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Algebraic Properties of the Shape Invariance Condition

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Based on the formalism of the Z_k-graded deformed oscillator algebra, we systematically construct, within the framework of supersymmetric quantum mechanics, general algebraic properties of four known classes of shape invariant potentials that are extended from the ordinary one step in the literature to arbitrary k steps.

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I. INTRODUCTION

Supersymmetry (SUSY) is the symmetry relating bosonic and fermionic degrees of freedom. Historically, supersymmetric quantum mechanics (SUSY QM) was proposed as a limiting theory to gain a better understanding of the dynamical SUSY breaking of quantum field theories [1, 2]. It was soon recognized that SUSY QM could be a very interesting topic in its own right, because, by the method of factorization [3], it enables us to build the SUSY partner Hamiltonian of a given nonrelativistic one. When the method is repeatedly used, we consequently construct an entire hierarchy of isospectral SUSY partner Hamiltonians. For complete reviews on SUSY QM, please refer to [4–7] and the references therein.

To be specific, let us consider two Hamiltonians $H^{(\pm)}(x, a_0) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(\pm)}(x, a_0)$, which are said to be SUSY partners, if the corresponding potentials $V^{(\pm)}(x, a_0)$ are related to each other by

$$V^{(\pm)}(x, a_0) = W^2(x, a_0) \pm W'(x, a_0)\,, \tag{1}$$

where $W(x, a_0)$ is the superpotential, $W'(x, a_0) \equiv \frac{\mathrm{d}}{\mathrm{d}x}W(x, a_0)$, and a_0 is a set of parameters. The SUSY partner Hamiltonians $H^{(\pm)}(x, a_0)$ defined through Equation (1) are found to be exactly isospectral, except for a zero-energy ground-state eigenfunction that can be completely determined by the asymptotic behavior of the superpotential $W(x \to \pm\infty, a_0) \equiv w_\pm$. For this, we define the topological Witten index by $\Delta = \frac{1}{2}[\mathrm{sgn}(w_+) - \mathrm{sgn}(w_-)]$, with $\mathrm{sgn}(w_\pm)$ being the sign of $w_\pm$. Then, $\Delta \neq 0$ indicates unbroken SUSY, thus the existence of the zero-energy eigenstate; otherwise $\Delta = 0$ signifies broken SUSY and the nonexistence of such an eigenstate [5].

One of the important conceptual breakthroughs in the study of SUSY QM is the introduction of the shape invariance condition (in one step) [8], which generically speaking

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is a discrete reparameterization and gives rise to an integrability condition on the solvable potentials [9–13]. It can be shown that when the shape invariance condition is fulfilled by the pair of partner Hamiltonians $H^{(\pm)}(x, a_0)$, the energy spectra and eigenfunctions are completely determined by algebraic means [8, 14]. Let us mention here that the underlying algebraic structure of shape invariance is actually a direct consequence of what is referred to as the potential algebra [15, 16]; thus all shape invariant potentials (in one step) can be studied by group theoretical methods [17, 18].

By 'shape invariance', we demand that the pair of partner potentials $V^{(\pm)}(x, a_0)$ in Equation (1) be similar in shape but differ only up to a change of parameters and additive constants. For this purpose, let us consider the general case of the shape invariance condition in arbitrary k steps that can be described by k arbitrary superpotentials $W_s(x, a_0)$, where k is a positive integer and $s = 0, 1, \cdots, k-1$. To preserve SUSY, the k superpotentials are chosen to fulfill both the asymptotic behavior: $\mathrm{sgn}(w_{0+}) = \mathrm{sgn}(w_{1+}) \cdots = \mathrm{sgn}(w_{(k-1)+})$ and the Witten index requirements: $\Delta_s \neq 0$, for $s = 0, 1, \cdots, k-1$. Mathematically, the shape invariance condition in k steps reads [11]

