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IMPULSE APPROXIMATION AND SCALING VIOLATION IN QCD*

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ABSTRACT

The nature of the impulse approximation in local field theory is clarified by dividing the interaction Hamiltonian into two parts V and W, where V contains only those interactions causing large energy transfers. Partons are introduced as eigenstates of $\mathcal{H}_0$ + V, where $\mathcal{H}_0$ is the free Hamiltonian. Their time development is governed by the soft operator W, thus making it possible to use the impulse approximation in deep inelastic processes. Application is made to deep inelastic electron scattering and the Drell-Yan process. Making some reasonable assumptions on the parton matrix elements, the variation of parton density functions with Q^2 is expressed in terms of a set of integrodifferential equations, which reduce to the known results when restricted to the longitudinal distributions. Explicit solutions of the scaling violation equations are obtained in some simple cases.

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I. INTRODUCTION

The parton model[1] has been a very useful guide in analyzing deep inelastic experiments involving a large momentum transfer Q. In this model the structure functions of the deep inelastic lepton scattering processes are identified with the longitudinal momentum distributions of partons inside the hadronic targets. The partons are assumed to be free at large Q, giving Bjorken scaling in rough agreement with experiment. In local field theories, however, the partons cannot be free and there is no reason to expect Bjorken scaling to occur. This dilemma was solved by the discovery of asymptotic freedom in non-Abelian gauge theories,[2] in which the scaling is violated only logarithmically. Furthermore, explicit calculations[3] based on quantum chromodynamics (QCD) give results which agree well with recent experimental data.

However, the reconciliation of the simple parton model with field theory does not seem to be completely satisfactory. First, the usual analysis[4] of scaling violations involves sophisticated mathematical techniques such as the operator product expansion and the renormalization group equations, whose physical meaning is not as transparent as the intuitive parton model. Second, the method has been successful only for the calculation of the longitudinal momentum distributions of partons, but not successful for the transverse momentum distributions. Finally, the usual treatment can not be generalized in a straightforward manner to other deep inelastic processes such as the Drell-Yan process.[5] This is because the Drell-Yan process is not light-cone dominated[6] so the operator product expansion does not apply.[7] In contrast, all deep inelastic processes are more or less on the same footing in the framework of the parton model.

The purpose of this paper is to provide a more satisfactory field theoretic foundation of the parton model in the context of QCD.[8] The starting point of the present approach is to recall that the concept of the parton is useful and natural only in connection with the impulse approximation.[9] Now the validity of the impulse approximation depends essentially on our choice of the basis states which are thought to interact with the external hard currents. Thus the impulse approximation may be applicable for scattering of a fast electron off a nucleus, but it will not in general work if one chooses the nucleons themselves as the basis states. In atomic physics the choices of the basis states is obvious because the length scales change discontinuously. In field theories, however, the change in the length scales is continuous and the identification of the basis states is not so straightforward. To identify the correct basis states in field theory, it is necessary to formulate quantum mechanically the classical notion that a system remains essentially the same during a short time interval Δt. In quantum mechanics, the time evolution of a system is described by the U-matrix. Therefore, it is natural to define the basis states to be such states in which $U(t+\Delta t,t)$ can be approximated by 1 for a small time interval Δt. In this paper, this will be achieved by defining the basis states (i.e., the partons) to be dressed quanta whose internal energy transfers are restricted to be larger than some given value which depends on Q. With this definition of the parton states, it is then possible to give a physical derivation of the parton model expressions of cross sections for deep inelastic lepton scattering and the Drell-Yan process. The scaling violations arise in the present approach simply because the parton states change as Q varies.

The physical basis of the scaling violation was originally treated on an intuitive level by Kogut and Susskind.[10,11] They argued that the partons probed in a deep inelastic process with momentum transfer Q are the dressed quanta whose internal transverse momenta are larger than Q. However, in their approach it is difficult to formulate the transverse momentum cut-off in a precise way. The cut-off in the energy transfer employed here is precise and its relation to the impulse approximation is straightforward.

The paper is organized as follows:

In Sect. II a precise definition of the parton states is given by dividing the interaction Hamiltonian into two parts, one containing the large energy transfers while the other contains the rest of the interactions. The matrix element between the parton states so defined is assumed to be governed by an effective coupling constant. This is quite reasonable in view of the usual renormalization group analysis in the Green's function theory. Sect. III discusses some properties of the parton states which play an important role later on. In particular, it is shown that hadrons have finite wave functions if expressed in terms of the parton states defined in Sect. II. In Sect. IV, the physics of the impulse approximation is clarified in terms of the present definition of the parton states. Although the impulse approximation fails in general for a local field theory, the approximation is justified in the so-called Λ-picture. In Sect. V, the concepts developed so far are applied to deep inelastic electron scattering and to the Drell-Yan process. For the former process, one obtains the usual parton model result, the only modification being the replacement of the naive parton distribution functions by the Q^2 dependent distribution functions. For the latter process, one obtains a formula identical to the one recently conjectured by Kogut[11] and Hinchliffe and Llleweylin Smith.[12] However, some more assumptions are necessary in arriving at this result. Sect. VI is devoted to the subject of scaling violation effects. An integro-differential equation is derived which describes the change of a general distribution function of partons as Q^2 varies. If restricted to the longitudinal distribution, the equation is identical to the one derived using the method of the operator product expansions and the renormalization group equations. This formalism is applied in Sect. VII to discuss the parton transverse momentum distributions. Explicit solutions are obtained for the parton's transverse momentum squared averaged over the longitudinal fraction x. Sect. VIII contains some concluding remarks. Finally in the Appendix, the explicit form of the QCD Hamiltonian used throughout this paper is derived.

II. DEFINITION OF THE PARTON STATES

In order to discuss wave functions of hadrons in terms of partons, it is necessary to employ time ordered perturbation theory. The rules of time ordered perturbation theory are simplest in the infinite momentum frame (IMF) because vacuum effects are absent there. Therefore I will be working with time ordered perturbation theory quantized in the IMF[13] throughout this paper. Thus the momentum p and the coordinate variable x have the following IMF decompositions:

$$p^\mu = (p^o, p_\perp, p^3) = (\eta, p_\perp, \mathscr{E})$$
$$x^\mu = (x^o, x_\perp, x^3) = (\tau, x_\perp, \mathscr{Z}) \quad , \tag{2.1}$$

where

$$\eta = \frac{1}{\sqrt{2}}(E + P_z) \quad , \qquad \mathscr{E} = \frac{1}{\sqrt{2}}(E - P_z)$$
$$\tau = \frac{1}{\sqrt{2}}(t + z) \qquad \text{and} \qquad \mathscr{Z} = \frac{1}{\sqrt{2}}(t - z) \tag{2.2}$$

Here E, P_z and $p_\perp$ are the components in the ordinary reference frame. For the particle on the mass shell, one has

$$\mathscr{E} = ({p_\perp}^2 + M^2)/2\eta \tag{2.3}$$

where M is the mass. One also has

$$p\cdot x = \mathscr{E}\tau + \eta\mathscr{Z} - p_\perp \cdot x_\perp \tag{2.4}$$

In the IMF, one identifies τ as the time variable. Then its conjugate variable is $\mathscr{E}$, which is identified as the energy variable. Finally, the vector $\underset{\sim}{p} = (\eta, p_\perp)$ will be used to specify the momentum of a state.

The discussions in this section are applicable to any theory, but I will work with QCD defined from the following Lagrangian density:

$$\mathcal{L} = \bar{\psi}\, \not{D}\, \psi - \frac{1}{4} G_{\mu\nu}^{\ a} G^{\mu\nu a} \qquad , \tag{2.5}$$

where

$$D_\mu = \partial_\mu - ig\, T^a A^a_\mu$$

$$G_{\mu\nu}^{\ a} = \partial_\mu A_\nu^{\ a} - \partial_\nu A_\mu^{\ a} + g\, f^{abc} A_\mu^{\ b} A_\nu^{\ c} \tag{2.6}$$

In the above, ψ and $A_\mu^{\ a}$'s are the field variables for the quarks and the gluons, respectively, and f^{abc}'s are the structure constants of the gauge group and T^a's are the group generators in the fermion representation. Notice that the quark masses are set to zero in the above Lagrangian. Although the following discussion can be generalized to incorporate the mass of quarks, it will be neglected for simplicity. To obtain the Hamiltonian it is necessary to impose a gauge condition. In the IMF, it is convenient to choose the infinite momentum (IM) gauge[13] defined as follows:

$$A^o(x) = 0 \tag{2.7}$$

In this gauge, no ghosts appear and the independent variables are the transverse components $A_\perp^{\ a}$ of the gluon fields and the two component Pauli spinor χ of the quarks. The derivation of the Hamiltonian $\mathcal{H}$ is well known[14] and the result is given in the Appendix. For the present purpose, it is sufficient to write the Hamiltonian $\mathcal{H}$ in terms of the free part $\mathcal{H}_o$ and the interacting part $\mathcal{H}_I$ as follows:

$$\mathcal{H} = \mathcal{H}_o + \mathcal{H}_I \tag{2.8}$$

$\mathcal{H}_I$ in the above is a sum over virtual processes such as those shown in Fig. (1). Each of these processes conserves the total momentum $\underset{\sim}{p} = (\eta, p_\perp)$ but causes the total energy to change from $\mathcal{E}_i$ to $\mathcal{E}_f$. It should then be possible to divide $\mathcal{H}_I$ into two parts so that the first part V contains only those interactions involving large energy transfers while the second part W contains only small energy transfers. If one defines the parton states as the eigenstates of the operator $\mathscr{H}_0 + V$, their time development will be governed by the soft interaction W only.

In lowest order, it is trivial to carry out the desired decomposition of $\mathscr{H}_I$. In higher order, however, the operators V and W cannot be expressed in a closed form because of the occurrence of divergences. One would like to have the wave functions of a hadron in terms of the dressed partons free of ultraviolet divergences. To meet these requirements, the operator V (or W) and the corresponding parton states will be defined in the following steps: First let there be operators V_Λ and W_Λ so that

$$\mathscr{H}_I = V_\Lambda + W_\Lambda \tag{2.9}$$

where Λ is an arbitrary parameter. Next introduce the parton Hamiltonian $\mathscr{H}_\Lambda$ as follows:

$$\mathscr{H}_\Lambda = \mathscr{H}_0 + V_\Lambda \tag{2.10}$$

Let $|n,\Lambda\rangle$ be the eigenstates of $\mathscr{H}_\Lambda$ with energy $\mathcal{E}_n$.[15] The operator W_Λ will now be specified in terms of its matrix elements $\langle n,\Lambda|W_\Lambda|m,\Lambda\rangle$ as follows:

$$\begin{aligned} \langle n,\Lambda|W_\Lambda|m,\Lambda\rangle &= \langle n,\Lambda|\mathscr{H}_I|m,\Lambda\rangle \text{for } |\mathcal{E}_n - \mathcal{E}_m| \leq \frac{\Lambda^2}{2\eta_0} \\ &= 0 \qquad \text{for } |\mathcal{E}_n - \mathcal{E}_m| > \frac{\Lambda^2}{2\eta_0} \end{aligned} \tag{2.11}$$

η_0 in the above is the η of the parent hadron whose partons are under study.

The definition of the operators W_Λ and V_Λ introduced above is not a simple one because they are defined in terms of the states $|n,\Lambda\rangle$

which in turn are defined in terms of W_Λ. Therefore V_Λ and W_Λ can only be determined perturbatively. Nevertheless, it is clear from the above definition that W_Λ is the operator which contains only small energy transfers. Since $V_\Lambda = \mathscr{H}_I - W_\Lambda$, it follows that V_Λ contains only large energy transfers. Since every particle appearing in the intermediate states is on the mass shell in time ordered perturbation theory, it follows from the mass shell condition Eq. (2.3) that large energy transfers correspond roughly to large transverse momenta if the longitudinal variable η is not too small. It is in this sense that the present definition of the parton states is qualitatively the same as the one introduced by Kogut and Susskind[10] in their intuitive analysis of the scaling violation effects.

