [50] and so a finite temperature would be required to produce a deconfined plasma.

In our general analysis, the coefficients c_i are treated as independent couplings. As described above, $N = 2$ supergravity in five dimensions provides a particularly interesting class of theories, since the super-graviton multiplet also contains a $U(1)$ vector. In this case, the bosonic action for the metric and this vector takes precisely the form given in (2.1) with $\kappa = 1/4\sqrt{3}$. However, in this case, the bulk supersymmetry is sufficiently restrictive to constrain all of these four-derivative couplings to be proportional to a single overall constant. Given the result in (2.4), this means that all of these coefficients are proportional to $(c - a)/c$ where a and c are the central charges of the dual supersymmetric CFT. In this case, the vector in the supergravity multiplet is dual to the CFT’s R -current, which is in the same supermultiplet as the stress tensor [27]. Hence the previous result is in agreement that the observation that two- and three-point functions of these two operators in the CFT (at zero temperature and chemical potential) are parameterized entirely by the two central charges [51].

After examining the supergravity action [26] in more detail in appendix A, we find that

$$c_2 = -\frac{1}{2}c_1 \simeq -\frac{1}{16} \frac{c - a}{c} . \quad (4.3)$$

Hence for these theories, from (3.25), the ratio η/s becomes

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[1 - 8c_1 - \frac{32\bar{\mu}^2 c_1}{3 \left(1 + \sqrt{1 + 2\bar{\mu}^2/3} \right)^2} \right] \quad (4.4)$$

in the presence of a chemical potential. It is clear that the sign of the third term is controlled by c_1 and in fact, this sign will be the same as that appearing in the second term. Hence if c_1 is positive, both of these contributions lead to a violation of the conjectured KSS bound [16]. So in this particular class of theories, introducing a chemical potential only makes the violation stronger.¹⁵ For example, we note that for large $\bar{\mu}$ (4.4) yields: $\eta/s \simeq 1/(4\pi) (1 - 24c_1 + O(c_1/\bar{\mu}^2))$. We should also add that in fact it was found that $c > a$ and hence $c_1 > 0$ for all of the examples of superconformal gauge theories examined in [14]. Hence it appears that such violations of the KSS bound should be considered generic rather the exception to rule. We return to this point below.

It is also interesting to examine the bounds conjectured in [20] for the conductivity in the case of supergravity. If we restrict our attention to $\mu = 0$, as was explicitly considered in [20], (3.46) and (3.48) reduce to

$$\left. \frac{\sigma T^2}{\eta e^2} \right|_{\mu=0} = 1 - \frac{10}{3}c_1 + 16c_2, \quad (4.5)$$

$$\left. \frac{\sigma}{\Xi} \right|_{\mu=0} = \frac{e^2}{2\pi T} (1 - 2c_1 + 16c_2). \quad (4.6)$$

Hence while the general expressions also depended on c_3 and c_4 , only c_1 and c_2 appear in both of these ratios when $\bar{\mu}$ vanishes. Now let us consider these results when we substitute the supergravity result (4.3), $c_2 = -c_1/2$, and assume that c_1 is positive, as found in specific examples [13, 14] but is plausibly a general result. In this case, the higher order corrections reduce the value of both of the ratios, (4.5) and (4.6). Hence the first result is in agreement with the conjecture that $\sigma T^2/(\eta e^2)|_{\mu=0} \leq 1$. However, the second ratio was conjectured to obey a lower bound $\sigma/\Xi|_{\mu=0} \geq e^2/(2\pi T)$ and so the correction produces a violation of this conjectured bound.

The fact that the sign of contribution coming from the chemical potential in (4.4) was controlled by c_1 is related, in part, to the fact that only even powers of the chemical potential everywhere in our analysis — of course, n_q has an overall factor of μ . This property arises naturally from the tensor structure of gravitational action (2.1) and the particular background that we are studying, i.e., all of the relevant interactions contain even powers of the field strength F_{ab} . Of course, the bulk vector appears with an odd power in the two Chern-Simons-like terms in (2.1) but these interactions did not play a role in the present calculations, e.g., the two couplings, κ and c_5 , appear nowhere in our results. However, the two derivative coupling κ is known to play a role when rotation is introduced [52]. Further, these couplings should play a role if one considers the magneto-hydrodynamics of the CFT plasma, i.e., if we introduce both bulk electric and magnetic fields, as has been

¹⁵A similar result appears in [53].

recently studied with the AdS/CFT correspondence for three-dimensional field theories [54]. Extending this work to four-dimensional CFT's as considered here may be an interesting direction for future research.

On the other hand, the behaviour noted above suggests that the properties of sQGP studied at RHIC, with $(\mu_B/T)^2 \lesssim 0.02$, should be almost unaffected by the baryon chemical potential. Of course, one may question whether or not this property of our holographic models carries over to the sQGP. However, it seems that the appearance of only even powers of μ should be a general feature emerging from the CPT invariance of the underlying gauge theory, irrespective of whether the latter has a holographic dual or is even conformal. For example, the behaviour of the plasma with a net ‘quark’ density must be identical to that of the charge-conjugate plasma with a net density of ‘anti-quarks’. Hence, one should expect that the thermal and hydrodynamic properties of the gauge theory plasma should generally be even functions of μ .

There was a striking difference in the computations in subsections 3.1 and 3.2. In particular, the shear viscosity could be framed in terms of quantities which were independent of the radius and so the latter could be evaluated at the horizon. Hence the shear viscosity of the CFT could also be interpreted as the shear viscosity associated to the stretched horizon in the membrane paradigm [55]. In contrast, the quantities determining conductivity evolved nontrivially in the radial direction and so the same connection could not be made to the membrane paradigm.

Our analysis of the conductivity contrasts with the discussion in [6] which considered the conductivity with a vanishing chemical potential. Note that in this case, the mixing observed (3.31) vanishes and one may set $h_t^x = 0$. Further with $\mu = 0$, as seen in (3.33), the effective mass $M(u)$ vanishes in the low-frequency limit and so the radial evolution (3.35) becomes trivial. Hence, in the absence of a chemical potential, there is a simple relation between the conductivity in the CFT and the universal conductivity of the stretched horizon, $\sigma_{\text{mb}} = e^2/g_5^2$ [6]:

$$\sigma_{\text{CFT}, \mu=0} = \sigma_{\text{mb}} \sqrt{g_{zz}(u_0)}. \quad (4.7)$$

Since the conductivity is a dimensionful quantity, the factor $\sqrt{g_{zz}(u_0)}$ appears to convert the length scale in the CFT to the corresponding proper length at the horizon. Of course, our conductivity (3.44) simplifies to reproduce this result in the limit that q (or μ) vanishes.