$$\begin{aligned} W_0^2(x, a_0) + W_0'(x, a_0) &= W_1^2(x, a_0) - W_1'(x, a_0) + R_0(a_0)\,, \\ W_1^2(x, a_0) + W_1'(x, a_0) &= W_2^2(x, a_0) - W_2'(x, a_0) + R_1(a_0)\,, \\ \cdots &= \cdots \\ W_{k-1}^2(x, a_0) + W_{k-1}'(x, a_0) &= W_0^2(x, a_1) - W_0'(x, a_1) + R_{k-1}(a_0)\,, \end{aligned} \tag{2}$$

where $a_1 = f(a_0)$ is a function of a_0 and k and the remainders $R_s(a_0)$ are arbitrary, independent of x. Via Equation (2), it is straightforward to show that the energy eigenvalues of the initial Hamiltonian $H_0^{(-)}(x, a_0) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0^{(-)}(x, a_0)$ is of the form

$$E_{nk+s}^{(-)} = \sum_{m=0}^{n-1} \sum_{t=0}^{k-1} R_t(a_m) + \sum_{t=0}^{s-1} R_t(a_n)\,, \tag{3}$$

where $n = 0, 1, 2, \cdots$, $s = 0, 1, \cdots, k-1$, and the convention $\sum_{t=0}^{-1} = 0$ is used. Here, we assume that the superpotentials $W_s(x, a_0)$ are constructed such that it is the Hamiltonian $H_0^{(-)}(x, a_0)$ that possesses the unique zero-energy ground state. In the literature, some solvable potentials based on the shape invariance condition in two or higher multi-steps (2) have been constructed [13, 14, 19–21].

With regard to the algebraic structures described by Equations (2) and (3), a simplified version of the potential algebra of the shape invariance condition in k steps has been proposed, under certain circumstances. For $k = 2$, it was shown that the corresponding simplified potential algebra is based on three angular-momentum-like generators [20, 22]. As for the case of arbitrary k steps, the simplified algebra can be realized by the generalized deformed oscillator algebra, which has a built-in Z_k-grading structure [23, 24]. In addition, this Z_k-graded deformed oscillator algebra automatically includes that of cyclic shape invariant potentials of period k [13, 25, 26].

The purpose of the present article is to extend the major findings of reference [24]. In that paper, we showed that the general algebraic properties of translational shape invariant potentials in k steps can be completely determined, when the remainders $R_s(a_m)$

in Equation (2) are analytic functions of the parameter a_m that is related to the other parameters by translation: $a_{m+1} = a_m + \delta$, in which δ is a constant. In this article, we turn our attention to the other three known classes of shape invariant potentials (in one step) in the SUSY literature, namely, (i) the scaling class, with the relationship between parameters $a_{m+1} = qa_m$, for $0 < q < 1$, (ii) the 'exotic'-I class, with $a_{m+1} = qa_m^p$, for $0 < q < 1$ and $p = 2, 3, ...$, and (iii) the 'exotic'-II class, the generalization of 'exotic'-I class, with $a_{m+1} = qa_m/(1 + ra_m)$, for $0 < q < 1$ and $ra_m \ll 1$. We shall show that the detailed algebraic properties for these three classes of shape invariant potentials extended to arbitrary k steps can be systematically determined as well, by using the Z_k-graded deformed oscillator algebra.

The article is organized as follows. In Section II, for the purpose of completeness, we briefly review on the equivalence between the simplified potential algebra of shape invariance in k steps and the algebra of Z_k-graded generalized deformed oscillators. In Section III, based on the main results of translational shape invariant potentials, we present in detail the general algebraic properties of the other three classes of shape invariant potentials in k steps. Finally, Section IV contains a discussion of the present article.

II. Z_k-GRADED SHAPE INVARIANCE CONDITION

In this section, we review how the simplified shape invariance condition in k steps is realized by the Z_k-graded generalized deformed oscillator algebra [23, 24]. Deformed oscillators have been studied in many different deformation schemes [28–31]. The Z_k-graded generalized deformed oscillators are the ordinary ones, but having an extra built-in Z_k-grading symmetry [32, 33].

