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On AdS/CFT Models

by

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Abstract

This thesis consists of two parts. In the first part we review the LLM's bubbling AdS_5 and its application to different cases, and bubbling AdS_3 ansatz from which we derive the conical defect solution and the Aichelburg-Sexl solution, which turns to be the superstar in AdS_3. In the second part we consider the AdS_6 black hole with D4-$\overline{\text{D4}}$ flavor branes. We study the phase structure of the system. We find that deconfinement and chiral symmetry breaking might not occur together and it depends on the ratio L/R. We also find a more general model that includes backgrounds given by near extremal limit of Dp critical and non-critical branes. We also study the mesonic spectrum of the system and the drag effect on quarks and mesons.

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Chapter 1

Bubbling AdS_5 space and 1/2 BPS geometries

1.1 Introduction

String theory is the most promising candidate to accommodate all the fundamental forces of the universe, including gravity. Unfortunately, our understanding of this theory is far from complete. In particular, how quantum gravity and the standard model are realized within string theory is still elusive. However, we do not need to solve in full string theory to learn important lessons: in some cases, even simple toy models produce amazing physical outputs. This is the case of the AdS/CFT conjecture, where we can study closed string theory (and therefore quantum gravity) as dual to super Yang-Mills theory (SYM). Actually what we really mean is that within this framework, we can concentrate on a sector of the theory where a lot of control is achieved but physical relevance is still present. A beautiful example is the bubbling construction of Lin, Lunin and Maldacena (LLM) [1], where they looked for solutions of type IIB supergravity theory with only the metric and the self dual Ramond- Ramond 5-form excited. Furthermore, it is required to have regular solutions with an $SO(4) \times SO(4)$ symmetry and at least 1/2 BPS supersymmetry. Once the above solutions are found, by means of the AdS/CFT duality, the corresponding dual operators in N=4 SYM theory are identified. One of the most interesting outcomes of the above studies is the simplicity of the dual field theory, a matrix quantum mechanics, that gives the possibility to obtain a deeper understanding on fundamental issues of quantum gravity: for example the role of closed time-like curves (CTC) [2], the study of black hole thermodynamics [3] and the appearance of stretched horizons [4] probing black hole entropy laws, the structure of the quantum phase space [5], [6] and many others [7].

In this chapter we consider a beautiful idea of LLM and its applications to several cases. We consider a class of 1/2 BPS states that are associated to chiral primary operators with conformal weight $\Delta = J$, where J is a particular $U(1)$ charge in the R-symmetry group. For small excitation energies $J \ll N$ these BPS states correspond to particular gravity modes propagating in the bulk [8]. As one increases the excitation energy so that $J \sim N$ one finds that some of the states can be described as branes in the internal sphere [9] or as branes in AdS [10]. These were called "giant gravitons". As we increase the excitation energy to $J \sim N^2$ we expect to find new geometries. The BPS states in question have a simple field theory description in terms of free fermions [11]. In a semiclassical limit we can characterize these states by giving the regions, or "droplets", in phase space occupied by the fermions. We can also picture the BPS states as fermions in a magnetic field on the lowest Landau level (quantum Hall problem). Geometries corresponding to these configurations that preserve 16 of the original 32 supersymmetries. The general form of the solution is given in terms of an equation whose boundary conditions are specified on a particular plane. There are two possible types of boundary conditions corresponding to either of two different spheres shrinking on this plane in an smooth fashion. This plane, and the corresponding regions are in direct correspondence with the regions in the phase space. Once the occupied regions are given on this plane, the solution is determined uniquely and the ten (or eleven) dimensional geometry is non-singular and does not contain horizons. For very symmetric cases solutions are pretty simple. On the other hand, for cases with less symmetries things become very complicated and the simple dual description does not help.

The topology of the solutions is fixed by the topology of the droplets on the plane. The actual geometry depends on the shape of the droplets. In the type IIB case we simply need to solve a Laplace equation. A circular droplet gives rise to the $AdS_5 \times S^5$ solution (figure 1.1).

Small ripples on the droplet correspond to small fluctuations corresponding to gravitons in AdS. A small droplet far away from the circular one corresponds to a group of D3

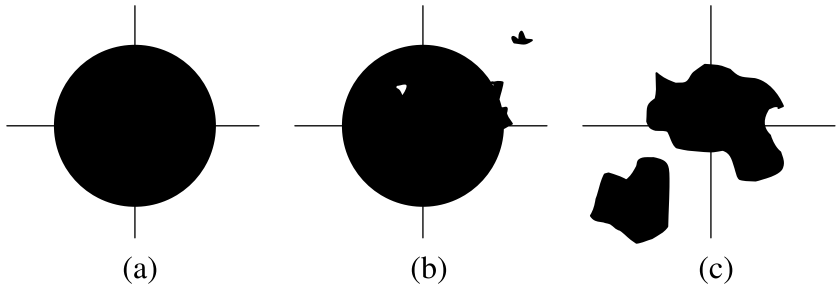

Figure 1.1: Droplets representing chiral primary states. In the field theory description these are droplets in phase space occupied by the fermions. In the gravity picture this is a particular two-plane in ten dimensions which specifies the solution uniquely. In (a) a droplet corresponding to the $AdS \times S$ ground state. (b) ripples on the surface corresponding to gravitons in $AdS \times S$. The separated black region is a giant graviton brane which wraps an S^3 in AdS_5 and the hole at the center is a giant graviton brane wrapping an S^3 in S^5. (c) a more general state.

branes wrapping an S^3 in AdS_5. A hole inside the circle corresponds to branes wrapping an S^3 in S^5. In the limit that the droplets become small these solutions reduce to the giant graviton branes. Some of our solutions smoothly interpolate between branes wrapping the sphere and branes wrapping AdS. We can also have solutions that correspond to new geometries which cannot be thought of as branes. In other words, when we put many branes together they back-react on the geometry and we get new geometries with new topologies that are determined by geometric transitions. The transition is that the sphere the branes are wrapping becomes contractible while the transverse sphere becomes non-contractible and the branes get replaced by flux.

We can also describe 1/2 BPS excitations of the plane wave geometry, which corresponds to a half filled plane. In this case the fermion becomes a relativistic Dirac fermion in 1+1 dimensions. The light-cone energy of the solution is the same as the usual energy for a Dirac fermion. Particle-hole duality corresponds to exchanging the 3-sphere in the first four of the eight transverse coordinates with a 3-sphere in the last four coordinates.

1.2 1/2 BPS states in the field theory

We consider $\mathcal{N} = 4$ super Yang Mills on $S^3 \times R$. We are interested in the class of states that preserves one half of the supersymmetries. These are the states associated to chiral primary operators that are built by taking products of traces of powers of a single chiral scalar field of $\mathcal{N} = 4$ Yang Mills. Denoting by ϕ^i the six scalars, we are interested in the field $Z = \phi^1 + i\phi^2$, and the operators $\prod_i (TrZ^{n_i})^{r_i}$. These BPS states can be described in a variety of ways. The one that will be most useful for our purposes will be the description in terms of free fermions discussed in [11]. We are interested in states with $\Delta - J = 0$. The only such state is the lowest Kaluza-Klein mode of the field Z on the S^3. This mode has a harmonic oscillator potential which arises from its conformal coupling to the curvature of S^3 [8]. So we are interested in the gauge invariant states of a matrix Z in a harmonic oscillator potential.

The system consists of a Hermitian $N \times N$ matrix Z, (or with explicit $U(N)$ indices Z^i_j) with potential $\frac{1}{2}\mathrm{tr}(Z^2)$, and kinetic term $\frac{1}{2}\mathrm{tr}(D_t Z)^2$, where

$$D_t(Z) = \dot{Z} + [A, Z]$$

and A is the gauge connection and acts as a lagrange multiplier (which is also a hermitian $N \times N$ matrix). We choose the matrix Z to be diagonal. Let us label the eigenvalues of Z as λ_i. Then, when we write wave functions for the Schrödinger equation, they will be functions of λ_i. There is a discrete subgroup of $U(N)$ which leaves the matrix Z diagonal. This is the permutation group of the eigenvalues, so the wave functions have to be invariant under this symmetry, and this means that we get totally symmetric wave functions on the eigenvalues.

Classically, the Lagrangian for the eigenvalue basis becomes

$$L = \sum \frac{1}{2}\dot{\lambda}_i^2 - \frac{1}{2}\lambda_i^2 \tag{2.1}$$

So the classical motion of the eigenvalues is that of a harmonic oscillator. However, quantum mechanically there is a change of measure from the matrix basis to the eigenvalue basis. This change of measure is the volume of the gauge orbit of the matrix Z, and it is

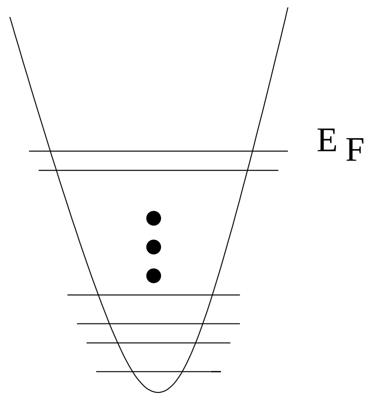

Figure 1.2: Filling the potential well.

equal to the square of the Van der Monde determinant of the λ_i, namely

$$\mu = \Delta(\lambda)^2 = \prod_{i \neq j} (\lambda_i - \lambda_j) \tag{2.2}$$

So that the Hamitonian in the quantum theory will be given by

$$H\psi = \frac{1}{2} \sum -\mu^{-1} \partial_{\lambda_i} (\mu \partial_{\lambda_i} \psi) + \lambda_i^2 \psi \tag{2.3}$$

with ψ the wave function of the eigenvalues.

The measure can be absorbed in the wave functions for the λ_i, by attaching a factor of the Van der Monde to the wave function. We define $\psi(\lambda) = \Delta^{-1}(\lambda)\tilde{\psi}(\lambda)$, where $\tilde{\psi}(\lambda)$ is the new wave function in the Z variables expressed in terms of the eigenvalues of Z (these are the λ_i), and the measure for $\tilde{\psi}$ is is just $\prod d\lambda_i$. This can be done for any one matrix model quantum mechanics with a single trace potential. This is a similarity transformation on the space of wave functions, so it affects the form of the Hamilltonian. The new Hamiltonian is

$$\tilde{H} = \frac{1}{2} \sum_i -\partial_{\lambda_i}^2 + \lambda_i^2 \tag{2.4}$$

so it becomes a Hamiltonian for N free particles in the harmonic oscillator potential well (figure 1.2). After this is done the wave functions are completely antisymmetric in the λ_i: the eigenvalues become fermions due to the Van Der Monde determinant. The system is reduced to N free fermions in a given potential, which is just $V(x) = x^2/2$.

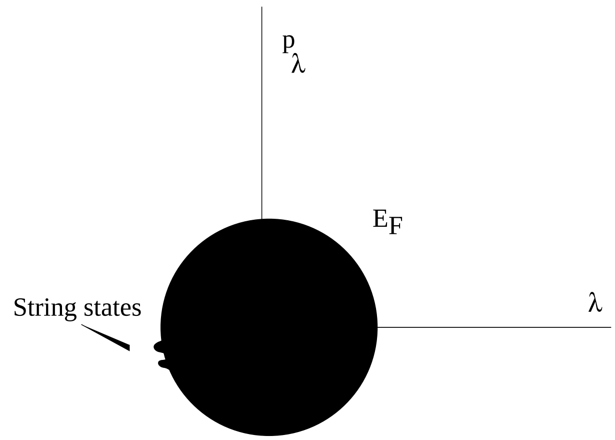

Figure 1.3: String states as small perturbations of the Fermi surface.

We can think of these fermions as forming droplets in phase space. The ground state corresponds to a circular droplet. Equivalently, we can say that we have a quantum hall fluid. We can form the new Hamiltonian $H' = H - J = \Delta - J$, where J is the angular momentum in the 12 plane. In terms of this new Hamiltonian we have a Landau level problem. The 1/2 BPS states are the ground states of H' and correspond to the lowest Landau level. The AdS ground state corresponds to a circular droplet. The conformal dimension $\Delta = J$ of any excitation is given by the angular momentum on the Hall plane, or the energy of the harmonic oscillator, above the ground state corresponding to the circular droplet. Small perturbations of the Fermi surface of the eigenvalue distribution in the phase space correspond to string states (figure 1.3).

These BPS states preserve 16 non-trivial supersymmetries as well as $SO(4)\times SO(4)\times R$ bosonic symmetries, where R corresponds to the Hamiltonian $H' = H - J$. This generator commutes with the preserved supercharges.

1.3 1/2 BPS geometries in type IIB supergravity

1.3.1 Type IIB solutions

We now look for the most general type IIB geometry that is invariant under $SO(4) \times SO(4) \times R$. This implies that the geometry will contain two three-spheres and a Killing vector. We only expect the five–form field strength to be excited. So we assume we have a geometry of the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\tilde{\Omega}_3^2 \\ F_{(5)} &= F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\tilde{\Omega}_3 \end{aligned} \tag{3.1}$$

where $\mu, \nu = 0, \cdots, 3$ and $d\Omega_3^2,\ d\tilde{\Omega}_3^2$ denote the metric on 3–spheres with unit radius. In addition, we assume that the dilaton and axion are constant and that the three-form field strengths are zero.

The self duality condition on the five-form field strength implies that $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ are dual to each other in four dimensions and we have only one independent gauge field:

$$F = e^{3G} *_4 \tilde{F}, \qquad \tilde{F} = -e^{-3G} *_4 F\ , \qquad F = dB\ , \qquad \tilde{F} = d\tilde{B} \tag{3.2}$$

We now demand that this geometry preserves the Killing spinor, i.e. we require that there are solutions to the Killing spinor equations

$$\nabla_M \eta + \frac{i}{480}\Gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}\Gamma_M \eta = 0 \tag{3.3}$$

By analyzing the Killing spinor equations one can relate the various functions appearing in the metric to a single function. This function ends up obeying a simple differential equation.

The conventions for normalizing differential forms are as follows:

$$A^{(k)} = \frac{1}{k!}\sum A_{i_1\ldots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k} \tag{3.4}$$

For example, $A^{(1)} = A_i dx^i,$, $F = dA^{(1)} = \partial_j A_i dx^j \wedge dx^i = \frac{1}{2}F_{ij}dx^i \wedge dx^j$. The dual $B =^\star F$ is defined by $F_{ij} = \epsilon_{ijk}B_k$, $B_k = \frac{1}{2}\epsilon_{ijk}F_{ij}$.

Using techniques similar to the ones developed in [12–14] one first write the 10-dimensional spinor as a product of 4-dimensional spinors and spinors on the sphere. Therefore we choose the following basis of gamma matrices

$$\Gamma_\mu = \gamma_\mu \otimes 1 \otimes 1 \otimes 1, \quad \Gamma_a = \gamma_5 \otimes \sigma_a \otimes 1 \otimes \hat{\sigma}_1, \quad \Gamma_{\tilde{a}} = \gamma_5 \otimes 1 \otimes \tilde{\sigma}_a \otimes \hat{\sigma}_2, \tag{3.5}$$

where $\sigma_a,\ \tilde{\sigma}_a,\ \hat{\sigma}_a$ are ordinary Pauli matrices. In this basis

$$\Gamma_{11} = \Gamma_0 \dots \Gamma_3 \prod \Gamma_a \prod \Gamma_{\tilde{a}} = -\gamma^5 \hat{\sigma}^3 \ , \qquad \gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 \tag{3.6}$$

The spinor obeys the chirality condition

$$\Gamma_{11}\eta = -\gamma^5 \hat{\sigma}_3 \eta = +\eta. \tag{3.7}$$

We begin with the spinor equation on the sphere [15, 16]. Suppose we have a spinor on a unit radius 3-sphere. We consider spinors obeying the equation

$$\nabla'_c \chi = a\frac{i}{2}\gamma_c\chi = \frac{ia}{2}E_c^{\underline{a}}\gamma_{\underline{a}}\chi \ , \qquad a = \pm 1 \tag{3.8}$$

where $E_c^{\underline{a}}$ are vielbeins on a unit sphere. The solutions of this equation transform in the spinor representation under the $SO(4)$ isometries of the sphere. The chirality of the $SO(4)$ spinor representation is correlated with the sign of a.

Now let us consider the full metric (3.1). The vielbeins are:

$$\begin{aligned} e^{\underline{\mu}} &= e^{\underline{\mu}}_{\mu} dx^\mu \\ e^{\underline{a}} &= e^{\frac{1}{2}(H+G)} E^{\underline{a}} \\ e^{\underline{\tilde{a}}} &= e^{\frac{1}{2}(H+G)} E^{\underline{\tilde{a}}} \end{aligned} \tag{3.9}$$

From the equation

$$de^{\underline{M}} + \omega^{\underline{M}}{}_{\underline{N}} \wedge e^{\underline{N}} = 0 \tag{3.10}$$

we find the spin connection

$$\omega_c^{\underline{a\mu}} = -\frac{1}{2}\partial_\mu(H+G)e^{\mu}_{\underline{\mu}} e^{\frac{1}{2}(H+G)}E^{\underline{a}} = -\frac{1}{2}\partial_\mu(H+G)e^{\mu}_{\underline{\mu}} e_c^{\underline{a}} \tag{3.11}$$

The warp factors lead to the following covariant derivatives in the sphere directions

$$\begin{aligned}\nabla_c =&\nabla'_c + \frac{1}{2}\omega_c^{\underline{a\mu}}\gamma_{\underline{a\mu}}\\ =& \frac{ia}{2}E_c^{\underline{a}}\gamma_{\underline{a}} - \frac{1}{4}\partial_\mu(H+G)e_{\underline{\mu}}^{\mu}e_c^{\underline{a}}\gamma_{\underline{a\mu}}\\ =& a\frac{i}{2}e^{-\frac{1}{2}(H+G)}\Gamma_c\gamma_5\hat{\sigma}_1 - \frac{1}{4}\partial_\mu(H+G)\Gamma_{c\mu}\end{aligned} \tag{3.12}$$

We now decompose the ten dimensional spinor as

$$\eta = \epsilon_{a,b}\otimes\chi_a\otimes\tilde{\chi}_b \tag{3.13}$$

where $\chi_a, \tilde{\chi}_b$ obey equation (3.8) with overall signs $a, b = \pm 1$. The spinor ϵ_{ab} is acted on by the four dimensional γ matrices and the matrices $\hat{\sigma}$. For simplicity we now drop the indices a, b on the spinor ϵ. We are interested in geometries that are asymptotically $AdS_5 \times S^5$ or plane waves which preserve a half of the original supersymmetries. Since the original supersymmetries have correlated chiralities under $SO(2,4)$ and $SO(6)$, and we are looking at supercharges with $H' = \Delta - J = 0$, we expect them to have chiralities $++$ or $--$ under the $SO(4)\times SO(4)$ generators.

The expression involving the five–form becomes ($\gamma_c = E_c^{\underline{a}}\Gamma_{\underline{a}}$):

$$\begin{aligned}M \equiv& \frac{i}{480}\gamma^{M_1M_2M_3M_4M_5}F^{(5)}_{M_1M_2M_3M_4M_5}\\ =& \frac{i}{480}\frac{5!}{3!2!}\left[\Gamma^{\mu\nu}F_{\mu\nu}\epsilon_{abc}\gamma^{abc} + \Gamma^{\mu\nu}\tilde{F}_{\mu\nu}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\gamma^{\tilde{a}\tilde{b}\tilde{c}}\right]\\ =& \frac{i}{48}\left[e^{-\frac{3}{2}(H+G)}\Gamma^{\mu\nu}F_{\mu\nu}\epsilon_{abc}\Gamma^{abc} + e^{-\frac{3}{2}(H-G)}\Gamma^{\mu\nu}\tilde{F}_{\mu\nu}\epsilon_{\tilde{a}\tilde{b}\tilde{c}}\Gamma^{\tilde{a}\tilde{b}\tilde{c}}\right]\\ =& -\frac{1}{8}\left[e^{-\frac{3}{2}(H+G)}\Gamma^{\mu\nu}F_{\mu\nu}\gamma_5\hat{\sigma}_1 + e^{-\frac{3}{2}(H-G)}\Gamma^{\mu\nu}\tilde{F}_{\mu\nu}\gamma_5\hat{\sigma}_2\right]\end{aligned} \tag{3.14}$$

Using the chirality condition (3.7), the self duality condition (3.2) and the equality

$$\Gamma^{\mu_1\ldots\mu_s}\Gamma = -\frac{(i)^{-k+s(s+1)}}{(d-s)!}\epsilon^{\mu_1..mu_d}\Gamma_{\mu_{s+1}\ldots\mu_d} \tag{3.15}$$

where d is the dimension of the spacetime and $k = (d-2)/2$, we get

$$
\begin{aligned}
M &= -\frac{1}{8}\left[e^{-\frac{3}{2}(H+G)}\Gamma^{\mu\nu}F_{\mu\nu}\gamma_5\hat{\sigma}_1 + e^{-\frac{3}{2}(H-G)}\left(-\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}\Gamma_{\rho\lambda}\gamma_5\right)\left(-e^{-3G}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}\right)\gamma_5\hat{\sigma}_2\right] \\
&= -\frac{1}{4}e^{-\frac{3}{2}(H+G)}\Gamma^{\mu\nu}F_{\mu\nu}\gamma^5\hat{\sigma}^1 \qquad (3.16)
\end{aligned}
$$

Using (3.12), (3.16) and the anticommutation relation of Γ-matrices $\Gamma^{\mu}_{a} = -2\gamma^{\mu}\Gamma_a$ the equation (3.3) becomes a system of equations:

$$(iae^{-\frac{1}{2}(H+G)}\gamma_5\hat{\sigma}_1 + \frac{1}{2}\gamma^{\mu}\partial_{\mu}(H+G))\epsilon + 2M\epsilon = 0 \qquad (3.17)$$

$$(ibe^{-\frac{1}{2}(H-G)}\gamma_5\hat{\sigma}_2 + \frac{1}{2}\gamma^{\mu}\partial_{\mu}(H-G))\epsilon - 2M\epsilon = 0 \qquad (3.18)$$

$$\nabla_{\mu}\epsilon + M\gamma_{\mu}\epsilon = 0 \qquad (3.19)$$

These equations are effectively four dimensional. The four dimensional system involves the four dimensional metric, one gauge field and two scalar fields.

It is now convenient to construct some spinor bilinears. An interesting set of spinor bilinears is

$$
\begin{aligned}
&K_{\mu} = -\bar{\epsilon}\gamma_{\mu}\epsilon, \qquad L_{\mu} = \bar{\epsilon}\gamma^5\gamma_{\mu}\epsilon\ , \qquad \bar{\epsilon} = \epsilon^{\dagger}\Gamma^0 \\
&f_1 = i\bar{\epsilon}\hat{\sigma}_1\epsilon, \quad f_2 = i\bar{\epsilon}\hat{\sigma}_2\epsilon, \qquad Y_{\mu\nu} = \bar{\epsilon}\gamma_{\mu\nu}\hat{\sigma}_1\epsilon \qquad (3.20)
\end{aligned}
$$

where $\bar{\epsilon} = \epsilon^{\dagger}\gamma^0$.