It is interesting to consider the radial flow found with $\mu \neq 0$ by ‘evaluating the conductivity’ at an arbitrary radius, i.e., removing the limit $u \rightarrow 0$ from (3.39). This yields

$$\sigma(u) = \frac{e^2 L_*^2}{\ell_p^3} \kappa_2^A(u_0) \left. \frac{\mathcal{N}(u_0)}{\mathcal{N}(u)} \right|_{\mathbf{k}=0} = \sigma_{\text{CFT}} \left(1 - \frac{3}{2} u \frac{\frac{q^2}{r_0^6}}{1 + \frac{q^2}{r_0^6}} \right)^{-2}, \quad (4.8)$$

where in the second expression, we have evaluated the result for the leading order background with (3.43) and denoted the conductivity in (3.44) as σ_{CFT} . Hence the radial evolution is such that $\sigma(u)$ decreases monotonically as the radius varies from $u = 1$ at the horizon to $u = 0$ at the asymptotic boundary. We also note that the boundary condition at horizon is precisely $\sigma(u = 1) = \sigma_{\text{CFT}, \mu=0} = \sigma_{\text{mb}} r_0/L$. Hence the membrane paradigm still

sets the inner boundary condition for the nontrivial radial evolution but in general then, $\sigma_{\text{CFT}} \leq \sigma_{\text{mb}} r_0/L$, where the equality is only achieved when $\mu = 0$ and there is no radial evolution.

In general, our discussion in subsection 3.2 generalizes the discussion of [6] to include both a finite chemical potential and the effect of higher derivative interactions. While the simplicity of the shear viscosity computation presented there is essentially not effected by these additional complications, there are in fact additional simplifications of the computation beyond our presentation in subsection 3.1. We will examine these further in an upcoming paper [48], as well giving a covariant Wald-type formula for η .

To close, we would like to comment on a possible implication of the ‘gravity as the weakest force’ conjecture, i.e., the recent conjecture in [21] that there should be a general upper bound on the strength of gravity relative to gauge forces in quantum gravity. This conjecture requires that there are always light ‘elementary particles’ with a mass-to-charge ratio smaller than the corresponding ratio for macroscopic extremal black holes and so allow the extremal black holes to decay. Recently, there has been some discussion of this conjecture in the context of the AdS/CFT correspondence and the implications for the spectrum of the CFT [56, 57] — see also footnote 4 in [20]. The relation which we would like to draw relies on the further corollary that higher derivative corrections should reduce the mass-to-charge ratio of extremal black holes in a consistent theory of the quantum gravity [21, 58]. In the present context, this suggests computing the ratio of the energy density to charge density ρ_E/n_q in the extremal limit (2.34)

$$\frac{\rho_E}{n_q} = \left(\frac{\rho_E}{n_q} \right)_0 \left(1 - \frac{47c_1 + 48c_2}{3} \right). \quad (4.9)$$

Here, the quantity $\left(\frac{\rho_E}{n_q} \right)_0 = \frac{3\sqrt{3}}{2\sqrt{2}} \frac{r_0}{\pi L^2}$ is the leading order ‘classical’ result. Hence the weak gravity conjecture would impose the constraint $47c_1 + 48c_2 > 0$. However, if we again consider the specific case of supergravity with $c_2 = -c_1/2$, this constraint becomes simply $c_1 > 0$.

While the above analysis is suggestive, it misses the intent of the discussion in [21, 58] which was phrased in terms of extremal black holes in asymptotically flat space. Their natural assumption was that the ‘quantum’ corrections coming from higher derivative terms in the gravitational action should decrease the mass-to-charge ratio but also do so in a way that the decrease becomes more pronounced for smaller (extremal) black holes. Hence a proper comparison requires repeating the analysis of section 2.2 for charged AdS black holes with spherical horizons. This is a straightforward exercise and the final result replacing (4.9) is

$$\frac{\rho_E}{n_q} = \left(\frac{\rho_E}{n_q} \right)_0 (1 - c_1 f(r_0/L)) \quad \text{with} \quad f(r_0/L) = \frac{7 + 34 \frac{r_0^2}{L^2} + 107 \frac{r_0^4}{L^4} + 138 \frac{r_0^6}{L^6}}{3 \frac{r_0^2}{L^2} \left(1 + 2 \frac{r_0^2}{L^2} \right) \left(2 + 3 \frac{r_0^2}{L^2} \right)}. \quad (4.10)$$

Above r_0 is the position of the horizon in coordinates where the area of the spherical horizon is $2\pi^2 r_0^3$. In fact, a full analysis yields a ratio which depends on all of the c_i but in

presenting (4.10), we have focussed on the supergravity case where all of these dimensionless parameters are proportional to c_1 . One easily verifies that in the limit of large black holes, i.e., $r_0/L \gg 1$, this result reduces to that for the planar black holes given in (4.9) when $c_2 = -c_1/2$. It is also evident that $f(r_0/L)$ is positive for all values of r_0/L and hence the higher derivative corrections always reduce the mass-to-charge ratio of these extremal black holes as long as $c_1 > 0$. As desired, this effect is also largest for small black holes because of the factor of r_0^2/L^2 in the denominator of $f(r_0/L)$. However, it is interesting that $f(r_0/L)$ does not decrease monotonically as r_0/L grows. Instead the function exhibits a local minimum near $r_0 \sim L$, which seems to be an effect of the asymptotic AdS geometry.

In certain cases, it is well understood that there are supersymmetric bound states of giant gravitons [59] carrying the same charges as these charged black holes with spherical horizons. Further that there is a gap in the spectrum between the extremal black hole and these bound states [60]. Hence one can understand the details of realizing the ‘gravity as the weakest force’ conjecture within this framework. However, our interest in these issues comes rather from the general constraint $c_1 > 0$. In particular with (2.4), we can infer that the weak gravity conjecture requires an inequality for the central charges of any four-dimensional conformal field theory with a gravitational dual, namely $c > a$. Of course, this is precisely the inequality which was observed for the broad class of superconformal gauge theories examined in [14]. Hence it seems there may be a deep connection between this provisional observation for superconformal gauge theories and the consistency of their holographic duals as a theories of quantum gravity.