The novel version of the shape invariance condition in k steps that simplifies Equation (2) can be established as follows. We first identify in Equation (2) the parameters as $a_m \equiv \boldsymbol{\alpha}(N_0 - m)$, for N_0 and m arbitrary integers. Then, we impose extra relations on the k superpotentials $W_s(x, \boldsymbol{\alpha}(N_0))$ and the k remainders $R_s(\boldsymbol{\alpha}(N_0))$, respectively, in the forms

$$W_s(x, \boldsymbol{\alpha}(N_0)) \equiv \mathcal{W}\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right), \qquad R_s(\boldsymbol{\alpha}(N_0)) \equiv \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right), \tag{4}$$

where the identification $a_{\frac{s}{k}} = \boldsymbol{\alpha}(N_0 - \frac{s}{k})$ is thus inferred. By using Equation (4), the k relations of Equation (2) can be rewritten into a unified one, in terms of both the unified superpotential $\mathcal{W}(x, \boldsymbol{\alpha}(N_0))$ and the unified remainder $\mathcal{R}(\boldsymbol{\alpha}(N_0))$, as

$$\begin{aligned}&\mathcal{W}^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) + \mathcal{W}'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) \\ &= \mathcal{W}^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) - \mathcal{W}'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) + \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right). \end{aligned} \tag{5}$$

In this way, the k relations in Equation (2) can be readily reproduced from Equation (5) by letting, one at a time, $s = 0, 1, \cdots, k-1$. Equation (5) consequently represents the simplified version of the shape invariance condition in k steps, as told. Actually, this equation admits an inherited Z_k-grading algebra that can be further realized by the Z_k-graded deformed oscillator algebra.

To show this, let us first recall the relevant definition of the algebra of generalized deformed oscillators, which has a built-in Z_k-grading structure. It is defined by a nonlinear algebra generated by the number operator $\mathcal{N}$, the lowering operator $\mathcal{A}$, the raising operator $\mathcal{A}^\dagger$, and, most importantly, the k projection operators Π_s (for $s = 0, 1, \cdots, k-1$). Altogether, they fulfill the Hermiticity conditions $(\mathcal{A})^\dagger = \mathcal{A}^\dagger$, $\mathcal{N}^\dagger = \mathcal{N}$, $\Pi_s^\dagger = \Pi_s$, and the following defining relations [32, 33]

$$[\mathcal{N}, \mathcal{A}^\dagger] = \frac{1}{k}\mathcal{A}^\dagger, \qquad [\mathcal{N}, \mathcal{A}] = -\frac{1}{k}\mathcal{A}, \qquad [\mathcal{N}, \Pi_s] = 0, \tag{6}$$

$$\mathcal{A}^\dagger\mathcal{A} = \mathcal{F}\big(\boldsymbol{\alpha}(\mathcal{N})\big), \qquad \mathcal{A}\mathcal{A}^\dagger = \mathcal{F}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right), \tag{7}$$

$$\sum_{s=0}^{k-1} \Pi_s = I, \quad \Pi_s\Pi_t = \delta_{s,t}\,\Pi_s, \quad \mathcal{A}^\dagger\Pi_s = \Pi_{s+1}\mathcal{A}^\dagger, \quad \mathcal{A}\,\Pi_s = \Pi_{s-1}\mathcal{A}. \tag{8}$$

Here, the convention is used: $\Pi_t = \Pi_s$, if $t - s = 0 \bmod k$. The Hermitian positive function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ in Equation (7) is usually called the structure function.