Explicit construction of the operator V_Λ or W_Λ, and the parton states $|n,\Lambda\rangle$, will not be attempted in this paper. Wilson[16] has carried out such constructions for some simpler models using a momentum cut-off rather than the cut-off in energy transfers. He found that the coupling between states at momentum scale μ is governed by an effective coupling constant $g(\mu^2)$. In this paper, I will simply assume that a similar analysis can be carried out for QCD with a cut-off in the energy transfer. More precisely, I assume that, for $\Lambda'^2 >> \Lambda^2 >> \Lambda'^2 - \Lambda^2$,

$$\begin{aligned} &\langle n,\Lambda' | (W_{\Lambda'} - W_\Lambda) | m,\Lambda \rangle \\ &= \langle n | \mathscr{H}_I | m \rangle \; (g \to g(\Lambda^2)) \quad , \quad \frac{\Lambda^2}{2\eta_0} \leq |\mathcal{E}_n - \mathcal{E}_m| \leq \Lambda'^2/2\eta_0 \quad , \\ &= 0 \qquad , \quad \text{otherwise.} \end{aligned} \tag{2.12}$$

In the above, $|n\rangle$'s are the bare states, i.e., the eigenstates of the free Hamiltonian $\mathscr{H}_0$, and $(g \to g(\Lambda^2))$ implies that the bare coupling constant g should be replaced by the effective coupling constant $g(\Lambda^2)$.

The restriction of the energy transfer appearing in Eq. (2.12) follows from Eq. (2.11). In the present case, $g(\Lambda^2)$ is given by

$$g(\Lambda^2) = g_0^2 \Big/ \left(1 + \frac{1}{8\pi^2}\, b\, g_0^2\, \ell n(\Lambda^2/\Lambda_0^2)\right) \quad , \tag{2.13}$$

where[17]

$$b = \frac{11}{6}\, Cg - \frac{2}{3}\, T_r \quad . \tag{2.14}$$

g_0 in Eq. (2.13) is the coupling constant at $\Lambda = \Lambda_0$; $g_0 = g(\Lambda_0^2)$. Throughout this paper, b will be taken to be positive so that the theory is asymptotically free.

A crude argument can be given which renders the statement made in Eq. (2.12) plausible. Consider a parton state $|n,\Lambda\rangle$. As Λ approaches infinity, $|n,\Lambda\rangle$ should approach the bare state $|n\rangle$. This is because the operator W_Λ must approach the entire interaction $\mathscr{H}_I$ so that $V = \mathscr{H}_I - W_\Lambda$ approaches zero in some sense. Consider Eq. (2.12) in the limit $\Lambda \to \infty$, keeping $\Lambda' >> \Lambda$. Then it is reasonable that the matrix element will be governed by the bare coupling constant g in this limit. The behavior specified by Eq. (2.12) is reasonable since the effective coupling $g(\Lambda^2)$ approaches the bare coupling constant g as $\Lambda \to \infty$.

III. PROPERTIES OF PARTON STATES

There are several important remarks concerning the nature of the parton states defined in the previous section. First, a parton state $|n,\Lambda\rangle$ depends on the property of the parent hadron through the appearance of the quantity η_0 in Eq. (2.12). Therefore, to completely specify a parton state, one should label the state in terms of the quantity Λ as well as η_0, i.e., $|n,\Lambda,\eta_0\rangle$. This dependence on η_0 means that the states are not invariant under longitudinal boosts[13] which transform a momentum $(\eta,p_\perp)$ into $(\lambda\eta,p_\perp)$. This is also clear from the fact that the quantity $\mathcal{E}_n - \mathcal{E}_m$ transforms into $(\mathcal{E}_n - \mathcal{E}_m)/\lambda$ under a longitudinal boost. More precisely, there exists no unitary operator which connects a state $|(\eta,p_\perp),\Lambda,\eta_0\rangle$ to the state $|(\lambda\eta,p_\perp),\Lambda,\eta_0\rangle$. This property is desirable because one should expect that the nature of parton states change under longitudinal boosts. The Lorentz invariance is not lost, however, because there exists a unitary operator which connects the state $|(\eta,p_\perp),\Lambda,\eta_0\rangle$ to $|(\lambda\eta,p_\perp),\Lambda,\lambda\eta_0\rangle$. In the following, the level η_0 will be suppressed when no confusion will occur.

It is possible to define parton states which are invariant under longitudinal boosts. This can be achieved if one replaces the inequality in Eq. (2.11) by $|\mathcal{E}_n - \mathcal{E}_m| < \Lambda^2/2\eta$, where η is the total longitudinal momentum entering the vertex. Then both sides of this inequality transform the same way under longitudinal boosts. Recently, Lam and Yan[18] have investigated the transverse momentum distribution of partons by generalizing the scaling violation equations to incorporate the transverse momentum distributions. Their analysis essentially amounts to introducing an energy cut-off which is invariant under longitudinal boosts as discussed above. However, it will be shown later in this paper that the impulse approximation cannot be established if one uses parton states which are invariant under longitudinal boosts. In contrast, the parton states introduced in this paper are quite well suited for the impulse approximation in deep inelastic processes.

On the other hand, the parton states defined above <u>are</u> invariant under Galilean boosts[13] which transform a momentum $(\eta, p_\perp)$ into $(\eta, p_\perp + \eta v_\perp)$. This is because the quantity $\mathcal{E}_n - \mathcal{E}_m$ is invariant under such transformations. Therefore, there exists a unitary operator which connects a state $|(\eta,p_\perp),\Lambda,\eta_0\rangle$ to $|(\eta,p_\perp + \eta v_\perp),\Lambda,\eta_0\rangle$. This invariance of the parton states under Galilean boosts will play an important role in deriving the Drell-Yan formula in Sect. V.

The definition in the previous section implies that the wave function of a hadronic state $|h\rangle$ expressed in terms of the states $|m,\Lambda\rangle$'s is well defined and free of ultraviolet divergences. This follows from the formula

$$|h\rangle = \sqrt{Z_h}\left[|n,\Lambda\rangle + \sum_m{}' \, |m,\Lambda\rangle \frac{\langle m,\Lambda|W_\Lambda|h\rangle}{\mathcal{E}_h - \mathcal{E}_m}\right] \tag{3.1a}$$

$$= \sqrt{Z_h}\left[|n,\Lambda\rangle + \sum_m{}' \, |m,\Lambda\rangle \frac{\langle m,\Lambda|W_\Lambda|n,\Lambda\rangle}{\mathcal{E}_n - \mathcal{E}_m}\right.$$

$$\left. + \sum_m{}' \sum_\ell{}' \, \frac{|m,\Lambda\rangle\langle m,\Lambda|W_\Lambda|\ell,\Lambda\rangle\langle \ell,\Lambda|W_\Lambda|n,\Lambda\rangle}{(\mathcal{E}_h - \mathcal{E}_m)(\mathcal{E}_h - \mathcal{E}_\ell)}\right] \tag{3.1b}$$

Here $|n,\Lambda\rangle$ is any state whose quantum numbers and energy are the same as those of the parent hadron, and the prime in the summation symbols implies that states which have the same energy as the parent hadron are to be excluded from the sum. The constant Z_h is the renormalization constant which can be computed by comparing the normalization of both sides of Eq. (3.1). Now the usual ultraviolet divergences arise from the intermediate state sums in Eq. (3.1b). However, the sums cannot give rise to divergences in the present case because the matrix elements appearing in Eq. (3.1b) vanish outside the finite regions of phase space specified by Eq. (2.12). Notice that as Λ approaches infinity, the region of the relevant phase space extends to the whole space and the terms in Eq. (3.1b) will in general blow up. This is precisely the usual ultraviolet divergence appearing in field theories. In the present approach, things are arranged so that all the divergences are contained in the definition of the states $|n,\Lambda\rangle$, so that the rest of the dynamics evolve in a finite way. It is perhaps worthwhile to emphasize the importance of the finiteness of the wave functions in connection with the parton interpretation of deep inelastic processes. If the hadronic wave functions in terms of partons contained divergences, then it would be meaningless to talk about the probability of finding the partons, etc. In fact, the elaborate definition of the parton state $|n,\Lambda\rangle$ introduced in the previous section is tailored to satisfy the requirement that the hadronic wave functions should be finite. (Notice that the above argument also implies that the expansion (3.1b) is free of divergences in the η-integration because η cannot be zero.)

Finally, it should be remarked that only those states with energy $\mathcal{E}_m \lesssim \Lambda^2/2\eta_0$ appear in the expansion of Eq. (3.1). This follows from Eq. (2.12), and will be relevant in the derivation of the scaling violation equations later in this paper.

IV. IMPULSE APPROXIMATION

In this section the states $|n,\Lambda\rangle$ introduced in Sect. II will be used to clarify the impulse approximation in deep inelastic processes. For this purpose, one must first define the meaning of the impulse approximation. Qualitatively, the impulse approximation is applicable if nothing much happens during a short time interval. Quantum mechanically, the evolution of a physical system in time is described by the U-matrix. This suggests that the impulse approximation should be identified with the approximation $U(\tau',\tau) \sim 1$ when $\tau' - \tau$ is small. Throughout this paper, the impulse approximation will be understood in this sense. In this regard, recall that approximating $U(\tau',\tau)$ by 1 was one of the most crucial steps in the original derivation of the parton model from cut-off field theory by Drell et al.[19]

Now the U-matrix will be constructed in the representation introduced in Sect. II. To do this, consider a Heisenberg operator O_H which develops in time as follows:

$$O_H(\tau) = e^{i\mathcal{H}\tau}\, O_s\, e^{-i\mathcal{H}\cdot\tau} \quad , \tag{4.1}$$

where $O_s = O_H(0)$ is the corresponding operator in the Schrödinger picture. Introduce the operator O_Λ which may be called the operator in the Λ-picture as follows:

$$O_\Lambda(\tau) = e^{i\mathcal{H}_\Lambda\cdot\tau}\, O_s\, e^{-i\mathcal{H}_\Lambda\cdot\tau} \quad . \tag{4.2}$$

Here $\mathcal{H}_\Lambda$ is the parton Hamiltonian defined by Eq. (2.10). The Heisenberg picture and the Λ-picture are connected by the formula

$$O_H(\tau) = U_\Lambda^{-1}(\tau,0)\; O_\Lambda(\tau)\; U_\Lambda(\tau,0) \quad . \tag{4.3}$$

Here U_Λ is the time evolution matrix in the Λ-picture, and given by

$$U_\Lambda(\tau_2,\tau_1) = e^{i\mathcal{H}_\Lambda\cdot\tau_1} e^{-i\mathcal{H}(\tau_1-\tau_2)} \; e^{-i\mathcal{H}_\Lambda\cdot\tau_2}$$

$$= T\ \mathrm{Exp}\left(-i\int_{\tau_1}^{\tau_2} W_\Lambda'(\tau)d\tau\right) \quad . \tag{4.4}$$

Here T is the τ-ordering symbol and

$$W_\Lambda'(\tau) = e^{i\mathcal{H}_\Lambda\cdot\tau} W_\Lambda \; e^{-i\mathcal{H}_\Lambda\cdot\tau} \quad . \tag{4.5}$$

Consider now the matrix element of U_Λ between the states $|n,\Lambda\rangle$. It has the expansion

$$\langle n,\Lambda|\; U_\Lambda(\tau,0)|m,\Lambda\rangle =$$

$$\delta nm - i\int_0^\tau d\tau_1 e^{i(\mathscr{E}_n-\mathscr{E}_m)\cdot\tau_1} \langle n,\Lambda|W_\Lambda|m,\Lambda\rangle$$

$$+ (-i)^2 \sum_\ell \int_0^\tau d\tau_1\; e^{i(\mathcal{E}_n-\mathcal{E}_\ell)\cdot\tau_1} \int_0^{\tau_1} d\tau_2\, e^{i(\mathcal{E}_\ell-\mathcal{E}_n)\cdot\tau} \tag{4.6}$$

$$\langle n,\Lambda|W_\Lambda|\ell,\Lambda\rangle\langle \ell,\Lambda|W_\Lambda|m,\Lambda\rangle$$

+ higher orders.

Suppose now $\tau \ll 2\eta_0/\Lambda^2$. Then since the matrix element of W_Λ is limited by Eq. (2.11), it follows that $|(\mathcal{E}_n - \mathcal{E}_m)\cdot\tau| \ll 1$. Therefore one has

$$\langle n,\Lambda|U_\Lambda(\tau,0)|m,\Lambda\rangle \sim \delta_{n,m} - i\tau\langle n,\Lambda|W_\Lambda|m,\Lambda\rangle$$

$$+ (-i\tau)^2 \sum_\ell \langle n,\Lambda|W_\Lambda|\ell,\Lambda\rangle\langle \ell,\Lambda|W_\Lambda|m,\Lambda\rangle + \ldots \tag{4.7}$$

In view of Eq. (2.12), one has

$$\tau\cdot\langle n,\Lambda|W_\Lambda|m,\Lambda\rangle \lesssim \frac{2\eta_0}{\Lambda^2}\, g(\Lambda^2)\; \Gamma \quad , \tag{4.8}$$

where Γ is some finite quantity independent of Λ as $\Lambda \to \infty$. From Eqs. (4.7) and (4.8), it is then clear that $U(\tau,0)$ can be approximated by 1 if $\tau << 2\eta_0/\Lambda^2$ and if $g(\Lambda^2)$ does not blow up like Λ^2 as $\Lambda \to \infty$. The latter condition is certainly satisfied in QCD where $g(\Lambda^2)$ vanishes logarithmically. Therefore, if the kinematics of the system are such that only small τ is relevant, one can always make the impulse approximation by suitably choosing the quantity Λ.