Acknowledgments

It is a pleasure to thank Alex Buchel, Sera Cremonini, Ben Freivogel, Sean Hartnoll and Uli Heinz for useful correspondence and conversations. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. RCM also acknowledges support from an NSERC Discovery grant and funding from the Canadian Institute for Advanced Research. MFP is supported by the Portuguese Fundacao para a Ciencia e Tecnologia, grant SFRH/BD/23438/2005.

A Field Redefinitions and the four-derivative action

In this appendix, we will demonstrate that our action (2.1) contains the most general four-derivative interactions involving a single Maxwell field. We should add that a similar analysis appeared in [30] but they began with a more restricted starting point. First, we consider the leading order two-derivative action

$$I_2 = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[\frac{12}{L^2} + R - \frac{1}{4}F^2 + \frac{\kappa_1}{3} \varepsilon^{abcde} A_a F_{bc} F_{de} \right]. \quad (\text{A.1})$$

Next the most general four-derivative action for gravity coupled to a single U(1) vector takes the form:

$$\begin{aligned}
 I_4 = \frac{L^2}{2\ell_p^3} \int d^5x \sqrt{-g} \Big[& \alpha_1 R^2 + \alpha_2 R_{ab} R^{ab} + \alpha_3 R_{abcd} R^{abcd} \\
 & + \beta_1 R F^2 + \beta_2 R^{ab} F_{ac} F_b{}^c + \beta_3 R_{abcd} F^{ab} F^{cd} + \beta_4 R_{abcd} F^{ac} F^{bd} \\
 & + \delta_1 (F^2)^2 + \delta_2 F^4 + \delta_3 \nabla^a F_{ab} \nabla^c F_c{}^b + \delta_4 \nabla_a F_{bc} \nabla^a F^{bc} \\
 & + \delta_5 \nabla_a F_{bc} \nabla^b F^{ac} + \delta_6 \nabla^2 F_{ab} F^{ab} + \delta_7 \nabla_a \nabla^b F_{bc} F^{ac} + \delta_8 \nabla^b \nabla_a F_{bc} F^{ac} \\
 & + \varepsilon^{abcde} \left(F_{ab} \left(\gamma_1 F^{cd} \nabla^f F_{fe} + \gamma_2 F_{cf} \nabla^f F_{de} + \gamma_3 F_{cf} \nabla_d F_e{}^f \right) + \kappa_2 A_a R_{bcfg} R_{de}{}^{fg} \right) \Big] ,
 \end{aligned} \tag{A.2}$$

where, as in the main text, we use $F^2 = F_{ab} F^{ab}$ and $F^4 = F^a{}_b F^b{}_c F^c{}_d F^d{}_a$. Here all of the coefficients, α_i , β_i , δ_i , γ_i and κ_2 are dimensionless constants, that we expect are generically very small. Many of the four-derivative terms above can be eliminated by simply integrating by parts. For example,

$$\begin{aligned}
 & \int d^5x \sqrt{-g} \nabla^a F_{ab} \nabla^c F_c{}^b \\
 & = \int d^5x \sqrt{-g} \left(\nabla_a F_{bc} \nabla^b F^{ac} - R^{ab} F_{ac} F_b{}^c + R_{abcd} F^{ac} F^{bd} \right) .
 \end{aligned} \tag{A.3}$$

Using integration by parts, as well as the identities $\nabla_{[a} F_{bc]} = 0 = R_{[abc]d}$, one can eliminate β_4 , $\delta_{4,5,6,7,8}$ and $\gamma_{2,3}$. In this way, the general four-derivative action can be reduced to

$$\begin{aligned}
 I_4 = \frac{L^2}{2\ell_p^3} \int d^5x \sqrt{-g} \Big[& \alpha_1 R^2 + \alpha_2 R_{ab} R^{ab} + \alpha_3 R_{abcd} R^{abcd} \\
 & + \beta_1 R F^2 + \beta_2 R^{ab} F_{ac} F_b{}^c + \beta_3 R_{abcd} F^{ab} F^{cd} + \delta_1 (F^2)^2 + \delta_2 F^4 \\
 & + \delta_3 \nabla^a F_{ab} \nabla^c F_c{}^b + \gamma_1 \varepsilon^{abcde} F_{ab} F_{cd} \nabla^f F_{fe} + \kappa_2 \varepsilon^{abcde} A_a R_{bcfg} R_{de}{}^{fg} \Big] .
 \end{aligned} \tag{A.4}$$

Now consider making field redefinitions: $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$ and $A_a \rightarrow A_a + \delta A_a$. The most general field redefinition involving two-derivative contributions can be written

$$\begin{aligned}
 \delta g_{ab} &= \mu_1 L^2 R_{ab} + \mu_2 L^2 F_{ac} F_b{}^c + (\mu_3 L^2 R + \mu_4 L^2 F^2 + \mu_5) g_{ab} , \\
 \delta A_a &= \lambda_1 A_a + \lambda_2 \nabla^b F_{ba} + \lambda_3 \varepsilon_{abcde} F^{bc} F^{de} .
 \end{aligned} \tag{A.5}$$

Note that μ_5 and λ_1 give a (constant) rescalings of the metric and vector, respectively, which will prove useful in the following. We also note that other than the rescaling the field redefinition of the vector involves covariant terms so that the modified action remains invariant with standard gauge transformations. In general, we might note that, aside from the two rescalings, the field redefinitions (A.5) contain six two-derivative terms while the four-derivative action contains eleven interactions. Hence, on general grounds, we expect that we will be left with five independent terms in our higher-order action. With the field redefinitions (A.5), the leading change in the action comes from the variation of (A.1)

$$\begin{aligned}
 \delta I_2 = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \Big\{ & \left[\left(\frac{6}{L^2} + \frac{1}{2} R - \frac{1}{8} F^2 \right) g^{ab} - R^{ab} + \frac{1}{2} F^{ac} F^b{}_c \right] \delta g_{ab} \\
 & + \left(\nabla_b F^{ba} + \kappa_1 \varepsilon^{abcde} F_{bc} F_{de} \right) \delta A_a \Big\} .
 \end{aligned} \tag{A.6}$$