Next, let us build the corresponding set of operators $\{\mathcal{N}, \Pi_s, \mathcal{A}, \mathcal{A}^\dagger\}$ that is implicitly presented in the simplified shape invariance condition (5). For short, the analogous number operator $\mathcal{N}$ and the projection operators Π_s are respectively constructed as

$$\mathcal{N} \equiv \frac{1}{i}\frac{\partial}{\partial\phi}, \qquad \Pi_s \equiv \frac{1}{k}\sum_{t=0}^{k-1} e^{2\pi \mathrm{i} t(\mathcal{N}+s/k)}. \tag{9}$$

Further, the analogous ladder operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ are built, using the unified superpotential $\mathcal{W}(x, \alpha(\mathcal{N}))$, by

$$\mathcal{A} = e^{-i\phi/k}\left[\frac{\partial}{\partial x} + \mathcal{W}(x, \boldsymbol{\alpha}(\mathcal{N}))\right], \qquad \mathcal{A}^\dagger = \left[-\frac{\partial}{\partial x} + \mathcal{W}(x, \boldsymbol{\alpha}(\mathcal{N}))\right]e^{i\phi/k}, \tag{10}$$

where the configuration space of the parameter is $\phi \in [0, 2\pi k]$, since $\mathcal{A}$ and $\mathcal{A}^\dagger$ remain invariant under the transformation: $\phi \to \phi + 2\pi k$.

Now, it is straightforward to verify that the set of operators $\{\mathcal{N}, \Pi_s, \mathcal{A}, \mathcal{A}^\dagger\}$ constructed in Equations (9) and (10) indeed satisfies all the required relations of Z_k-graded deformed oscillator algebra, provided that the following remainder-structure-function relation, via Equation (5), holds

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right) - \mathcal{F}(\boldsymbol{\alpha}(\mathcal{N})) = \mathcal{R}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right). \tag{11}$$

In other words, the identification (4) simplifies the original shape invariance condition in k steps (2) to the relatively simple version (5), which turns out to be identical to the well-established Z_k-graded deformed oscillator algebra.

The Fock space for the Z_k-graded deformed oscillator algebra is symbolically denoted by the direct sum $\mathcal{H} = \sum_{s=0}^{k-1} \oplus \mathcal{H}_s$, consisting of k distinct Fock subspaces

$$\mathcal{H}_s \equiv \left\{ \left| N_0 - \frac{mk+s}{k} \right\rangle \middle| m = 0, 1, 2, \cdots \right\} . \tag{12}$$

As usual, the number eigenstates in the Fock space are simultaneous eigenstates of $\mathcal{N}$ and Π_s, fulfilling the respective eigenvalue equations (for $s, t = 0, 1, \cdots, k-1$)

$$\mathcal{N} \left| N_0 - \frac{mk+s}{k} \right\rangle = \left(N_0 - \frac{mk+s}{k} \right) \left| N_0 - \frac{mk+s}{k} \right\rangle , \tag{13}$$

$$\Pi_t \left| N_0 - \frac{mk+s}{k} \right\rangle = \delta_{t,s} \left| N_0 - \frac{mk+s}{k} \right\rangle . \tag{14}$$

When acting on these number eigenstates, the ladder operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ in Equation (10), as anticipated, change the eigenvalues of $\mathcal{N}$ by $-\frac{1}{k}$ and $+\frac{1}{k}$, respectively,

$$\mathcal{A} \left| N_0 - \frac{n}{k} \right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k} \right) \right)} \left| N_0 - \frac{n+1}{k} \right\rangle , \tag{15}$$

$$\mathcal{A}^\dagger \left| N_0 - \frac{n}{k} \right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n-1}{k} \right) \right)} \left| N_0 - \frac{n-1}{k} \right\rangle , \tag{16}$$

where $n = 0, 1, 2, \cdots$. Further, if the spectrum exhibits a lowest-weight eigenstate: $\mathcal{A} \left| N_0 - \frac{n_0}{k} \right\rangle = 0$, for a given integer n_0, then $\mathcal{F}\big(\boldsymbol{\alpha}(N_0 - \frac{n_0}{k})\big) = 0$. Otherwise, if it exhibits a highest-weight eigenstate: $\mathcal{A}^\dagger \left| N_0 - \frac{n_0}{k} \right\rangle = 0$, then $\mathcal{F}\big(\boldsymbol{\alpha}(N_0 - \frac{n_0 - 1}{k})\big) = 0$.