Notice that the above arguments do not go through if one introduces parton states which are invariant under longitudinal boosts as described in the second paragraph of the previous section. In this case, the energy differences appearing in Eq. (4.6) are restricted as follows:

$$|\mathcal{E}_n - \mathcal{E}_\ell| < \Lambda^2/2\eta \quad . \tag{4.9}$$

The variable η appearing in the above can be made as small as possible so that the quantity $|(\mathcal{E}_n - \mathcal{E}_\ell)\cdot\tau|$ can always be made larger than one, however small τ may be.

V. APPLICATION TO DEEP INELASTIC PROCESSES

In this section, the ideas developed so far will be applied to deep inelastic electron scattering and the Drell-Yan process,[5] and obtain the parton model expressions with scaling violation effects incorporated.

A. Deep Inelastic Electron Scattering

The kinematics of this process are shown in Fig. (2). The cross section can be computed from the following well known tensor:

$$W^{\mu\nu}(q,P_i) \propto \int dx\ e^{iq\cdot x} \langle h|J^{\mu}(x)J^{\nu}(0)|h\rangle \quad , \tag{5.1}$$

where $|h\rangle$ is the physical hadronic state with momentum P_i, and J^{μ} is the electromagnetic current in the Heisenberg picture. Choose the coordinate frame so that

$$\begin{aligned} q &= (0,Q_{\perp},\nu/\eta_0) \\ P_i &= (\eta_0,0,\eta_0) \quad , \end{aligned} \tag{5.2}$$

where $\eta_0 = M_h/\sqrt{2}$, $\nu = q\cdot P_i$ and $q^2 = -{Q_{\perp}}^2 \equiv -Q^2$. In this frame, one has

$$q\cdot x = \nu\tau/\eta_0 - Q_{\perp}\cdot x_{\perp} \quad . \tag{5.3}$$

In the Bjorken limit $\nu \to \infty$ and $Q^2 \to \infty$ with $Q^2/2\nu \equiv x$ held fixed, the τ-integration in Eq. (5.1) is only appreciable in the range

$$|\tau| \lesssim \eta_0/\nu \leq 2\eta_0/Q^2 \quad . \tag{5.4}$$

The second inequality in the above follows because $x \leq 1$. In view of the discussions in the previous section, the inequality (5.4) suggests that one identify Λ with Q. Let us then undress the Heisenberg operator J^{μ} into the Q-picture as follows:

$$J^{\mu}(x) = U_Q^{-1}(\tau,0)\ J_Q^{\mu}\ (x)\ U_Q(\tau,0) \quad , \tag{5.5}$$

where J_Q^{μ} is the current in Q-picture, whose time development is given by

$$J_Q^{\mu}(\tau,x_{\perp},\mathscr{y}) = e^{i\mathscr{H}_Q\tau}\ J_Q^{\mu}(0,x_{\perp},\mathscr{y})\,e^{-i\mathscr{H}_Q\tau} \quad . \tag{5.6}$$

From Eq. (5.4) and the discussions in Sect. IV, the matrix $U_Q(\tau,0)$ appearing in Eq. (5.5) can be approximated by 1, the correction terms being of order m^2/Q^2 where m is some finite, dimensionful parameter. Eq. (5.1) can then be approximated as follows:

$$W^{\mu\nu} \propto \int dx\ e^{iq\cdot x}\ \langle h|J_Q^{\mu}(x)\ J_Q^{\nu}(0)|h\rangle + 0(m^2/Q^2) \quad . \tag{5.7}$$

At this point, the reader must have noticed that the present derivation parallels closely the derivation of Drell et al.[19] The only difference is that they have used the interaction picture in a cut-off field theory, while the present derivation uses the Q-picture in a full theory. The ideas of undressing the Heisenberg current and of approximating the U matrix by 1 originated in their papers. The rest of the steps are then clear: one sandwiches the identity

$$\sum_n |n,Q\rangle\langle n,Q| = 1 \tag{5.8}$$

between the hadronic state $|h\rangle$ and the operator J_Q^{μ} in Eq. (5.7), and uses the constraint of momentum conservation. The resulting expression is especially simple if one considers the quantity W^{oo} because the charge density $J^o(x)$ is simply given by Eq. (A.14a). In this way, one finds that $\nu W_2(x,Q^2) = F(x,Q^2)$ is the probability of finding a Q-parton of longitudinal fraction x: In equations, this means

$$F(x,Q^2) = \sum_{n,P_{\perp}} |\langle h|(\eta_0 x,P_{\perp}),n,Q\rangle|^2 \quad . \tag{5.9}$$

This function changes as Q changes because the state $|n,Q\rangle$ changes, giving rise to the scaling violations. The effects of scaling violation will be studied in detail in Sect. VI. Notice that the function F depends only on the ratio $\eta/\eta_0 = x$ because of invariance under longitudinal boosts. Of course, the parton states change as remarked in Sect. III, but the existence of the unitary operator which connects the state $|(\eta,P_\perp),\Lambda,\eta_0\rangle$ to $|(\lambda\eta,P_\perp),\Lambda,\lambda\eta_0\rangle$ is sufficient for the boost invariance of F.

B. Drell-Yan Process

Now consider the process hadron a + hadron b $\rightarrow \mu^- + \mu^+$ + anything as shown in Fig. (3.a). In the parton model,[5,9] this process goes via the annihilation of parton-antiparton pair into massive photons, as shown in Fig. (3.b). Perturbative calculations[20] show that bremsstrahlung gives a correction of order $(g(Q^2))^4$ to the annihilation term. In this paper, only the annihilation diagrams as shown in Fig. (3.b) will be considered. The main purpose here is to investigate the modification of the naive parton model result coming from the scaling violation effect.

Fig. (3.a) also specifies the coordinate system adopted in the present derivation. Notice that the longitudinal direction, the z-direction, is chosen to be perpendicular to the collision axis. This is necessary if one would like to treat the two incoming hadrons on the same footing. If one chooses the collision axis to be the z-direction, then it is necessary to consider two IMF's, one associated with the hadron moving along the +z direction, the other associated with the hadron moving along the -z direction. The coordinate system shown in

Fig. (3.a) was proposed by Drell and Yan[9] in their original derivation of the cross section in the cut-off field theory.

The production cross section is proportional to the quantity

$$W = \int dx\, e^{-iq\cdot x} \langle P,P'|J_\mu(x)J^\mu(x)|P,P'\rangle \quad , \tag{5.10}$$

where $|P,P'\rangle$ is the physical state of two incoming hadrons. In the present coordinate system, the vectors P,P' and q have the following IMF components:

$$P = (\eta_0, \sqrt{2}\,\eta_0, 0, \eta_0) \tag{5.11a}$$

$$P' = (\eta_0, -\sqrt{2}\,\eta_0, 0, \eta_0) \tag{5.11b}$$

and

$$q = (\eta_q, q_\perp, (Q^2 + q_\perp{}^2)/2\eta_q) \quad , \tag{5.11c}$$

where

$$\eta_0 = \frac{1}{2}\sqrt{S/2} \ , \qquad S = (P + P')^2 \quad . \tag{5.12}$$

From (5.11c) one obtains

$$q\cdot x = \tau(Q^2 + q_\perp{}^2)/2\eta_q + \eta_q\cdot \mathscr{z} - q_\perp\cdot x_\perp \quad . \tag{5.13}$$

Now it will be argued that the partons measured in the process described by Fig. (3) are the Q-partons, i.e., the states $|n,Q\rangle$'s defined in Sect. II. It does not seem to be easy to demonstrate this in a straightforward way. Therefore, I will first assume that it is indeed the case, and then show that the impulse approximation as described in Sect. IV works for the process. To do this, it is necessary to limit the kinematics as follows:

$$S >> Q^2 \to \infty \quad . \tag{5.14}$$

Most present experiments are in the region specified by (5.14). With these assumptions, let us estimate the magnitude of the vectors $\underset{\sim}{P_1}$ and

$\underset{\sim}{P}_2$, whose components are

$$\underset{\sim}{P}_1 = (\eta_1, P_{1\perp}) \quad \text{and} \quad \underset{\sim}{P}_2 = (\eta_2, P_{2\perp}) \quad . \tag{5.15}$$

Since $\underset{\sim}{P}_1$ ($\underset{\sim}{P}_2$) is the parton momentum appearing in the hadron a (b), and the η-variables are conserved and positive, one can write

$$\eta_i = \eta_0 x_i \quad , \qquad 0 \leq x_1, \quad x_2 \leq 1 \quad . \tag{5.16}$$

To estimate the magnitude of the transverse components, it is convenient to perform a Galilean boost[13] with the boost parameter $\upsilon_\perp = (-\sqrt{2}, 0)$. In the boosted frame, the hadron a has only a longitudinal momentum component, and the magnitude of the transverse momentum of the Q-partons in the hadron a can be at most of order Q. This follows from the discussion given in the last paragraph of Sect. III. By transforming back to the laboratory frame, one finds that

$$P_{1\perp} = P'_{1\perp} + \sqrt{2}\, \eta_0\, x_1\, \underset{\sim}{e}_x \quad , \tag{5.17}$$

where $\underset{\sim}{e}_x$ is a unit vector along the x-direction. In Eq. (5.17), the first term is of order Q while the second is of order $\sqrt{S}$, so the second term dominates this expression for finite x_1 in the region given by Eq. (5.14). Similarly, one obtains

$$P_{2\perp} = P'_{2\perp} - \sqrt{2}\, \eta_0\, x_2\, \underset{\sim}{e}_x \quad . \tag{5.18}$$

Again, the first term in the R.H.S. of Eq. (5.18) is negligible compared to the second. From Eqs. (5.17) and (5.18), one obtains

$$\eta_q = \eta_0 (x_1 + x_2) \quad , \tag{5.19}$$

$$q_\perp = P'_{1\perp} + P'_{2\perp} + \sqrt{2}\, \eta_0\, (x_1 - x_2)\, \underset{\sim}{e}_x \quad . \tag{5.20}$$

and

$$q_\perp^2 \sim 2\eta_0^2 (x_1 - x_2)^2 \quad . \tag{5.21}$$

The coefficient of τ in Eq. (5.13) can now be computed. It is

$$\frac{Q^2 + q_\perp{}^2}{2\eta_q} \simeq \frac{Q^2 + 2\eta_0^2(x_1 - x_2)^2}{2\eta_0(x_1 + x_2)} \quad . \tag{5.22}$$

From this equation, one sees that the τ-integral in Eq. (5.10) is limited to the following region:

$$\tau < \frac{2\eta_0(x_1 + x_2)}{Q^2 + 2\eta_0^2(x_1 - x_2)^2} < \frac{2\eta_0}{Q^2} \quad . \tag{5.23}$$

The second inequality follows because the product $x_1 \cdot x_2$ is small if Q^2/S is small, so that $x_1 + x_2$ can be at most 1. Notice that the range of τ given by (5.23) is identical to that appearing in (5.4). Therefore if one undresses the Heisenberg operator J_μ appearing in (5.10) into the Q-picture, then the impulse approximation will be valid in exactly the same manner as in the previous subsection.

There is one complication to be dealt with in obtaining the cross section. In the case of νW_2, it was only necessary to consider the charge density J^o which is simple. In the present case, one is dealing with the product

$$J_\mu J^\mu = 2J^o J^3 - J_\perp \cdot J_\perp \quad . \tag{5.24}$$

From Eq. (A.14), one sees that the currents J^3 and $J_\perp$ involve the covariant derivative $D_\perp = \partial_\perp + igA_\perp$. In the present derivation, the terms involving the gluon fields $A_\perp$'s will be dropped without a detailed justification. They could contribute correction terms of order $g^2(Q^2)$.