Of course, one should note that we have integrated by parts to produce the expressions in (A.6). Now we will divide this variation into two parts, examining separately the contributions to the four- and two-derivative actions. Beginning with the former, one finds:

$$\begin{aligned} \delta I_4 = \frac{L^2}{2\ell_p^3} \int d^5x \sqrt{-g} \Big[& \left(\frac{\mu_1}{2} + \frac{3}{2}\mu_3 \right) R^2 - \mu_1 R_{ab} R^{ab} + \left(-\frac{\mu_1}{8} + \frac{\mu_2}{2} - \frac{\mu_3}{8} + \frac{3}{2}\mu_4 \right) R F^2 \\ & + \left(\frac{\mu_1}{2} - \mu_2 \right) R^{ab} F_{ac} F_b{}^c + \left(-\frac{\mu_2}{8} - \frac{\mu_4}{8} + 8\kappa_1 \lambda_3 \right) (F^2)^2 \\ & + \left(\frac{\mu_2}{2} - 16\kappa_1 \lambda_3 \right) F^4 + \lambda_2 \nabla^a F_{ab} \nabla^c F_c{}^b + (\kappa_1 \lambda_2 + \lambda_3) \varepsilon^{abcde} F_{ab} F^{cd} \nabla^f F_{fe} \Big]. \end{aligned} \quad (\text{A.7})$$

An obvious choice of the field redefinition parameters is then

$$\begin{aligned} \mu_1 &= \alpha_2, \quad \mu_2 = \beta_2 + \alpha_2/2, & \mu_3 &= -(2\alpha_1 + \alpha_2)/3, \\ \mu_4 &= -(24\beta_1 - 12\beta_2 + 2\alpha_1 - 11\alpha_2)/36, & \lambda_2 &= -\delta_3, & \lambda_3 &= -\gamma_1 + \kappa_1 \delta_3. \end{aligned} \quad (\text{A.8})$$

These choices eliminate six of the four-derivative interactions leaving

$$\begin{aligned} I_4 = \frac{L^2}{2\ell_p^3} \int d^5x \sqrt{-g} \Big[& \alpha_3 R_{abcd} R^{abcd} + \beta_3 R_{abcd} F^{ab} F^{cd} \\ & + \tilde{\delta}_1 (F^2)^2 + \tilde{\delta}_2 F^4 + \kappa_2 \varepsilon^{abcde} A_a R_{bcfg} R_{de}{}^{fg} \Big], \end{aligned} \quad (\text{A.9})$$

where

$$\begin{aligned} \tilde{\delta}_1 &= \delta_1 + \frac{1}{288} (2\alpha_1 - 29\alpha_2) + \frac{1}{12} (\beta_1 - \beta_2) - 8\kappa_1 \gamma_1 + 8\kappa_1^2 \delta_3, \\ \tilde{\delta}_2 &= \delta_2 + \frac{\alpha_2}{4} + \frac{\beta_2}{2} + 16\kappa_1 \gamma_1 - 16\kappa_1^2 \delta_3. \end{aligned} \quad (\text{A.10})$$

One may try to be more clever with the field redefinitions in the present case. In particular, as seen in [45] (or our calculations in sections 2 and 3), when the background only contains a radial electric field, the four- F terms only couple to the graviton and gauge field equations for motion with the combination $2\tilde{\delta}_1 + \tilde{\delta}_2$. Hence might try to make a field redefinition that sets this combination to zero. If we go back to (A.7), we find that the field redefinitions yield the following variation of this linear combination:

$$\Delta [2\tilde{\delta}_1 + \tilde{\delta}_2] = \frac{1}{4} (\mu_2 - \mu_4). \quad (\text{A.11})$$

Hence we can indeed arrange to set this combination of couplings to zero. However, the result will be that we would not be able to eliminate all of the R^2 and RF^2 interactions involving the Ricci tensor and Ricci scalar. Hence such a field redefinition will simply replace the complications of accounting for the four- F interactions with those of accounting for another set of interactions. With respect to the dual CFT, these two four- F interactions define two characteristic parameters that would appear in the four-point function of the dual current. So it seems to natural not to field redefine them away, as the four-point function must be invariant and the previous parameters would just appear from exchanges

between bulk three-point interactions rather than being manifest in a four-point contact interaction. However, given this discussion, we could arrange the four- F terms in (A.9) as

$$\frac{L^2}{2\ell_p^3} \int d^5x \sqrt{-g} \left[\frac{1}{2} (2\tilde{\delta}_1 + \tilde{\delta}_2) (F^2)^2 + \tilde{\delta}_2 \left(F^4 - \frac{1}{2} (F^2)^2 \right) \right], \quad (\text{A.12})$$

in which case, the second term should not contribute in the present calculations.

At this point, we have not yet commented on fixing the values of μ_5 and λ_1 . In this regard, let us consider the effect of the field redefinitions on varying the two-derivative action (A.1):

$$\begin{aligned} \delta I_2 = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} & \left[\frac{12}{L^2} \frac{5}{2} \mu_5 + R \left(6\mu_1 + 30\mu_3 + \frac{3}{2} \mu_5 \right) \right. \\ & \left. + F^2 \left(6\mu_2 + 30\mu_4 - \frac{\mu_5}{8} - \frac{\lambda_1}{2} \right) + \kappa_1 \lambda_1 \varepsilon^{abcde} A_a F_{bc} F_{de} \right]. \end{aligned} \quad (\text{A.13})$$

Hence the convenient choice which we make in fixing these parameters is to set

$$\mu_5 = -4(\mu_1 + 5\mu_3), \quad \lambda_1 = \mu_1 + 12\mu_2 + 5\mu_3 + 60\mu_4. \quad (\text{A.14})$$