Applying (3.19) on f_1 we get:

$$
\begin{aligned}
\nabla_{\mu}f_1 &= i\nabla\bar{\epsilon}\hat{\sigma}_1\epsilon + i\bar{\epsilon}\hat{\sigma}_1\nabla\epsilon \\
&= \frac{i}{4}e^{-\frac{3}{2}(H+G)}\left[\bar{\epsilon}\gamma^{\mu\nu}F_{\mu\nu}\gamma^5\gamma_{\mu}\epsilon - \bar{\epsilon}(-\gamma^{\mu})\epsilon\gamma^{\mu\nu}F_{\mu\nu}\gamma^5\right] \\
&= \frac{i}{2}e^{-\frac{3}{2}(H-G)}\epsilon^{\mu}_{\nu}K^{\nu}\gamma^{\mu\nu}(-e^{-3G})F_{\mu\nu}\gamma^5 \\
&= -\frac{1}{2}e^{-\frac{3}{2}(H+G)}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}K^{\nu} \\
&= -e^{-\frac{3}{2}(H-G)}\tilde{F}_{\mu\nu}K^{\nu} \qquad (3.21)
\end{aligned}
$$

The same way we get the folowing equations:

$$\begin{aligned}\nabla_\mu f_2 &= -e^{-\frac{3}{2}(H+G)} F_{\mu\nu} K^\nu && (3.22)\\ \nabla_\nu K_\mu &= -e^{-\frac{3}{2}(H+G)} \left[F_{\mu\nu} f_2 - \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} f_1 \right] \\ &= -e^{-\frac{3}{2}(H+G)} F_{\mu\nu} f_2 - e^{-\frac{3}{2}(H-G)} \tilde{F}_{\mu\nu} f_1 && (3.23)\\ \nabla_\nu L_\mu &= e^{-\frac{3}{2}(H+G)} \left[-\frac{1}{2} g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho} - F_\mu{}^\rho Y_{\rho\nu} - F_\nu{}^\rho Y_{\rho\mu} \right] && (3.24)\end{aligned}$$

Another interesting set of spinor bilinears involves taking the the spinor and its transpose, e.g. the one-form which obeys a useful equation

$$\begin{aligned}\omega_\mu &= \epsilon^t \Gamma^2 \gamma_\mu \epsilon \ , && (3.25)\\ d\omega &= 0 && (3.26)\end{aligned}$$

where in our conventions $\Gamma^2 \gamma_\mu^t \Gamma^2 = -\gamma_\mu$, and the last equation says that the exterior derivative vanishes.

By Fierz rearrangement identities we find

$$K \cdot L = 0 \ , \qquad L^2 = -K^2 = f_1^2 + f_2^2 \tag{3.27}$$

We now use all these facts to constrain the metric and the gauge fields. First we observe that K^μ is a Killing vector and $L_\mu dx^\mu$ is a (locally) exact form. We begin by choosing a coordinate y through

$$\gamma dy = L_\mu dx^\mu \ , \qquad \gamma = \pm 1 \tag{3.28}$$

We will later determine the sign of γ. We choose the other three coordinates in the subspace orthogonal to y

$$ds^2 = h^2 dy^2 + \hat{g}_{\alpha\beta} dx^\alpha dx^\beta \tag{3.29}$$

Let us now look at the vector K^μ. Using the relation

$$0 = K^\mu L_\mu = K^y L_y = \gamma K^y \tag{3.30}$$

we find that K^α is a vector in three dimensional space spanned by x^α. Choosing one of the coordinates along K^α (we will call it t), we find the metric

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \tilde{h}_{ij}dx^i dx^j) \tag{3.31}$$

were i, j take values $1, 2$. We have used the equation $K^2 = -L^2$ to link the g_{tt} and the g^{yy} coefficients of the metric. We also pulled out a factor of h^2 out of the remaining two dimensions for later convenience.

Now we look at equation (3.22). Since K^μ has only one component $K^t = 1$ (we can always choose this normalization of t), and B_i is independent of t, that equation becomes

$$\partial_\mu f_2 = -e^{-\frac{3}{2}(H+G)}\partial_\mu B_t, \quad \text{i.e.} \quad df_2 = -e^{-\frac{3}{2}(H+G)}dB_t \tag{3.32}$$

We now compute

$$\partial_\mu B_t = F_{\mu\nu}K^\nu = -F_{\mu\nu}\bar{\epsilon}\gamma^\nu\epsilon = -\frac{1}{4}\bar{\epsilon}[\gamma_\mu, \not{F}]\epsilon \tag{3.33}$$

where $\not{F} = F_{\mu\nu}\gamma^{\mu\nu}$. Now we recall the equation coming from the sphere (3.17) and its adjoint

$$\begin{aligned}
\frac{1}{2}e^{-\frac{3}{2}(H+G)}\not{F}\epsilon &= (iae^{-\frac{1}{2}(H+G)} + \frac{1}{2}\gamma_5\not{\partial}(H+G)\hat{\sigma}_1)\epsilon, \\
\frac{1}{2}e^{-\frac{3}{2}(H+G)}\bar{\epsilon}\not{F} &= \bar{\epsilon}(iae^{-\frac{1}{2}(H+G)} + \frac{1}{2}\gamma_5\not{\partial}(H+G)\hat{\sigma}_1)
\end{aligned}$$

Using this in (3.33) we obtain

$$\begin{aligned}
\partial_\mu B_t &= -\frac{1}{4}\bar{\epsilon}\gamma_\mu \not{F}\epsilon + \frac{1}{4}\bar{\epsilon}\not{F}\gamma_\mu\epsilon = e^{\frac{3}{2}(H+G)}\frac{1}{2}\partial_\mu(H+G)\ \epsilon^\dagger\Gamma^0\gamma_5\hat{\sigma}_1\epsilon \\
&= -e^{\frac{3}{2}(H+G)}\frac{1}{2}\partial_\mu(H+G)\ f_2
\end{aligned} \tag{3.34}$$

We now get an equation which involves only f_2 and $H + G$

$$\partial_\mu f_2 = \frac{1}{2}f_2\partial_\mu(H+G), \tag{3.35}$$

which can be easily solved

$$f_2 = 4\alpha e^{\frac{1}{2}(H+G)}, \qquad B_t = -\alpha e^{2(H+G)} \tag{3.36}$$

In the same way, starting from equations (3.21), (3.18), we can prove that

$$f_1 = 4\beta e^{\frac{1}{2}(H-G)}, \qquad \tilde{B}_t = -\beta e^{2(H-G)}\ , \qquad 4\beta = 1 \tag{3.37}$$

Here we have set $4\beta = 1$ by choosing the overall sign of the five–form field strength and an appropriate rescaling of the Killing spinor. We will fix α below.

We will now show that H has a simple coordinate dependence. We begin with the equation coming from the sum of (3.17) plus (3.18) and its adjoint

$$\hat{\sigma}_1 \partial\!\!\!/ H \epsilon = (-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)})\epsilon \tag{3.38}$$

$$\bar{\epsilon}\hat{\sigma}_1 \partial\!\!\!/ H = -\bar{\epsilon}(-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)}) \tag{3.39}$$

We find

$$\begin{aligned} \partial_\mu H f_1 &= i\partial_\mu H \bar{\epsilon}\hat{\sigma}_1\epsilon = \frac{i}{2}\bar{\epsilon}[\gamma_\mu, (-iae^{-\frac{1}{2}(H+G)}\gamma_5 + be^{-\frac{1}{2}(H-G)})]\epsilon = \\ &= -ae^{-\frac{1}{2}(H+G)}\bar{\epsilon}\gamma_5\gamma_\mu\epsilon = -ae^{-\frac{1}{2}(H+G)}L_\mu \end{aligned} \tag{3.40}$$

so H is a function of y only. Using (3.37) we can determine this function

$$\begin{aligned} &\frac{\partial H}{\partial x^\mu}e^H = -aL_\mu \\ &dHe^H = -a\gamma dy \\ &e^H = -a\gamma y = y\ , \qquad \qquad \gamma = -a \end{aligned} \tag{3.41}$$

where we have fixed the sign of γ. We now fix α by multiplying (3.38) by $\bar{\epsilon}\gamma^5\hat{\sigma}^1$ and using (3.36),(3.37) and (3.27) we get

$$
\begin{aligned}
\bar{\epsilon}\gamma^5\hat{\sigma}^1\hat{\sigma}_1 \partial\!\!\!/ H\epsilon &= -\bar{\epsilon}\gamma^5\hat{\sigma}^1 iae^{-\frac{1}{2}(H+G)}\gamma_5\epsilon + \bar{\epsilon}\gamma^5\hat{\sigma}^1 be^{-\frac{1}{2}(H-G)}\epsilon \\
L_\mu\partial_\mu H &= -ae^{-\frac{1}{2}(H+G)} f_1 - be^{\frac{1}{2}(H-G)} f_2 \\
L_\mu\partial_\mu He^H &= -a\left(f_1^2 + \frac{b}{4a\alpha} f_2^2\right) \\
f_1^2 + f_2^2 &= f_1^2 + \frac{b}{4a\alpha} f_2^2
\end{aligned} \tag{3.42}
$$

We see that we need to have $ab4\alpha = 1$. We can now choose $4\alpha = 4\beta$ (the sign choice in $\alpha = \pm\beta$ corresponds to whether we look at chiral primaries with $\Delta \mp J = 0$). Note that with these choices only supersymmetries with $b = a$ are preserved, but still we have both choices of sign for a.

We now go back to (3.38). Using (3.41) we find ($\hat{\sigma}_1\gamma^\mu\partial_\mu H = -a\hat{\sigma}_1\Gamma^3 L_y \frac{1}{y}$)

$$
\left(\gamma\frac{1}{y}\hat{\sigma}_1\Gamma^3 L_y + iae^{-\frac{1}{2}(H+G)}\gamma_5 - be^{-\frac{1}{2}(H-G)}\right)\epsilon = 0 \tag{3.43}
$$

Using (3.27), (3.36), (3.37), this reduces to the projector

$$
\begin{aligned}
&\left[\gamma e^{\frac{1}{2}(H-G)}\frac{1}{y} L_y\hat{\sigma}_1\Gamma^3 + iae^{-G}\gamma_5 - a\right]\epsilon = 0 \\
&\left[\gamma e^{\frac{1}{2}(H-G)} e^H e^{\frac{1}{2}(H+G)}\sqrt{1+e^{-2G}}\hat{\sigma}_1\Gamma^3 + iae^{-G}\gamma_5 - a\right]\epsilon = 0 \\
&\left[\gamma\sqrt{1+e^{-2G}}\hat{\sigma}_1\Gamma^3 + aie^{-G}\gamma_5 - a\right]\epsilon = 0
\end{aligned} \tag{3.44}
$$

The definitions (3.20) and the equations $K^t = 1$, $L_y = -a$ imply that $\epsilon^\dagger\epsilon = 1$ and $\epsilon^\dagger\Gamma^0\Gamma^5\Gamma^3\epsilon = -a$. Since $\Gamma^0\Gamma^5\Gamma^3$ is a unitary operator we conclude that we must also have the following projection condition

$$
\left[1 + a\Gamma^0\Gamma^5\Gamma_3\right]\epsilon = 0, \quad \text{or} \quad \left[1 + ai\Gamma_1\Gamma_2\right]\epsilon = 0 \tag{3.45}
$$

The two projectors (3.44) and (3.45) imply that the Killing spinor has the form

$$
\epsilon \ = \ e^{i\delta\gamma^5\Gamma^3\hat{\sigma}^1}\epsilon_1\ , \qquad \Gamma^3\hat{\sigma}^1\epsilon_1 = a\epsilon_1, \qquad \sinh 2\delta = ae^{-G} \tag{3.46}
$$

We can fix the scale of ϵ_1 by inserting (3.46) in the expression for f_2 which gives

$$\epsilon_1 = e^{\frac{1}{4}(H+G)}\epsilon_0 \ , \qquad \epsilon_0^\dagger\epsilon_0 = 1 \tag{3.47}$$

We can set the phase of ϵ_0 to zero by performing a local Lorentz rotation in the 12 plane. Then ϵ_0 is a constant spinor.

We can now insert this expression for the Killing spinor in the definition of the one form (3.25) to find that

$$\begin{aligned}
\omega_{\hat{2}} &= \epsilon^t\Gamma^2\Gamma^2\epsilon = e^{\frac{1}{2}(H+G)}\cosh 2\delta\,\epsilon_0^t\epsilon_0 = h^{-1}\epsilon_0^t\epsilon_0 \\
\omega_{\hat{1}} &= \epsilon^t\Gamma^2\Gamma^1\epsilon = -iah^{-1}\epsilon_0^t\epsilon_0 \\
\omega_\mu &= \omega_{\hat{c}}e_\mu^{\hat{c}}dx^\mu = (\text{constant})(\tilde{e}_i^{\hat{1}} + ia\tilde{e}_i^{\hat{2}})dx^i
\end{aligned} \tag{3.48}$$

Where $\tilde{e}_i^{\hat{c}}$ is the vielbein of the metric $\tilde{h}_{ij} = \tilde{e}_i^{\hat{c}}\tilde{e}_j^{\hat{c}}$ and $e_i^{\hat{i}} = h\tilde{e}_j^{\hat{i}}$ is the full vielbein for the four dimensional metric in the directions 1,2. Equation (3.26) implies that these vielbeins are independent of y and that the two dimensional metric is flat. So we choose coordinates such that $\tilde{h}_{ij} = \delta_{ij}$.

We now use equation (3.2) to write an expression for the gauge field

$$\begin{aligned}
B &= B_t(dt+V) + \hat{B} \ , \\
d\hat{B} + B_t dV &= -L^2 e^{3G} \star_3 d\tilde{B}_t
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
\tilde{B} &= \tilde{B}_t(dt+V) + \hat{\tilde{B}} \\
d\hat{\tilde{B}} + \tilde{B}_t dV &= L^2 e^{-3G} \star_3 dB_t
\end{aligned} \tag{3.50}$$

where $\hat{B}$, $\hat{\tilde{B}}$ have no components along the time direction and $*_3$ it the flat space epsilon symbol in the directions y, x_1, x_2. It is now possible to obtain an expression for the vector V. We start from the antisymmetric part of the equation for the Killing spinor (3.23)

$$-\frac{1}{2}d[L^2(dt+V)] \;=\; \frac{1}{2}dK = e^{-(H+G)}F + e^{-(H-G)}\tilde{F} \tag{3.51}$$

This equation splits into two equations, one gives no new information, the equation giving new information is

$$
\begin{aligned}
\frac{1}{2}L^2 dV &= -e^{-(H+G)}(d\hat{B} + B_t dV) - e^{-(H-G)}(d\hat{\tilde{B}} + \tilde{B}_t dV) \\
&= L^{-2}(e^{-H+2G} *_3 d\tilde{B}_t - e^{-H-2G} *_3 dB_t) && (3.52) \\
dV &= 2L^{-4}e^{-H}(e^{2G} *_3 d\tilde{B}_t - e^{-2G} *_3 dB_t) && (3.53) \\
&= 2L^{-4} y *_3 dG && (3.54)
\end{aligned}
$$

From (3.27) we find

$$
L^2 = h^{-2} = f_1^2 + f_2^2 = y(e^G + e^{-G}) \tag{3.55}
$$

We define

$$
\begin{aligned}
z &\equiv \frac{1}{2}\tanh G \\
dz &= \frac{1}{2}\mathrm{sech}^2 G dG = \frac{1}{2}\left(\frac{2}{e^G + e^{-G}}\right)^2 dG = 2y^2 h^4 dG && (3.56)
\end{aligned}
$$

Therefore we get

$$
dV = \frac{1}{y} *_3 dz \tag{3.57}
$$

The consistency condition $d(dV) = 0$ gives the equation

$$
\frac{1}{y}\partial_i^2 z + \partial_y(\frac{1}{y}\partial_y z) = 0 \tag{3.58}
$$

From equations (3.49), (3.50) and (3.53) we can determine the gauge fields

$$
\begin{aligned}
d\hat{B} &= -\frac{1}{4}y^3 *_3 d(\frac{z+\frac{1}{2}}{y^2}) && (3.59) \\
d\hat{\tilde{B}} &= -\frac{1}{4}y^3 *_3 d(\frac{z-\frac{1}{2}}{y^2}) && (3.60)
\end{aligned}
$$

In summary, we have derived the full form of the metric and gauge fields described in (3.61)–(3.65). In addition we found the expression (3.46), (3.47) for the Killing spinor. It is possible to show that this killing spinor obeys all other equations, so that we have a consistent solution.

The end result is:

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 && (3.61)\\
h^{-2} &= 2y \cosh G, && (3.62)\\
y\partial_y V_i &= \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z && (3.63)\\
z &= \frac{1}{2}\tanh G && (3.64)\\
F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B}\ , &&\\
\tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} && (3.65)\\
B_t &= -\frac{1}{4}y^2 e^{2G}, \qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} && (3.66)\\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d(\frac{z+\frac{1}{2}}{y^2})\ , \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d(\frac{z-\frac{1}{2}}{y^2}) && (3.67)
\end{aligned}
$$

where $i = 1, 2$ and $*_3$ is the flat space epsilon symbol in the three dimensions parameterized by y, x_1, x_2.

1.3.2 Solution to the differential equation

We see that the full solution is determined in terms of a single function z. This function obeys the linear equation

$$\partial_i\partial_i z + y\partial_y(\frac{\partial_y z}{y}) = 0 \qquad (3.68)$$

Since the product of the radii of the two 3-spheres is y, we would have singularities at $y = 0$ unless z has a special behavior. It turns out that the solution is non-singular as long as $z = \pm\frac{1}{2}$ on the $y = 0$ plane spanned by x_1, x_2. Let us consider the case $z = \frac{1}{2}$ at $y = 0$. Then we see that z will have an expansion $z \sim \frac{1}{2} - e^{-2G} = \frac{1}{2} - y^2 f(x) + \cdots$, where $f(x)$ will be positive with our boundary conditions. From this we find that $e^{-G} \sim yc(x)$. So we see that the metric in the y direction and the second 3-sphere directions becomes

$$h^2 dy^2 + ye^{-G} d\tilde{\Omega}_3^2 \sim c(x)(dy^2 + y^2 d\tilde{\Omega}_3^2) \qquad (3.69)$$

In addition we see that h remains finite and the radius of the first sphere also remains finite. One can also show that V remains finite by using the explicit expression we write below. When $z = -\frac{1}{2}$ the discussion is similar. In fact the transformation $z \to -z$ and an exchange of the two three–spheres is a symmetry of the equations. This corresponds to a particle hole transformation in the fermion system. This will not be a symmetry of the solutions if the fermion configuration itself is not particle-hole symmetric, or the asymptotic boundary conditions are not particle-hole symmetric (as in the $AdS_5 \times S^5$ case). We will explain below that the solution is non-singular at the boundary of the two regions. So in order to determine the solution we need to specify regions in the x_1, x_2 plane where $z = \pm\frac{1}{2}$. These two signs corresponds to the fermions and the holes, and the x_1, x_2 plane corresponds to the phase space. After defining $\Phi = z/y^2$ the equation (3.68) becomes the Laplace equation in six dimensions for Φ with spherical symmetry in four of the dimensions, y is then the radial variable in these four dimensions. The boundary values of z on the $y = 0$ plane are charge sources for this equation in six dimensions. It is then straightforward to write the general solution once we specify the boundary values. We find

$$
\begin{aligned}
z(x_1, x_2, y) &= \frac{y^2}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0) dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2} = -\frac{1}{2\pi} \int_{\partial\mathcal{D}} dl \; n_i' \frac{x_i - x_i'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]} + \sigma \qquad (3.70) \\
V_i(x_1, x_2, y) &= \frac{\epsilon_{ij}}{\pi} \int_{\mathcal{D}} \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{[(\mathbf{x} - \mathbf{x}')^2 + y^2]^2} = \frac{\epsilon_{ij}}{2\pi} \oint_{\partial\mathcal{D}} \frac{dx_j'}{(\mathbf{x} - \mathbf{x}')^2 + y^2} \qquad (3.71)
\end{aligned}
$$

where in the second expressions for z, V_i we have used that $z(x_1', x_2', 0)$ is locally constant and we have integrated by parts to convert integrals over droplets $\mathcal{D}$ into the integrals over the boundary of the droplets $\partial\mathcal{D}$. In these expressions n_i is the unit normal vector to the droplet pointing towards the $z = \frac{1}{2}$ regions, σ is a contribution from infinity which arises in the case that z is constant outside a circle of very large radius (asymptotically $AdS_5 \times S^5$ geometries). $\sigma = \pm\frac{1}{2}$ when we have $z = \pm\frac{1}{2}$ asymptotically. The contour integral in (3.71) is oriented in such a way that the $z = -\frac{1}{2}$ region is to the left. We see from the second expression for V in (3.71) that V is finite as $y \to 0$ in the interior of a droplet. We also see from (3.71) that V is a globally well defined vector field.

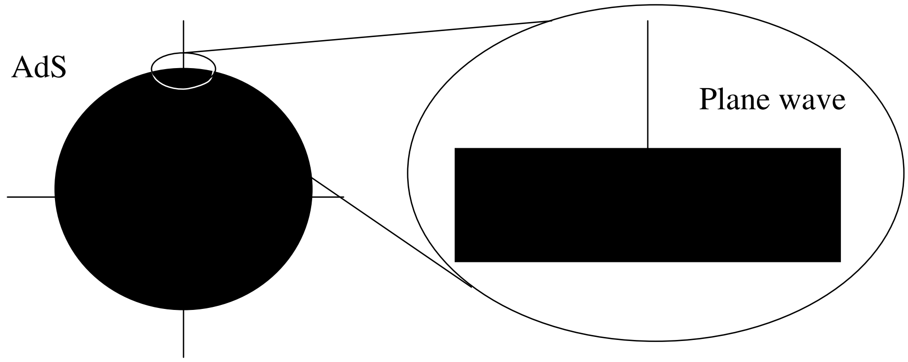

Figure 1.4: Plane wave configurations correspond to filling the lower half plane. This can be understood from the fact that the plane wave solution is a limit of the $AdS \times S$ solution.

1.4 Solutions for special cases

1.4.1 Plane wave solution

Let us now consider a simple solution which is associated to the half filled plane (figure 1.4). We have the boundary conditions

$$z(x_1', x_2', 0) = \frac{1}{2}\text{sign } x_2' \tag{4.1}$$

From this data we can compute the entire function $z(x_2, y)$ using (3.70), (3.71)

$$z(x_2, y) = \frac{1}{2}\frac{x_2}{\sqrt{x_2^2 + y^2}} \tag{4.2}$$

$$V_1 = \frac{1}{2}\frac{1}{\sqrt{x_2^2 + y^2}} \ , \qquad V_2 = 0 \tag{4.3}$$

Inserting this into the general ansatz (3.61) and performing the change of coordinates

$$y = r_1 r_2 \tag{4.4}$$

$$x_2 = \frac{1}{2}(r_1^2 - r_2^2) \tag{4.5}$$

we obtain the usual form of the metric for the plane wave [17]

$$ds^2 = -2dtdx_1 - (r_1^2 + r_2^2)dt^2 + d\vec{r}_1^{\,2} + d\vec{r}_2^{\,2} \tag{4.6}$$

We see that the final solution is smooth, despite the fact that on the $y = 0$ plane V diverges at the boundary between two regions ($x_2 = 0$ in this case). In fact, this computation shows that, in general, the boundary between two regions is smooth. The reason is that locally the boundary between two regions looks like the plane wave and therefore we will get a non-singular metric.