With these remarks, it is now straightforward to compute the cross section. One obtains

$$\omega_q \frac{d\sigma}{d\underset{\sim}{q}dQ^2} = \frac{4\pi\alpha^2}{3} \cdot \quad S\sum_i \int e_i^2 \, \tilde{F}_{ia}(\underset{\sim}{P},\underset{\sim}{P}_1,Q^2)\tilde{F}_{\bar{i}b}(\underset{\sim}{P}',\underset{\sim}{P}_1,Q^2) \tag{5.25}$$

$$\frac{1}{Q^2}\,\delta(Q^2 - S_{12})\;\omega_q\delta^3(q - P_1 - P_2)\;\; dx_1dx_2\;d^2P_{\perp 1}\;d^2P_{\perp 2}$$

Here ω_q and $\underset{\sim}{q}$ are the energy and momentum components of q, respectively, in the ordinary reference frame, $S_{12} = (P_1 + P_2)^2$. $\delta^{(3)}$ appearing in the above is the δ-function in the ordinary reference frame, i.e.,

$$\delta^{(3)}(q - P_1 - P_2) = \delta(q_z - P_{1z} - P_{2z})\delta(q_x - P_{1x} - P_{2x}) \tag{5.26}$$
$$\delta(q_y - P_{1y} - P_{2y}) \quad .$$

Finally $\tilde{F}_{ia}(\underset{\sim}{P},\underset{\sim}{P}_1,Q^2)$ is the probability of finding a Q-parton of quantum number i inside the hadron a whose momentum is $\underset{\sim}{P}$, i.e.,

$$\tilde{F}_{ia}(\underset{\sim}{P},\underset{\sim}{P}_1,Q^2) = \sum_n \left|\langle a,\underset{\sim}{P}|(\eta_1,P_{1\perp})^i,n,Q\rangle\right|^2 \quad . \tag{5.27}$$

By a simple Galilean boost, this quantity can be related to the probability $F_{ia}(x,p_\perp{}^2,Q^2)$ of finding a Q-parton of momentum $(x\eta_0,P_\perp)$ in the hadron of momentum $(\eta_0,0)$ as follows:

$$\tilde{F}_{ia}\;(\underset{\sim}{P},\underset{\sim}{P}_1,Q^2) = F_{ia}(x_1,P'^{\,2}_{\perp 1},Q^2) \quad , \tag{5.28}$$

where $P_\perp{}'$ is related to $P_\perp$ by Eq. (5.17). Similarly one has

$$\tilde{F}_{\bar{i}b}(\underset{\sim}{P}',P_2,Q^2) = F_{\bar{i}a}(x_2,P'^{\,2}_{\perp 2},Q^2) \quad . \tag{5.29}$$

Notice that the functions F_{ia} depend only on the longitudinal fraction x as discussed in the last paragraph of the previous subsection. Also, the function depends on $P_\perp$ only through $P_\perp{}^2$ by rotational invariance.

Now consider the δ-functions appearing in the integrand of (5.25). From (5.17) and (5.18), taking into account the fact that the $P'_{i\perp}$'s are small, one has

$$\delta(Q^2 - S_{12}) \sim \frac{1}{S}\,\delta(Q^2/S - x_1x_2) \tag{5.30}$$

and

$$\delta(q_x - P_{1x} - P_{2x}) \sim \frac{2}{\sqrt{S}}\;\delta(\frac{2q_x}{\sqrt{S}} - x_1 + x_2) \quad . \tag{5.31}$$

Next, consider the z-component. One has

$$P_{iz} = \frac{1}{\sqrt{2}}\,(\eta_i - \frac{P_{ix}{}^2 + P_{iy}{}^2}{2\eta_i}) \quad . \tag{5.32}$$

From (5.32), (5.17) and (5.18), it follows:

$$P_{1z} \sim -P'_{1x} \quad \text{and} \quad P_{2z} \sim P'_{2x} \quad . \tag{5.33}$$

Therefore

$$\delta(q_z - P_{1z} - P_{2z}) \sim \delta(q_z + P'_{1x} - P'_{2x}) \quad . \tag{5.34}$$

Putting these results back into Eq. (5.25), one obtains

$$\omega_q \frac{d\sigma}{d\underset{\sim}{q}dQ^2} = \frac{4\pi\alpha^2}{3}\sum_i \int e_i^2\, F_{ia}(x_1, P_{1\perp}{}^2, Q^2)$$

$$F_{\bar{i}b}(x_2, P_{2\perp}{}^2, Q^2)\,\frac{1}{Q^2}\,\delta(Q^2/S - x_1x_2)\,\delta(q_z + P_{1x} - P_{2x}) \tag{5.35}$$

$$\delta(q_y - P_{1y} - P_{2y})\cdot\frac{2\omega_q}{\sqrt{S}}\;\delta\left(\frac{2qx}{\sqrt{S}} - x_1 + x_2\right) dx_1\,dx_2\,d^2P_{1\perp}\,d^2P_{2\perp}$$

The above formula is somewhat peculiar in that the z-direction and the x-direction appear mixed in the δ-function. This can be cured by the following observation: First notice that P_{1x} and P_{2x} appear as integration variables, so one may call them $-P_{1z}$ and P_{2z}, respectively. Under this substitution, the quantity $F_{ia}(x_1, P_{1\perp}{}^2, Q^2)$ becomes $F_{ia}(x_1, P_{1y}{}^2 + P_{1z}{}^2, Q^2)$, which can be interpreted as the probability of finding a Q-parton of longitudinal fraction x_1 and the transverse momentum (P_{1y}, P_{1z}) inside the hadron a moving along the x-direction. The same can be repeated for $F_{\bar{i}b}(x_2,P_\perp{}^2,Q^2)$. Finally, one obtains

$$\omega_q \frac{d\sigma}{d\underset{\sim}{q}dQ^2} = \frac{4\pi\alpha^2}{3} \sum_i \int e_i^2 \, F_{ia}(x_1,P_{1\perp}{}^2,Q^2)$$

$$\cdot F_{\bar{i}b}(x_2,P_{2\perp}{}^2,Q^2) \, \frac{1}{Q^2} \, \delta(Q^2/S - x_1x_2) \, \delta^{(2)}(q_\perp - P_{1\perp} - P_{2\perp}) \qquad (5.36)$$

$$\cdot \frac{2\omega_q}{\sqrt{2}} \, \delta\left(\frac{2q_x}{\sqrt{S}} - x_1 + x_2\right) dx_1 \, dx_2 \, d^2P_{1\perp} d^2P_{2\perp}$$

Exactly the same formula as Eq. (5.36) was conjectured by Kogut[11] and also by Hinchliffe and Lllewellyn-Smith.[12] Kogut and Shigemutsu[21] have given a numerical analysis of this formula.

VI. SCALING VIOLATIONS

In this section, the variation of the quantities F_{ai} appearing in the cross sections of the deep inelastic electron scattering and the Drell-Yan processes will be discussed. They are defined by

$$F_{ai}(\underset{\sim}{P},Q^2) = \sum_n |\langle a|(\underset{\sim}{P},i),n,Q\rangle|^2 \quad , \tag{6.1}$$

where

$$\underset{\sim}{P} = (\eta_0 x, P_\perp) \quad . \tag{6.2}$$

As is clear from Eq. (6.1), $F_{ai}(\underset{\sim}{P},Q^2)$ is the probability of finding a Q-parton of momentum $\underset{\sim}{P}$ and quantum number i inside the hadron with momentum $(\eta_0,0)$ and the quantum number a. Given $F_{ai}(\underset{\sim}{P},Q^2)$, let us compute $F_{ai}(\underset{\sim}{P},Q'^2)$ where

$$Q'^2 >> Q^2 >> Q'^2 - Q^2 = \Delta Q^2 \quad . \tag{6.3}$$

For this purpose, it is necessary to compute the quantity $|\langle a|\underset{\sim}{P},n,Q'\rangle|^2$ for arbitrary n. By sandwiching the complete set of states $|m,Q\rangle$, one obtains

$$F_{ai}(\underset{\sim}{P},Q'^2) = \sum_n |\sum_m \langle a|m,Q\rangle\langle m,Q|\underset{\sim}{P},n,Q'\rangle|^2 \quad . \tag{6.4}$$

Therefore the problem reduces to computing the matrix element $\langle mQ|P,n,Q'\rangle$. To do this, consider the following expansion:

$$|\underset{\sim}{P},n,Q'\rangle = \sqrt{Z_p^{\Delta} Z_n^{\Delta}} \left\{ |\underset{\sim}{P},n,Q\rangle + \sum_\ell{}' |\ell,Q\rangle \frac{\langle \ell,Q|\Delta W|P,n,Q'\rangle}{\Delta\mathcal{E}_\ell} \right\} \tag{6.5}$$

where

$$\Delta W = W_{Q'} - W_Q \quad \text{and} \quad \Delta\mathcal{E}_\ell = \mathcal{E}_\ell - \mathcal{E}_p - \mathcal{E}_n \quad .$$

The sum over ℓ in (6.5) excludes the states which have the same energy with the state $|\underset{\sim}{P},n,Q'\rangle$. Z^Δ's in the above are the wave function renormalization constants which can be computed from the normalization condition

$$\langle m,Q|n,Q\rangle = \langle m,Q'|n,Q'\rangle \ . \qquad (6.6)$$

From Eq. (6.5), one obtains

$$\langle m,Q|\underset{\sim}{P},n,Q'\rangle = \sqrt{Z_p^\Delta Z_n^\Delta}\left[\delta_{m,p+n} + \frac{\langle mQ|\Delta W|\underset{\sim}{P},n,Q'\rangle}{\Delta\mathscr{E}m}\right] \qquad (6.7)$$

The matrix element $\langle m,Q|\Delta W|\underset{\sim}{P},n,Q'\rangle$ can be computed from Eq. (2.12). To order $g(Q^2)$, only the 1-particle processes as shown in Fig. (4) need to be considered. As discussed in Eq. (2.12), the matrix element is non-vanishing only in the region

$$Q^2/2\eta_0 < |\Delta\mathscr{E}_m| < Q'^2/2\eta_0 \quad . \qquad (6.8)$$

From Eq. (6.8) and from the discussions in the last paragraph of Sect. III, it follows that the diagram shown in Fig. (4.b) does not contribute, since the state $|m,Q\rangle$ does not have enough energy. For a given configuration n, the configurations m that contribute to Fig. (4.a) and the one contributing to Fig. (4.c) are distinct. For Fig. (4.a), one has

$$\langle m,Q|\Delta W|\underset{\sim}{P},n,Q'\rangle = \langle P',Q|\Delta W|P,q,Q'\rangle + 0(g^2(Q^2)) \quad . \qquad (6.9)$$

And for Fig. (4.c), one has

$$\langle m,Q|\Delta W|\underset{\sim}{P},n,Q'\rangle = \langle m',Q|\Delta W|n\rangle + 0(g(Q^2)) \quad . \qquad (6.10)$$

Now substitute the results (6.7), (6.9) and (6.10) into Eq. (6.4). Considering the incoherence, one obtains

$$\begin{aligned} F(\underset{\sim}{P},Q'^2) = &\sum_n Z_p^\Delta Z_n^\Delta \, |\langle a|P,n,Q\rangle|^2 \\ &+ \sum_n |\langle a|P',n,Q\rangle|^2 \cdot \left|\frac{\langle P',Q|\Delta W|p,q,Q'\rangle}{\Delta\mathscr{E}}\right|^2 \qquad (6.11) \\ &+ \sum_{n,n'} |\langle a|p,n,Q\rangle|^2 \cdot \left|\frac{\langle n,Q|\Delta W|n',Q'\rangle}{\Delta\mathscr{E}}\right|^2 + 0(g^4(Q^2)) \quad . \end{aligned}$$

Now up to order $g^2(Q^2)$, one has

$$\frac{1}{Z_n^\Delta} - 1 = 1 - Z_n^\Delta + 0(g^4(Q^2))$$

$$= \sum_{n'} \left|\frac{\langle n,Q|\Delta W|n',Q'\rangle}{\Delta\mathscr{E}}\right|^2 + 0(g^4(Q^2)) \quad . \tag{6.12}$$

By means of Eq. (6.12), it is easy to show that the last and the first term in Eq. (6.11) combine to yield the final result:

$$F(P,Q'^2) = Z_p^\Delta F(\underset{\sim}{P},Q^2)$$
$$+ \sum_{n,\underset{\sim}{P}',q} |\langle a|P',n,Q\rangle|^2 \left|\frac{\langle P',Q|\Delta W|P,q,Q'\rangle}{\Delta\mathscr{E}}\right|^2 \quad . \tag{6.13}$$