This choice leaves the Planck scale fixed, as well as the coefficient for the vector kinetic term. Hence after the field redefinitions, the two-derivative action (A.1) becomes

$$I_2 = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[\frac{12}{\tilde{L}^2} + R - \frac{1}{4} F^2 + \frac{\tilde{\kappa}_1}{3} \varepsilon^{abcde} A_a F_{bc} F_{de} \right] \quad (\text{A.15})$$

where

$$\begin{aligned} \frac{1}{\tilde{L}^2} &= \frac{1}{L^2} (1 - 10(\mu_1 + 5\mu_3)) \\ \tilde{\kappa}_1 &= \kappa_1 (1 + 3(\mu_1 + 12\mu_2 + 5\mu_3 + 60\mu_4)). \end{aligned} \quad (\text{A.16})$$

Next we would like to apply this analysis to compare the four-derivative supergravity action given in [26] to the effective action (2.1) studied in the present paper.¹⁶ Of course, the leading two-derivative terms in the supergravity action take precisely the form given in (A.1) with $\kappa_1 = 1/4\sqrt{3}(1 - 32c_1)$ — here and in the following, we present the results in terms of $c_1 = (c - a)/(8c)$, as in (2.4). The four-derivative supergravity action can certainly be described in terms of our general action (A.2) where the dimensionless coefficients are all assigned specific values proportional to c_1 . However, the supergravity action is naturally described in terms of the Weyl tensor and so to make this matching, we must first re-express the following:

$$\begin{aligned} C_{abcd} C^{abcd} &= R_{abcd} R^{abcd} - \frac{4}{3} R_{ab} R^{ab} + \frac{1}{6} R^2, \\ C_{abcd} F^{ab} F^{cd} &= R_{abcd} F^{ab} F^{cd} - \frac{4}{3} R^{ab} F_{ac} F_b{}^c + \frac{1}{6} R F^2, \end{aligned} \quad (\text{A.17})$$

¹⁶Note that this comparison requires care since [26] adopts the supergravity conventions of [28], with, e.g., the mostly minus convention for the signature of the metric. We thank Sera Cremonini for her detailed explanation of these conventions.

which applies for the five-dimensional Weyl tensor. Now as described above, integration by parts can be used to eliminate many terms in the general action (A.2), reducing it to the form given in (A.4). Putting the supergravity action in this particular form yields:

$$\begin{aligned}\alpha_1 &= \frac{1}{6}c_1, & \alpha_2 &= -\frac{4}{3}c_1, & \alpha_3 &= c_1, \\ \beta_1 &= \frac{1}{4}c_1, & \beta_2 &= -\frac{4}{3}c_1, & \beta_3 &= -\frac{1}{2}c_1, \\ \delta_1 &= -\frac{41}{288}c_1, & \delta_2 &= \frac{5}{8}c_1, & \delta_3 &= 2c_1, \\ \gamma_1 &= \frac{5}{8\sqrt{3}}c_1, & \kappa_2 &= \frac{1}{2\sqrt{3}}c_1.\end{aligned}\tag{A.18}$$

$$\begin{aligned}\delta_1 &= -\frac{41}{288}c_1, & \delta_2 &= \frac{5}{8}c_1, & \delta_3 &= 2c_1, \\ \gamma_1 &= \frac{5}{8\sqrt{3}}c_1, & \kappa_2 &= \frac{1}{2\sqrt{3}}c_1.\end{aligned}\tag{A.19}$$

Now applying the field redefinitions (A.5) with the parameters fixed as in (A.8), we are able to further reduce the four-derivative action to the canonical form (A.9). While these field redefinitions leave the values of α_3 , β_3 and κ_2 unchanged from those given above, (A.10) yields

$$\tilde{\delta}_1 = \frac{1}{24}c_1, \quad \tilde{\delta}_2 = -\frac{5}{24}c_1.\tag{A.20}$$

As a final step, we give the Einstein and Maxwell kinetic terms their standard normalization with (A.14) yielding

$$\frac{1}{\tilde{L}^2} = \frac{1}{L^2} \left(1 - \frac{10}{3}c_1\right), \quad \tilde{\kappa}_1 = \kappa_1 (1 - 256c_1) = \frac{1}{4\sqrt{3}} (1 - 288c_1).\tag{A.21}$$

Hence, after all of these manipulations, the supergravity action takes the form given in (2.1) with

$$c_2 = -\frac{1}{2}c_1, \quad c_3 = \frac{1}{24}c_1, \quad c_4 = -\frac{5}{24}c_1, \quad c_5 = \frac{1}{2\sqrt{3}}c_1,\tag{A.22}$$

as well as $\kappa = 1/4\sqrt{3}(1 - 288c_1)$ and $c_1 = (c - a)/(8c)$, as in (2.4). As discussed, this latter combination of the central charges in the (supersymmetric) CFT is seen to explicitly fix the coefficients all of these four-derivative corrections to the supergravity action.

References

- [1] See, for example, W. Zajc, *Quark gluon plasma at RHIC (and in QCD and string theory)*, presented at *PASCOS08*, June 2–6, Waterloo, Canada (2008), see <http://pirsa.org/08060040>; K. Rajagopal, *Quark gluon plasma in QCD, at RHIC, and in string theory*, presented at *PASCOS08*, June 2–6, Waterloo, Canada (2008), see <http://pirsa.org/08060041>; D. Mateos, *String theory and quantum chromodynamics*, *Class. Quant. Grav.* **24** (2007) S713 [[arXiv:0709.1523](#)] [[SPIRES](#)]; S.S. Gubser, *Heavy ion collisions and black hole dynamics*, *Gen. Rel. Grav.* **39** (2007) 1533 [*Int. J. Mod. Phys. D* **17** (2008) 673] [[SPIRES](#)]; D.T. Son, *Gauge-gravity duality and heavy-ion collisions*, *AIP Conf. Proc.* **957** (2007) 134 [[SPIRES](#)].
- [2] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[SPIRES](#)].