1.4.2 $AdS_5 \times S^5$ geometry

Now we are able to recover the familiar $AdS_5 \times S^5$ geometry. In this case it is convenient to introduce a function $\tilde{z} = z - \frac{1}{2}$. The Laplace equation for $\tilde{z}/y^2$ has sources on a disk of radius r_0 (figure 1.5). We choose polar coordinates r, ϕ in the x_1, x_2 plane. We obtain

$$\begin{aligned}
\tilde{z}(r, y) &= -\frac{y^2}{\pi}\int_{\text{Disk}} \frac{r'dr'd\phi}{[r^2 + r'^2 - 2rr'\cos\phi + y^2]^2} \\
\tilde{z}(r, y; r_0) &\equiv \frac{r^2 - r_0^2 + y^2}{2\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2r_0^2}} - \frac{1}{2} && (4.7) \\
V_\phi &= -r\sin\phi V_1 + r\cos\phi V_2 = -\frac{1}{2\pi}\int_{\partial\mathcal{D}} \frac{rr'\cos\phi' d\phi'}{r^2 + r'^2 + y^2 - 2rr'\cos\phi'} \\
V_\phi(r, y; r_0) &\equiv -\frac{1}{2}\left(\frac{r^2 + y^2 + r_0^2}{\sqrt{(r^2 + r_0^2 + y^2)^2 - 4r^2r_0^2}} - 1\right) && (4.8)
\end{aligned}$$

Inserting this into the general ansatz and performing the change of coordinates

$$y = r_0 \sinh\rho\sin\theta \tag{4.9}$$

$$r = r_0 \cosh\rho\cos\theta \tag{4.10}$$

$$\tilde{\phi} = \phi - t \tag{4.11}$$

we see that we get the standard $AdS_5 \times S^5$ metric

$$ds^2 = r_0[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + d\theta^2 + \cos^2\theta d\tilde{\phi}^2 + \sin^2\theta d\tilde{\Omega}_3^2] \tag{4.12}$$

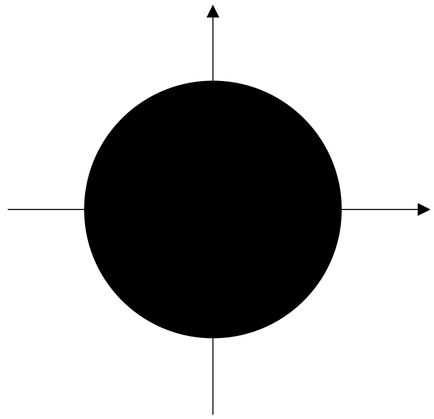

Figure 1.5: A droplet corresponding to the $AdS_5 \times S^5$ ground state.

So we see that $r_0 = R^2_{AdS} = R^2_{S^5}$. In fact, under an overall scaling of the coordinates $(x_i, y) \to \lambda(x_i, y)$ the metric scales by a factor λ. This is what we expect since the total area of the droplets is equal to the number of branes, a fact which we will demonstrate later. By comparing the value of the AdS radius we obtained in (4.12) and the standard answer, $R^4_{AdS} = 4\pi l_p^4 N$, we can write the precise quantization condition on the area of the droplets in the 12 plane as (we define $l_p = g^{\frac{1}{4}}\sqrt{\alpha'}$)

$$(\text{Area}) = 4\pi^2 l_p^4 N \ , \qquad \text{or} \quad \hbar = 2\pi l_p^4 \tag{4.13}$$

where N is an integer, and we have defined an effective $\hbar$ in the x_1, x_2 plane, where we think of the x_1, x_2 plane as phase space.

We can check that the flux equals exactly to N. We can consider a surface that ends on the $y = 0$ plane on a region with $z = -\frac{1}{2}$. Looking at the expressions for the field strength (3.1) in terms of the four dimensional gauge field (3.67), (3.65) we find that the spatial components are given by $F|_{spatial} = d(B_t V) + d\hat{B}$. Since $B_t V$ is a globally well defined vector field the flux is given by

$$N = \frac{1}{2\pi^2 l_p^4}\int d\hat{B} = -\frac{1}{8\pi^2 l_p^4}\int_{\Sigma_2} y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right) = \frac{(\text{Area})_{z=\frac{1}{2}}}{4\pi^2 l_p^4} \tag{4.14}$$

where Σ_2 is the two surface in the three dimensional space spanned by y, x_1, x_2. This

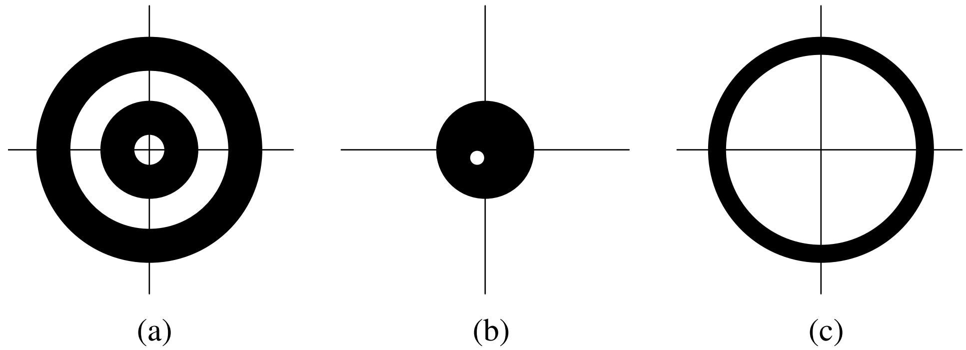

Figure 1.6: Various configurations whose solutions can be easily constructed as superpositions of the $AdS_5 \times S^5$ solution: (a) generic configurations that lead to solutions which have two Killing vectors and lead to static configurations in AdS, (b) the solution corresponding to a superposition of D3 branes wrapping the $\tilde{S}^3$ in S^5, (c) the configuration resulting from many branes, which can be thought of as a superposition of branes on the S^3 of AdS_5 uniformly distributed along the angular coordinate $\tilde{\phi}$ of S^5.

expression gives the total charge inside this region for the Laplace equation, which in turn is equal to the total area with $z = -\frac{1}{2}$ contained within the contour on which Σ_2 ends at $y = 0$and it leads to the quantization of area.

As long as $y \neq 0$ we have two S^3s: S_3 contained in AdS_5 and $\tilde{S}_3$ contained in S^5. At the $y = 0$ plane the first sphere S^3 shrinks in a non-singular fashion if $z = -\frac{1}{2}$, i.e. inside the circular droplet, while the second sphere $\tilde{S}^3$ shrinks if $z = \frac{1}{2}$, i.e. outside the circle.

We can also construct in a trivial way the solutions that are superpositions of circles, see figure 1.6.

Among these the ones corresponding to concentric circles have an extra Killing vector. These lead to time independent configurations in AdS. All other solutions will depend on $\phi = t + \tilde{\phi}$ where t is the time in AdS and $\tilde{\phi}$ is an angle on the asymptotic S^5 (see (4.12)). The solutions corresponding to concentric circles are therefore superpositions of (4.7) and (4.8)

$$\tilde{z} = \sum_i (-1)^{i+1} \tilde{z}(r, y; r_0^{(i)}), \qquad V_\phi = \sum_i (-1)^{i+1} V_\phi(r, y; r_0^{(i)}) \tag{4.15}$$

Here $r_0^{(1)}$ is the radius of the outermost circle, $r_0^{(2)}$ the next one, etc (see figure 1.6(a)).

The solution corresponding to a single black ring, when the white hole in the center is very small, can be viewed as branes wrapping a maximal $\tilde{S}^3$ in S^5 (figure 1.6(b)). When the area of this hole, N_h, is smaller than the original area, N, of the droplet ($N_h \ll N$), the solution will locally look like an $AdS_5 \times S^5$ solution near the hole. When we increase the number of branes wrapped on $\tilde{S}^3$ in S^5 the area of the holes becomes very large and in the limit we get a rather thin ring, which could be viewed as a superposition of D3 branes wrapping an S^3 in AdS_5, see figure 1.6(c).

1.4.3 Bubbling Orientifolds

$\frac{1}{2}$-BPS bubbling geometries associated to orientifolds of type IIB string theory correspond to excited states of the $SO(N)/Sp(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It was found in [18] that these geometries are in correspondence with free fermions moving in a harmonic oscillator potential on the half-line. Besides being of intrinsic interest, these solutions may also occur as local geometries in flux compactifications where orientifold planes are present to ensure global charge cancellation.

We are interested in an $AdS_5 \times RP^5$ ground state, arising from a Z_2 orientifold projection of the S^5. Introducing an orientifold plane changes the gauge group on a stack of D-branes to $SO(2N)$, $SO(2N+1)$ or $Sp(2N)$, instead of previously $SU(N)$. The orientifolded theory has no fixed points on the S^5, so there is no open string sector. The spectrum consists only of those $AdS_5 \times S^5$ states which are invariant under the orientifold projection.

$AdS_5 \times S^5$ is parametrised as (4.12):

$$(ds)^2 = r_0 \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_3^2 + d\theta^2 + \cos^2 \theta \, d\tilde{\phi}^2 + \sin^2 \theta \, d\tilde{\Omega}_3^2 \right) \quad (4.16)$$

with $\theta \in [0, \frac{\pi}{2}], \tilde{\phi} \in [0, 2\pi]$. In terms of embedding coordinates in an R^6, one can write:

$$
\begin{aligned}
X_1 &= R \cos\gamma \sin\alpha \sin\theta \\
X_2 &= R \sin\gamma \sin\alpha \sin\theta \\
X_3 &= R \cos\beta \cos\alpha \sin\theta \\
X_4 &= R \sin\beta \cos\alpha \sin\theta \\
X_5 &= R \cos\tilde{\phi} \cos\theta \\
X_6 &= R \sin\tilde{\phi} \cos\theta
\end{aligned} \tag{4.17}
$$

with $\alpha, \theta \in [0, \frac{\pi}{2}]$ and $\beta, \gamma, \tilde{\phi} \in [0, 2\pi]$. The embedding condition is $\sum_i X_i^2 = R^2 \equiv r_0$.

The orientifold action on this R^6 is $X_I \to -X_I,\ I = 1, 2, \cdots, 6$. In terms of the angular variables this amounts to

$$
\begin{aligned}
\beta &\to \beta + \pi \\
\gamma &\to \gamma + \pi \\
\tilde{\phi} &\to \tilde{\phi} + \pi.
\end{aligned} \tag{4.18}
$$

Going to the $x_1 - x_2$ plane of the bubbling geometry, we recall that it is described by polar coordinates (r, ϕ) where

$$
\begin{aligned}
r &= r_0 \cosh\rho \cos\theta \\
\phi &= \tilde{\phi} + t,
\end{aligned} \tag{4.19}
$$

and $\rho, \tilde{\phi}$ are the coordinates appearing in Eq.(4.16) above. Therefore, under the orientifolding operation, the $x_1 - x_2$ plane undergoes the involution $\phi \to \phi + \pi$, which is the same as the reflection $(x_1, x_2) \to (-x_1, -x_2)$.

The precise picture is a little more complicated because at $y = 0$, the full spatial geometry is not really 2-dimensional. In the regions where $z = \pm\frac{1}{2}$ (the white and black regions) the geometry is 5-dimensional, and consists of the (x_1, x_2) plane together with one of the 3-spheres S^3 or $\tilde{S}^3$, parametrised respectively by $d\Omega_3$ or $d\tilde{\Omega}_3$, while the other

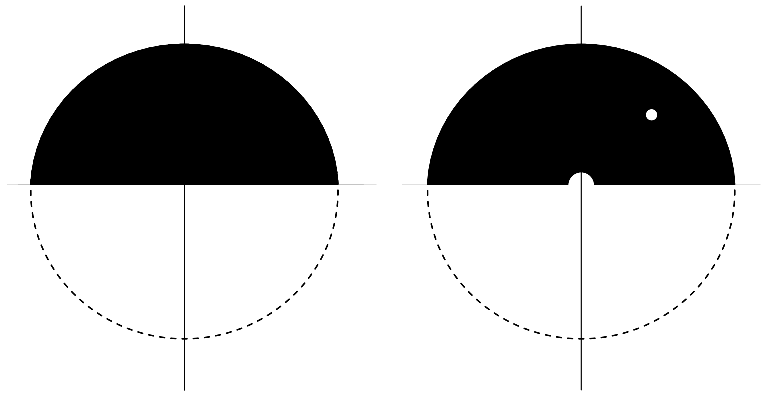

Figure 1.7: (a) Profile describing $AdS_5 \times RP^5$ and (b) two types of giant gravitons in the $AdS_5 \times RP^5$

3-sphere has shrunk to zero size. The sphere $\tilde{S}^3$ lies inside S^5 (and is parametrised by the angles α, β, γ in Eq.(4.17)). Thus it is inverted by the orientifold action, while the other 3-sphere S^3 that lies in AdS_5 remains unaffected. Thus, at a generic point of the (x_1, x_2) plane, the orientifold involution acts by reflecting (x_1, x_2) and simultaneously inverting $\tilde{S}^3$. In the white regions, where $z = \frac{1}{2}$, the $\tilde{S}^3$ shrinks to zero size, while in the black regions, where $z = -\frac{1}{2}$, it is S^3 that shrinks. It follows that in the white regions, the orientifolding operation acts solely by inverting the (x_1, x_2) plane and turning it into $\mathcal{C}/\mathcal{Z}_2$. The same is true on boundaries between the black and white regions with $z = -\frac{1}{2}, z = +\frac{1}{2}$ respectively (where both $S^3, \tilde{S}^3$ shrink). In the black region, however, one has to keep in mind that the $\tilde{S}^3$ above a given point of the (x_1, x_2) plane is identified with a reversed $\tilde{S}^3$ above the diametrically opposite point. Finally, at the origin $x_1 = x_2 = 0$ which is the fixed point of $\phi \to \phi + \pi$, the $\tilde{S}^3$ gets an antipodal identification and becomes RP^3.

The profile describing $AdS_5 \times RP^5$ is drawn in figure 1.7 (a). "Giant gravitons" in the bubbling ansatz are small holes inside an area which is much bigger than the area of the hole.In the orientifolded case there are two types of such holes, and correspondingly two types of giant gravitons (figure 1.7(b)).

If the small hole is in a generic location then we have giant gravitons wrapping a

3-sphere $\tilde{S}^3$ in RP^5. On the other hand if the hole surrounds the origin, we have giant gravitons wrapping an RP^3 cycle of RP^5. As the hole around the origin becomes large enough to interpret this as a new back-reacted geometry, the RP^3 cycle has disappeared because the black region does not enclose the origin. Also, when the hole expands further so that the black region forms a thin semicircular ring (stuck to the horizontal axis), we can interpret the configuration as consisting of dual giant gravitons wrapping an S^3 of AdS_5, and uniformly distributed around an equator of RP^5.

The orientifolded geometries themselves have no orientifold plane and therefore the underlying string theory has no open-string sector. These backgrounds have the same amount of supersymmetry and, upto discrete factors, the same $SO(4)\times SO(4)$ symmetry as the original bubbling geometries. They are associated to free fermions in a half-oscillator potential, which in turn arise as the eigenvalues of matrices in the Lie algebra of $SO(2N), Sp(2N)$ and $SO(2N+1)$.

1.4.4 Superstar in $AdS_5 \times S^5$

The BPS solutions of 5-dimensional black holes, called superstars, in $AdS_5 \times S_5$ were first introduced in [19] and were derived using the LLM ansatz [1] in [2], where the corresponding function $z(R,\phi,y;R_0)$ was found

$$\begin{aligned}
z &= \frac{1}{2(1+\frac{Q}{L^2})}\left[\frac{y^2+R^2-R_0^2}{\sqrt{(y^2+R^2+R_0^2)^2-4R^2R_0^2}}+\frac{Q}{L^2}\right] \\
C_\phi &= -\frac{1}{2(1+\frac{Q}{L^2})}\left[\frac{y^2+R^2+R_0^2}{\sqrt{(y^2+R^2+R_0^2)^2-4R^2R_0^2}}-1\right] \\
C_R &= 0
\end{aligned} \tag{4.20}$$

with

$$R_0^2 = L^4 + L^2 Q \tag{4.21}$$

Inserting these functions in to the LLM ansatz (3.61) and using the coordinate transformation

$$y = Lr\cos\theta \qquad R = L^2\sqrt{f}\sin\theta \qquad t \to Lt \tag{4.22}$$

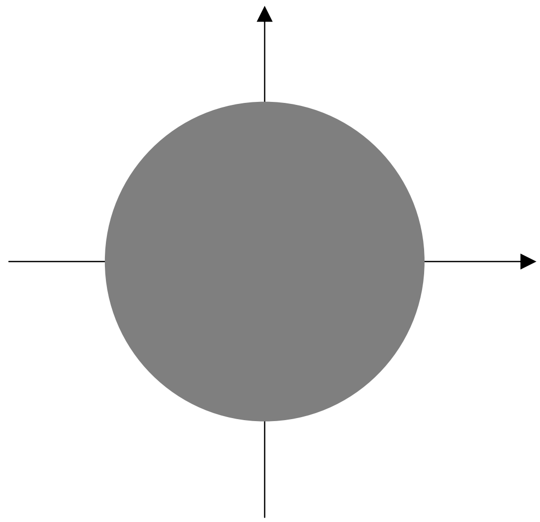

Figure 1.8: A "grey" disk that corresponds to the superstar solution.

we get the following metric for the superstar:

$$ds^2 = -\frac{1}{\Delta}\left(\cos^2\theta + \frac{r^2}{L^2}\Delta\right)dt^2 + \frac{L^2 H}{\sqrt{\Delta}}\sin^2\theta d\phi^2 + \frac{2L}{\sqrt{\Delta}}\sin^2\theta dt d\phi$$
$$+\sqrt{\Delta}\left(r^2 d\Omega_3^2 + \frac{dr^2}{f}\right) + L^2\sqrt{\Delta}d\theta^2 + \frac{L^2}{\sqrt{\Delta}}\cos^2\theta d\tilde{\Omega}_3^2 \tag{4.23}$$

with

$$H = 1 + \frac{Q}{r^2} \qquad f = 1 + \frac{Hr^2}{L^2} \qquad \Delta = \sin^2\theta + H\cos^2\theta \tag{4.24}$$

The S^5 factor of the metric in these coordinates is given by $d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\tilde{\Omega}_3^2$. For $Q = 0$ we recover standard $AdS_5 \times S^5$. For $Q > 0$ case exhibits a naked singularity at the origin $r = 0$ of AdS, where a condensate of giant gravitons growing in the five-sphere sits, and acts as a source for the supergravity fields [19]. The corresponding fermion distribution in the dual CFT can be interpreted as a dilute gas of holes in the Fermi sea and it gives a "grey" disk on the phase space (figure 1.8).

On the $y = 0$ plane radii of both spheres shrink to zero inside the circle, that is "grey disk" gives us singular geometry, and one of the radii remains constant outside the circle:

$$z = \begin{cases} \frac{1}{2} & R > R_0 \\ \frac{Q-L^2}{2(L^2+Q)} & R < R_0 \end{cases}$$

$$R^2_{\Omega_3} = ye^G = y\sqrt{\frac{\frac{1}{2}+z}{\frac{1}{2}-z}} \propto \begin{cases} 0\cdot\frac{1}{0}\Big|_{y=0} = const\Big|_{y=0} & R > R_0 \\ 0\cdot const\Big|_{y=0} = 0\Big|_{y=0} & R < R_0 \end{cases}$$

$$R^2_{\tilde{\Omega}_3} = ye^{-G} = y\sqrt{\frac{\frac{1}{2}-z}{\frac{1}{2}+z}} \propto \begin{cases} 0\cdot 0 = 0\Big|_{y=0} & R > R_0 \\ 0\cdot const\Big|_{y=0} = 0\Big|_{y=0} & R < R_0 \end{cases}$$

We can define the fermion distribution $\rho = 1/2 - z(R,0)$:

$$\rho(R) = \begin{cases} \dfrac{1}{1+Q/L^2} & R < R_0 \\ \\ 0 & R > R_0\,. \end{cases} \tag{4.25}$$

The vacuum $AdS_5 \times S^5$ is represented by a Fermi droplet of density $\rho = 1$ and radius L^2. Its total area πL^4 consists therefore exactly of N Fermi cells of area $\hbar = 2\pi l_p^4$. By turning on the R-charge Q, the probability density spreads to a disk of radius $L^2\sqrt{1+Q/L^2}$, but with lower density, in such a way that the correct number of fermions $\frac{1}{\hbar}\int \rho\, da = N$ is recovered. This fact supports the interpretation of ρ as a density distribution of fermions. The fermion system represents a uniform gas of holes delocalized in the Fermi sea, and since such holes correspond to giant gravitons growing on the five-sphere [11], the superstar can be thought of as the backreaction on spacetime produced by a condensate of giant gravitons [19].

Chapter 2

Bubbling AdS_3 space

2.1 General solution of bubbling AdS_3

2.1.1 Introduction

The results of the previous part provide the most general supersymmetric solutions of Type IIB supergravity, in the presence of a five-form flux admitting an $SO(4) \times SO(4)$ group of isometries. In this part we study solutions that correspond to $\frac{1}{2}$-BPS deformations of $AdS_3 \times S^3$ (or its pp-wave limit) and the shock wave solution, which are dual to chiral primaries in the boundary CFT. The existence of a free fermion description of primaries in the two-dimensional CFT [22] suggests that bubbling solutions should find room in six-dimensional supergravity. Here we show that this is indeed the case. Much is already known about the supergravity description of chiral primaries of the two-dimensional CFT [21–23], and we will ask ourselves whether these results can be reinterpreted in terms of bubblings of AdS_3.

The AdS3 case is interesting, as it has a somewhat richer structure, harbors many well-known solutions such as conical defect metrics, the BTZ black hole and black rings, and plays an important role in most of the microscopic derivations of black hole entropy.

Half BPS geometries associated to excitations around $AdS_3 \times S^3$ are dual to chiral primaries in the two-dimensional D1D5 (or FNS) CFT. The spectrum of chiral primaries and its dual KK descendants in supergravity have been worked out in [24–27]. States in the CFT are classified by four charges $h, \bar{h}, j, \bar{j}$ describing the quantum numbers under the isometry group $SO(2,2) \times SO(4) \sim SL(2,R)_L \times SL(2,R)_R \times SU(2)_L \times SU(2)_R$. $h, \bar{h}$

describe the conformal dimension of the two-dimensional CFT and $j, \bar{j}$ the R-symmetry charges. In the case of minimal $\mathcal{N} = (1,0)$ supergravity in $D = 6$, the CFT has $\mathcal{N} = (4,0)$ supersymmetry. This sector is universal to any supergravity in $D = 6$ and solutions are shared by supergravities following from reductions on T^4, $K3$ and orientifolds.

In analogy with the ten-dimensional case we start by decomposing the isometry group as $SO(2)^2 \times SO(2)^2_{\theta_1,\theta_2}$ and consider states with zero $SO(2)^2_{\theta_i}$ charges i.e. $h = \bar{h}$ and $j = \bar{j}$. $\frac{1}{2}$-BPS states correspond to chiral primaries $h = j$ and therefore we look for states with $h = \bar{h} = j = \bar{j} = \frac{m}{2}$. There is a single state of this type for each m in the spectrum of KK descendants of the gravity multiplet and one for each tensor multiplet. We therefore look for solutions in the pure $\mathcal{N} = (1,0)$ supergravity and its minimal extension by adding a tensor multiplet.

Notice that, contrary to the ten-dimensional case studied in [1], requiring the solution to admit an $SO(2) \times SO(2)$ group of isometries *does not* fix uniquely the form of the internal space. In particular, on the two-torus T^2 we could have a non-diagonal metric, and generically a non-trivial fibration structure.By working in the minimal supergravity and mimicking the ansatz used in [1] with $T^2 = S^1 \times S^1$ we find an almost identical set of equations describing our solutions. In particular, it turns out that a function z obeys the same equation as in [1]. However, unlike in the LLM case, the Bianchi identity translates into a further harmonic condition on the function h^2, related via a non-linear equation to z. The important property of linearity of the equations is in this way lost and solutions are rare: $AdS_3 \times S^3$, the pp-wave and the multi-center string.