It is interesting to note that based on Equation (11), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ now can be expressed solely in terms of the structure function $\mathcal{F}(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ by

$$E_n^{(-)} = \sum_{m=0}^{n-1} \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{m}{k} \right) \right) = \mathcal{F}\big(\boldsymbol{\alpha}(N_0)\big) - \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k} \right) \right) , \tag{17}$$

where $n = 0, 1, 2, \cdots$. Equation (17) represents the energy spectrum of the simplified shape invariance condition in k steps, in contrast to the originally more complicated energy spectrum in Equation (3).

Two remarks are in order. (i) For the case $k = 2$, the simplified potential algebra is known as the Calogero-Vasiliev oscillator algebra [34], in which the Z_2-grading structure is characterized by the Klein operator. It is also related to the R-deformed Heisenberg algebra [29] that has found many interesting applications, recently [35]. (ii) The remainder $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k}))$ in Equation (17) is the energy gap between two adjacent eigenstates. To prevent the energy levels from crossing, we must have $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k})) > 0$, for positive integer m. If $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k})) \leq 0$ happens, it simply means that the associated shape invariant potential in k steps contains only a finite number of bound states, thus is of finite depth.

III. SHAPE INVARIANT POTENTIALS IN k STEPS

In this section, the detailed algebraic properties of four known classes of shape invariant potentials (in one step) in the literature that retain SUSY will be extended to the general case of k steps, which include the translational class, the scaling class, the 'exotic'-I class, and the 'exotic'-II class.

To proceed, let us consider an arbitrary shape invariant potential in k steps, in which the k unrelated remainders $R_s(a_m)$ in Equation (2) admit the respective power series expansions ($m = 0, 1, 2, \cdots$)

$$R_s(a_m) \;=\; \sum_{i=0}^{\infty} \alpha_{s,i} \, (a_m)^i \,, \tag{18}$$

where $\alpha_{s,i}$ ($s = 0, 1, \cdots, k-1$) are expansion coefficients. Next, according to Equation (4), we can identify the k remainders $R_s(a_m)$ to be the unified remainder $\mathcal{R}(a_{m+\frac{s}{k}})$ at different values of the parameter $a_{m+\frac{s}{k}}$. That is,

$$R_s(a_m) \;\equiv\; \mathcal{R}\Big(a_{m+\frac{s}{k}}\Big) \;=\; \sum_{i=0}^{\infty} \omega_{s,i} \, \Big(a_{m+\frac{s}{k}}\Big)^i \tag{19}$$

where $\omega_{s,i}$ are another set of expansion coefficients that are introduced to characterize the inherited Z_k-grading structure of $\mathcal{R}(a_{m+\frac{s}{k}})$. They are expressible in terms of $\alpha_{s,i}$, if the function $f(a_m) = a_{m+1}$ that relates different shape invariant parameters is given.

Furthermore, if we denote $n = mk + s$ in Equation (19), the unified remainder $\mathcal{R}\big(a_{\frac{n}{k}}\big)$ consequently admits the very similar expansion

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) \;=\; \sum_{s=0}^{k-1} \left[\sum_{i=0}^{\infty} \omega_{s,i} \, \Big(a_{\frac{n}{k}}\Big)^i \right] \Delta_{n,s} \,, \tag{20}$$

where the symbol $\Delta_{n,s}$ is introduced for the purpose of singling out the unique term in the index s summation, when letting $n = mk + s$. It is defined as the analogous Kronecker delta for the cyclic group of order k:

$$\Delta_{n,s} \;=\; \begin{cases} 1\,, & \text{for} \;\; n = s \mod k\,, \\ 0\,, & \text{for} \;\; n \neq s \mod k\,. \end{cases} \tag{21}$$

Finally, an equivalent operator expression of Equation (20) can be obtained, using the number operator $\mathcal{N}$ and projection operators Π_s, as

$$\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big) \;=\; \sum_{s=0}^{k-1} \left[\sum_{i=0}^{\infty} \omega_{s,i} \, \big(a_{N_0 - \mathcal{N}}\big)^i \right] \Pi_s \,. \tag{22}$$

It is easily seen that Equation (20) can be readily recovered by applying the operator relation (22) directly on the number eigenstate $\left|N_0 - \frac{n}{k}\right\rangle$.