- [3] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large- N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)] [[SPIRES](#)].
- [4] D.T. Son and A.O. Starinets, *Viscosity, black holes and quantum field theory*, *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95 [[arXiv:0704.0240](#)] [[SPIRES](#)].
- [5] P. Kovtun, D.T. Son and A.O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [[hep-th/0405231](#)] [[SPIRES](#)];
A. Buchel and J.T. Liu, *Universality of the shear viscosity in supergravity*, *Phys. Rev. Lett.* **93** (2004) 090602 [[hep-th/0311175](#)] [[SPIRES](#)];
A. Buchel, J.T. Liu and A.O. Starinets, *Coupling constant dependence of the shear viscosity in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **707** (2005) 56 [[hep-th/0406264](#)] [[SPIRES](#)];
A. Buchel, *On universality of stress-energy tensor correlation functions in supergravity*, *Phys. Lett. B* **609** (2005) 392 [[hep-th/0408095](#)] [[SPIRES](#)];
P. Benincasa, A. Buchel and R. Naryshkin, *The shear viscosity of gauge theory plasma with chemical potentials*, *Phys. Lett. B* **645** (2007) 309 [[hep-th/0610145](#)] [[SPIRES](#)];
D. Mateos, R.C. Myers and R.M. Thomson, *Holographic viscosity of fundamental matter*, *Phys. Rev. Lett.* **98** (2007) 101601 [[hep-th/0610184](#)] [[SPIRES](#)];
K. Landsteiner and J. Mas, *The shear viscosity of the non-commutative plasma*, *JHEP* **07** (2007) 088 [[arXiv:0706.0411](#)] [[SPIRES](#)].
- [6] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, *Phys. Rev. D* **79** (2009) 025023 [[arXiv:0809.3808](#)] [[SPIRES](#)].
- [7] For example, see, D. Teaney, *Effect of shear viscosity on spectra, elliptic flow and Hanbury Brown-Twiss radii*, *Phys. Rev. C* **68** (2003) 034913 [[nucl-th/0301099](#)] [[SPIRES](#)];
PHENIX collaboration, A. Adare et al., *Energy loss and flow of heavy quarks in Au + Au collisions at $\sqrt{s_{NN}} = 200$ GeV*, *Phys. Rev. Lett.* **98** (2007) 172301 [[nucl-ex/0611018](#)] [[SPIRES](#)];
M. Luzum and P. Romatschke, *Conformal relativistic viscous hydrodynamics: applications to RHIC results at $\sqrt{s_{NN}} = 200$ GeV*, *Phys. Rev. C* **78** (2008) 034915 [[arXiv:0804.4015](#)] [[SPIRES](#)].
- [8] A. Buchel, J.T. Liu and A.O. Starinets, *Coupling constant dependence of the shear viscosity in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **707** (2005) 56 [[hep-th/0406264](#)] [[SPIRES](#)].
- [9] A. Buchel, *Resolving disagreement for η/s in a CFT plasma at finite coupling*, *Nucl. Phys. B* **803** (2008) 166 [[arXiv:0805.2683](#)] [[SPIRES](#)].
- [10] R.C. Myers, M.F. Paulos and A. Sinha, *Quantum corrections to η/s* , *Phys. Rev. D* **79** (2009) 041901 [[arXiv:0806.2156](#)] [[SPIRES](#)].
- [11] A. Buchel, R.C. Myers, M.F. Paulos and A. Sinha, *Universal holographic hydrodynamics at finite coupling*, *Phys. Lett. B* **669** (2008) 364 [[arXiv:0808.1837](#)] [[SPIRES](#)].
- [12] M. Brigante, H. Liu, R.C. Myers, S. Shenker and S. Yaida, *Viscosity bound violation in higher derivative gravity*, *Phys. Rev. D* **77** (2008) 126006 [[arXiv:0712.0805](#)] [[SPIRES](#)]; *The viscosity bound and causality violation*, *Phys. Rev. Lett.* **100** (2008) 191601 [[arXiv:0802.3318](#)] [[SPIRES](#)].

- [13] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, *JHEP* **01** (2009) 044 [[arXiv:0712.0743](#)] [[SPIRES](#)].
- [14] A. Buchel, R.C. Myers and A. Sinha, *Beyond $\eta/s = 1/4\pi$* , *JHEP* **03** (2009) 084 [[arXiv:0812.2521](#)] [[SPIRES](#)].
- [15] M. Mia, K. Dasgupta, C. Gale and S. Jeon, *Five easy pieces: the dynamics of quarks in strongly coupled plasmas*, [arXiv:0902.1540](#) [[SPIRES](#)].
- [16] P. Kovtun, D.T. Son and A.O. Starinets, *Holography and hydrodynamics: diffusion on stretched horizons*, *JHEP* **10** (2003) 064 [[hep-th/0309213](#)] [[SPIRES](#)].
- [17] D.T. Son and A.O. Starinets, *Hydrodynamics of r-charged black holes*, *JHEP* **03** (2006) 052 [[hep-th/0601157](#)] [[SPIRES](#)].
- [18] J. Mas, *Shear viscosity from R-charged AdS black holes*, *JHEP* **03** (2006) 016 [[hep-th/0601144](#)] [[SPIRES](#)];
K. Maeda, M. Natsuume and T. Okamura, *Viscosity of gauge theory plasma with a chemical potential from AdS/CFT*, *Phys. Rev. D* **73** (2006) 066013 [[hep-th/0602010](#)] [[SPIRES](#)];
O. Saremi, *The viscosity bound conjecture and hydrodynamics of M2-brane theory at finite chemical potential*, *JHEP* **10** (2006) 083 [[hep-th/0601159](#)] [[SPIRES](#)].
- [19] STAR collaboration, J. Adams et al., *Experimental and theoretical challenges in the search for the quark gluon plasma: the STAR collaboration's critical assessment of the evidence from RHIC collisions*, *Nucl. Phys. A* **757** (2005) 102 [[nucl-ex/0501009](#)] [[SPIRES](#)];
B.B. Back et al., *The PHOBOS perspective on discoveries at RHIC*, *Nucl. Phys. A* **757** (2005) 28 [[nucl-ex/0410022](#)] [[SPIRES](#)];
PHENIX collaboration, K. Adcox et al., *Formation of dense partonic matter in relativistic nucleus nucleus collisions at RHIC: experimental evaluation by the PHENIX collaboration*, *Nucl. Phys. A* **757** (2005) 184 [[nucl-ex/0410003](#)] [[SPIRES](#)];
BRAHMS collaboration, I. Arsene et al., *Quark Gluon Plasma an Color Glass Condensate at RHIC? The perspective from the BRAHMS experiment*, *Nucl. Phys. A* **757** (2005) 1 [[nucl-ex/0410020](#)] [[SPIRES](#)].
- [20] P. Kovtun and A. Ritz, *Universal conductivity and central charges*, *Phys. Rev. D* **78** (2008) 066009 [[arXiv:0806.0110](#)] [[SPIRES](#)].
- [21] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The string landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [[hep-th/0601001](#)] [[SPIRES](#)].
- [22] J.P. Gauntlett, R.C. Myers and P.K. Townsend, *Black holes of $D = 5$ supergravity*, *Class. Quant. Grav.* **16** (1999) 1 [[hep-th/9810204](#)] [[SPIRES](#)].
- [23] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)] [[SPIRES](#)]; *Holography and the Weyl anomaly*, *Fortsch. Phys.* **48** (2000) 125 [[hep-th/9812032](#)] [[SPIRES](#)].
- [24] S. Nojiri and S.D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*, *Int. J. Mod. Phys. A* **15** (2000) 413 [[hep-th/9903033](#)] [[SPIRES](#)];
M. Blau, K.S. Narain and E. Gava, *On subleading contributions to the AdS/CFT trace anomaly*, *JHEP* **09** (1999) 018 [[hep-th/9904179](#)] [[SPIRES](#)].
- [25] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[SPIRES](#)];
D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d + 1) correspondence*, *Nucl. Phys. B* **546** (1999) 96 [[hep-th/9804058](#)] [[SPIRES](#)];