Relaxing the initial metric ansatz, namely, considering a torus which is not any more rectangular we can get a more general form of $\frac{1}{2}$-BPS solutions of minimal supergravity. In this case, the non-linear relation between z and h^2 is lifted, and one is able to freely superpose different solutions in a fashion similar to [1]. The resulting geometries are given in terms of harmonics generated by lines of charges distributed along the boundary of droplets in a two-dimensional plane. The cycles of the torus degenerate along this plane, while crossing the charged strings the corresponding pinching cycles get flipped.

By adding a tensor multiplet to the minimal theory, namely an anti–self–dual three-form, and a scalar field we derive a wider class of D1D5 classical geometries like, for instance, giant gravitons that have been systematically studied in [21]. The familiar string profiles describing these solutions are reinterpreted here as the boundaries of the droplet configurations in the two-dimensional plane, i.e. the classical string profile corresponds to a certain parametrized curve $\mathbf{F}(s) \subset \mathcal{R}^4$. The 1/2-BPS geometries are obtained by dualizing solutions describing classical string profiles in [22].

2.1.2 General solution of bubbling AdS_3

In [28–34] the AdS/CFT conjecture for the D1/D5 case was studied, focusing on the simplest 1/4 BPS sector (from the ten dimensional point of view) with an $SO(2)\times SO(2)$ symmetry, that is known as bubbling AdS_3. Half-BPS states in the dual CFT_2 correspond to chiral primaries with $h = \bar{h} = j = \bar{j}$. This states appear only in the KK towers descending from the gravity and tensor multiplets of $\mathcal{N} = (1, 0)$ superravity.

We can get a general solution in the bubbling ansatz if we consider a tilted torus (4 compact dimensions). It reads

$$\begin{aligned} ds^2 =& h^{-2}(dt + C)^2 - h^2(dy^2 + dx_1^2 + dx_2^2) - \big(h^2y^2 + h^{-2}(z + \frac{1}{2})^2\big)d\theta_1^2 - \\ & - \big(h^2y^2 + h^{-2}(z - \frac{1}{2})^2\big)d\theta_2^2 + 2\big(h^2y^2 - h^{-2}(\frac{1}{4} - z^2)\big)d\theta_1 d\theta_2 \end{aligned} \tag{6.1}$$

with $dC = \frac{1}{y} *_3 dz$ and the requirement $d(dC) = 0$ implies

$$\Delta_3 z - \frac{1}{y}\partial_y z = 0 \tag{6.2}$$

The Bianchi identity and Einstein equation reduce to

$$\Delta_3 h^2 - \frac{1}{y}\partial_y h^2 = 0 \tag{6.3}$$

which can be expressed as Laplacian equations in 6 and 4 dimensions respectively

$$\begin{aligned} &\Delta_6\left(\frac{z}{y^2}\right) = 0 \\ &\Delta_4 h^2 = 0 \end{aligned} \tag{6.4}$$

$$
\begin{aligned}
z(x_1,x_2,y) &= \frac{y^2}{\pi}\int_{\mathcal{D}}\frac{z(x_1',x_2',0)dx_1'dx_2'}{(|\mathbf{x}-\mathbf{x}'|^2+y^2)^2} = \sigma - \frac{1}{2\pi}\int_{\mathcal{C}} dv \frac{|\partial_v \mathbf{x}'(v)|\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}'(v))}{|\mathbf{x}-\mathbf{x}'(v)|^2+y^2} \\
C_i(x_1,x_2,y) &= \frac{\epsilon_{ij}}{\pi}\int_{\mathcal{D}}\frac{z(x_1',x_2',0)(x_i-x_i')dx_1'dx_2'}{(|\mathbf{x}-\mathbf{x}'|^2+y^2)^2} = \frac{\epsilon_{ij}}{2\pi}\int_{\mathcal{C}} dv \frac{\partial_v x_j'(v)}{|\mathbf{x}-\mathbf{x}'(v)|^2+y^2} \\
h^2(x_1,x_2,y) &= \int_{\mathcal{D}}\frac{\rho(x_1',x_2')dx_1'dx_2'}{|\mathbf{x}-\mathbf{x}'|^2+y^2} = \frac{1}{2\pi}\int_{\mathcal{C}} dv \frac{|\partial_v \mathbf{x}'(v)|}{|\mathbf{x}-\mathbf{x}'(v)|^2+y^2}
\end{aligned} \tag{6.5}
$$

$\sigma = \pm\frac{1}{2}$ is a contribution coming from infinity arising for solutions for which $z = \pm\frac{1}{2}$ outside some circle of large radius [1]. Note that the functions z and h^2 are in generally independent, but if we reduce them to line integrals using Stokes theorem and considering regularity conditions [29] ($\rho(x_1', x_2') = \frac{1}{2\pi}$), we get that both functions are integrals along a curve $\mathcal{C}$ that divides the two-dimensional phase space into two regions - black and white. We parameterize the curve $\mathcal{C}$ with parameter v in the $\{x_1, x_2, 0\}$ plane. $\mathbf{x}'(v)$ is a derivative with respect to v, therefore we can call it "velocity".

2.1.3 Solution for a rectangular torus

Only in 3 cases (AdS_3, pp-wave and multi-center string) the tilted torus can be reduced to a rectangular one. Then we recover the equation that relates h^2 and z as it was in the case of AdS_5:

$$h^2 = \frac{1}{y}\sqrt{\frac{1}{4} - z^2} \tag{6.6}$$

Then the metric is specified by a single function G and is given by

$$
\begin{aligned}
& ds^2 = h^{-2}(dt+C)^2 - h^2(dy^2 + dx_1^2 + dx_2^2) - ye^{G}d\theta_1^2 - ye^{-G}d\theta_2^2 \\
& h^{-2} = 2y\cos G \\
& z = \frac{1}{2}\tanh G \\
& dC = \frac{1}{y} *_3 dz
\end{aligned} \tag{6.7}
$$

where $*_3$is the Hodge star in $\{y, x_1, x_2\}$. The 3-form is given by

$$
\begin{aligned}
&F = dB_t \wedge (dt + C) + B_t dC + d\hat{B} \\
&\tilde{F} = d\tilde{B}_t \wedge (dt + C) + \tilde{B}_t dC + d\hat{\tilde{B}} \\
&B_t = \frac{1}{2} y e^{G} \qquad \tilde{B}_t = \frac{1}{2} y e^{-G} \\
&d\hat{B} = -d\hat{\tilde{B}} = \frac{1}{2} y *_3 dh^2
\end{aligned} \tag{6.8}
$$

The requirements $d(dC) = 0$ and $d(d\hat{B}) = 0$ impose the equations (6.4) to be satisfied by h^2 and z.

Here we write explicitly the form of z and h^2 for the 3 known cases.

pp-wave

$$
\begin{aligned}
z &= \frac{1}{2} \frac{x_2}{\sqrt{x_2^2 + y^2}} \\
h^2 &= \frac{1}{2} \frac{1}{\sqrt{x_2^2 + y^2}} \ .
\end{aligned} \tag{6.9}
$$

This corresponds to dividing the $y = 0$ plane in two regions (filled and empty), separated by the x_1 axis [1]. The functions z and h^2 can be written as the integrals (6.5) over the x_1-axis dividing the two regions:

$$
\mathbf{x}'(v) = (v, 0) \qquad\qquad -\infty < v < \infty \ . \tag{6.10}
$$

It is depicted in the figure 2.1(a).

$\mathbf{AdS_3} \times \mathbf{S^3}$

$$
\begin{aligned}
z &= \frac{1}{2} \frac{\mathbf{x}^2 + y^2 - a^2}{\sqrt{(\mathbf{x}^2 + y^2 + a^2)^2 - 4\, a^2\, \mathbf{x}^2}} \\
h^2 &= \frac{a}{\sqrt{(\mathbf{x}^2 + y^2)^2 - 2a^2(\mathbf{x}^2 - y^2) + a^4}} \ .
\end{aligned} \tag{6.11}
$$

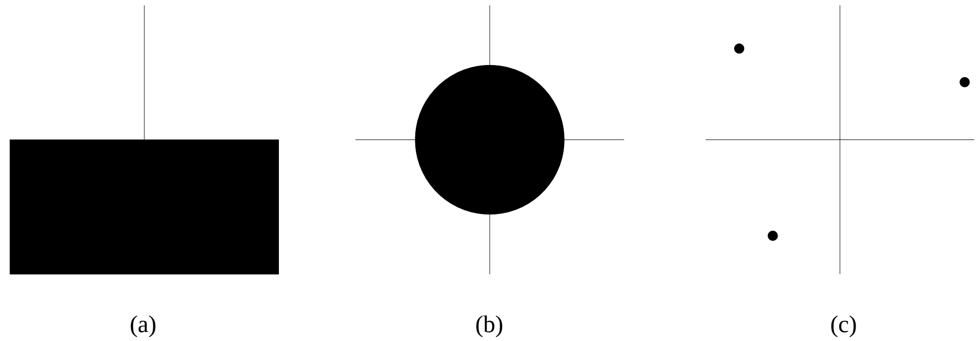

Figure 2.1: The basic figures in minimal supergravity. (a) The pp-wave, (b) $AdS_3 \times S^3$, (c) Multi-center string.

This corresponds to a round disk of radius a centered in the origin (figure 2.1(b)). The functions z and h^2 can be written as the integrals over the droplet boundary (a circle of radius a) parametrized by v:

$$\mathbf{x}'(v) = (a\cos\, v, a\sin\, v) \qquad 0 < v < 2\pi\ . \tag{6.12}$$

Multi-center string

$$\begin{aligned} h^2 &= \frac{1}{\lambda^2}\, H \qquad H = \sum_i \frac{Q_i}{(\vec{x} - \vec{x}_{0,i})^2 + y^2} \\ z &= \pm\sqrt{\frac{1}{4} - \frac{1}{\lambda^4}\, H^2 y^2} \qquad \lambda \to \infty\ . \end{aligned} \tag{6.13}$$

In this case the equation for z is satisfied in the limit $\lambda \to \infty$. Suppose we take the plus sign in (6.13), then in the limit, $e^{-G} \sim yH/\lambda^2$. Substituting these into the metric one finds $C \approx 0$ and

$$ds^2 = H^{-1}(d\tilde{t}^2 - dw^2) - H(d\tilde{\vec{x}}^2 + d\tilde{y}^2 + \tilde{y}^2 d\theta_2^2) \tag{6.14}$$

where we have rescaled $\tilde{t} = \lambda\, t, w = \lambda\, \theta_1, \tilde{y} = \lambda^{-1}\, y$, $\tilde{\vec{x}} = \lambda^{-1}\, \vec{x}$. The resulting solution corresponds to a multi-center string in $D = 6$ (figure 2.1(c)). Obviously, the same solution is obtained choosing the minus sign in (6.13), upon exchanging θ_1 and θ_2.

Notice that also in this case, the harmonic function h^2 can be thought of as arising from a profile, but now the boundary of the droplets are points $\partial\mathcal{D} = \{\mathbf{x_{0,i}}\}$. The profile function reads:

$$\mathbf{x}'(v_i) = \mathbf{x_{0,i}} \qquad 0 < v_i < Q_i \ .$$

2.1.4 Connection to the solutions for a regular distribution of D1-D5 branes

In fact, half-BPS solutions, we are interested in, lie within the general class of solutions for a regular distribution of D1-D5 branes. To show this we can rewrite the metric in terms of new angular variables (α, ϕ) as

$$\begin{aligned} ds^2 &= h^{-2}((dt + C)^2 - (d\alpha + B)^2) - h^2(dx_1^2 + dx_2^2 + dy^2 + y^2 d\phi^2) \\ dB &= - *_4 dC \end{aligned} \tag{6.15}$$

We can write $B = zd\phi$ and hence we reproduce the equation $dC = \frac{1}{y} *_3 dz$ and the equation (6.4) $\Delta_6\left(\frac{z}{y^2}\right) = 0$ as well. Therefore we get the same functions h^2, C_i and z as in (6.5) here. To get the metric (6.1) with a general tilted torus fibration we need to change variables

$$\theta_1 = \alpha + \frac{1}{2}\phi \qquad \theta_2 = \alpha - \frac{1}{2}\phi \tag{6.16}$$

This solution (6.15) is a more general one than (6.1) because it includes tensor multiplet in addition to minimal sugra.

Now we turn our attention to the general near horizon metrics of the D1-D5 system. It is possible to construct the metric for the D1-D5 system by mapping it by a set of S, T dualities to the FP system [21]. The F string has a vibration profile in the non-compact spatial directions described by the profile $\{F_1(v), F_2(v), F_3(v), F_4(v)\}$. The metric of the

D1-D5 system is [23, 28]

$$
\begin{aligned}
ds^2 &= \frac{1}{\sqrt{f_1 f_2}}((dt+A)^2 - (dy+B)^2) - \sqrt{f_1 f_2}d\mathbf{x}^2 + \sqrt{\frac{f_1}{f_5}}d\mathbf{z}^2 \\
dB &= - *_4 \, dA \qquad\qquad e^{2\Phi} = \frac{f_1}{f_5} \\
f_1 &= 1 + \frac{Q_5}{L}\int_0^L dv \frac{|\partial_v \mathbf{F}(v)|^2}{|\mathbf{x}-\mathbf{F}(v)|^2 + y^2} \qquad f_5 = 1 + \frac{Q_5}{L}\int_0^L dv \frac{1}{|\mathbf{x}-\mathbf{F}(v)|^2 + y^2} \\
A_i &= \frac{Q_5}{L}\int_0^L dv \frac{\partial_v F_i(v)}{|\mathbf{x}-\mathbf{F}(v)|^2 + y^2}
\end{aligned} \tag{6.17}
$$

The solutions are asymptotically $\mathcal{R}^{1,4} \times S^1 \times T^4$, y parametrizes the S^1, which has the radius R, and $\mathbf{z}$ are the coordinates on the T^4, which volume is V_4. The Hodge duals $*_4$ are defined with respect to the four non-compact transversal coordinates $\mathbf{x}$. We can take a decoupling limit which simply amounts to erasing the 1 from the harmonic functions. The resulting metric will then be asymptotically equal to $AdS_3 \times S^3 \times T^4$.

The number of D1 and D5-branes is denoted by N_1 and N_5, and they are related to the charges Q_i by

$$
Q_5 = g_s N_5 \qquad Q_1 = \frac{g_s}{V_4} N_1 \tag{6.18}
$$

The parameter L has to satisfy

$$
L = \frac{2\pi Q_5}{R} \tag{6.19}
$$

Besides, the curve has to satisfy the following relation

$$
Q_1 = \frac{Q_5}{L}\int_0^L |\partial_v \mathbf{F}(v)|^2 dv \tag{6.20}
$$

Actually we can directly relate our curve $\mathbf{x}(v)$ in (6.5) parameterized by v to the profile of the F string in 2 dimensions $\{F_1(v), F_2(v), 0, 0\}$. If we identify

$$
h^2 \equiv \sqrt{f_1 f_2} \qquad\qquad \mathbf{x}'(v) \equiv \mathbf{F}(v) \qquad\qquad R = 1 \tag{6.21}
$$

and omit the T^4 coordinates, then we get the metric (6.15) with $C_i \equiv A_i$ and $\alpha \equiv y$. Now in stead of one harmonic function h^2 we have two, which have to obey the Laplacian

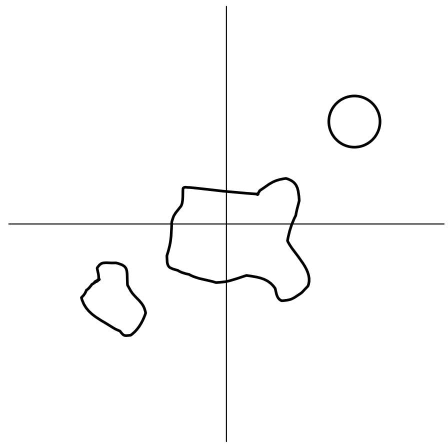

Figure 2.2: Generic bubbling profile: fixing the length we can draw any set of disconnected closed curves.

equation

$$\Delta_4 f_1 = 0 \qquad\qquad \Delta_4 f_5 = 0 \tag{6.22}$$

but when the velocity $\mathbf{x}'(v)$ is constant we come back to the single function h^2 as in (6.5).

From (6.17) we see that when the velocity $\partial_v \mathbf{F}(v)$ is not constant we get a non-constant dilaton. This refers to including a tensor multiplet in addition to the minimal sugra. That is the minimal sugra solution refers to taking the velocity to be constant.

An interesting fact that the most general solution (6.15) is specified not just by the boundaries in the $\{x_1, x_2, 0\}$ plane, but also by the velocity $\{x_1'(v), x_2'(v)\}$ in contrast to the AdS_5 1/2 BPS solutions. The classical phase space of gravitational solutions of AdS_3 is the set of curves $\mathbf{x}'(s)$ of fixed length, which is defined by fixing the flux (6.20)

$$N = \frac{1}{2\pi} \int dv |\partial_v \mathbf{x}'(v)|^2 \tag{6.23}$$

and therefore it does not make much sense to look at black and white parts of the phase space (figure 2.2). In the AdS_5 case the regularity condition was to fix the flux, which was equivalent to fixing the area of the droplet, but in the AdS_3 case fixing the flux means a fixed length but with an arbitrary number of disconnected parts of any shape.

The Fourier modes of $\mathbf{F}$ correspond to standard free bosonic string oscillators without the zero mode, with the length corresponding to the energy or L_0 eigenvalue of a state. The Hilbert space in question is therefore the set of states of level N in the Hilbert space of four free bosons.

2.2 The general conical defect metric

In the next two sections we get the conical defect and the shock wave metrics from certain distributions on the phase space.

By taking a specific distribution of curves $\mathbf{x}'(v)$ on the phase space and computing the harmonic functions h^2 and C we can get the conical defect metric with an arbitrary open angle [20]. According to [21] the source for the conical defect metric has to be contained in a circle of radius a on the phase space. The most general source curve satisfying this requirement is

$$x_1'(v) = a\cos\left[f(v)\right] \qquad\qquad x_2'(v) = a\sin\left[f(v)\right] \tag{7.1}$$

where $f(v)$ is some arbitrary function that has to satisfy

$$\int_0^{2\pi} e^{if(v)}dv = 0 \tag{7.2}$$

because $\mathbf{x}'$ does not contain a zero mode. The source (7.1) also should be invariant under rotations in the x_1, x_2-plane, therefore we need to coarse grain over all U(1) rotations of (7.1). This is mostly easily done by introducing polar coordinates

$$x_1 + ix_2 = ue^{i\varphi} \qquad\qquad x_3 + ix_4 = we^{i\psi} \tag{7.3}$$

so that the average over rotations can be expressed as

$$\begin{aligned} h^2 &= a\frac{1}{(2\pi)^2}\int_0^{2\pi} d\xi \int_0^{2\pi} dv \frac{\partial_v f(v)}{|ue^{i\varphi} - ae^{if(v)+i\xi}|^2 + w^2} \\ C &= -a\frac{1}{(2\pi)^2}\int_0^{2\pi} d\xi \int_0^{2\pi} dv \frac{i\partial_v f(v) e^{if(v)+i\xi}}{|ue^{i\varphi} - ae^{if(v)+i\xi}|^2 + w^2} \end{aligned} \tag{7.4}$$

The flux (6.23) reads

$$N = \frac{a^2}{2\pi} < (\partial_v f)^2 > \qquad \text{with} \qquad < (\partial_v f)^2 > \equiv \frac{1}{2\pi}\int_0^{2\pi} (\partial_v f)^2 dv \tag{7.5}$$

It is straight forward to evaluate the integrals in (7.4) to get

$$\begin{aligned} h^2 &= \frac{a}{(u^2+w^2+a^2)^2-4a^2u^2} \\ C_\varphi &= \frac{a^3}{2\pi}\frac{u^2+w^2+a^2-(u^2+w^2+a^2)^2-4a^2u^2}{(u^2+w^2+a^2)^2-4a^2u^2} < \partial_v f > \end{aligned} \tag{7.6}$$

In order to put it in a form which resembles the conical defect one as much as possible, one has to make the following change of coordinates

$$u^2 = (r^2+a^2)\sin^2\theta \qquad\qquad w = r\cos\theta \tag{7.7}$$

Using these new coordinates, the various ingredients of (6.15) become

$$\begin{aligned} h^2 &= \frac{a}{r^2+a^2\cos^2\theta} \\ C_\varphi &= \beta\frac{a^2}{r^2+a^2\cos^2\theta}\sin^2\theta \\ B_\psi &= -\beta\frac{a^2}{r^2+a^2\cos^2\theta}\cos^2\theta \end{aligned} \tag{7.8}$$

where

$$\beta^2 = \frac{1}{2\pi}\frac{< \partial_v f >^2}{< (\partial_v f)^2 >} \tag{7.9}$$

is a constant introduced for later convenience.

Inserting the functions (7.8) into (6.15) we get the following metric

$$ds^2 = \frac{r^2+a^2\cos^2\theta}{r^2+a^2}dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + r^2\cos^2\theta d\phi^2 + (r^2+a^2)\sin^2\theta d\varphi^2 \tag{7.10}$$

Next we define $a = \sqrt{Q_1Q_2}$ and $\gamma = 2\pi\beta / < \partial_v f >$, and also rescale r by a factor of a/R. Then after some straightforward algebraic manipulations we end up with

$$\begin{aligned} ds^2 = \sqrt{Q_1Q_2}\Big[&-(r^2+\gamma^2)\frac{dt^2}{R^2} + r^2\frac{d\alpha^2}{R^2} + \frac{dr^2}{r^2+\gamma^2} \\ &+ \left(d\theta^2 + \sin^2\theta(d\varphi - \beta\gamma\frac{dt}{R})^2 + \cos^2\theta(d\phi - \beta\gamma\frac{d\alpha}{R})^2\right) \\ &+ \frac{(1-\beta^2)\gamma^2}{r^2+\gamma^2\cos^2\theta}(\sin^2\theta d\Upsilon_1^2 + \cos^2\theta d\Upsilon_2^2)\Big] \end{aligned} \tag{7.11}$$

where we defined

$$
\begin{aligned}
d\Upsilon_1^2 &= \sin^2\theta d\varphi^2 + (r^2+\gamma^2\cos^2\theta)\frac{dt^2}{R^2} \\
d\Upsilon_2^2 &= -\cos^2\theta d\psi^2 + (r^2+\gamma^2\cos^2\theta)\frac{d\alpha^2}{R^2}
\end{aligned} \tag{7.12}
$$

This metric is a conical defect metric for $\beta = 1$ so the question is which values of γ are compatible with $\beta = 1$. To analyze this, we recast the constraints on $f(v)$ for $\beta = 1$ here

$$
\int_0^{2\pi} e^{if(v)} dv = 0 \tag{7.13}
$$

$$
\left(\int_0^{2\pi} \partial_v f(v) dv\right)^2 = 2\pi \int_0^{2\pi} (\partial_v f(v))^2 dv = \frac{4\pi^2}{\gamma^2} \tag{7.14}
$$

However, according to Schwarzs inequality,

$$
\left(\int_0^{2\pi} \partial_v f(v) dv\right)^2 \leq 2\pi \int_0^{2\pi} (\partial_v f(v))^2 dv \tag{7.15}
$$

for integrable functions $\partial_v f(v)$ with equality if and only if $\partial_v f(v)$ is a constant. Thus, $\beta \leq 1$ and $\beta \geq 1$ only if $\partial_v f(v) = \text{const}$. Interestingly, the metric (7.11) is in general a perfectly acceptable metric, since $\beta \leq 1$ is precisely the condition for the absence of CTC's as one can derive using the results in [35]. If $\beta = 1$ then $\partial_v f(v) = \text{const}$ together with (7.13) imply that $f(v) = kv$ for some nonzero integer k, and $\gamma = 1/k$. We can therefore indeed only construct conical defect metrics with $\gamma = 1/k$ and k integer. For k non-integer, we find a bound on β

$$
\beta^2 \leq \left[\frac{1}{\gamma^2}\right]\gamma^2 \tag{7.16}
$$

with $[x]$ the largest integer less than or equal to x. Indeed, we cannot come arbitrarily close to a non-integer conical defect metric in this way.