III-1. The translational shape invariant potentials

We are at a position to present the detailed algebraic results for shape invariant potentials in k steps. Let us first recall some of the main findings of [24]. So, we consider the translational class of shape invariant potentials in k steps, in which the shape invariant parameters are related to others by translation [9, 10]: $a_{m+1} = a_m + \delta$. In the general case, we have $a_{m+\frac{s}{k}} = a_m + \frac{s}{k}\delta = a_0 + (m + \frac{s}{k})\delta$, where $m = 0, 1, 2, \cdots$ and $s = 0, 1, \cdots, k-1$. Here, a_0 and δ are constants.

In the translational class, Equations (19) therefore read as

$$\mathcal{R}\Big(a_{m+\frac{s}{k}}\Big) = \sum_{i=0}^{\infty} \omega_{s,i}\Big(a_m + \frac{s}{k}\,\delta\Big)^{i} . \tag{23}$$

Now, by comparing the power series expansions in a_m in both Equations (18) and (23), we obtain

$$\alpha_{s,i} = \sum_{j=i}^{\infty} \binom{j}{i}\Big(\frac{s}{k}\,\delta\Big)^{j-i} \omega_{s,j} \, , \tag{24}$$

where $\binom{j}{i} = j!/i!(j-i)!$ is the binomial coefficient.

In the same vein, if we denote $n = mk + s$, Equation (**??**) can be expressed in the power series of $\frac{n}{k}$ as

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1}\Big[\sum_{i=0}^{\infty} \omega_{s,i}\Big(a_0 + \frac{n}{k}\,\delta\Big)^{i}\Big]\Delta_{n,s} = \sum_{s=0}^{k-1}\Big[\sum_{i=0}^{\infty} \varepsilon_{s,i}\Big(\frac{n}{k}\Big)^{i}\Big]\Delta_{n,s} \, . \tag{25}$$

Here, the Kronecker delta $\Delta_{n,s}$ singles out the term that in the index s summation satisfies the condition $n - s = 0 \bmod k$. The expansion coefficients $\varepsilon_{s,i}$ are then found to be expressible as

$$\varepsilon_{s,i} = \delta^{\,i} \sum_{j=i}^{\infty} \binom{j}{i} (a_0)^{j-i} \omega_{s,j} = \delta^{\,i} \sum_{j=i}^{\infty} \binom{j}{i}\Big(a_0 - \frac{s}{k}\,\delta\Big)^{j-i} \alpha_{s,i} \, , \tag{26}$$

where the second equality is derived from the first one by inverting the system of linear relations of Equation (24). Therefore, in the translational class the associated equivalent operator relation (25) can be established, in terms of $\mathcal{N}$ and Π_s, in the form as

$$\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big) = \sum_{s=0}^{k-1}\Big[\sum_{i=0}^{\infty} \varepsilon_{s,i}\,\big(N_0 - \mathcal{N}\big)^{i}\Big]\Pi_s \, . \tag{27}$$

Clearly, Equation (25) is easily reproduced when Equation (27) acts on the number eigenstate $\left|N_0 - \frac{n}{k}\right\rangle$.

Note that the remainder $\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big)$ shown in Equation (27) is expanded in the power series of $(N_0-\mathcal{N})$. We hence expect that the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of the associated Z_k-graded deformed oscillator algebra can also be obtained by the same power series of $(N_0-\mathcal{N})$. The expectation is correct. Recently, the detailed order-by-order derivations of the structure function in powers of $(N_0-\mathcal{N})$ were carried out by the author [24]. Hence, we will not present the detailed calculations, but only quote the relevant facts of the article.