Eq. (6.13) is the desired relation describing the scaling violation effects. Restoring the quantum number indices and writing

$$F(\underset{\sim}{P},Q^2) = F_{ai}(x,P_\perp{}^2,Q^2) \quad , \tag{6.14}$$

Eq. (6.13) can be written in the following form:

$$F_{ai}(x,P_\perp{}^2,Q^2+\Delta Q^2) = Z_i^\Delta F_{ai}(x,P_\perp{}^2,Q^2)$$
$$+\int_x^1 \frac{dy}{y}\int d^2P_\perp{}' \; d^2P_\perp \; f_{ij} \; (\underset{\sim}{P},\underset{\sim}{P}')\cdot F_{aj}(y,P_\perp{}'^2,Q^2) \quad , \tag{6.15}$$

where $f_{ij}(\underset{\sim}{P},\underset{\sim}{P}')$ is the probability of a Q-parton of momenta $\underset{\sim}{P}' = (\eta_0 y,P_\perp{}')$ going into a Q'-parton of momenta $\underset{\sim}{P} = (\eta_0 x,P_\perp)$ via the action of the ΔW vertex, as shown in Fig. (5). Notice that, by longitudinal and Galilean boost invariance, the function $f_{ij}(\underset{\sim}{P},\underset{\sim}{P}')$ depends only on the combinations x/y and $\overline{P_\perp}$ where

$$\overline{P_\perp} = P_\perp - x/y\cdot P_\perp{}' \quad . \tag{6.16}$$

It is convenient to rewrite Eq. (6.15) in terms of the following quantities:

$$Z_i^{\Delta} = 1 - \frac{\alpha(Q^2)}{2\pi} \frac{\Delta Q^2}{Q^2} \tilde{Z}_i \quad , \tag{6.17}$$

where

$$\Delta Q^2 = Q'^2 - Q^2 \quad , \tag{6.18}$$

and

$$f_{ij}(\underset{\sim}{P},\underset{\sim}{P}') = \frac{\alpha(Q^2)}{2\pi^2} \frac{1}{\overline{P}_\perp^{\,2}} \tilde{f}_{ij}(x/y) \quad . \tag{6.19}$$

Here

$$\alpha(Q^2) = \frac{g^2(Q^2)}{4\pi} \quad . \tag{6.20}$$

The factorization implied by Eq. (6.19) holds in lowest order. Eq. (6.15) now becomes

$$\begin{aligned} & F_i(x,P_\perp^{\,2},Q^2+\Delta Q^2) - F_i(x,P_\perp^{\,2},Q^2) \\ &= \frac{\alpha(Q^2)}{2\pi} \Bigg\{ - \tilde{Z}_i \frac{\Delta Q^2}{Q^2} F_i(x,P_\perp^{\,2},Q^2) \\ &\quad + \int_x^1 \frac{dy}{y} \int d^2P_\perp \frac{d^2P_\perp'}{\pi \overline{P}_\perp^{\,2}} \tilde{f}_{ij}\left(\frac{x}{y}\right) F_j(y,P_\perp^{\,2},Q^2) \Bigg\} \quad . \end{aligned} \tag{6.21}$$

Here and in the following, the hadronic quantum number a will be suppressed. Let us now work out the restriction of phase space implied by Eq. (2.12). For this purpose it is only necessary to compute the energy difference $\Delta\mathscr{E}$ between the initial and the final states of the diagram shown in Fig. (5). It is

$$\Delta\mathscr{E} = \frac{1}{2\eta_0} \overline{P}_\perp^{\,2} \cdot \frac{y}{x} \frac{}{(y-x)} \quad . \tag{6.22}$$

From this and Eq. (2.12), one finds

$$\frac{x}{y}(y - x)Q^2 \leq \overline{P}_\perp^{\,2} \leq \frac{x}{y}(y - x)(Q^2 + \Delta Q^2) \quad , \tag{6.23}$$

where $\overline{P_\perp}$ is defined by Eq. (6.16). The inequality (6.23) must be imposed in the second term in the R.H.S. of Eq. (6.21). The restriction of the phase space integral on the wave function renormalization constant Z_i^Δ is already taken into account by Eq. (6.17).

It remains to compute the quantities $\tilde{Z}_i$ and $\tilde{f}_{ij}$'s. In the lowest order, they can be computed from the knowledge of the bare vertices shown in Fig. (1.a) and Fig. (1.b), the corresponding matrix elements being specified by Eqs. (A.12) and (A.13) in the Appendix. The calculation is straightforward and already reported elsewhere,[22] and need not be repeated here. To write down the result, let us first discuss the indexing of the quantum numbers. In QCD, there appear color and flavor indices. Since the color group is an exact symmetry, the probabilities of finding a given parton and the probability of finding another parton which differs from the first one only in its color index must be identical. This implies that one need only consider the transition probabilities $\tilde{f}$ which are averaged over the initial colors and summed over the final colors. It is not hard to see that these quantities then become independent of the flavor indices. The distribution functions F, however, still depend on the flavor indices. Let F_{qi} and F_g be the color summed probability distributions of finding a quark of flavor i and a gluon, respectively. Eq. (6.21) can then be rewritten as follows:

$$F_{qi}(x,P_\perp{}^2,Q^2+\Delta Q^2) - F_{qi}(x,P_\perp{}^2,Q^2)$$

$$= \frac{\alpha(Q^2)}{2\pi} \Bigg\{ - \tilde{Z}_q \frac{\Delta Q^2}{Q^2} F_{qi}(x,P_\perp{}^2,Q^2)$$

$$+\int_x^1 \frac{dy}{y} \int d^2P_\perp \frac{d^2P_\perp{}'}{\pi \overline{P}^2{}_\perp} \Bigg[\tilde{f}_{qq}\left(\frac{x}{y}\right) F_{qi}(y,P_\perp'^2,Q^2) \tag{6.24a}$$

$$+ \frac{1}{2N} \tilde{f}_{qg}\left(\frac{x}{y}\right) F_g(y,P_\perp'^2,Q^2)\Bigg]\Bigg\}$$

and

$$F_g(x,P_\perp{}^2,Q^2+\Delta Q^2) - F_g(x,P_\perp{}^2,Q^2)$$

$$= \frac{\alpha(Q^2)}{2\pi}\left\{ - \tilde{Z}_g \frac{\Delta Q^2}{Q^2} F_g(x,P_\perp{}^2,Q^2) + \int_x^1 \frac{dy}{y}\int \frac{d^2P_\perp d^2P_\perp'}{\pi\bar{P}_\perp^2} \right.$$

$$\left.\left[\tilde{f}_{gq}\left(\frac{x}{y}\right)\sum_{i=1}^{2N} F_{qi}(y,P_\perp'^2,Q^2) + \tilde{f}_{gg}\left(\frac{x}{y}\right) F_g(y,P_\perp'^2,Q^2)\right]\right\}, \qquad (6.24b)$$

where

$$\begin{aligned}
\tilde{f}_{qq}(x) &= C_r[2/(1-x) - (1+x)] \quad , \\
\tilde{f}_{gq}(x) &= \tilde{f}_{qq}(1-x) \quad , \\
\tilde{f}_{qg}(x) &= 2T_r\Big((1-x)^2 + x^2\Big) \quad , \\
\tilde{f}_{gg} &= C_g[2/(1-x) + 2/x - 4 + 2x(1-x)]
\end{aligned} \qquad (6.25)$$

and

$$\tilde{Z}_q = C_r(2I_\infty - 3/2) \quad \text{and} \quad \tilde{Z}_g = 2I_\infty C_g - b \quad . \qquad (6.26)$$

N appearing in Eq. (6.25) is the number of the quark fields. The flavor index i runs from 1 to 2N to include anti-flavors. The quantity I_∞ appearing in Eq. (6.26) is an infinite constant defined by

$$I_\infty = \int_0^1 dx/(1-x) \quad . \qquad (6.27)$$

However, the infinities in $\tilde{Z}$'s are cancelled by the integral terms in Eq. (6.24), which diverge because the functions $\tilde{f}$ are singular at x = 1. Eqs. (6.24) - (6.26), together with the inequality (6.23), constitute the main results of this section. Equations of the same general structure were first proposed by Kogut and Susskind,[10] and Kogut.[11]

If one integrates both sides of Eq. (6.24) w.r.t. the transverse variable $P_\perp$, then one obtains a simpler set of equations which we will call the longitudinal equations. The longitudinal equations were first written down by Parisi.[23] He obtained them by taking the inverse Mellin transform of results obtained from the operator product expansion and the renormalization group equations. A derivation of the longitudinal equations in a spirit closer to the present paper was given by Altarelli and Parisi,[24] and also independently by Kim and Schilcher.[22]

For actual calculation, it is convenient to express Eq. (6.24) in a slightly different form. Define

$$F_q = \sum_{i=1}^{2N} F_{qi} \quad . \tag{6.28}$$

Also, let δ be the set (q_i, q_j), and define

$$F_\delta = F_{qi} - F_{qj} \quad . \tag{6.29}$$

Eq. (6.24) can then be reduced to the following set of equations:

$$F_\delta(x,P_\perp{}^2,Q^2+\Delta Q^2) - F_\delta(x,P_\perp{}^2,Q^2) = \frac{\alpha(Q^2)}{2^2}\left\{-\tilde{Z}_q \frac{\Delta Q^2}{Q^2} F_\delta(x,P_\perp{}^2,Q^2)\right.$$

$$\left. + \int_x^1 \frac{dy}{y}\int \frac{d^2P_\perp d^2P_\perp{}'}{\pi \bar{P}_\perp{}^2}\ \tilde{f}_{qq}(x/y)\ F_\delta(y,P_\perp'^2,Q^2)\right\} \ , \tag{6.30a}$$

$$F_q(x,P_\perp{}^2,Q^2+\Delta Q^2) - F_q(x,P_\perp{}^2,Q^2) = \frac{\alpha(Q^2)}{2\pi}\left\{- \tilde{Z}_q \frac{\Delta Q^2}{Q^2} F_q(x,P_\perp{}^2,Q^2) + \right.$$

$$\left. \int_x^1 \frac{dy}{y}\int \frac{d^2P_\perp d^2P_\perp{}'}{\pi \bar{P}_\perp{}^2}\left[\tilde{f}_{qq}(\tfrac{x}{y})\ F_q(y,P_\perp{}'^2,Q^2) + \tilde{f}_{qg}(\tfrac{x}{y})\ F_g(y,P_\perp{}'^2,Q^2)\right]\right\} \tag{6.30b}$$

and

$$F_g(x,P_\perp{}^2,Q^2+\Delta Q^2) - F_g(x,P_\perp{}^2,Q^2) = \frac{\alpha(Q^2)}{2\pi}\Bigg\{$$

$$- \tilde{Z}_g \frac{\Delta Q^2}{Q^2} F_g(x,P_\perp{}^2,Q^2) +$$

$$\int_x^1 \frac{dy}{y}\int\frac{d^2P_\perp d^2P_\perp{}'}{\pi\bar{P}_\perp{}^2}\left[\tilde{f}_{gq}(\frac{x}{y})\; F_q(y,P_\perp{}'^2,Q^2) + \tilde{f}_{gg}(\frac{x}{y})\; F_g(y,P_\perp{}'^2,Q^2)\right]\Bigg\}. \quad (6.30c)$$

VII. SOLUTION OF SCALING VIOLATION EQUATIONS AND PARTON TRANSVERSE MOMENTUM DISTRIBUTIONS

A. General Introduction

Let us first consider the general form of the equation given by Eq. (6.21). In order to impose the phase space cut given by the inequality (6.23), it is convenient to consider the transverse moments $F_i^{(n)}$ defined as follows:

$$F_i^{(n)}(x,Q^2) = \int (P_\perp{}^2)^n \; F(x,P_\perp{}^2,Q^2) \; d^2P_\perp \quad . \tag{7.1}$$

Multiplying both sides of Eq. (6.21) by $(P_\perp{}^2)^n$ and integrating over $P_\perp$ keeping the inequality (6.23) in mind, one obtains the following differential equation:

$$Q^2 \frac{\partial F_i^{(n)}(x,Q^2)}{\partial Q^2} = \frac{\alpha(Q^2)}{2\pi} \left\{ - \tilde{Z}_i \; F_i^{(n)}(x,Q^2) + \sum_{r=0}^{n} Q^{2r} \cdot \frac{n!}{r!(n-r)!} \int_x^1 \frac{dy}{y} \left(\frac{x}{y}\right)^{2n-r} \left(1 - \frac{x}{y}\right)^r \tilde{f}_{ij}\left(\frac{x}{y}\right) y^r F_j^{(n-r)}(y,Q^2) \right] \tag{7.2}$$

The quantities $\tilde{Z}_i$'s diverge as shown in Eq. (6.26). However, it is easy to see from the explicit forms of the f's given in Eq. (6.25) that these divergences cancel the divergences arising from the r = 0 term in the integrals appearing in the above equation. Therefore the function $F^{(n)}(x,Q^2)$'s must be finite if they were finite for some given value of Q^2. One expects also that the large n behavior of the $F^{(n)}(x,Q^2)$ dictate the large $P_\perp{}^2$ behavior of $F_i(x,P_\perp{}^2,Q^2)$. This point should be investigated further. For any given n, Eq. (7.2) can be solved numerically. This is presently under investigation.