2.3 Superstar in $AdS_3 \times S^3$

Now we are interested in finding a solution analogous to the superstar solution in $AdS_5 \times S^5$ (section 1.4.4). Since in AdS_3 (in contrast to the AdS_5 case) configurations on the phase space do not completely define the corresponding metrics on the gravity side (there are only 3 solution when z is enough and from it we find h) and we should in addition tell what the velocities of the configurations are. It turns out that there is a map between the bubbling AdS_3 ansatz and the black rings supertube solution. From the alternative description we find that the analog of the superstar is not a kind of a black hole, as one would expect, but it is a shock wave, which is the Aichelburg-Sexl solution.

2.3.1 A map between bubbling AdS_3 and the black supertube solution

The solutions corresponding to supersymmetric black rings with three charges and three dipoles in 10 dimensions can be realized as D1-D5-P black supertubes, carrying the usual charges of the D1-D5-P system [36]. In addition the solution carries dipole charges of D1 and D5 branes as well as KK-monopoles. The d5-branes wrap a 4-torus parameterized by $z^1, ..., z^2$, ψ parameterizes a contractible torus (the direction of the ring) and the solution carries momentum in the z-direction, which is the coordinate that describes the $U(1)$ fiber of the KK-monopoles. The string frame metric of the black supertube is given by (we omit the torus coordinates)

$$\begin{aligned} ds^2 &= -\frac{1}{H_3\sqrt{H_1H_2}}(dt+\omega)^2 + \frac{H_3}{\sqrt{H_1H_2}}(dz+A^3)^2 + \sqrt{H_1H_2}dx_4^2 \\ H_1 &= 1+\frac{Q_1}{\Sigma} - \frac{q_2q_3r_0^2\cos 2\theta}{\Sigma^2} \\ H_2 &= 1+\frac{Q_2}{\Sigma} - \frac{q_1q_3r_0^2\cos 2\theta}{\Sigma^2} \\ H_3 &= 1+\frac{Q_3}{\Sigma} - \frac{q_1q_2r_0^2\cos 2\theta}{\Sigma^2} \end{aligned} \tag{8.1}$$

where $\Sigma = r^2+r_0^2\cos^2\theta$ and the base space dx_4^2 a flat space written in a peculiar coordinate system

$$dx_4^2 = \Sigma\left(\frac{dr^2}{r^2+r_0^2} + d\theta^2\right) + (r^2+r_0^2)\sin^2\theta d\psi^2 + r^2\cos^2\theta d\phi^2 \tag{8.2}$$

The one form $\omega = \omega_\phi d\phi + \omega_\psi d\psi$ and gauge potential A^i are given by

$$
\begin{aligned}
\omega_\phi &= -\frac{r^2\cos^2\theta}{2\Sigma^2}Y \\
\omega_\psi &= -(q_1+q_2+q_3)\frac{r_0^2\sin^2\theta}{2\Sigma^2} - \frac{(r^2+r_0^2)\sin^2\theta}{2\Sigma^2}Y \\
A^i &= H_i^{-1}(dt+\omega) + \frac{q_i R^2}{\Sigma}(\sin^2\theta d\psi - \cos^2\theta d\phi)
\end{aligned}
\tag{8.3}
$$

where we defined

$$
Y = q_1 Q_1 + q_2 Q_3 + q_3 Q_3 - q_1 q_2 q_3 (1 + \frac{2r_0^2\cos 2\theta}{\Sigma})
\tag{8.4}
$$

Defining γ as

$$
\gamma = \sqrt{H_3}\frac{q_3 r_0^2}{\Sigma}(\sin^2\theta d\psi - \cos^2\theta d\phi)
\tag{8.5}
$$

we can rewrite the metric as

$$
ds^2 = -\frac{1}{\sqrt{H_1H_2}}[(\sqrt{H_3}dz+\gamma)^2 + \frac{2}{\sqrt{H_3}}(dt+\omega)(\sqrt{H_3}dz+\gamma)] + \sqrt{H_1H_2}dx_4^2
\tag{8.6}
$$

For the case $H_3 = 1$ ($Q_3 = q_2 = 0$) we find a direct map to the Bubbling AdS_3 metric (6.15). To do so we define $d\chi \equiv dz + dt$ and we get

$$
ds^2 = \frac{1}{\sqrt{H_1H_2}}\left[(dt+\omega)^2 - (d\chi + (\gamma+\omega))^2\right] - \sqrt{H_1H_2}dx_4^2
\tag{8.7}
$$

Looking at the bubbling AdS_3 metric (6.15) we see that it is the same as (8.7) if we do the following identifications:

$$
h^2 = \sqrt{H_1H_2} \qquad C = \omega \qquad B = \gamma + \omega \qquad \chi = \alpha
\tag{8.8}
$$

and we get (6.15)

$$
ds^2 = h^{-2}((dt+C)^2 - (d\alpha + B)^2) - h^2(dx_1^2 + dx_2^2 + dy^2 + y^2 d\phi^2)
\tag{8.9}
$$

2.3.2 Aichelburg-Sexl solution

To find the superstar solution the simplest guess is to take h^2 with constant velocity and C ($dC = *_3 dz$) as in the AdS_5 superstar:

$$
\begin{aligned}
h^2 &= \frac{R_0}{\sqrt{(R^2+y^2+R_0^2)^2 - 4R_0^2R^2}} \\
C_\psi &= -q_3\left[\frac{y^2+R^2+R_0^2}{\sqrt{(y^2+R^2+R_0^2)^2-4R^2R_0^2}} - 1\right]
\end{aligned}
\tag{8.10}
$$

Then using the transformation of coordinates $R = R_0\sqrt{1+r^2}\sin\theta$ and $y = R_0 r\cos\theta$ we get

$$
\begin{aligned}
h^2 &= \frac{\sqrt{Q_1Q_2}}{r^2+r_0^2\cos^2\theta} \\
C_\psi &= -q_3\frac{r_0^2\sin^2\theta}{r^2+r_0^2\cos^2\theta}
\end{aligned}
\tag{8.11}
$$

where $r_0 = \sqrt{Q_1Q_2}$. This is exactly the solution of supertube with three out of six charges set to zero $q_1 = q_2 = Q_3 = 0$. Then using the identifications (8.8) and inserting them into (8.7) we get

$$
ds^2 = \frac{\Sigma}{\sqrt{Q_1Q_2}}\left[\left(dt - q_3\frac{r_0^2\sin^2\theta}{\Sigma}d\psi\right)^2 - \left(d\chi - q_3\frac{r_0^2\cos^2\theta}{\Sigma}d\phi\right)^2\right] - \frac{\sqrt{Q_1Q_2}}{\Sigma}dx_4^2 \tag{8.12}
$$

Making the transformations

$$
dt \to \frac{q_3 r_0}{\sqrt{Q_1Q_2}}dt \qquad\qquad d\chi \to \frac{q_3 r_0}{\sqrt{Q_1Q_2}}d\chi \tag{8.13}
$$

and doing some algebra we get

$$
\begin{aligned}
ds^2 = \frac{1}{r_0}&\Big[(r^2+r_0^2)dt^2 - r_0^2\sin^2\theta(dt+d\psi)^2 - r^2d\chi^2 - r_0^2\cos^2\theta(d\phi-d\chi)^2\Big] - \frac{r_0}{r^2+r_0^2}dr^2 \\
&- r_0 d\theta^2 - \frac{q}{\Sigma r_0}\Big[(r^2+r_0^2)dt^2 + r_0^4\sin^4\theta(dt+d\psi)^2 - 2(r^2+r_0^2)r_0^2\sin^2\theta(dt+d\psi)dt \\
&- r^4d\chi^2 - r_0^4\cos^4\theta(d\phi-d\chi)^2 + 2r_0^2r^2\cos^2\theta d\chi(d\phi-d\chi)\Big]
\end{aligned}
\tag{8.14}
$$

where we defined $q_3 = \sqrt{1-q}$. Performing the coordinate change

$$\tilde{\psi} \to \psi - t \qquad\qquad \tilde{\phi} \to \phi - \chi \tag{8.15}$$

that gives spectral flow and writing $r' = r/r_0$ we get the metric

$$\begin{aligned} ds^2 &= r_0 \left[(r'^2+1)dt^2 - \sin^2\theta d\tilde{\psi}^2 - r'^2 d\chi^2 - \cos^2\theta d\tilde{\phi}^2 - \frac{1}{r'^2+1} dr'^2 - d\theta^2 \right] \\ &- \frac{q r_0}{r'^2 + \cos^2\theta} \left[((r'^2+1)dt - \sin^2\theta d\tilde{\psi})^2 - (r'^2 d\chi - \cos^2\theta d\tilde{\phi})^2 \right] \end{aligned} \tag{8.16}$$

which is the Aichelburg-Sexl solution. Near the singular line $\theta = \pi/2$, $r' = 0$ it behaves as a shock wave

$$ds^2 = -dt^2 + dz^2 + \frac{q}{x_i x_i}(dt - dz)^2 + \sum_{i=1}^{4} dx_i dx_i \tag{8.17}$$

and it goes over to $AdS_3 \times S^3$ at large r.

In the $AdS_5 \times S^5$ case the superstar solution could be interpreted as a condensate of giant gravitons. In AdS_3 the dipoles in the supertube solution actually correspond to giant gravitons. But this dipole charges q_1, q_2 we had to set to zero to get the superstar solution, and the only non zero charges are the the usual charges Q_1, Q_2 and the KK-monopole charge q_3 but if set $q_3 = 0$ the solution does not change. Therefore it is unclear how to interpret the result. We expected that the superstar would be a kind of a black hole solution but the shock wave solution we get is not what we expected. Looking for a superstar solution we assumed the velocity of the profile on the phase space was constant, but, on the other hand, we can thing about a more general configuration, e.g. with a nonconstant velocity, since the only constraint on h^2 (6.21) that it is a solution of the equations (6.22), but then the solution is very complicated and it is very hard to identify it with any known background.

Chapter 3

Deconfinement and Chiral Symmetry Restoration in the Non-critical Flavored AdS_6 System

3.1 Introduction

The AdS/CFT duality first was realized as a correspondence between a string theory in $AdS_5 \times S_5$ space and a 4-dimensional conformal field theory [8, 37–39]. Since then there were many attempts to extend it to more general cases for non-conformal and non-supersymmetric theories. The aim is to get more realistic QCD from the string/gauge duality with such features as confinement and spontaneous chiral symmetry breaking. This approach is often called the holographic QCD. One of the interesting recent developments in the holographic approach to QCD is the D4/D8-$\overline{\mathrm{D8}}$ model proposed by Sakai and Sugimoto [40, 41]. In this model probe D8-branes were introduced in such a way that strings stretching between them and the original branes have the properties of flavored fundamental quarks. In [39] it was observed that in the SS model at finite temperature deconfinement and chiral symmetry restoration can happen at different temperatures and it depends on the ratio L/R.

In this chapter, we look at the non-critical AdS_6 black hole solution [43]. This model was shown [44] to reproduce some properties of the 4-dimensional non-supersymmetric YM theory like an area law for the Wilson loop, a mass gap in the glueball spectrum etc. At high energies the theory is dual to a thermal gauge theory at the same temperature as the black hole temperature and at low energies the dual theory is effectively 4-dimensional pure YM. For any non-critical model, the curvature is of order one in units of α'. But taking large N_c limit guarantees small string coupling and one expects that stringy corrections will not affect calculations on the non-critical gravity side.

In this work we introduce flavor in this setup by adding D4,$\overline{\text{D4}}$ probe branes. This is similar to the D8,$\overline{\text{D8}}$ SS model in critical dimension and it inherits from it many qualitative features. As in the SS model there are three different phases. In the *low-temperature* phase the background is the Euclidean continuation of a Lorentzian background. Gluons are confined in this phase. After the confinement/deconfinement transition for the gluons, there is the *intermediate-temperature* phase. In this phase gluons are deconfined, but chiral symmetry is still broken. Mesonic bound states still exist, as the D4-brane embedding is not yet touching the horizon. At sufficiently high temperature, the lowest-energy configuration of the D4-branes is the one in which they are parallel and fall down to the horizon. This is the *high-temperature* phase, in which chiral symmetry is restored. If the ratio $L/R > 1.06$, there is no intermediate-temperature phase, so that the confinement/deconfinement and the chiral symmetry breaking transition coincide.

We also analyze the spectrum of low-spin and high-spin mesons. Low-spin mesons correspond on the string theory side to fluctuations of the massless fields on the probe branes. We identify the Goldstone boson associated with the chiral symmetry breaking. High-spin mesons, as for the critical case, can be described as classical spinning open strings. The temperature dependence for both low-spin and high-spin mesons is similar, that is the masses of mesons go down as the temperature goes up. For high-spin mesons there is a maximum value of angular momentum beyond which mesons cannot exist and have to melt. We find the drag force that a quark experiences moving through a hot gluon plasma, and we also find that high-spin mesons do not experience any drag force because for high-spin mesons at finite temperature, one can find generalized solutions where the meson moves with linear velocity, rigidly, with free boundary conditions in the direction of motion. Hence one does not need to apply any force to maintain this motion.

The main interest was to find what are the changes in the non-critical theory comparatively to the critical one. Adding the CS term to the action would be the direct naive generalization from the 10-dimensional theory, but it is not well understood what should be written in this two-derivative approximation to the non-critical setup. We had to omit the CS term because if we leave it a parallel brane configuration cannot be a solution of the equation of motion and we will not get a chiral symmetry restoration. Without the CS term our non-critical theory is similar in the phase structure to the critical SS model.

We begin in section 2 with a short review of the SS model and its behavior at finite temperature. In section 3 we describe the AdS_6 model at zero temperature. In section 4 we discuss the behavior of this theory at finite temperature. We discuss the bulk thermodynamics, which leads to the confinement and deconfinement phases, and chiral symmetry restoration at a certain temperature. In section 5 we have a close look on the spectrum of low-spin as well as high-spin mesons in different phases and also discuss the drag force on quarks and mesons.

3.2 Review of Sakai-Sugimoto model at finite temperature

The Sakai-Sugimoto model [40, 41] is based on a D4/D8-$\overline{\text{D8}}$ brane system consisting of N_c D4-branes compactified on S^1 and N_f D8-$\overline{\text{D8}}$-brane pairs transverse to the S^1. The brane configuration of the system is

	t	x_1	x_2	x_3	x_4	x_5	θ_1	θ_2	θ_3	θ_4
D4	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$					
D8-$\overline{\text{D8}}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$		$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$

with x_4 and θ's being coordinates of S^1 and S^4 respectively.

We look at the D4-branes in the large N_c and near horizon limits. In these limits they are classical solutions of the type IIA supergravity in ten dimensions. This gravitational background is dual to a five-dimensional gauge theory, which looks four-dimensional at energy scale below the compactification scale. Imposing periodic boundary conditions on the bosons and antiperiodic ones on the fermions along the compactified direction, supersymmetry is explicitly broken. The scalars and the fermions on the D4-branes become massive and are decoupled from the system at low energy. Thus one obtains a $U(N_c)$ pure gauge theory. To describe quarks in the fundamental representation of the gauge group $U(N_c)$ one introduces flavor N_f $D8-\overline{D8}$ pairs into the D4 background. We assume $N_f << N_c$ and it allows us to treat the N_f $D8-\overline{D8}$ branes as a probes.

The finite temperature behavior of the Sakai-Sugimoto model was discussed in [42,45–47]. As opposed to the zero temperature case there are two solutions at finite temperature, because one Wick-rotates the metric (generates a black hole solution) and an asymptotic symmetry between compactified Euclidean time coordinate (with periodicity $\beta = 1/T$) and x_4 (with periodicity $2\pi R$) appears. In [42] it was shown that one of them dominates at low temperatures and the other one at high temperatures. A phase transition between these backgrounds occurs at the temperature $T_c = 1/2\pi R$. This phase transition is of the first order and represents a confinement/deconfinement transition [48].

The bulk background geometry at low temperature is represented by the following metric

$$ds^2 = \left(\frac{u}{R_{D4}}\right)^{\frac{3}{2}} \left(dt^2 + \delta_{ij}dx^i dx^j + f(u)dx_4^2\right) + \left(\frac{R_{D4}}{u}\right)^{\frac{3}{2}} \left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right),$$
$$e^{\phi} = g_s \left(\frac{u}{R_{D4}}\right)^{\frac{3}{4}}, \qquad F_4 = dC_3 = \frac{2\pi N_c}{V_4}\epsilon_4, \qquad f(u) = 1 - \frac{u_T^3}{u^3}, \tag{8.1}$$

where $d\Omega_4^2$ is the metric of S^4 and $R_{D4}^3 = \pi g_s N_c l_s^3$ with g_s and l_s being the string coupling and the string length. ϵ_4 and V_4 are the volume form and the volume of S^4. The x_4-u submanifold has a cigar-like form with a tip at $u = u_T$. To avoid singularity at the tip of the cigar x_4 should be periodic with periodicity

$$\delta x_4 = \frac{4\pi}{3}\left(\frac{R_{D4}^3}{u_T}\right)^{1/2} = 2\pi R \tag{8.2}$$

The effective action of the D8-branes consists of the DBI action and the Chern-Simons term

$$S_{\mathrm{D8}} = T_8 \int d^9x\, e^{-\phi}\, \mathrm{Tr}\sqrt{\det(g_{MN} + 2\pi\alpha' F_{MN})} - \frac{i}{48\pi^3}\int_{\mathrm{D8}} C_3\, \mathrm{Tr}F^3, \tag{8.3}$$

where g_{MN} and F_{MN} are the induced metric and the field strength on the D8-brane and T_8 is the tension of the D8-brane. The CS term in the D8-action does not affect the solution of the equation of motion of the gauge field since it has a classical solution of a vanishing gauge field.

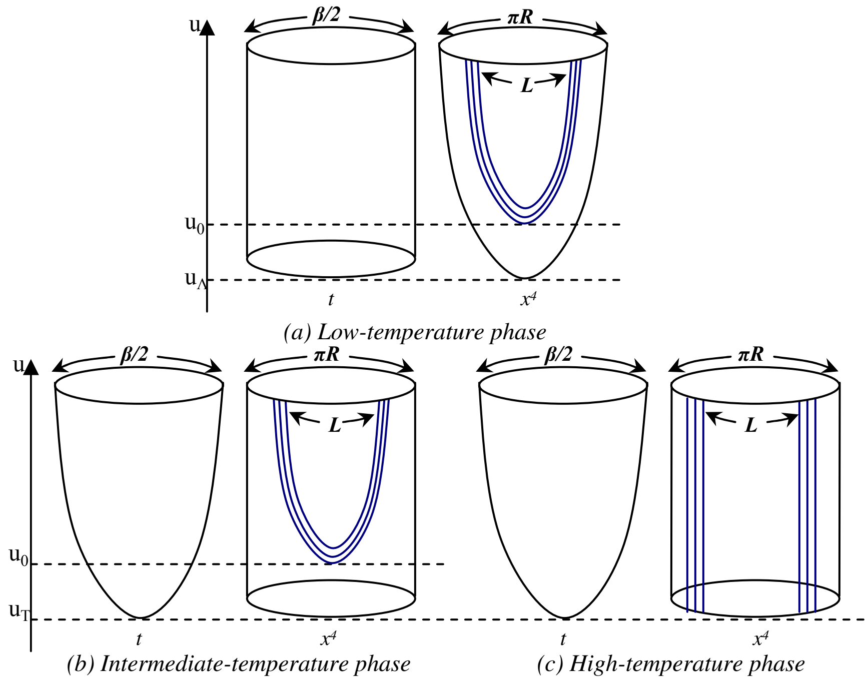

Figure 3.1: The configuration of flavor branes at three phases: (a) low-temperature phase, (b) intermediate and (c) high-temperature phases.

The Hamiltonian of the action does not depend on du/dx_4 (a slope of the profile of D8-branes) and therefore the equation of motion equals to a constant. To solve it we assume that there is a point u_0 where the profile $u(x_4)$ has a minimum ($u'|_{u=u_0} = 0$). The form of the profile is drawn in figure 3.1(a).

The x_4 circle shrinks to zero at $u = u_T$. Therefore D8 branes and antibranes have no place to end and should stay all the time connected. Because of this configuration the chiral symmetry $U(N_f)_L \times U(N_f)_R$ on the probe D8-D8 pairs is always broken to a diagonal subgroup $U(N_f)_V$ in the low temperature phase.

In the high temperature phase the preferred background is the one with the interchanged role of the t and x_4 circles (by moving the factor of $f(u)$ in (8.1) from the dx_4^2 term to the dt^2 term). Now the t-circle shrinks to zero at u_T (which is now related to T rather that to R), while the x_4 circle never shrinks. While the configuration with connected flavor branes is still possible, a new configuration with parallel branes appears (see figure 3.1(b) and 3.1(c)). This new configuration is also a solution of the equation of motion of the new background DBI action and it means that chiral symmetry $U(N_f)_L \times U(N_f)_R$ is restored.

Both configuration are possible at the high temperature phase (deconfinement phase). To see when they are preferred we need to compute their free energy (the one with the lower free energy is preferred in the given temperature range). In [42] it was found that in the range $T_c \leq T < T_{\chi SB}$ ($T_{\chi SB} = 0.154/L$, L is the separation distance between the flavor branes at $u \to \infty$) the preferred configuration is with connected branes and at $T \geq T_{\chi SB}$ - with parallel branes. Therefore deconfinement and chiral symmetry restoration do not occur together but there is an intermediate phase with deconfinement and broken chiral symmetry.

3.3 Near extremal AdS_6 model with flavor branes at zero temperature

We are interested in the non-critical flavored version of the ten-dimensional black hole background, which was considered in [43], [49]. The starting point is to consider unflavored conformal AdS_6 background, which is the dual of a fixed point non-supersymmetric 5-dimensional gauge theory without fundamental quarks. The construction of the model can be made by either first taking the near extremal limit of the AdS_6 background and then adding flavors, or by adding flavors first and then taking the near extremal limit of the flavored AdS_6. We follow the former one. The non-critical version of the near horizon limit of N_c near extremal $D4$-branes wrapped over a circle with anti-periodic boundary conditions takes the form of a static black hole embedded inside AdS_6. The only surviving fermionic degrees of freedom are excited Kaluza-Klein modes because the anti-periodic boundary conditions project massless fermions out of the spectrum. At high energies the system is dual to a thermal gauge theory at the same temperature as the black hole temperature and at low energies the KK modes can not be excited and the dual theory is effectively 4-dimensional pure YM.