- O. Aharony, J. Pawełczyk, S. Theisen and S. Yankielowicz, *A note on anomalies in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 066001 [[hep-th/9901134](#)] [[SPIRES](#)].
- [26] S. Cremonini, K. Hanaki, J.T. Liu and P. Szepietowski, *Black holes in five-dimensional gauged supergravity with higher derivatives*, [arXiv:0812.3572](#) [[SPIRES](#)].
- [27] D.M. Hofman and J. Maldacena, *Conformal collider physics: energy and charge correlations*, *JHEP* **05** (2008) 012 [[arXiv:0803.1467](#)] [[SPIRES](#)].
- [28] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric completion of an R^2 term in five-dimensional supergravity*, *Prog. Theor. Phys.* **117** (2007) 533 [[hep-th/0611329](#)] [[SPIRES](#)];
For notation, see T. Kugo and K. Ohashi, *Supergravity tensor calculus in 5D from 6D*, *Prog. Theor. Phys.* **104** (2000) 835 [[hep-ph/0006231](#)] [[SPIRES](#)];
T. Fujita and K. Ohashi, *Superconformal tensor calculus in five dimensions*, *Prog. Theor. Phys.* **106** (2001) 221 [[hep-th/0104130](#)] [[SPIRES](#)].
- [29] S. Kobayashi, D. Mateos, S. Matsuura, R.C. Myers and R.M. Thomson, *Holographic phase transitions at finite baryon density*, *JHEP* **02** (2007) 016 [[hep-th/0611099](#)] [[SPIRES](#)].
- [30] J.T. Liu and P. Szepietowski, *Higher derivative corrections to R-charged AdS₅ black holes and field redefinitions*, [arXiv:0806.1026](#) [[SPIRES](#)].
- [31] I. Racz and R.M. Wald, *Extension of space-times with Killing horizon*, *Class. Quant. Grav.* **9** (1992) 2643 [[SPIRES](#)].
- [32] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev. D* **60** (1999) 064018 [[hep-th/9902170](#)] [[SPIRES](#)];
Holography, thermodynamics and fluctuations of charged AdS black holes, *Phys. Rev. D* **60** (1999) 104026 [[hep-th/9904197](#)] [[SPIRES](#)].
- [33] C.S. Peca and P.S. Lemos, Jose, *Thermodynamics of Reissner-Nordström-anti-de Sitter black holes in the grand canonical ensemble*, *Phys. Rev. D* **59** (1999) 124007 [[gr-qc/9805004](#)] [[SPIRES](#)];
M. Cvetič and S.S. Gubser, *Phases of R-charged black holes, spinning branes and strongly coupled gauge theories*, *JHEP* **04** (1999) 024 [[hep-th/9902195](#)] [[SPIRES](#)]; *Thermodynamic stability and phases of general spinning branes*, *JHEP* **07** (1999) 010 [[hep-th/9903132](#)] [[SPIRES](#)].
- [34] For example, see S.W. Hawking, *The path-integral approach to quantum gravity*, in *General relativity: an einstein centenary survey*, S.W. Hawking and W. Israel eds., Cambridge University Press, Cambridge U.K. (1979).
- [35] V. Balasubramanian and P. Kraus, *A stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [[hep-th/9902121](#)] [[SPIRES](#)];
R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 104001 [[hep-th/9903238](#)] [[SPIRES](#)];
C.R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, [math/9909042](#).
- [36] R.M. Wald, *Black hole entropy is the Nöther charge*, *Phys. Rev. D* **48** (1993) 3427 [[gr-qc/9307038](#)] [[SPIRES](#)];
V. Iyer and R.M. Wald, *Some properties of Nöther charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [[gr-qc/9403028](#)] [[SPIRES](#)];