Eq. (7.2) can be simplified further in terms of the following Mellin-transformed quantity:

$$M_{\alpha i}^{(n)}(Q^2) = \int_0^1 dx\, x^{\alpha-1}\, F_i^{(n)}(x,Q^2) \quad . \tag{7.3}$$

One obtains

$$Q^2 \frac{\partial M_{\alpha i}^{(n)}}{\partial Q^2} = \frac{\alpha(Q^2)}{2\pi} \left[- \tilde{Z}_i\, M_{\alpha i}^{(n)}(Q^2) \right.$$

$$\left. + \sum_{r=0}^{n} Q^{2r} \frac{n!}{r!(n-r)!}\, m_\alpha(n,r)_{ij}\, M_{\alpha+r}^{(n-r)} \right] , \tag{7.4}$$

where

$$m_\alpha(n,r)_{ij} = \int_0^1 \frac{dy}{y}\, y^{\alpha+2n-r}(1-y)^r\, \tilde{f}_{ij}(y) \quad . \tag{7.5}$$

Eq. (7.4) is considerably simpler than Eq. (7.2). Still it is sufficiently complicated so that an analytical solution does not seem feasible at the present time.

B. Average Transverse Momentum Squared of Partons

To give an explicit example of the solution of equations derived in the previous subsection, let us now compute the partons average momentum squared $\langle P_\perp^2 \rangle_i \equiv T_i(Q^2)$. In terms of the M's defined in Eq. (7.3), it is

$$T_i(Q^2) = M_{1,i}^{(1)}(Q^2) \tag{7.6}$$

It is also necessary to know the partons average longitudinal momentum $\langle x \rangle_i \equiv N_i(Q^2)$, which is

$$N_i(Q^2) = M_{2,i}^{(0)}(Q^2) \quad . \tag{7.7}$$

The N's and T's satisfy the following coupled set of equations:

$$\frac{\partial N_\delta}{\partial \ln Q^2} = - \frac{\alpha(Q^2)}{2\pi}\, b\, a_1\, N_\delta \quad , \tag{7.8}$$

$$\frac{\partial T_\delta}{\partial \ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \cdot b \left\{ -b_1 \ T_\delta + Q^2 \ C_1 \ N_\delta \right\} , \tag{7.9}$$

$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} N_q \\ N_g \end{pmatrix} = - \frac{\alpha(Q^2)}{2\pi} \ bA \begin{pmatrix} N_q \\ N_g \end{pmatrix} , \tag{7.10}$$

and

$$\frac{\partial}{\partial \ln Q^2} \begin{pmatrix} T_q \\ T_g \end{pmatrix} = \frac{\alpha(Q^2)}{2\pi} \cdot b \cdot \left\{ -B \begin{pmatrix} T_q \\ T_g \end{pmatrix} + Q^2 C \begin{pmatrix} N_q \\ N_g \end{pmatrix} \right\} . \tag{7.11}$$

In the above b is given by Eq. (2.14), and a_1, b_1 and c_1 are the (1,1) elements of the matrices A, B and C, respectively. These matrices are given as follows:

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 4/3 \ Cr & -2/3 \ Tr \\ -4/3 \ Cr & 2/3 \ Tr \end{pmatrix} , \tag{7.12}$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 25/12 \ Cr & -7/15 \ Tr \\ -7/12 \ Cr & 7/5 \ Cg + 2/3 \ Tr \end{pmatrix} \tag{7.13}$$

and

$$C = B - A = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \tag{7.14}$$

It is straightforward to solve Eqs. (7.8) - (7.11). One writes

$$\alpha^o = \alpha(Q_0^2), \ N_i(Q_0^2) = N_i^o \ \text{and} \ T_i(Q_0^2) = T_i^o \ . \tag{7.15}$$

Also, it is convenient to define the function

$$\gamma(Q^2) = \alpha(Q^2)/\alpha^o = 1/\left(1 + \frac{\alpha^o}{2\pi} \ln(Q^2/Q^2)\right) . \tag{7.16}$$

The solutions of (7.8) and (7.9) are

$$N_\delta(Q^2) = N_\delta{}^o[\gamma(Q^2)]^{a1} \tag{7.17}$$

and

$$T_\delta(Q^2) = [\gamma(Q^2)]^{b_1} \Bigg\{ T_\delta{}^o + \frac{C_1\alpha^o}{2\pi} b \cdot N_\delta{}^o \int_{Q_0^2}^{Q^2} d\mu^2 [\gamma(\mu^2)]^{1-C_1} \Bigg\} \quad . \tag{7.18}$$

Eq. (7.10) has the following solutions:

$$N_q = K - [\gamma(Q^2)]^\lambda \quad [K - 1 + N_g^o] \quad \text{and} \tag{7.19}$$

$$N_g = 1 - K + [\gamma(Q^2)]^\lambda \ [K - 1 + N_g^o] \quad . \tag{7.20}$$

Here λ is the non-vanishing eigenvalue of the matrix A and is given by

$$\lambda = \frac{1}{3b}\left[2Cr + Tr + \sqrt{(2Cr - Tr)^2 + 8Cr \cdot Tr}\ \right] \quad , \tag{7.21}$$

and

$$K = Tr/(Tr + 2Cr) \quad . \tag{7.20}$$

Eqs. (7.17), (7.18) and (7.19) describe the Q^2-variation of the average longitudinal momentum of the partons, and are well known from the usual method employing the operator product expansion and the renormalization group equations. Their explicit form is necessary to solve Eq. (7.11). To solve the latter equation, let us introduce the following eigen modes of the matrix B:

$$T_\pm(Q^2) = \xi_\pm \ T_q(Q^2) + T_g \ (Q^2) \quad , \tag{7.21}$$

where

$$\xi_\pm = \frac{1}{2b_1}\left[(b_1 - b_4) \pm \sqrt{(b_1 - b_4)^2 + 4b_2b_3}\right] . \tag{7.22}$$

The corresponding eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left[(b_1 + b_4) \pm \sqrt{(b_1 - b_4)^2 + 4b_2b_3}\right] . \tag{7.23}$$

The solutions are

$$T_\pm(Q^2) = [\gamma(Q^2)]^{\lambda\pm}\left\{T_\pm{}^o + \frac{\alpha^o}{2\pi}\int_{Q_0^2}^{Q^2} d\mu^2\left[\gamma(\mu^2)\right]^{1-\lambda\pm}(u_\pm - v_\pm[\gamma(\mu^2)]^\lambda)\right\}, \tag{7.24}$$

where

$$u_\pm = b\left\{\xi_\pm C_2 + C_4 + K\cdot[\xi_\pm(C_1 - C_2) + C_3 - C_4)]\right\} \tag{7.25}$$

and

$$v_\pm = b\cdot[\xi_\pm(C_1 - C_2) + C_3 - C_4]\cdot[K - 1 + N_g^o] \quad . \tag{7.26}$$

From $T_\pm$, one recovers

$$T_q(Q^2) = [T_+(Q^2) - T_-(Q^2)]/(\xi_+ - \xi_-) \tag{7.27}$$

and

$$T_g(Q^2) = [\xi_+T_-(Q^2) - \xi_-T_+(Q^2)]/(\xi_+ - \xi_-) \quad . \tag{7.28}$$

Eqs. (7.18), (7.27) and (7.28) describe the Q^2-variation of the parton's average transverse momentum squared. To analyze these results numerically, consider QCD with color SU(3) $\otimes$ flavor SU(4) in which case

$$C_g = 3, \quad C_r = 4/3 \quad \text{and} \quad T_r = 2 \quad . \tag{7.29}$$

Also, it will be assumed that the parton distributions inside the nucleon can be divided into a valence contribution, an SU(3) symmetric sea of quarks and anti-quarks and a charmed sea. For an iso-spin zero target, one has

$$\begin{gathered} F_p = \frac{1}{2}F_v + F_s,\ F_n = \frac{1}{2}F_v + F_s,\ F_\lambda = F_s,\ F_{p'} = F_c \\ F_{\overline{p}} = F_s,\ F_{\overline{n}} = F_s,\ F_{\overline{\lambda}} = F_s \text{ and } F_{\overline{p}'} = F_c \quad . \end{gathered} \tag{7.30}$$

The notations in the above are self-explanatory: p, n, λ and p' denote the proton, the neutron, the strange and the charmed quarks, respectively while the barred ones denote the antiquarks. From Eq. (7.30), it is clear that the quantities $T_v = T_p - T_\lambda$ and $T_{c-s} = T_c - T_s = T_\lambda - T_{p'}$ belong to T_δ. Thus their Q^2-development can be determined if the initial values T_v^o, T_{c-s}^o, N_v^o and N_{c-s}^o are known. Also the Q^2-development of T_q and T_g are determined if the initial values T_q^o, T_g^o and $N_g^o = 1 - N_q^o$ are known. Furthermore, it follows from (7.30) that

$$T_q = \sum_{I+L}^{2N} T_i = T_v + 2T_c + 6T_s \tag{7.31}$$

and

$$N_q = \sum_{i=1}^{2N} N_i = N_v + 2N_c + 6N_s \quad . \tag{7.32}$$

Therefore, if one knows the Q^2-development of T_v, T_{c-s} and T_q, then one can determine the behavior of T_c and T_s separately.

Before going into a detailed numerical analysis, let us first discuss some general properties of the function $T_i(Q^2)$. Schematically it can be written as follows:

$$T_i(Q^2) = C_i(T^o)\ A_i(Q^2) + D_i(N^o)\ B_i(Q^2) \quad . \tag{7.33}$$

The first (second) term in the above corresponds to the first (second) term of Eq. (7.18) or (7.24). At Q^2 near Q_0^2, one has the behavior

$$A_i(Q^2) \propto [\gamma(Q^2)]^{a_i} \tag{7.34a}$$

and

$$B_i(Q^2) \propto (Q^2 - Q_0^2) \quad , \tag{7.34b}$$

where a_i is a positive constant. As Q^2 becomes large, one has

$$A_i(Q^2) \sim 1/(\log Q^2)^{a_i} \tag{7.35a}$$

and

$$B_i(Q^2) \sim Q^2/(\log Q^2)^{b_i} \quad , \tag{7.35b}$$

where b_i is another positive constant. From Eqs. (7.33) - (7.35), one sees that the function $T_i(Q^2)$ behaves roughly as follows: At Q^2 near Q_0^2, it is mainly governed by $A_i(Q^2)$ which decreases as Q^2 increases. At large Q^2, on the other hand, it is mainly governed by the function $B(Q^2)$ whose behavior is given by (7.35b). Furthermore, the coefficient of $B(Q^2)$ involves only the initial values N_i^o as shown in Eq. (7.33). Therefore, the behavior of the partons' transverse momentum at large Q^2 is determined if the initial values of the parton's average longitudinal momentum are known.

Consider now Eq. (7.24) in the ultra high Q^2 region. Integrating by parts, one obtains

$$\int_{Q_o^2}^{Q^2} d\mu^2[\gamma(\mu^2)]^a = Q^2\,\gamma^a(Q^2) - Q_0^2\,\gamma^a(Q_0^2) - \int_{Q_0^2}^{Q^2} d\mu^2\,\frac{[\gamma(\mu^2)]^a}{d\mu^2} = Q^2\gamma^a(Q^2) + O[Q^2\gamma^a(Q^2)/\log(Q^2/Q_0^2)] \;. \tag{7.36}$$

If $\log(Q^2/Q_0^2) >> 1$, one may retain only the first term in Eq. (7.36). In the same limit, the term involving $v_\pm$ in the integral in (7.24) may also be neglected, and one gets

$$T_\pm(Q^2) \xrightarrow{Q^2\to\infty} \frac{\alpha^o}{2\pi}\,U_\pm\cdot Q^2 \quad . \tag{7.37}$$

From (7.37), (7.27) and (7.28), one obtains

$$r(Q^2) = T_q(Q^2)/T_g(Q^2) \xrightarrow[Q^2\to\infty]{} r_\infty = 23T_r/(56\,Cg + 15T_r) \quad . \tag{7.38}$$

However, the approach to r_∞ should be very slow, the correction terms being logarithmic.