The 6-dimensional background metric, 6-form field strength and constant dilaton are given by

$$
\begin{aligned}
ds_6^2 &= \left(\frac{u}{R_{AdS}}\right)^2(-dt^2+\delta_{ij}dx^i dx^j + f(u)dx_4^2) + \left(\frac{R_{AdS}}{u}\right)^2\frac{du^2}{f(u)} \\
F_{(6)} &= Q_c\left(\frac{u}{R_{AdS}}\right)^4 dx_0\wedge dx_1\wedge dx_2\wedge dx_3\wedge du\wedge dx_4 \\
e^{\phi} &= \frac{2\sqrt{2}}{\sqrt{3}Q_c} \qquad R_{AdS}^2 = \frac{15}{2} \qquad f(u) = 1-\left(\frac{u_\Lambda}{u}\right)^5
\end{aligned}
\tag{8.4}
$$

The space spanned by u and x_4 has a topology of a cigar with the minimum value u_Λ at its tip. To avoid a conical singularity at the origin, x_4 needs to be periodic with periodicity

$$x_4 \sim x_4 + \frac{4\pi R_{AdS}^2}{5u_\Lambda} = x_4 + 2\pi R \tag{8.5}$$

The typical mass scale below which the theory is effectively 4-dimensional is

$$M_\Lambda = \frac{2\pi}{\delta x_4} = \frac{5}{2}\frac{u_\Lambda}{R_{AdS}^2} \tag{8.6}$$

Since the gauge theory is not supersymmetric there seems to be two ways to add flavor to the AdS_6 black hole background - by adding D4- or D5-probe branes. But it seems natural to include probe $D4$-branes and antibranes extended along the Minkowski directions and stretching to infinity in the radial direction since then the low energy limit of the gauge theory will contain massless fundamental quarks, while adding D5-branes, which need to wrap the S^1, due to the antiperiodic boundary conditions on S^1 will generate mass to the quarks of the 4-dimensional gauge theory. When all the quarks are massless there is an ability to reproduce a spontaneous chiral symmetry breaking in terms of the string dual theory. The brane configuration looks the following way

	t	x_1	x_2	x_3	x_4	x_5
D4	$\diamond$	$\diamond$	$\diamond$	$\diamond$	$\diamond$	
D4-$\overline{\text{D4}}$	$\diamond$	$\diamond$	$\diamond$	$\diamond$		$\diamond$

In the limit of large N_c and very small g_s (with fixed g_sN_c), $N_f \ll N_c$ and $L \gg l_s$, the coupling of the strings stretching between two D4-probe branes, two D4-probe antibranes or between a D4-probe brane and an antibrane goes to zero and they become non-dynamical sources. Hence, the degrees of freedom in the low energy limit and in the above limiting case are described by the strings stretching between color branes or between a color brane and a probe brane/antibrane. The gauge symmetry of the flavor branes $U(N_f) \times U(N_f)$ becomes a global symmetry of QCD and represents a chiral symmetry of the quarks. The fermions that appear from the color - D4-probe branes intersection transform as $(\bar{N}_f, 1)$ of the global symmetry and that from the color - $\overline{D4}$-probe branes intersection transform as $(1, \bar{N}_f)$. Both fermions transform in the fundamental N_c representation of the color group.

The picture is similar to the Sakai-Sugimoto model [41, 42] but the background we consider is non-critical. For any non-critical model, the curvature is of order one in units of α'. But taking large N_c limit guarantees small string coupling and one expects that stringy corrections will not affect calculations on the non-critical gravity side. The results in [44] are at least of the same order of magnitude as those given by experiments or lattice calculations, showing therefore that this assumption is not meaningless.

We consider the action

$$S_{D4} = T_4 \int d^5 x e^{-\phi} \sqrt{-det\hat{g}} - \tilde{a} T_4 \int \mathcal{P}(C_{(5)}) \qquad (8.7)$$

where $\hat{g}$ is the induced metric over D4-brane worldvolume and $\mathcal{P}(C_{(5)})$ is the pull-back of the RR 5-form potential over the D4-brane worldvolume. Taking $\tilde{a}$ is a constant which fixes the relative strength of the DBI and CS terms in (8.7). $\tilde{a}$ should be taken equal to zero if WZ coupling is not present at all. $\tilde{a}$ equal to one would be the direct naive generalization from the 10-dimensional theory, but it is not well understood what should be written in this two-derivative approximation to the non-critical setup. We leave $\tilde{a}$ general here but we will see that we should set CS term to zero at finite temperature.

The induced metric on the D4-branes is

$$ds_6^2 = \left(\frac{u}{R_{AdS}}\right)^2 (-dt^2 + \delta_{ij} dx^i dx^j) + \left(\frac{u}{R_{AdS}}\right)^2 \left(f(u) + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}\right) dx_4^2 \qquad (8.8)$$

Substituting the determinant of the metric and the pullback of the RR 5-form potential

$C_{(5)}$ into the action (8.7),we get

$$S_{D4} = \hat{T}_4 e^{-\phi} \int dx_4 \left(\frac{u}{R_{AdS}}\right)^5 \left[\sqrt{f(u) + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}} - a\right] \tag{8.9}$$

where $\hat{T}_4$ includes the outcome integration over all coordinates apart from dx_4 and $a \equiv \frac{2}{\sqrt{5}}\tilde{a}$.

The action does not depends explicitly on x_4 therefore the Hamiltonian will be conserved:

$$\left(\frac{u}{R_{AdS}}\right)^5 \left(\frac{f(u)}{\sqrt{f(u) + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}}} - a\right) = \left(\frac{u_0}{R_{AdS}}\right)^5 (\sqrt{f(u_0)} - a) \tag{8.10}$$

where u_0 is a point of a vanishing profile $u'(u)\big|_{u_0} = 0$.

Defining $y \equiv \frac{u}{u_0}$, $y_\Lambda = \frac{u_\Lambda}{u_0}$, $f(y) \equiv 1 - \left(\frac{y_\Lambda}{y}\right)^5$, the profile reads

$$u' = \left(\frac{u}{R_{AdS}}\right)^2 f(y) \sqrt{\frac{f(y)}{(y^{-5}\sqrt{f(1)} + a(1 - y^{-5}))^2} - 1} \tag{8.11}$$

At $u \to \infty$ we want N_f D4-branes to be localized at $x_4 = 0$ and N_f $\overline{D4}$-branes at $x_4 = L$. These branes can't go to the interior of the space because they don't have where to end inside the "cigar". Therefore they should smoothly connect at some point $u = u_0$ ($u_0 \leq u_\Lambda$) and therefore at zero temperature chiral symmetry is broken.

We can express L as a function of u_0, u_Λ and R_{AdS}:

$$\begin{aligned} L &= \int_0^L dx_4 = 2\int_{u_0}^{\infty} \frac{du}{u'} \\ &= 2u_0 \int_1^{\infty} dy \left(\frac{R_{AdS}}{u}\right)^2 \frac{1}{f(y)} \frac{1}{\sqrt{\frac{f(y)}{(y^{-5}\sqrt{f(1)} + a(1-y^{-5}))^2} - 1}} \end{aligned} \tag{8.12}$$

Setting $z \equiv y^{-5}$, we get

$$L = \frac{2R_{AdS}^2}{5u_0} \int_0^1 dz \frac{1}{z^{\frac{4}{5}}(1 - y_\Lambda^5 z)} \frac{z(1 - y_\Lambda^5) + a(1 - z)}{\sqrt{1 - y_\Lambda^5 z - (z(1 - y_\Lambda^5) + a(1 - z))^2}} \tag{8.13}$$

From here we see that small values of L correspond to large values of u_0 and to $y_\Lambda << 1$.

In this limit $L \propto \frac{R_{AdS}^2}{u_0}$. The general dependence of L on u_0 is more complicated.

3.4 Near extremal AdS_6 model with flavor $D4$-$\overline{D4}$ branes at finite temperature

3.4.1 Bulk thermodynamics

We consider flavor branes as probes with $N_f \ll N_c$ and therefore we can analyze the thermodynamics of our model at finite temperature by considering only background geometry and then add probe D4-branes to the dominant bulk background at each temperature. In the gravity approximation (large N_c limit) we should look at Euclidean backgrounds, which are asymptotically (8.4), but with Euclidean and periodic time with a periodicity $t = 1/T = \beta$ and with anti-periodic boundary conditions for the fermions along the time direction in addition to the x_4 direction. This background is just a Euclidian continuation of the background (8.4) with the x_4 compact direction with a periodicity $2\pi R$ (with R related to u_Λ by (8.5)) that shrinks to zero at $u = u_0$ and with the time direction that remains always finite with an arbitrary periodicity equal to β (see figure 3.1(a)).

But now we can consider another solution with the same asymptotics, which is given by exchanging the behavior of the t and x_4 circles (i.e. by moving $f(u)$ in the metric (8.4) from the dx_4^2 term to the dt^2 term). Then now the time direction shrinks to zero size at $u = u_T$ (u_T is related to β), while the x_4 circle never shrinks (see figure 3.1(b)).

3.4.2 The bulk free energies of the low and high temperature phases

In order to decide which background dominates at a given temperature we need to compute their free energies. We look at the difference between the free energies, which is proportional to the difference between the actions of the backgrounds times the temperature (in the gravitational approximation), because it turns out to be finite despite that classical actions might diverge. In our calculations we use the notations and the results for the action computed in [43].

The class of Euclidean metrics that we are looking on can be parameterized as

$$l_s^{-2} ds^2 = d\tau^2 + e^{2\lambda(\tau)} dx_{||}^2 + e^{2\tilde{\lambda}(\tau)} dx_c^2 \tag{8.14}$$

where x_c is either x_4 or t (the one whose circle shrinks to zero size at the minimal value of u at a certain temperature), $x_{||}$ are the other four coordinates of $R^{4,1}$ (one of which is also compactified), τ is the radial direction and

$$e^{2\lambda} = \left(\frac{u}{R_{AdS}}\right)^2 \qquad e^{2\tilde{\lambda}} = \left(\frac{u}{R_{AdS}}\right)^2 \left(1 - \left(\frac{u_\Lambda}{u}\right)^5\right) \tag{8.15}$$

The functions λ and $\tilde{\lambda}$ depend only on the radial coordinate. The color D4-brane wrap the circle x_c, while the flavor D4-brane are points on the circle. The background also includes a constant dilaton ϕ_0 and a 6-form RR field strength. We define a deformed dilaton φ and a new radial coordinate ρ as

$$\begin{aligned} \varphi &= 2\phi_0 - 4\lambda - \tilde{\lambda} \\ d\tau &= -e^{-\varphi} d\rho \end{aligned} \tag{8.16}$$

Since the background depends only on a radial direction, the sugra action reduces to the following (0+1)-dimensional action:

$$\begin{aligned} S &= V \int d\rho \big(- 4(\lambda')^2 - (\tilde{\lambda}')^2 + (\varphi')^2 + 4e^{-2\varphi} - Q_c^2 e^{4\lambda + \tilde{\lambda} - \varphi} \big) \\ &= -V \int_{u_\Lambda}^{\infty} du \Big[\big(- 4\dot{\lambda}^2 - \dot{\tilde{\lambda}}^2 + \dot{\varphi}^2 \big) \frac{du}{d\rho} + \Big(4e^{-4\phi_0} - Q_c^2 e^{-2\phi_0} \Big) e^{8\lambda + 2\tilde{\lambda}} \frac{d\rho}{du} \Big] \end{aligned} \tag{8.17}$$

where V is the volume of all other directions except ρ in string units and Q_c is a constant that corresponds to the contribution of the RR flux. After rewriting the action in terms of integrals over u the minus sign arises because $d\rho/du$ is negative (dots denote derivatives with respect to u). Here we wrote the solution of the low temperature phase, for the high temperature phase one should replace u_Λ with u_T.

The equations of motion associated with the action (8.17) are

$$\lambda'' - \frac{1}{2} Q_c^2 e^{2(4\lambda + \tilde{\lambda} - \phi_0)} = 0 \tag{8.18}$$
$$\tilde{\lambda}'' - \frac{1}{2} Q_c^2 e^{2(4\lambda + \tilde{\lambda} - \phi_0)} = 0$$

The most general solution of this system is ([43]):

$$\lambda = -\frac{1}{5} \ln(\sinh(-5b\rho)) + 4b\rho \tag{8.19}$$
$$\tilde{\lambda} = -\frac{1}{5} \ln(\sinh(-5b\rho)) - b\rho$$

where $b = -\frac{1}{\sqrt{10}} Q_c e^{-\phi_0}$.

Substituting (8.15) into (8.17) we get

$$\begin{aligned} S = V \int_{u_\Lambda}^{\infty} du \Bigg[\left(\frac{20}{u^2} \frac{1}{1 - \left(\frac{u_\Lambda}{u}\right)^5} \right) \frac{du}{d\rho} \\ + \left\{ \left(4e^{-4\phi_0} - Q_c^2 e^{-2\phi_0}\right) \left(\frac{u}{R_{AdS}} \right)^{10} \left(1 - \left(\frac{u_\Lambda}{u} \right)^5 \right) \right\} \frac{d\rho}{du} \Bigg] \end{aligned} \tag{8.20}$$

Using the solutions of the equations of motion associated with the above action (8.19) and (8.15)

$$\tilde{\lambda} - \lambda = 5b\rho \tag{8.21}$$
$$\frac{e^{2\tilde{\lambda}}}{e^{2\lambda}} = 1 - \left(\frac{u_\Lambda}{u} \right)^5 = e^{10b\rho}$$

we find that

$$\frac{d\rho}{du} = \frac{1}{2bu} \left(\frac{u_\Lambda}{u} \right)^5 \frac{1}{1 - \left(\frac{u_\Lambda}{u}\right)^5} \tag{8.22}$$

Substituting the expression into the action we find that the divergence at large u is independent of u_Λ, so it makes sense to subtract the expressions with u_Λ and with u_T to obtain a finite answer. The result for the difference between the action densities in the low temperature phase and in the high temperature phase is given by (defining $\hat{b} = b/u_\Lambda^5$

which is constant independent of u_Λ)

$$\frac{\Delta S}{V_3} \equiv \frac{S_{low} - S_{high}}{V_3} = \frac{2\pi R\beta}{l_s^2}\left(2\hat{b} + \left(4e^{-2\phi_0} - Q_c^2\right)e^{-2\phi_0}\frac{1}{10\hat{b}R_{AdS}^{10}}\right)(u_T^5 - u_\Lambda^5) \tag{8.23}$$

Using (8.5) and (8.31) we find

$$u_\Lambda = \frac{2R_{AdS}^2}{5R} \quad \text{and} \quad u_T = \frac{4\pi R_{AdS}^2}{5\beta} \tag{8.24}$$

Therefore the action is proportional to

$$\Delta S \propto N_c^2\left(\frac{1}{(\beta/2\pi)^5} - \frac{1}{R^5}\right) \tag{8.25}$$

Both backgrounds have equal free energy when both circles are equal, i.e. $\beta = 2\pi R$. When temperature is less than $T_d = 1/2\pi R$ the background in which x_4 circle shrinks to zero size dominates and when temperature is greater than $T_d = 1/2\pi R$ the background with t circle shrinking to zero dominates. There is a phase transition of first order here since two different configurations are possible at the transition point. If we compute the quark-antiquark potential (using the methods of [50–52]) , which is proportional to $\sqrt{g_{tt}g_{xx}}$, in the two backgrounds, we find that in the low-temperature background it is finite at u_0 and linear, corresponding to a confined phase, and in the high-temperature it decays, corresponding to a deconfined phase.

3.4.3 Low temperature phase

As described above the background corresponding to the low temperature phase is the one with the x_4 circle shrinking to zero at $u = u_0$. The only difference from the zero temperature case is that time direction is Euclidean and compactified with a circumference $\beta = 1/T$. Hence, at low temperatures the dual gauge theory is in the confining phase. When we add flavor branes and anti-brains to the background they have no other possibility than to connect because x_4 shrinks to zero size. Therefore chiral symmetry is broken at least until the temperatures corresponding to deconfinement.

The metric is

$$\begin{aligned}
ds_6^2 &= \left(\frac{u}{R_{AdS}}\right)^2 (dt^2 + \delta_{ij} dx^i dx^j + f(u) dx_4^2) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{f(u)} \\
f(u) &= 1 - \left(\frac{u_\Lambda}{u}\right)^5 \\
x_4 &\sim x_4 + 2\pi R = x_4 + \frac{4\pi R_{AdS}^2}{5u_\Lambda} \quad \text{and} \quad t \sim t + \beta \ (\beta \text{ arbitrary})
\end{aligned} \tag{8.26}$$

Setting $\tilde{a}$ to zero, we find from (8.11)

$$u' = \left(\frac{u}{R_{AdS}}\right)^2 f(y) \sqrt{y^{10} \frac{f(y)}{f(1)} - 1} \tag{8.27}$$

Substituting (8.27) into (8.7) and using the definitions $y \equiv \frac{u}{u_0}$, $y_\Lambda = \frac{u_\Lambda}{u_0}$, $f(y) \equiv 1 - \left(\frac{y_\Lambda}{y}\right)^5$ and $z \equiv y^{-5}$, we get the following DBI action:

$$\begin{aligned}
S_{DBI} &= \hat{T}_4 e^{-\phi} \int dx_4 \left(\frac{u}{R_{AdS}}\right)^5 \sqrt{f(u) + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}} \\
&= \frac{2\hat{T}_4 e^{-\phi} u_0^4}{5R_{AdS}^3} \int_0^1 dz \frac{1}{z^{\frac{9}{5}}} \frac{1}{\sqrt{1 - y_\Lambda^5 z - z^2(1 - y_\Lambda^5)}}
\end{aligned} \tag{8.28}$$

The relation between L and u_0 is the same as at zero temperatures. For small L the dependence of the action on L is:

$$S_{DBI} \propto \frac{\hat{T}_4 e^{-\phi}}{R_{AdS}^3 L^4} \tag{8.29}$$

3.4.4 Intermediate and high temperature phases

In the high temperature phase the background metric takes the form

$$\begin{aligned}
ds_6^2 &= \left(\frac{u}{R_{AdS}}\right)^2 (f(u) dt^2 + \delta_{ij} dx^i dx^j + dx_4^2) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{du^2}{f(u)} \\
f(u) &= 1 - \left(\frac{u_T}{u}\right)^5
\end{aligned} \tag{8.30}$$

Now the time circle shrinks to zero at the minimal value of $u = u_T$ and to avoid a singularity there the time direction should be identified with the periodicity

$$t \sim t + \beta = t + \frac{4\pi R_{AdS}^2}{5u_T} \tag{8.31}$$

On the other hand the periodicity of x_4 is now arbitrary:

$$x_4 \sim x_4 + 2\pi R \tag{8.32}$$

D4-branes span the same coordinates as previously and are described by some profile $u(x_4)$. The induced metric and the DBI action now takes the form

$$ds_6^2 = \left(\frac{u}{R_{AdS}}\right)^2 (f(u)dt^2 + \delta_{ij}dx^i dx^j) + \left(\frac{u}{R_{AdS}}\right)^2 \left(1 + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}\right) dx_4^2 \tag{8.33}$$

We get the following action:

$$\begin{aligned} S_{D4} &= T_4 \int d^5x e^{-\phi}\sqrt{-det\hat{g}} - \frac{\sqrt{5}a}{2} T_4 \int \mathcal{P}(C_{(5)}) \\ &= \hat{T}_4 e^{-\phi} \int dx_4 \left(\frac{u}{R_{AdS}}\right)^5 \left[\sqrt{f(u)}\sqrt{1 + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}} - a\right] \\ &= 2\hat{T}_4 e^{-\phi}\left[\int_{u_0}^{\infty} \frac{du}{u'}\left(\frac{u}{R_{AdS}}\right)^5 \sqrt{f(u)}\sqrt{1 + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}} \right. \\ &\quad \left. - a\int_{u_0}^{\infty} \frac{du}{u'}\left(\frac{u}{R_{AdS}}\right)^5\right] \end{aligned} \tag{8.34}$$

Conservation of the Hamiltonian of (8.34) implies that

$$\left(\frac{u}{R_{AdS}}\right)^5 \left(\frac{\sqrt{f(u)}}{\sqrt{1 + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}}} - a\right) = const \tag{8.35}$$

There is a solution for a vanishing profile at some point $u_0 \geq u_T$, where the velocity $u'(u)\big|_{u_0} = 0$ (see figures 3.2(a) and 3.2(b)). This is a solution for branes and anti-branes connected at $u = u_0$, where the velocity u' should be zero. At low temperatures this was the only possible configuration, but since in the high temperature phase the x_4 circle never shrinks to zero size we can consider a configuration of non-intersecting branes and

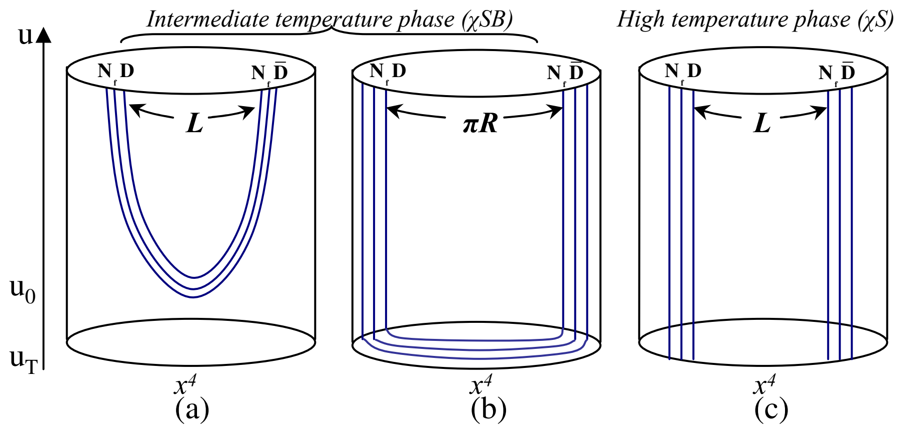

Figure 3.2: The three possible configurations of the flavor D4-branes and antibranes at intermediate and high temperatures: (a) a configuration of connected branes at the minimum $u = u_0$ with an asymptotic separation L at $u \to \infty$ (intermediate temperature phase), (b) a configuration of connected branes in the case $u_0 = u_T$ (intermediate temperature phase), (c) parallel branes configuration (high temperature phase).

antibranes that end on the horizon of the black hole (see figure 3.2(c)). The branes and antibranes stay disconnected in the $u - x_4$ submanifold with constant values $x_4(u) = 0, L$, e.g chiral symmetry is restored. Since branes are now parallel to each other the velocity of their profile u' should be infinite at u_0 (which now equals to u_T). Then there should exist a solution of (8.35) also for the case $u' \to \infty$. But there is no such a solution here because the term $u^5 a$ cannot equal to a constant. A solution exists only if we set the CS term to zero from the beginning (i.e. $\tilde{a} = 0$). [1]

Instead of the parallel brane configuration we could consider taking a configuration of connected branes and allowing $u_0 < u_T$ but restricting ourselves only to the region $u > u_T$ and keep the CS term. If we had a configuration of branes connecting at some point $\bar{u}_0 < u_T$ with a certain length L at infinity and another configuration of branes connecting at $u_0 > u_T$ with the same length L at infinity then we could subtract the two actions of this configuration in order to get a finite difference. But from the figure 3.3 we see that $y_T = u/u_0$ can never be bigger than 1 and therefore we cannot have the same length L for u_0 and $\bar{u}_0$. So we should just omit the CS term in our calculations.

[1] If leave $\tilde{a} \neq 0$ we get an infinity in the difference between the free energies. *A priory* we do not expect that a difference of two actions should be finite, but here it shows an additional anomalous behavior of the CS term.

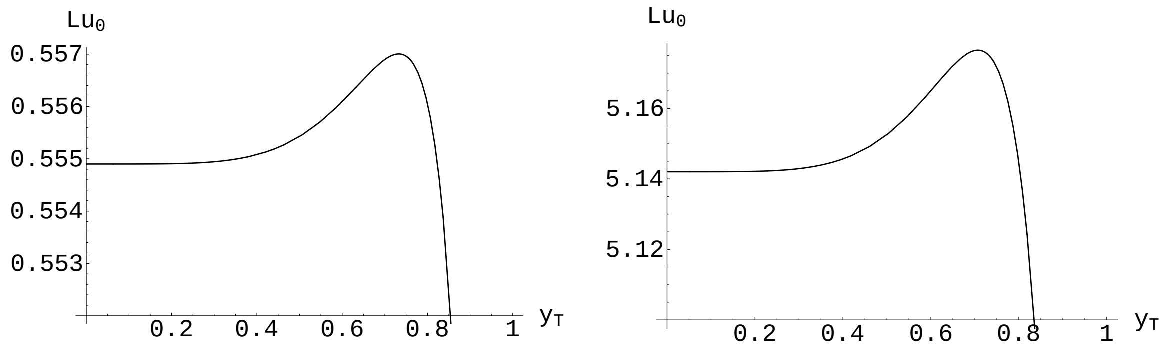

Figure 3.3: Lu_0 as a function of y_T. The left graph is for the case $\tilde{a} = 0$ and the right one is for the case $\tilde{a} = 1$.