- T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev. D* **49** (1994) 6587 [[gr-qc/9312023](#)] [[SPIRES](#)].
- [37] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, *JHEP* **09** (2002) 043 [[hep-th/0205052](#)] [[SPIRES](#)];
D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [[hep-th/0205051](#)] [[SPIRES](#)].
- [38] N. Banerjee and S. Dutta, *Higher derivative corrections to shear viscosity from graviton's effective coupling*, *JHEP* **03** (2009) 116 [[arXiv:0901.3848](#)] [[SPIRES](#)].
- [39] S. Caron-Huot, P. Kovtun, G.D. Moore, A. Starinets and L.G. Yaffe, *Photon and dilepton production in supersymmetric Yang-Mills plasma*, *JHEP* **12** (2006) 015 [[hep-th/0607237](#)] [[SPIRES](#)].
- [40] X.-H. Ge, Y. Matsuo, F.-W. Shu, S.-J. Sin and T. Tsukioka, *Density dependence of transport coefficients from holographic hydrodynamics*, *Prog. Theor. Phys.* **120** (2008) 833 [[arXiv:0806.4460](#)] [[SPIRES](#)].
- [41] R.-G. Cai and Y.-W. Sun, *Shear viscosity from AdS Born-Infeld black holes*, *JHEP* **09** (2008) 115 [[arXiv:0807.2377](#)] [[SPIRES](#)].
- [42] X.-H. Ge, Y. Matsuo, F.-W. Shu, S.-J. Sin and T. Tsukioka, *Viscosity bound, causality violation and instability with stringy correction and charge*, *JHEP* **10** (2008) 009 [[arXiv:0808.2354](#)] [[SPIRES](#)].
- [43] R.-G. Cai, Z.-Y. Nie and Y.-W. Sun, *Shear viscosity from effective couplings of gravitons*, *Phys. Rev. D* **78** (2008) 126007 [[arXiv:0811.1665](#)] [[SPIRES](#)].
- [44] R.-G. Cai, Z.-Y. Nie, N. Ohta and Y.-W. Sun, *Shear viscosity from Gauss-Bonnet gravity with a dilaton coupling*, *Phys. Rev. D* **79** (2009) 066004 [[arXiv:0901.1421](#)] [[SPIRES](#)].
- [45] D. Anninos and G. Pastras, *Thermodynamics of the Maxwell-Gauss-Bonnet Anti-de Sitter black hole with higher derivative gauge corrections*, [arXiv:0807.3478](#) [[SPIRES](#)].
- [46] A. Ritz and J. Ward, *Weyl corrections to holographic conductivity*, [arXiv:0811.4195](#) [[SPIRES](#)].
- [47] R. Brustein and A.J.M. Medved, *The ratio of shear viscosity to entropy density in generalized theories of gravity*, *Phys. Rev. D* **79** (2009) 021901 [[arXiv:0808.3498](#)] [[SPIRES](#)];
R. Brustein and A.J.M. Medved, *The shear diffusion coefficient for generalized theories of gravity*, *Phys. Lett. B* **671** (2009) 119 [[arXiv:0810.2193](#)] [[SPIRES](#)]; *The sound damping constant for generalized theories of gravity*, [arXiv:0901.2191](#) [[SPIRES](#)].
- [48] R.C. Myers, M.F. Paulos and A. Sinha, to appear.
- [49] A.A. Tseytlin, *On non-abelian generalisation of the Born-Infeld action in string theory*, *Nucl. Phys. B* **501** (1997) 41 [[hep-th/9701125](#)] [[SPIRES](#)].
- [50] For example, see the single-charge black holes in K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes of five dimensional $N = 2$ AdS supergravity*, *Nucl. Phys. B* **553** (1999) 317 [[hep-th/9810227](#)] [[SPIRES](#)].
- [51] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Universality of the operator product expansions of $SCFT(4)$* , *Phys. Lett. B* **394** (1997) 329 [[hep-th/9608125](#)] [[SPIRES](#)]; *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys. B* **526** (1998) 543 [[hep-th/9708042](#)] [[SPIRES](#)];

- D. Anselmi, J. Erlich, D.Z. Freedman and A.A. Johansen, *Positivity constraints on anomalies in supersymmetric gauge theories*, *Phys. Rev. D* **57** (1998) 7570 [[hep-th/9711035](#)] [[SPIRES](#)].
- [52] N. Banerjee et al., *Hydrodynamics from charged black branes*, [arXiv:0809.2596](#) [[SPIRES](#)];
J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055 [[arXiv:0809.2488](#)] [[SPIRES](#)];
S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, *JHEP* **09** (2008) 054 [[arXiv:0708.1770](#)] [[SPIRES](#)].
- [53] S. Cremonini, K. Hanaki, J.T. Liu and P. Szepietowski, *Higher derivative effects on eta/s at finite chemical potential*, [arXiv:0903.3244](#) [[SPIRES](#)].
- [54] See, for example S.A. Hartnoll and P. Kovtun, *Hall conductivity from dyonic black holes*, *Phys. Rev. D* **76** (2007) 066001 [[arXiv:0704.1160](#)] [[SPIRES](#)];
S.A. Hartnoll, P.K. Kovtun, M. Muller and S. Sachdev, *Theory of the Nernst effect near quantum phase transitions in condensed matter and in dyonic black holes*, *Phys. Rev. B* **76** (2007) 144502 [[arXiv:0706.3215](#)] [[SPIRES](#)];
E.I. Buchbinder, A. Buchel and S.E. Vazquez, *Sound waves in $(2+1)$ dimensional holographic magnetic fluids*, *JHEP* **12** (2008) 090 [[arXiv:0810.4094](#)] [[SPIRES](#)];
E.I. Buchbinder and A. Buchel, *The fate of the sound and diffusion in holographic magnetic field*, [arXiv:0811.4325](#) [[SPIRES](#)]; *Relativistic conformal magneto-hydrodynamics from holography*, [arXiv:0902.3170](#) [[SPIRES](#)];
J. Hansen and P. Kraus, *Nonlinear magnetohydrodynamics from gravity*, *JHEP* **04** (2009) 048 [[arXiv:0811.3468](#)] [[SPIRES](#)].
- [55] K.S. Thorne, R.H. Price and D.A. Macdonald, *Black holes: the membrane paradigm*, Yale University Press, New Haven U.S.A. (1986).
- [56] F. Denef and S.A. Hartnoll, *Landscape of superconducting membranes*, [arXiv:0901.1160](#) [[SPIRES](#)].
- [57] S. Hellerman, *A universal inequality for CFT and quantum gravity*, [arXiv:0902.2790](#) [[SPIRES](#)].
- [58] Y. Kats, L. Motl and M. Padi, *Higher-order corrections to mass-charge relation of extremal black holes*, *JHEP* **12** (2007) 068 [[hep-th/0606100](#)] [[SPIRES](#)].
- [59] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from Anti-de Sitter space*, *JHEP* **06** (2000) 008 [[hep-th/0003075](#)] [[SPIRES](#)];
M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [[hep-th/0008015](#)] [[SPIRES](#)];
A. Hashimoto, S. Hirano and N. Iizuka, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [[hep-th/0008016](#)] [[SPIRES](#)].
- [60] R.C. Myers and O. Tafjord, *Superstars and giant gravitons*, *JHEP* **11** (2001) 009 [[hep-th/0109127](#)] [[SPIRES](#)].