For numerical evaluation, it is necessary to specify the initial values α^o, N^o's and T^o's at $Q^2 = Q_0^2 = 1$ Gev^2. I use

$$\alpha^o = 0.5 \tag{7.39}$$

$$N_v^{\ o} = 0.46,\ N_c^{\ o} = 0,\ N_s^{\ o} = 0.01,\ N_g^{\ o} = 0.48, \tag{7.40}$$

$$T_v^{\ o} = 0.75\ \text{Gev}^2,\ T_c^{\ o} = 0,\ T_s^{\ o} = 0.25\ \text{Gev}^2 \text{ and } T_g^{\ o} = 0.25\ \text{Gev}^2. \tag{7.41}$$

In the above, (7.39) and (7.40) are the standard results that follow from the analysis[25] of deep inelastic lepton scattering data. They are thus presumably reliable. In writing (7.41), it was assumed that the average transverse momentum per quark in the nucleon is 0.5 Gev at $Q^2 = 1$ Gev^2. Since there are three valence quarks, one obtains $T_v^{\ o} = 3\cdot(0.5)^2 = 0.75$ Gev^2. $T_c^{\ o}$ is set to zero since no charm should be present at low Q^2. The choice $T_g^{\ o} = 0.25$ Gev^2 was made by naively assuming that the quarks and the gluons have the same transverse momenta. The $T^o_{\ i}$'s given by (7.41) are at most speculative, and it is conceivable that the present estimate could be wrong by a factor of 2. According to the discussion in the previous paragraph, however, the behavior of the function $T_i(Q^2)$ at large Q^2 is insensitive to the precise value of the $T^o_{\ i}$'s.[26]

Table I presents the result of the calculation in the Q^2 range from 1 Gev^2 to 120 Gev^2, which covers most of the present experiments. The gluonic contribution rises rapidly, while the quark contribution changes very slowly in the Q^2 range investigated. The general trend of the $T_i(Q^2)$'s agree with the qualitative analysis carried out above.

At $Q^2 = 120$ Gev^2, $r(Q^2) \sim 2/5$ which should approach $r_\infty = 0.232$ according to Eq. (7.38). At $Q^2 = 10^5$ and 10^7 Gev^2, $r(Q^2) = 0.276$ and 0.265, respectively. Such a slow approach to r_∞ is to be expected.

In interpreting the result shown in Table I, it should be kept in mind that $T_V(Q^2)$ is the average transverse momentum squared summed over the three valence quarks. Thus the average transverse momentum squared of a valence quark is ~0.56 Gev^2 at $Q^2 = 120$ Gev^2. By the same token, the rapid rise of the gluonic transverse momentum squared may simply mean that the average number of gluons increases rapidly. The average transverse momentum of quarks obtained in this section seems reasonable in view of the recent experiments. However, more assumptions are necessary in order to compare the result of Table I with experiment. This is beyond the scope of the present paper.

VIII. CONCLUDING REMARKS

In this paper, it was shown how to reconcile the apparently contradictory concepts of field theoretic local interactions and the impulse approximation. It was observed that the impulse approximation requires the time development matrix U to be close to unity during a short time interval. To accomplish this, the interacting part of the Hamiltonian was separated into two parts, one containing only those terms involving large energy transfer, the other containing the remaining small energy transfer pieces. The parton states were introduced to be eigenstates of the large energy transfer Hamiltonian. The evolution of such a state is then governed by the soft operators with small energy transfer, thus enabling one to make the impulse approximation. It was then possible to give a physically transparent derivation of the usual formula for the cross section of the deep inelastic electron scattering. With some additional assumptions, it was also possible to confirm the conjecture that the cross section for the Drell-Yan process can be obtained from the naive parton model result by replacing the naive parton density functions with the Q^2-dependent density functions. The variation of the density functions with Q^2 was determined in terms of coupled integro-differential equations. The equations reduce to the usual ones when restricted to the longitudinal distributions only. As for the transverse distributions, explicit solutions were obtained for the simple case of the average transverse momentum squared.

There are still many questions to be cleared up. First, it must be rigorously proved that the separation of $\mathcal{H}_I$ into W_Λ and V_Λ as defined in Sect. II works so that Eq. (2.12) follows, although they are highly reasonable in view of our intuition gained from the renormalization theory of the Green's functions. The problem here is essentially to find a consistent renormalization procedure within the framework of the Hamiltonian approach. Solving this problem and thus reformulating the usual renormalization group theory in terms of the Hamiltonian language should be very useful, since the Hamiltonian approach has a strong intuitive appeal.

Second, the discussion of the Drell-Yan process in Sect. V is still incomplete, because the covariant derivatives occurring in the electromagnetic currents were replaced by the usual derivative without any justification. These are indications that the results in Sect. V are correct from the explicit calculations in lowest order perturbation theory[20] and from the calculation[27] in one time one space dimensions. However, the problem should be investigated further.

Finally, the scaling violation equations were only solved for the simplest case. Extending the solutions to a more general case is presently under study. At any rate, the problem here is a technical one and not one of physics.

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Added Note

Since this work was submitted for typing (Oct. 77), the theory of the impulse approximation presented in this paper is developed further as follows:

Eq. (2.12) is not completely satisfactory on dimensional ground. Instead, it should be replaced by

$$\langle n,\Lambda' | (W_{\Lambda'} - W_{\Lambda}) | m,\Lambda \rangle$$

$$= \langle_n | \mathscr{H}_I | m \rangle \; (g \rightarrow g(\bar{\Lambda}^2)) \, , \; \frac{\Lambda^2}{2\eta_o} \leq |\mathscr{E}_n - \mathscr{E}_m| \; \langle \frac{\Lambda'^2}{2\eta_o} \rangle$$

$$= o \; , \; \text{otherwise} \; , \qquad \text{(N.1)}$$

where $\bar{\Lambda}^2$ is a quantity of order Λ^2. To determine $\bar{\Lambda}^2$, note that Λ^2 enters into W_Λ only in the combination $\Lambda^2/2\eta_o$. Therefore the most general structure that $\bar{\Lambda}^2$ can have is

$$\bar{\Lambda}^2 = \frac{\Lambda^2}{\eta_o} \, \eta \, \xi_{nm} \quad . \qquad \text{(N.2)}$$

Here η is the o-component of the total momentum of state n. $\xi_{n,m}$ is a dimensionless quantity depending on momenta and quantum number of the states n and m. It must be invariant under longitudinal boosts as well as Galilean boosts. For the discussions in this paper, it is only necessary to consider the three point coupling shown in Fig. (5), in which case it is easy to see that $\xi_{n,m}$ can only depend on the ratio x/y (in the notation of Fig. (5)), being independent of $\underset{\sim}{P}_\perp$ or $\underset{\sim}{P}'_\perp$. Assuming that $\xi_{n,m}$ is independent of the quantum number of the states n and m, one has

$$\xi_{n,m} = \xi(x/y) \qquad \text{(N.3)}$$

The scaling violation equation (6.21) can be straight forwardly generalized to incorporate (N.2) and (N.3). One obtains

$$F_i(x,P_\perp,\Lambda^2+\Delta\Lambda^2) - F_i(x,P_\perp^2,\Lambda^2)$$

$$= \frac{1}{2\pi} \Big\{ - \tilde{Z}_i(x) \frac{\Delta\Lambda^2}{\Lambda^2} - F_i(x,P_\perp^2,\Lambda^2)$$

$$+ \int_x^1 \frac{dy}{y} \alpha(\Lambda^2 y \, \xi(\frac{x}{y})) \, f_{ij}(\frac{x}{y}) \int d^2P_\perp \frac{d\,P'_\perp}{\pi\bar{P}_\perp^2} F_j(y,P_\perp^2,\Lambda^2) \quad . \qquad \text{(N.4)}$$

Here

$$\tilde{Z}_i(x) = \int_0^1 \mathcal{J}_i(z) \, \alpha(\Lambda^2 x \, \xi(z)) dz \quad , \qquad \text{(M.5)}$$

with

$$\mathcal{J}_q(z) = C_r\left(\frac{2}{1-z} - \frac{3}{2}\right) \text{ and } \mathcal{J}_q(z) = \frac{2C_g}{1-z} - b \quad . \qquad \text{(N.6)}$$

The $P'_\perp$-integral in (N.4) is limited, of course, by the inequality (6.23). In the limit $\Delta\Lambda^2 \to 0$, Eq. (N.4) can be written in the following differential form:

$$2\pi\Lambda^2 \frac{\partial F_i(x,P_\perp^2,\Lambda^2)}{\partial\Lambda^2} = - \tilde{Z}_i(x) \, F_i(x,P_\perp^2,\Lambda^2)$$

$$+ \int_x^1 \frac{dy}{y} \alpha(\Lambda^2 y \, \xi(\frac{x}{y})) \, (\frac{x}{y})^2 \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \tilde{f}_{ij}(\frac{x}{y}) \, F_i(y,P_\Lambda^2,\Lambda^2) \quad , \qquad \text{(N.7)}$$

where

$$P_\Lambda^2 = (\frac{x}{y})^2 \left[P_\perp^2 + \Lambda^2\frac{x}{y}(y-x) - 2\cos\theta \, |\underset{\sim}{P_T}| \Lambda\sqrt{\frac{x}{y}(y-x)}\right] \quad . \qquad \text{(N.8)}$$

In Section V, it is shown that if one choose $\Lambda^2 \lesssim Q^2/x$ for deep inelastic electron scattering and $\Lambda^2 \lesssim (Q^2 + s(x_1-x_2)\ /4)/(x_1+x_2)$ for the Drell-Yan process, then $U_\Lambda(\tau,o)$ can effectively be approximated by 1. On the other hand, J^μ_Λ approaches the free current as $\Lambda \to \infty$. Therefore Λ^2 should be chosen as large as possible. These considerations determine the relevant Λ^2 for deep inelastic electron scattering ($\Lambda^2_{E.S.}$) and the Drell-Yan process ($\Lambda^2_{D.Y.}$) as follows:

$$\Lambda^2_{E.S.} = \frac{Q^2}{x} \text{ and } \Lambda^2_{D.Y.} = \frac{Q^2+s(x_1-x_2)^2/4}{x_1+x_2} \quad . \tag{N.9}$$

Therefore deep inelastic electron scattering measures the quantity

$$G_i(x,P_1^2,Q^2) \qquad F_i(x,P_1^2,Q^2/x) \quad , \tag{N.10}$$

while the Drell-Yan process measures

$$F_i(x,P_1^2, \frac{Q^2+(x_1-x_2)^2 s/4}{x_1+x_2} \quad . \tag{N.11}$$

Eqs. (N.8) and (N.9) then determine completely deep inelastic cross sections. In a recent paper (K. J. Kim, Impulse Approximation in QCD, Parton's p_T-Distribution and Application to Deep Inelastic Process, MZ-TH 78/1), phenomenological consequences of these equations were studied. Agreement with recent data on $R = \sigma_L/\sigma_T$ in deep inelastic electron scattering and the average transverse momentum of the Drell-Yan pairs is found to be very good.

APPENDIX

In this Appendix, the explicit form of the Hamiltonian $\mathscr{H}$ corresponding to the Lagrangian density (2.5) will be derived. Since the procedure is well known,[13,14] it is not necessary to go into the details of the derivation. For the present purpose, it is convenient to represent the Dirac matrices in the ordinary reference frame, $\hat{\gamma}\mu$, as follows:[28]

$$\hat{\gamma}^o = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \hat{\gamma}^3 = \begin{pmatrix} 0 & i \\ i & \end{pmatrix}, \; \hat{\gamma}^j = \begin{pmatrix} -i\sigma_j & 0 \\ 0 & i\sigma_j \end{pmatrix} . \tag{A.1}$$

The IMF components corresponding to the above representation are

$$\gamma^o = \sqrt{2}\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \; \gamma^3 = \sqrt{2}\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \text{ and } \gamma^j = \hat{\gamma}^j \quad . \tag{A.2}$$

In the IM gauge specified by Eq. (2.7), the independent degrees of freedom are the transverse components $A^a_\perp$'s of the gluon fields and the two component Pauli spinor χ. The latter is related to the 4-component quark field ψ as follows:

$$\psi(x) = \begin{pmatrix} \chi \\ \xi \end{pmatrix} , \tag{A.3}$$

where

$$\xi = \frac{1}{\sqrt{2}} \frac{1}{\partial^o} \; \vec{\sigma}\cdot\vec{D}\,\chi \quad . \tag{A.4}$$

Here $1/\partial^o$ is the inverse of the differential operator $\partial^o = \partial_3 = \partial/\partial \mathfrak{z}$ and $\vec{D}$ is the covariant derivative given by

$$\vec{D} = \vec{\partial} + ig\vec{A}^a \; T^a \quad . \tag{A.5}$$

In this appendix, the superscript $\rightarrow$ is used to denote two dimensional transverse vectors. The fundamental equal τ commutation relations are

$$\left\{\chi(x),\chi(x')\right\}_{E\cdot\tau} = \frac{1}{\sqrt{2}}\,\delta(\mathfrak{z} - \mathfrak{z}')\;\delta^{(2)}(\vec{x} - \vec{x}') \tag{A.6}$$

and

$$[A_i^a(x), \gamma^o A_j^b(x)]_{E\cdot\tau} = \frac{i}{2}\,\delta ab\;\delta ij\;\delta(\mathscr{z}-\mathscr{z}')\;\delta^{(2)}(\vec{x}-\vec{x}')\quad .$$