Defining $y \equiv \frac{u}{u_0}$, $y_T = \frac{u_T}{u_0}$, $f(y) \equiv 1 - \left(\frac{y_T}{y}\right)^5$, the profile velocity now reads

$$u' = \left(\frac{u}{R_{AdS}}\right)^2 \sqrt{f(y)} \sqrt{y^{10} \frac{f(y)}{f(1)} - 1} \tag{8.36}$$

Then x_4 as a function of u becomes

$$x_4(u) = \int_{u_0}^{u} d\hat{u} \frac{1}{\hat{u}'} = \int_{u_0}^{u} d\hat{u} \frac{1}{\left(\frac{\hat{u}}{R_{AdS}}\right)^2 \sqrt{1 - (\frac{u_T}{\hat{u}})^5} \sqrt{\left(\frac{\hat{u}}{u_0}\right)^5 \frac{1-(\frac{u_T}{\hat{u}})^5}{1-u_T^5} - 1}} \tag{8.37}$$

It is shown in figure 3.4. At the beginning when $u \approx u_0$ the profile of the branes growth very rapidly and when $u \to \infty$ the profile is almost straight. For $u_0 \approx u_T$ the profile is drawn at figure 3.2(b).

By substituting the outcome of the equation of motion into the action, we get

$$S_{DBI}^{high} = \frac{2\hat{T}_4 e^{-\phi} u_0^4}{R_{AdS}^3} \int_1^{\infty} dy \frac{y^3}{\sqrt{1 - \frac{f(1)}{f(y) y^{10}}}} \tag{8.38}$$

The velocity of the profile of a parallel branes configuration is always $u' \to \infty$ and

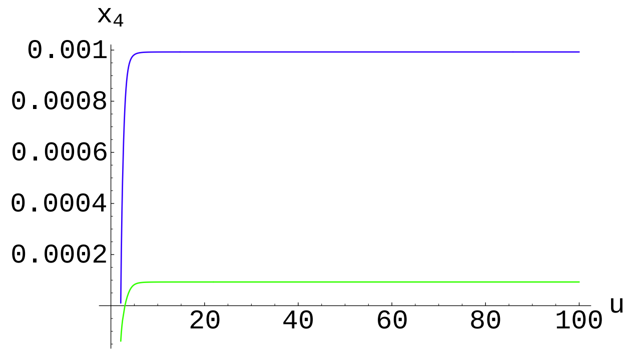

Figure 3.4: Profile x_4 as a function of u. The lower curve (green) is for $u_T = 2$, $u_0 = 3$. The upper curve (blue) is for the case $u_0 \approx u_T = 2$ ($R_{AdS} = 1$).

therefore the action reads

$$
\begin{aligned}
S^{high}_{DBI u' \to \infty} &= 2\hat{T}_4 e^{-\phi} \int_{u_T}^{\infty} du \left(\frac{u}{R_{AdS}} \right)^5 \left(\frac{R_{AdS}}{u} \right)^2 \\
&= \frac{2\hat{T}_4 e^{-\phi} u_0^4}{R_{AdS}^3} \left[\int_1^{\infty} dy y^3 + \int_{y_T}^{1} dy y^3 \right]
\end{aligned} \tag{8.39}
$$

To find whether a configuration with χSB or with a restored χS is preferred we can compute the difference between the actions of the two configuration that is proportional to free energy. Configuration that has a lower free energy is preferred.

$$
\begin{aligned}
\Delta S &\equiv \frac{R_{AdS}^3}{2\hat{T}_4 e^{-\phi} u_0^4} \left(S^{high}_{DBI} - S^{high}_{DBI u' \to \infty} \right) \\
&= \int_1^{\infty} dy y^3 \left[\frac{1}{\sqrt{1 - \frac{f(1)}{f(y) y^{10}}}} - 1 \right] - \int_{y_T}^{1} dy y^3 \\
&= \frac{1}{5} \int_0^1 dz \frac{1}{z^{\frac{9}{5}}} \left[\sqrt{\frac{1 - y_T^5 z}{1 - y_T^5 z - z^2 (1 - y_T^5)}} - 1 \right] - \frac{1}{4}(1 - y_T^4)
\end{aligned} \tag{8.40}
$$

where was introduced $z = y^{-5}$ change of variables. ΔS as a function of y_T is drawn in figure 3.5. When $y_T > 0.8$ ΔS is positive, i.e. $S^{high}_{DBI u' \to \infty}$ has a lower free energy and is preferred. In this phase D4-brane are disconnected and chiral symmetry is restored, while when $0 < y_T < 0.8$ D4-branes are smoothly connected and chiral symmetry is

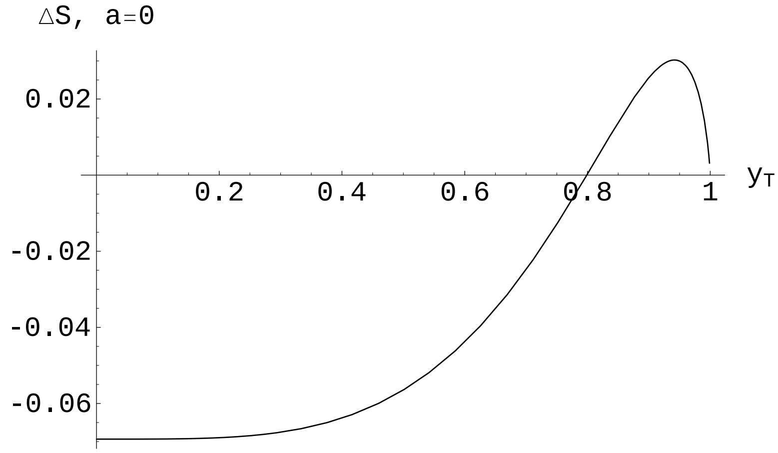

Figure 3.5: ΔS as a function of y_T, in units of $2\hat{T}_4 e^{-\phi} u_0^4/R_{AdS}^3$ in the case $\tilde{a} = 0$.

broken.

We would like to express the critical point in terms of physical quantities. For a certain value of y_T we can compute an integral that relates the minimal point of the connected branes configuration u_0 to the asymptotic distance between branes and antibranes L.

$$L = 2\int_{u_0}^{\infty} \frac{du}{u'} = \frac{2R_{AdS}^2}{u_0}\int_1^{\infty} dy \frac{1}{y^2\sqrt{f(y)}} \frac{1}{\sqrt{y^{10}\frac{f(y)}{f(1)} - 1}}$$
$$= \frac{2R_{AdS}^2}{5u_0}\sqrt{1-y_T^5}\int_0^1 dz \frac{1}{\sqrt{1-y_T^5 z}} \frac{z^{\frac{1}{5}}}{\sqrt{1-y_T^5 z - z^2(1-y_T^5)}} \tag{8.41}$$

For small values of L $u_0 \propto R_{AdS}^2/L$. At the transition temperature $y_T^c = 0.8$ the integral (8.41) gives $L = 0.53(R_{AdS}^2/u_0)$. From the equation (8.31) we find

$$T_{\chi SB} = \frac{5y_T u_0}{4\pi R_{AdS}^2} = \frac{5y_T 0.53}{4\pi L} = \frac{0.169}{L} \tag{8.42}$$

while the deconfinement phase transition happens at the temperature

$$T_d = \frac{1}{2\pi R} = \frac{0.159}{R} \tag{8.43}$$

Both temperatures are equal when $L = 1.06R$. For $L/R > 1.06$ and $T \cdot R > T_d \cdot R$

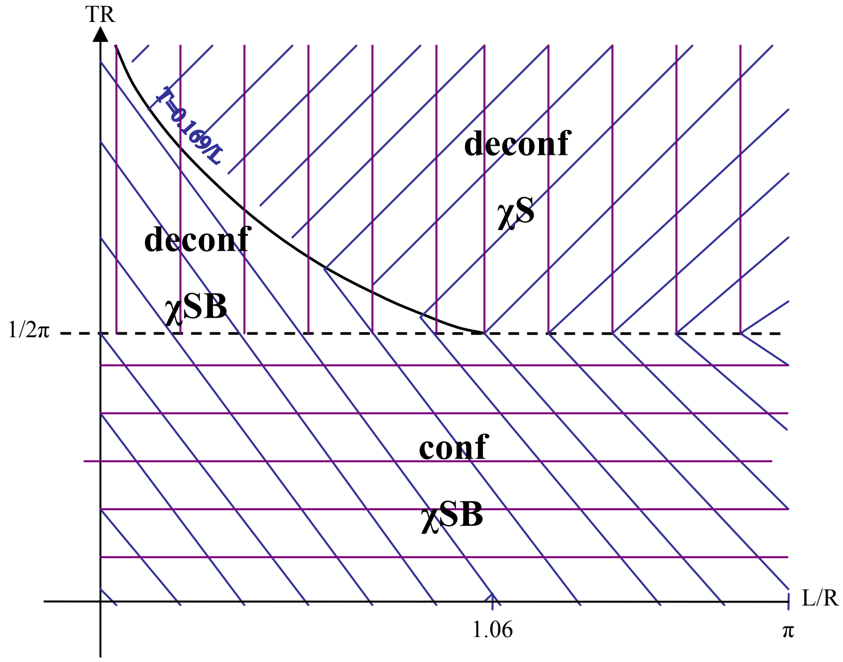

Figure 3.6: The phase diagram of the AdS_6 model with flavor D4-branes. The phase structure depends only on the two dimensionless parameters TR and L/R. For $L/R <$ 1.06 the deconfinement and chiral symmetry restoration transitions happens at different temperatures, while for $L/R > 1.06$ they occur together .

the system is deconfined and chiral symmetry is restored, while for $L/R < 1.06$ and temperatures bigger than the temperature of deconfinement the system is deconfined but chiral symmetry restoration happens separately: at $T \cdot R < T_{\chi SB} \cdot R$ chiral symmetry is still broken and at $T \cdot R > T_{\chi SB} \cdot R$ chiral symmetry is restored. The full phase diagram of the theory is drawn in figure 3.6.

3.4.5 General model

From the similarity of the result of the previous section to the SS model we can derive a general model, but it is not necessary that all three phases will be present. We indeed find a different behavior of some metrics.

We consider a n-dimensional Wick rotated black hole background and insert into it (n-2)-probe branes, that extend along all directions except x_4. We take the following general form of the metric at low temperatures:

$$ds_n^2 = H_2 dt^2 + \frac{1}{H_1} du^2 + H_1 dx_4^2 + ds_k^2 \tag{8.44}$$

and then the metric at high temperatures becomes:

$$ds_n^2 = H_1 dt^2 + \frac{1}{H_1} du^2 + H_2 dx_4^2 + ds_k^2 \tag{8.45}$$

where H_1 is a singular function of u with a horizon at $u = u_H$, H_2 is a non-singular function of u, x_4 is compact with a periodic that depends on u_H and ds_k^2 is any k-dimensional metric of the rest of coordinates, which components can depend on u. The condition of the singular H_1 is necessary to have a horizon on which (n-2)-branes can end. At low temperature the $u - x_4$ submanifold is cigar-shaped, while at high temperature the $u - t$ submanifold is cigar-shaped and the x_4 circle does not shrink to zero.

From (8.45) we find the induced metric on the (n-2)-probe brane

$$ds_n^2 = H_1 dt^2 + (H_2 + \frac{1}{H_1} u'^2) dx_4^2 + ds_k^2 \tag{8.46}$$

with $u' = du/dx_4$.

The DBI action is given by (without the CS term)

$$S_{DBI} = T_n \int d^{n-1}x e^{-\phi} \sqrt{g} \sqrt{H_1 (H_2 + \frac{1}{H_1} u'^2)} = \hat{T}_n \int dx_4 e^{-\phi} \sqrt{g} \sqrt{H_1 (H_2 + \frac{1}{H_1} u'^2)} \tag{8.47}$$

where $\hat{T}_n$ includes the outcome integration over all coordinates apart from dx_4, e^{ϕ} is a dilaton and g is the determinant of the ds_k^2 metric. Then from the conservation of the Hamiltonian we find that the equation of motion is

$$\frac{e^{-\phi} \sqrt{g} \sqrt{H_1} H_2}{\sqrt{H_2 + \frac{1}{H_1} u'^2}} = const \tag{8.48}$$

If we suppose that there is a solution with $u'(u = u_0) = 0$, then the constant is equal to $e^{-\phi_0} \sqrt{g^0 H_1^0 H_2^0}$ and

$$u' = \sqrt{H_1 H_2} \sqrt{\frac{e^{-2\phi} g H_1 H_2}{e^{-2\phi_0} g^0 H_1^0 H_2^0} - 1} \tag{8.49}$$

The case $u'(u=0) \to \infty$ is also a solution of the equation of motion and gives $const = 0$.

Inserting the expression for u' into (8.47) we find

$$S_{DBI} = 2\hat{T}_n \int_{u_0}^{\infty} due^{-\phi}\sqrt{g}\frac{1}{\sqrt{1 - \frac{e^{-2\phi_0}g^0 H_1^0 H_2^0}{e^{-2\phi}gH_1H_2}}} \tag{8.50}$$

At $u' \to \infty$ the actions reads

$$S_{DBI}^{u'\to\infty} = 2\hat{T}_n \int_{u_0}^{\infty} due^{-\phi}\sqrt{g} \tag{8.51}$$

The action of the parallel brane configuration actually depends only on the x^k coordinates and the dilaton. The difference between the actions is

$$\Delta S = 2\hat{T}_n \int_{u_0}^{\infty} due^{-\phi}\sqrt{g}\left[\frac{1}{\sqrt{1 - \frac{e^{-2\phi_0}g^0 H_1^0 H_2^0}{e^{-2\phi}gH_1H_2}}} - 1\right] - \int_{u_T}^{u_0} due^{-\phi}\sqrt{g} \tag{8.52}$$

To evaluate it we need to insert the explicit expressions of the functions H_1, H_2, g and the dilaton.

The existence of the solution with $u'(u=u_0) = 0$ and $u_0 \neq 0$ at low temperatures guarantees confinement, but to see whether we get chiral symmetry restoration or not we need an explicit form of the functions ϕ, g, H_1 and H_2. Let us look at the same family of metrics to which SS and AdS_6 metrics belong. That is:

$$\begin{aligned}
H_1 &= \left(\frac{u}{R}\right)^m \left(1 - \left(\frac{u_T}{u}\right)^n\right) \\
H_2 &= \left(\frac{u}{R}\right)^m \\
e^{-\phi} &= const_1 \cdot u^l \\
g &= const_2 \cdot u^j
\end{aligned} \tag{8.53}$$

$const_1$ and $const_1$ does not effect δS and therefore we them arbitrary. Substituting the functions (8.53) into the action's difference (8.52) and defining $y = u/u_0$, we get:

$$\Delta S \propto \int_1^{\infty} dyy^{l+\frac{j}{2}}\left(\frac{1}{\sqrt{1 - y^{-(2l+j+2m)}\frac{1-y_T^n}{1-(y_T/y)^n}}} - 1\right) - \int_{y_T}^{1} dyy^{l+\frac{j}{2}} \tag{8.54}$$

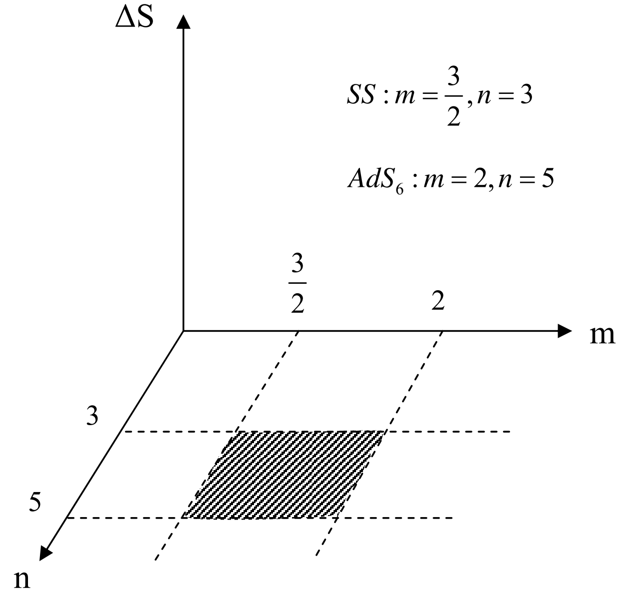

Figure 3.7: Models with values of n between 3 and 5, m between 3/2 and 2, and arbitrary j and l should have intermediate and high temperature phases.

We did the numerical computations for different values of j, l, m, n (since we cannot solve the first integral analytically). We checked that for n between 3 and 5, for m between 3/2 and 2, and arbitrary j and l ΔS is negative and then positive as in the SS and AdS_6 models, as expected (see figure 3.7).

Also we can see whether we get a different phase structure for general critical and non-critical versions of near extremal Dp-branes. The metric for the critical near extremal Dp-branes is [53]:

$$\begin{aligned} ds^2 &= \frac{u^{(7-p)/2}}{R_{Dp}^2}\left(-\left(1-\frac{u_T^{7-p}}{u^{7-p}}\right)dt^2 + dx_4^2\right) + \frac{R_{Dp}^2}{u^{(7-p)/2}}\frac{1}{1-\frac{u_T^{7-p}}{u^{7-p}}}du^2 \\ &+ \frac{u^{(7-p)/2}}{R_{Dp}^2}\delta_{ij}dx^i dx^j + R_{Dp}^2 u^{(p-3)/2} d\Omega_{8-p}^2 \qquad i,j = 1,...,p-1 \\ e^{-\phi} &= \frac{1}{(2\pi)^{2-p} g_{YM}^2 R_{Dp}^{3-p}} u^{(7-p)(3-p)/4} \qquad R_{Dp}^2 = g_{YM}\sqrt{N} \end{aligned} \tag{8.55}$$

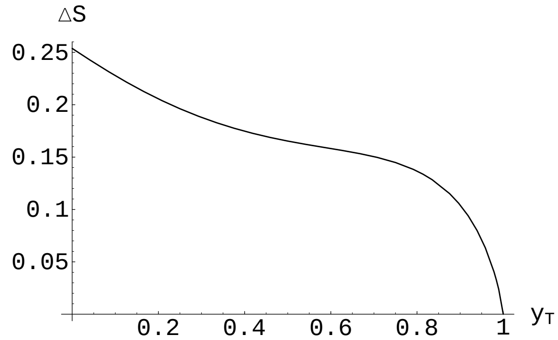

Figure 3.8: ΔS as a function of y_T for the near extremal D6-branes in 10 dimensions.

Therefore all the powers depend only on p and we find:

$$
\begin{aligned}
m &= \frac{7-p}{2} \\
n &= 7-p \\
l &= \frac{(7-p)(3-p)}{4} \\
j &= \frac{(7-p)(p-1)+(p-3)}{2}
\end{aligned}
\tag{8.56}
$$

Since the solutions are in 10 dimensions, we insert D8-probe branes because if there are directions along which probe branes do not extend, except the x_4 direction, the massive quarks appear and we will not get chiral symmetry. Drawing the ΔS (8.54) numerically for different values of p we find that for $p \leq 5$ we get the same behavior as in the SS model (two phases - chiral symmetry breaking and restoration), but for $p = 6$ we get a positive ΔS that means that chiral symmetry restoration and deconfinement occur together (see figure 3.8). For $p > 6$ we cannot get the solution of (8.54) numerically.

Near extremal solutions of Dp-branes in non-critical dimensions are given by [43]:

$$
\begin{aligned}
ds^2 &= \left(\frac{u}{R_{AdS}}\right)^2 \left(-\left(1-\frac{u_T^{p+1}}{u^{p+1}}\right)dt^2 + dx_4^2\right) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{1}{1-\frac{u_T^{p+1}}{u^{p+1}}} du^2 \\
&+ \left(\frac{u}{R_{AdS}}\right)^2 \delta_{ij} dx^i dx^j + R_{S^q}^2 d\Omega_q^2 \qquad i,j \text{ arbitrary but smaller than } 7-q \\
e^{-\phi_0} &= \left[\frac{1}{p+2-q}\left(\frac{(p+2-q)(q-1)}{c}\right)^q \frac{2c}{Q^2}\right]^{-1/2} \qquad R_{AdS}^2 = \frac{(p+1)(p+2-q)}{c}
\end{aligned}
\tag{8.57}
$$

where

$$
\frac{c}{\alpha'} = \frac{10-d}{\alpha'} \tag{8.58}
$$

is the non-criticality central charge term.

We see that again all the powers depend only on dimension of the branes p:

$$
\begin{aligned}
m &= 2 \\
n &= p+1 \\
l &= 0 \\
j &= 6
\end{aligned}
\tag{8.59}
$$

The probe branes that we insert into the backgrounds are (d-2)-branes and antibranes (d - is the dimension of a non-critical metric). For different values of $n = p+1$ we find that ΔS (8.54) always has the same behavior as in the AdS_6 BH model, i.e. chiral symmetry can be restored at a higher temperature then the temperature of deconfinement, but the value of the ratio L/R will be different for different Dp-branes.

From the above analysis we see that the phase structure of a model depends on the basic structure and dimensionality of its metric and it can be different for different models.

3.5 Spectrum of mesons

The mesons of our model are described by strings ending on the probe D4-branes. Low-spin mesons are described via modes of the massless fields living on the D4-branes and high-spin mesons are associated with string configurations that fall from the D4-branes down to the wall at $u = u_\Lambda$, stretch along the wall and then go back up again. The mesonic spectrum in the low-temperature phase is unchanged as the temperature is increased because in the confining phase the theory behaves effectively as a gas of non-interacting glueballs and mesons [54,55]. However, the mesonic spectrum at intermediate temperature might be not connected to the spectrum in the low-temperature regime since the phase transition is first-order and such a jump should be expected. Now we turn our attention to the meson spectrum at the intermediate and high temperature phases.

3.5.1 Low-spin mesons at intermediate temperature

Low-spin mesons correspond on the string theory side to fluctuations of the massless fields on the probe branes. The fluctuations of the gauge fields on the branes give pseudo-vector and scalar mesons and pions, and the fluctuations of the scalar field describing the embedding of the branes give massive scalar mesons. Using the analysis of the fluctuations performed in [49] we describe the modes coming from the components of the gauge field living on the D4-branes.

The spectrum of low-spin mesons in the low-temperature phase is unmodified with respect to zero temperature since the Euclidean metric is globally unmodified.

The spectrum in the intermediate temperature phase is discrete because the probe does not intersect the horizon. Also computations in [45] for the SS model show that given that the effective tension of strings near the brane decreases with the increase of temperature, the masses of mesons decrease as the temperature is increased. This behavior is also true for our model.

We start from the background metric describing the hot gluonic plasma (8.30).The induced metric on the D4-brane worldvolume at intermediate temperature reads

$$\mathrm{d}s^2_{\text{interm}} = \left(\frac{u}{R_{AdS}}\right)^2 \left(f(u)\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \left[\left(\frac{R_{AdS}}{u}\right)^2 \frac{1}{f(u)} + \left(\frac{\mathrm{d}x^4}{\mathrm{d}u}\right)^2 \left(\frac{u}{R_{AdS}}\right)^2\right]\mathrm{d}u^2 \quad (8.60)$$

We are interested in computing the spectrum of vector mesons, by considering small fluctuations on the worldvolume gauge fields of the probe D4-brane. We expand the gauge field as [40]

$$A_\mu(x^\mu, u) = \sum_n B^{(n)}_\mu(x^\mu)\psi_{(n)}(u) \quad (8.61)$$

$$A_u(x^\mu, u) = \sum_n \varphi^{(n)}_\mu(x^\mu)\phi_{(n)}(u) \quad (8.62)$$

and therefore the field strength reads

$$\begin{aligned} F_{\mu\nu} &= \sum_n F^{(n)}_{\mu\nu}(x^\rho)\,\psi_n(u)\,, \\ F_{\mu u} &= \sum_n \partial_\mu\varphi^{(n)}\,\phi_n(u) - B^{(n)}_\mu\partial_u\psi_n(u) \\ &= \partial_\mu\varphi^{(0)}\,\phi_0 + \sum_{n\geq 1}\left(\partial_\mu\varphi^{(n)} - B^{(n)}_\mu\right)\partial_u\psi_{(n)}\,. \end{aligned} \quad (8.63)$$

where the last line is obtained by taking $\phi_{(n)} = m_n^{-1}\partial_u\psi_{(n)}(u)$. To simplify the consideration, we furthermore go to the $A_0 = 0$ gauge and consider only spatially homogeneous modes, i.e. we consider the equation of motion for fields satisfying $\partial_i A_j = 0$. Then the probe brane action is

$$\hat{S}_{\text{trunc}} = \int \mathrm{d}^4x\mathrm{d}u\; u^4\gamma^{1/2}\, f(u)^{1/2} \left[\frac{1}{u^2\gamma\, f(u)}(\partial_0\varphi^{(0)})^2\, \phi^{(0)}\phi^{(0)} \right.$$

$$\left. - \frac{1}{f(u)}\left(\frac{R_{AdS}}{u}\right)^4 \partial_0 B_i^{(m)}\partial_0 B^i_{(n)}\, \psi_{(m)}\psi_{(n)} + \frac{1}{u^2\gamma}B_i^{(m)}B^i_{(n)}\,\partial_u\psi_{(m)}\partial_u\psi_{(n)}\right],$$

$$\text{with} \qquad \gamma \equiv \frac{u^8}{u^{10}f(u) - u_0^{10}f(u_0)} \tag{8.64}$$

After a partial integration with respect to the u-coordinate, the equation of motion for the field $B_i^{(m)}$ becomes

$$\frac{u^2}{\gamma^{1/2}f(u)^{1/2}}\partial_0^2 B_i^{(n)}\psi_{(n)} - \partial_u\left(u^2\gamma^{-1/2}f(u)^{1/2}\;\partial_u\psi_{(n)}\right)B_i^{(n)} = 0 \tag{8.65}$$

This equation will reduce to the canonical form

$$\partial_0^2 B_i^{(n)} = -m_n^2 B_i^{(n)}\,, \tag{8.66}$$

if the modes $\psi_{(n)}$ satisfy the equation

$$-\gamma^{-1/2}\, f(u)^{1/2}\,\partial_u\left(u^2\gamma^{-1/2}f(u)^{1/2}\partial_u\psi_{(n)}\right) = R_{AdS}^4\, m_n^2\,\psi_{(n)}\,. \tag{8.67}$$

This equation is very similar to the equation in the zero temperature case computed in [49], the only difference is the appearance of the factor $f(u)^{1/2}$ in the term on the left-hand side. The modes should also satisfy the normalization conditions

$$\begin{aligned}
&\int_{u_0}^{\infty}\mathrm{d}u\,\gamma^{1/2}f(u)^{-1/2}\,\psi_{(m)}\psi_{(n)} = \delta_{mn}\,,\\
&\int_{u_0}^{\infty}\mathrm{d}u\,\frac{u^2}{R_{AdS}^4}\gamma^{-1/2}f(u)^{-1/2}\,\phi^{(0)}\phi^{(0)} = 1\,.
\end{aligned} \tag{8.68}$$

The zero mode $\phi^{(0)} = u^{-2}f(u)^{-1/2}\gamma^{1/2}$ is normalizable with this norm (there is no problem at the horizon because $u_0 > u_T$), and therefore there is a massless pion $\pi^{(0)}$ present in the intermediate-temperature phase. The fields $\pi^{(0)}$ are the Goldstone bosons associated with the spontaneous breaking of the $U(N_f)_L \times U(N_f)_R$ global chiral symmetry to the diagonal $U(N_f)$.

In the limit of $u_0 \gg u_T$ the spectrum simplifies and one can easily determine the scale of the meson masses. In this limit, which corresponds to a small separation distance between the stacks of branes and anti-branes $L \ll R$, the thermal factor $f(u) \to 1$ and in particular also $f(u_0) \to 1$.Therefore,

$$\gamma \equiv \frac{u^8}{u^{10} f(u) - u_0^{10} f(u_0)} \to \frac{1}{u^2} \frac{1}{1 - y^{-10}} \tag{8.69}$$

where the dimensionless quantity $y \equiv u/u_0$. Then we can rewrite (8.67) in terms of y in the following form

$$-\gamma^{-1/2}(y)\partial_y \left(y^2 \gamma^{-1/2}(y)\partial_y \psi_{(n)}\right) = \frac{R_{AdS}^4}{u_0^2}\, m_n^2\, \psi_{(n)}\,. \tag{8.70}$$

Now since the left-hand side is expressed in terms of the dimensionless quantity y, the right-hand side should also be dimensionless which implies that

$$m_n^2 \sim \frac{u_0^2}{R_{AdS}^4} \tag{8.71}$$

From (8.41) we know that $u_0 \sim 1/L$. Therefore the mass of "short" mesons scales as

$$M_{\text{meson}} \sim \frac{1}{L}\,. \tag{8.72}$$

The explicit mass spectrum of the vector mesons can be found by looking for normalizable eigenfunctions of (8.67) and using numerical methods (e.g. a shooting technique), but the qualitative behavior of the spectrum is that the masses of mesons decrease as temperature increases. This behavior is a direct consequence of the fact that the constituent quark mass is related to the distance of the tip of the probe brane to the horizon. If the distance is increased, a meson of the same spin will correspond to an excitation of the brane which is further away from the horizon and hence less affected by the temperature. This behavior is common for all gravitational backgrounds that contain a horizon.

3.5.2 Low-spin mesons at high temperature

In the high-temperature phase the profile of the left and right stacks of branes is characterized by $u' \equiv \mathrm{d}u/\mathrm{d}x^4 \to \infty$ and the induced metric on the probe branes and probe anti-branes takes the form

$$\mathrm{d}\hat{s}^2_{\text{high}} = \left(\frac{u}{R_{AdS}}\right)^2 \left[-f(u)\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] + \left(\frac{R_{AdS}}{u}\right)^2 \frac{1}{f(u)}\mathrm{d}u^2 \tag{8.73}$$

The differential equation for the modes is now

$$-\,f(u)^{1/2}\,\partial_u\left(u^2 f(u)^{1/2}\partial_u\psi_{(n)}\right) = R^4_{AdS}\, m_n^2\,\psi_{(n)} \tag{8.74}$$

i.e. it is similar to the intermediate temperature phase case, but with $\gamma = 1$. Then the normalization conditions are now

$$\begin{aligned}
&\int_{u_0}^{\infty} \mathrm{d}u\, f(u)^{-1/2}\,\psi_{(m)}\psi_{(n)} = \delta_{mn}\,, \\
&\int_{u_0}^{\infty} \mathrm{d}u\,\frac{u^2}{R^4_{AdS}} f(u)^{-1/2}\,\phi^{(0)}\phi^{(0)} = 1\,.
\end{aligned} \tag{8.75}$$

The mode, which would be given by $\phi^{(0)} = u^{-2}f(u)^{-1/2}$, is no longer normalizable. Computation of its norm leads to the integral

$$\int_{u_T}^{\infty} \mathrm{d}u\, u^2 f(u)^{-1/2}\left|u^{-2}f(u)^{-1/2}\right|^2\,, \tag{8.76}$$

which, while convergent at the upper boundary, is divergent at the lower boundary because $f(u) \sim \sqrt{u - u_T}$ for $u \sim u_T$. In accordance with the fact that chiral symmetry is restored in the high-temperature phase, we see that the Goldstone boson has disappeared. In the high temperature phase the spectrum of vector mesons is continuous.

3.5.3 High-spin mesons at intermediate temperature

To describe higher-spin mesons we need to look at more general string configurations that start and end on the probe brane. For large spin these strings can be described semiclassically. The relevant string configuration can be decomposed into three parts: a segment from the probe brane at $u = u_0$ to the wall at $u = u_T$, then a segment that

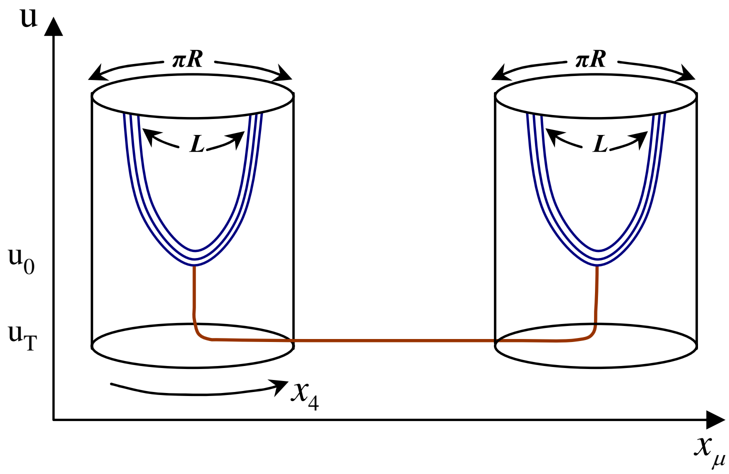

Figure 3.9: A high-spin meson at intermediate temperatures represented as a semiclassical string starting at the lowest point of the probe brane $u = u_0$, going down to the wall at $u = u_T$, stretching horizontally in the space along the wall, and then going back up vertically to the probe brane at $u = u_0$.

stretches along the wall in the spacial direction, and then another vertical part stretching from the wall back to the probe brane, as depicted at figure 3.9.

The relevant part of the background metric that represents this configuration is

$$\mathrm{d}s^2 = \left(\frac{u}{R_{AdS}}\right)^2 \left(-f(u)\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\varphi^2\right) + \left(\frac{R_{AdS}}{u}\right)^2 \frac{\mathrm{d}u^2}{f(u)} \tag{8.77}$$

We go to the static gauge for the string action and make the following ansatz for the rotating configuration,

$$t = \tau\,, \quad \rho = \rho(\sigma)\,, \quad u = u(\sigma)\,, \quad \varphi = \omega\tau \tag{8.78}$$

This ansatz has the same form as in the zero-temperature case [56]. Hence, the only effect of finite temperature will be in the change of the shape of $u(\sigma)$ as the temperature is increased.

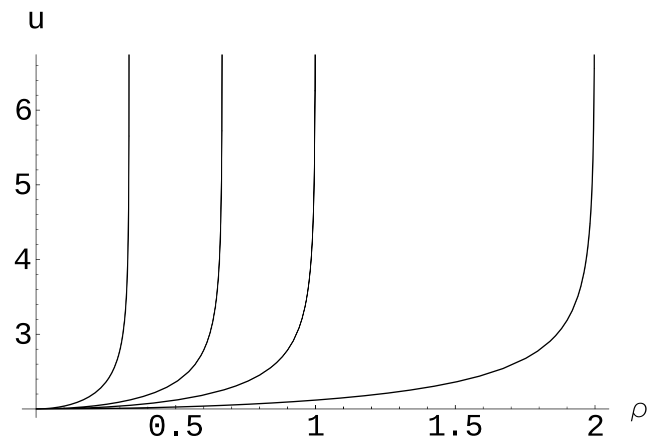

Figure 3.10: The boundary curve. Rotating strings have to lie above this curve in order for their action to be real. The curves correspond to the following values of the frequency ω (from right to left): 0.5, 1, 1.5 and 3. The horizon is located at $u_T = 2$.

With this ansatz the metric now reads

$$\mathrm{d}s^2 = \left(\frac{u}{R_{AdS}}\right)^2 (-f(u) + \rho^2\omega^2)\mathrm{d}\tau^2 + \left(\frac{u}{R_{AdS}}\right)^2 \left(\rho'^2 + \left(\frac{R_{AdS}}{u}\right)^4 \frac{u'^2}{f(u)}\right) d\sigma^2 \qquad (8.79)$$

and it leads to the following string (Polyakov) action

$$S = \int \mathrm{d}\tau\, \mathrm{d}\rho \sqrt{\left(\frac{u}{R_{AdS}}\right)^4 \left(\rho'^2 + \frac{u'^2}{f(u)} \frac{R_{AdS}^4}{u^4}\right) (f(u) - \rho^2\omega^2)} \qquad (8.80)$$

Positivity of the argument of the square root in (8.80) requires that $f(u) > \rho^2\omega^2$. This means that for a given angular frequency ω, the string solution $u(\rho)$ has to lie above the curve

$$u(\rho) \geq \frac{u_T}{(1 - \rho^2\omega^2)^{1/5}} \qquad (8.81)$$

In figure 3.10 these curves are depicted for various values of ω. We see that any string is allowed to touch the horizon u_T for any angular frequency ω and as ω decreases (i.e. the spin of the mesons increases) the string endpoints get more and more separated, the U-shaped string penetrates deeper to the horizon, and it becomes more and more rectangular. For a given ω, the maximal allowed extent of the string is determined by the intersection of the curve with u_0, and is given by

$$\rho_{\max} = \frac{1}{\omega}\sqrt{1-\left(\frac{u_T}{u_0}\right)^5} \tag{8.82}$$

The equation of motion following from the action (8.80) is given by

$$-2\sqrt{\ldots}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{\sqrt{\ldots}}\frac{u'}{f(u)}(f(u)-\rho^2\omega^2)\right)+f'(u)\left(\frac{u'^2\rho^2\omega^2}{f(u)^2}+\frac{u^4}{R_{AdS}^4}(\rho')^2\right)$$
$$+\frac{4u^3}{R_{AdS}^4}\left((\rho')^2 f(u)-(\rho')^2\rho^2\omega^2\right)=0 \tag{8.83}$$

where $\sqrt{\ldots}$ is the density of Nambu-Goto action (8.80).

The expressions for the energy and the angular momentum carried by the string are given by

$$E=\int \mathrm{d}\sigma\,\frac{1}{\sqrt{\ldots}}\left(\left(\frac{u}{R_{AdS}}\right)^4 f(u)(\rho')^2+u'^2\right) \tag{8.84}$$

$$J=\int \mathrm{d}\sigma\,\frac{1}{\sqrt{\ldots}}\omega\rho^2\left(\left(\frac{u}{R_{AdS}}\right)^4 \rho'^2+\frac{u'^2}{f(u)}\right) \tag{8.85}$$

The analysis of meson spectrum in [45] has showed that the meson spectrum of the SS model does not follow the well known Regge trajectories. For high-spin mesons at a fixed temperature there is a maximum value of angular momentum beyond which mesons cannot exist and have to dissociate. That is the temperature at which mesons melt is spin dependent. As the temperature increases, the maximal value of the spin that a meson can carry decreases. This behavior is also true for high-spin mesons in the AdS_6 background. Also for high mesons of fixed angular momentum, as for low-spin mesons, the energy decreases as a function of temperature.

3.5.4 Drag effects for quarks

In the deconfined phase the background contains a horizon and we can have a string starting on a flavor D4-brane/antibrane and going into the horizon. This string corresponds to a deconfined quark/anti-quark.

Because a strictly vertical string moving rigidly through the background would not have a real action (8.80), the string has to be bent when it is “pushed” through the plasma. In addition the bent string does not end anymore orthogonally on the brane. This means that one has to apply a force on the string endpoint, or in other words, one has to “drag” the string in order to keep it moving [57–63]. A suitable ansatz to describe the behavior of the string that moves with speed v_x in the x direction is (in static gauge)

$$t = \tau\,, \quad u = \sigma\,, \quad x = v_x t + \xi(u) \tag{8.86}$$

Inserting (8.86) into the Nambu-Goto Lagrangian we find

$$\begin{aligned} S &= \int d^2\sigma \sqrt{-det(G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu)} \\ &= \int \mathrm{d}\tau\, \mathrm{d}\rho \sqrt{1 - \frac{v_x^2}{f(u)} + \left(\frac{u}{R_{AdS}}\right)^4 f(u)\xi'^2} \end{aligned} \tag{8.87}$$

The corresponding equation for ξ implies that the conjugate momentum is a constant:

$$\pi_\xi = \frac{\partial \mathcal{L}}{\partial \xi'} = -\left(\frac{u}{R_{AdS}}\right)^4 \frac{f(u)\xi'}{\sqrt{-g}} \tag{8.88}$$

where g is the determinant of the induced metric. Inverting this relation we obtain

$$\xi' = \pi_\xi \left(\frac{R_{AdS}}{u}\right)^4 \frac{1}{f(u)} \sqrt{\frac{f(u) - v_x^2}{f(u) - \pi_\xi^2 (\frac{R_{AdS}}{u})^4}} \tag{8.89}$$

We must require that $\xi(u)$ is everywhere real, but the square root on the right hand side is in general not everywhere real. The function $f(u)$ interpolates between 1 at the boundary of AdS_6 to 0 at the horizon, so at some intermediate radius $f(u) - v_x^2$ switches sign at some intermediate point u_v, which is by definition such that $u_v^5 = u_T^5/(1 - v_x^2)$. The only way we can prevent ξ from becoming imaginary for $u < u_v$ is by choosing a value of π_ξ such that the denominator also vanishes at u_v:

$$\pi_\xi^2 = f(u_v)\left(\frac{u_v}{R_{AdS}}\right)^4 = \left(\frac{u_T}{R_{AdS}}\right)^4 \frac{v_x}{(1 - v_x^2)^{\frac{4}{5}}} \tag{8.90}$$

Plugging this back into (8.89) we find

$$\xi' = \frac{vR_{AdS}^2}{u^4}\frac{u_T^2}{f(u)} \tag{8.91}$$

Now we want to compute the σ component of the current associated with spacetime translations along x

$$P_x^u = -G_{x\nu}g^{u\alpha}\partial X^\nu = -\frac{f(u)\xi'}{g}\left(\frac{u_T}{R_{AdS}}\right)^4 \tag{8.92}$$

where $G_{\mu\nu}$ and $g_{\alpha\beta}$ denote respectively the spacetime and induced worldsheet metric. Together with (8.87) and (8.91) it yields the drag force

$$\frac{dp}{dt} = \sqrt{-g}P_x^u = -\frac{u_T^2}{R^2}\frac{v_x^2}{(1-v_x^2)^{\frac{4}{5}}} \tag{8.93}$$

We see that there are two effects happening as one tries to move a single string in the hot background: the string shape is modified in a way which depends on the temperature and velocity, and in order to preserve the motion one needs to apply a force.

3.5.5 Drag effects for mesons

Now we are interested if there is a drag force on a rotating meson at finite temperature. From the condition (8.81) we can see that a simple rotating motion does not experience a drag effect because the rotating string is always sufficiently high above the curve beyond which the action would turn to be imaginary. On the other hand the bending of the rotating string does depend on the angular velocity and on the temperature.

We can also consider a linear motion of the meson in a direction orthogonal to the plane of rotation. A suitable ansatz for this motion is

$$t = \tau\,, \quad \rho = \sigma\,, \quad u = u(\rho)\,, \quad \varphi = \omega\tau\,, \quad y = v_y\,\tau \tag{8.94}$$

In this case the string action becomes

$$S = \int \mathrm{d}\tau\,\mathrm{d}\rho\sqrt{\left(\frac{u}{R_{AdS}}\right)^4\left(1+\frac{u'^2}{f(u)}\frac{R_{AdS}^4}{u^4}\right)\left(f(u)-\rho^2\omega^2-v_y^2\right)}\,. \tag{8.95}$$

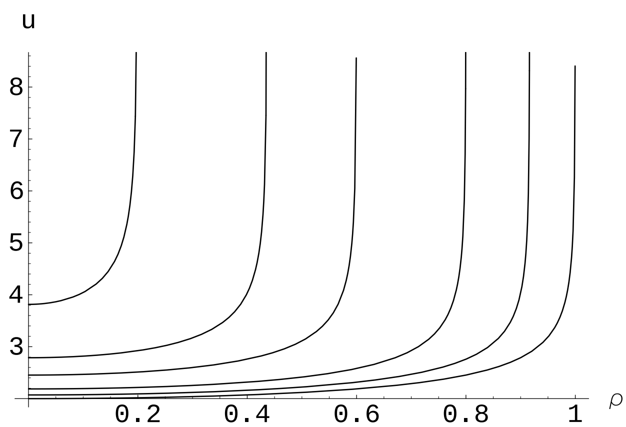

Figure 3.11: Analysis of the effect of a transverse velocity on the shape of spinning U-shaped strings keeping the quark masses and spin fixed. The curves display results for increasing values (from right to left) $v_y = 0, 0.4, 0.6, 0.8, 0.9, 0.98$ and $\omega = 1$. The horizon is located at $u_T = 2$.

The only modification with respect to the rotating meson is the addition of a term "$-v_y^2$" to the last factor under the square root. The condition for the action to be real is now

$$u \geq \frac{u_T}{(1 - \rho^2\omega^2 - v_y^2)^{1/5}} \, . \tag{8.96}$$

The curves are depicted in figure 3.11 for various values of v_y. For any high-spin meson at finite temperature one can find a generalized solution to the equation of motion such that the spinning configuration lies entirely above the curves (3.11). Thus, the mesons do not experience any drag effect that means that they do not experience any energy loss when propagating through the quark-gluon plasma - no force is necessary to keep them moving with a fixed velocity. In the dual language, this reflects the fact that if the quark gluon plasma is not hot enough to dissociate mesons, then these color singlets will not experience a drag force generated by a monopole interactions with the medium. However, the shape of the string in the (ρ, u) plane can be modified as it starts moving.

3.6 Summary

In this chapter we were mainly interested in the difference in QCD models in critical and non-critical backgrounds. We looked at the non-critical AdS_6 black hole background with D4-probe branes and antibranes and found that since the CS does not contribute to the model (with the CS term there is no solution of equations of motions at high temperature), it has a similar behavior with the critical Sakai-Sugimoto model. But neglecting the CS term altogether is a problem since it is needed to get the WZ term in the Skyrme model [40].

We found that the difference in free energy between Euclidean background at low an high temperatures scales as N_c^2 in the 't Hooft large N_c limit, as expected. In our model it is proportional to $(2\pi T)^5 - (1/R)^5$, while in the SS model it is proportional to $(2\pi T)^6 - (1/R)^6$. Also in the SS model dilaton goes to infinity as $u \to \infty$, where, in principle, sugra approximation is not valid and one should go to the M-theory. In the AdS_6 case dilaton is constant and we should not worry about the behavior of the model when $u \to \infty$. On the other hand in the non-critical case there is a problem with high curvature corrections of order α'. But we expect that for the symmetric case of AdS_6 they will not contribute.

In our model we got the chiral phase transition at the ratio of the separation distance between branes and antibranes at infinity and the radius of the x_4 circle $L/R = 1.06$, comparatively to the SS model $L/R = 0.97$. Spectrum of the low-spin mesons is discrete at low temperatures and continuous at high temperatures and we can identify the Goldstone pion associated with the spontaneous breaking of the $U(N_f)_L \times U(N_f)_R$ global chiral symmetry to the diagonal $U(N_f)$. Quarks and high-spin mesons experience a drag force at finite temperature.

In the section 4.5 we saw that the AdS_6 and the SS model can be unified into a family of metrics that have a similar phase structure.

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