These commutation relations can be realized by introducing the Fourier decomposition

$$\chi^{(x)} = \int \frac{dp_\perp}{(2\pi)^3}\int_0^\infty \frac{d\eta}{\sqrt{2\eta\sqrt{2}}}\left\{W(s)\;e^{-\ell p\cdot x}\;b(\underset{\sim}{p},s) + W(-s)e^{ip\cdot x}\;d^+(\underset{\sim}{p},s)\right\} \tag{A.7}$$

and

$$A_i(x) = \int \frac{d^2p_\perp}{(2\pi)^3}\int_0^\infty \frac{d\eta}{2\eta}\left\{\varepsilon_i(\lambda)e^{-ip\cdot x}\;a(\underset{\sim}{p},\lambda) + e^{ip\cdot x}\;\varepsilon_i^*(\lambda)\;a^+(\underset{\sim}{p},\lambda)\right\}. \tag{A.8}$$

The Hamiltonian can be obtained by the standard procedure, and is given as follows:

$$\mathscr{H}_0 = \int dzdx_\perp\left\{-\frac{1}{2}\vec{A}_a\nabla^2\vec{A}_a + \frac{i}{\sqrt{2}}\chi^+\nabla^2\frac{1}{\partial^o}\chi\right\} \tag{A.9}$$

and

$$\begin{aligned}\mathscr{H}_I = \int dzdx_\perp\Bigg\{&\frac{g^2}{z}J_a^o\left(\frac{1}{\partial^o}\right)^2 J_a{}^o - g\left(\frac{1}{\partial^o}J_a^o\right)\vec{\nabla}\cdot\vec{A}_a\\ &+\frac{i}{\sqrt{2}}\chi^+[-ig\;T_a\cdot\vec{A}_a\cdot\vec{\sigma}\frac{1}{\partial^o}\vec{\sigma}\cdot\vec{\nabla} - ig\;\vec{\sigma}\cdot\vec{\nabla}\frac{1}{\partial^o}T_a\cdot\vec{A}_a\cdot\vec{\sigma}\\ &\qquad - g^2\;T_a\cdot\vec{A}_a\cdot\vec{\sigma}\frac{1}{\partial^o}T_b\cdot\vec{A}_b\cdot\vec{\sigma}]\chi\\ &+ gf_{abc}\;\partial^i A_a^j\;A_b^i\;A_c^j + \frac{g^2}{4}f_{abc}\;f_{ade}\;A_b^i\;A_c^j\;A_d^i\;A_e^j\Bigg\}\quad,\end{aligned} \tag{A.10}$$

where J_a^o is the color-charge density given by

$$J_a^o = \sqrt{2}\;\chi^+\;T_a\;\chi + f_{abc}\;\vec{A}_b\cdot\partial^o\vec{A}_c \quad . \tag{A.11}$$

The matrix element $\langle m|\mathscr{H}_I|n\rangle$ in the momentum space can be obtained from (A.11), (A.7) and (A.8). For the purpose of the present paper, it is only necessary to consider the matrix elements shown in Fig. (1.a) and (1.b) and those which can be obtained from them by the substitution rule. The matrix element corresponding to Fig. (1.a) is

$$M_1 = gT_a\left\{\frac{q_i}{\eta_q} - \frac{\sigma_i\vec{\sigma}\cdot\vec{p}}{2\eta} - \frac{\vec{\sigma}\cdot\vec{p}\,'\sigma_i}{2\eta'}\right\}\quad, \tag{A.12}$$

and the one corresponding Fig. (1.b) is

$$M_2 = igf_{abc} \left\{ \left[\frac{\eta_2 - \eta_3}{\eta_1} P_{1i} \right] - (p_2 - p_3)_i \, \delta jk \right. \tag{A.13}$$

$$\left. + \text{ cyclic permutations} \right\} .$$

Finally the electromagnetic current $J^\mu(x) = \bar{\psi}\gamma^\mu\psi$ will be expressed in terms of the independent field variables χ and $A_\perp$'s. From (A.2), (A.3) and (A.4), it follows:

$$J^o = \sqrt{2}\, \chi^+ \chi \quad , \tag{A.14a}$$

$$J^3 = \sqrt{2}\, \xi^+ \xi \quad = \frac{1}{\sqrt{2}} \left(\frac{1}{\partial^o} \vec{\sigma} \cdot \vec{D}\chi \right)^+ \left(\frac{1}{\partial^o} \vec{\sigma} \cdot \vec{D}\chi \right) \tag{A.14b}$$

and

$$J = \xi^+ \sigma_\perp \chi + \chi^+ \sigma_\perp \xi \quad . \tag{A.14c}$$

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TABLE I

Q^2-dependence of $T_i(Q^2) = \langle p_\perp^2 \rangle_i$ (all units in GeV^2)

Q^2	$\langle p_\perp^2 \rangle_V$	$\langle p_\perp^2 \rangle_C$	$\langle p_\perp^2 \rangle_S$	$\langle p_\perp^2 \rangle_g$
1.1	0.738	0	0.245	0.272
1.4	0.712	0.002	0.235	0.329
2.6	0.667	0.005	0.214	0.504
4.6	0.653	0.010	0.202	0.735
7.4	0.662	0.017	0.197	1.023
15.4	0.727	0.032	0.200	1.763
26.6	0.832	0.052	0.213	2.715
41.0	0.968	0.076	0.235	3.867
70.0	1.230	0.122	0.279	6.065
120.0	1.657	0.196	0.357	9.640

FIGURE CAPTIONS

1. Examples of virtual processes contained in $\mathscr{H}_I$. a, b and c are the group indices and i, j and k are the polarization indices of gluons.
2. Deep Inelastic Electron Scattering.
3. Drell-Yan Process.
4. Lowest order diagrams contributing to the matrix element $\langle m,Q|\Delta W|\underset{\sim}{p},n,Q'\rangle$. Δ indicates the action of ΔW vertex.
5. A diagram contributing to f_{ij}. The dotted line can either be a gluon or a quark.

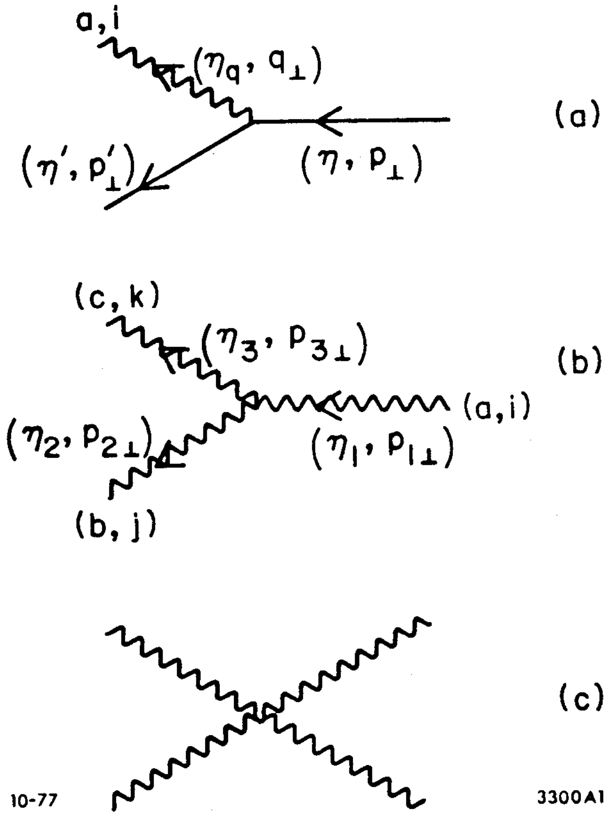

a, i
$(\eta_q, q_\perp)$
$(\eta', p'_\perp)$
$(\eta, p_\perp)$
(a)
(c, k)
$(\eta_3, p_{3\perp})$
(a, i)
$(\eta_2, p_{2\perp})$
$(\eta_1, p_{1\perp})$
(b, j)
(b)
(c)
10-77
3300A1

Fig. 1

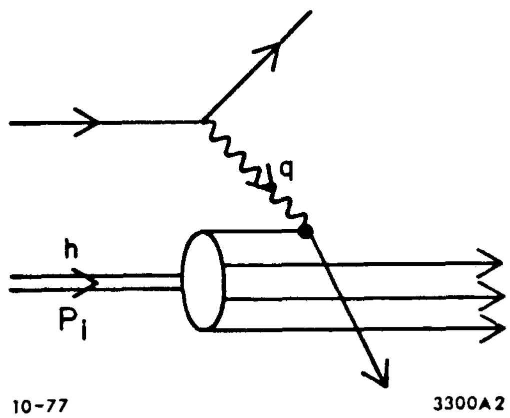
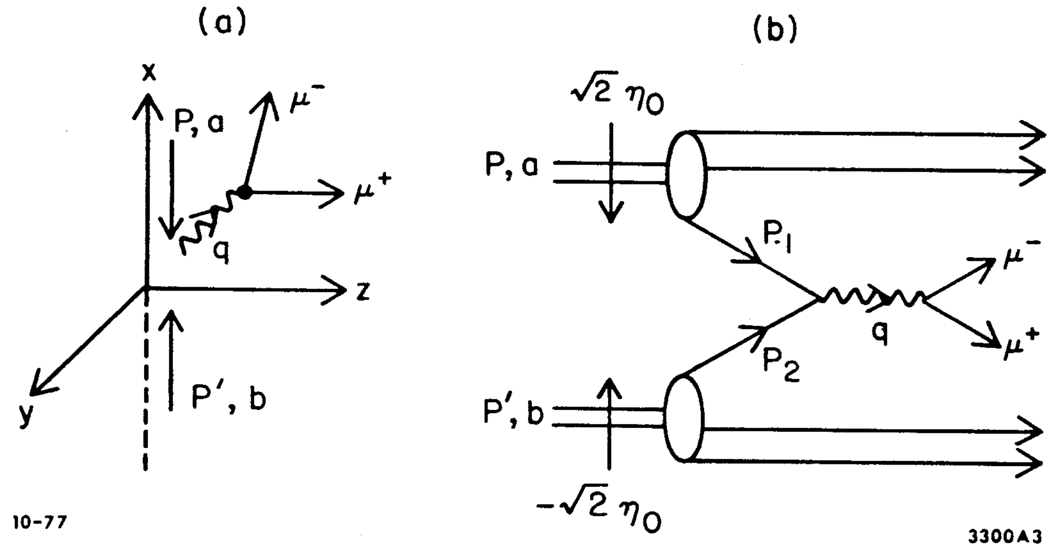


Fig. 2



(a)
x
P, a
μ^-
μ^+
q
z
y
P′, b
10-77
(b)
$\sqrt{2}\,\eta_0$
P, a
P_1
μ^-
q
μ^+
P_2
P′, b
$-\sqrt{2}\,\eta_0$
3300A3

Fig. 3

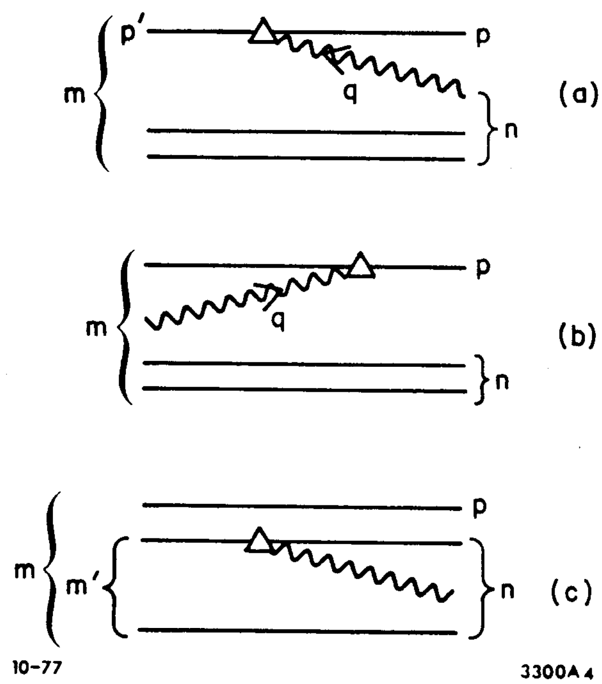


Fig. 4

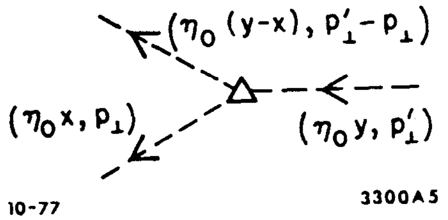


Fig. 5