Given the unified remainder (27) of the translational shape invariant potentials in k steps, we learned that the corresponding structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ is given, via the remainder-structure-function relation (11), by

$$\mathcal{F}\Big(a_{\frac{n}{k}}\Big) \;=\; C - \sum_{s,t=0}^{k-1}\left[\sum_{i=0}^{\infty}\frac{\varepsilon_{s+t,i}}{i+1}\,B_{i+1}\Big(\frac{t+n}{k}\Big)\right]\Delta_{n,s}\,, \tag{28}$$

where $n=0,1,2,\cdots$, $B_{i+1}(x)$ is the $(i+1)$th Bernoulli function, the convention $\varepsilon_{s+t,i}\equiv\varepsilon_{s+t \bmod k,i}$ is used, and C is a constant to render the structure function positive definite. As a brief remark, the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ (28) and the set of operators $\{\mathcal{N},\Pi_s,\mathcal{A},\mathcal{A}^\dagger\}$ in Equations (9) and (10) altogether thus define the simplified potential algebra of the translational shape invariance condition in arbitrary k steps.

In addition, the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x,\boldsymbol{\alpha}(N_0))$ in the translational class can be determined, according to Equation (17),

$$E_n^{(-)} \;=\; \sum_{t=0}^{k-1}\sum_{i=0}^{\infty}\left\{\sum_{s=0}^{k-1}\left[\frac{\varepsilon_{s+t,i}}{i+1}\,B_{i+1}\Big(\frac{t+n}{k}\Big)\right]\Delta_{n,s} \;-\; \left[\frac{\varepsilon_{t,i}}{i+1}\,B_{i+1}\Big(\frac{t}{k}\Big)\right]\right\}, \tag{29}$$

where $n=0,1,2,\cdots$.

The eigenenergies for the ordinary translational shape invariant potentials in SUSY QM can be readily obtained, that is, for the special case $k=1$. When $k=1$, there is no grading structure in the associated potential algebra, so that the grading indices are set to be $s=t=0$. By denoting $\varepsilon_{0,i}=\varepsilon_i$, we have

$$E_n^{(-)} \;=\; \sum_{i=0}^{\infty}\frac{\varepsilon_i}{i+1}\Big[B_{i+1}(n)-1\Big]\,, \tag{30}$$

which yields $E_0^{(-)}=0$, as is necessary by the requirement of unbroken SUSY. The energy difference between two adjacent eigenstates is

$$\mathcal{R}(a_n) \;=\; E_{n+1}^{(-)}-E_n^{(-)} \;=\; \sum_{i=0}^{\infty}\frac{\varepsilon_i}{i+1}\Big[B_{i+1}(n+1)-B_{i+1}(n)\Big] \;=\; \sum_{i=0}^{\infty}\varepsilon_i\,n^i. \tag{31}$$

Consistency of Equations (31) and (25) (when letting $k=1$ in the latter one) is therefore manifest.

To further show the usefulness of Equation (28), we continue to consider a typical SUSY QM system, in which the associated remainder function, i.e., the energy gap between two adjacent number eigenstates, takes the following exponential form

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1} \gamma_s \, e^{-\frac{n}{k}\sigma_s} \, \Delta_{n,s} = \sum_{s=0}^{k-1} \gamma_s \Bigg[\sum_{i=0}^{\infty} \frac{(-\sigma_s)^i}{i!} \left(\frac{n}{k}\right)^i \Bigg] \Delta_{n,s} \,, \tag{32}$$

where γ_s and σ_s are positive constants. The purpose of studying the remainder (32) is twofold. The first is to demonstrate the closed-form algebraic properties of this particular translational shape invariant potential in k steps. The second is to provide the key formulas in determining the algebraic properties for the other three known classes of shape invariant potentials in k steps.

When comparing (32) with (25), we obtain $\varepsilon_{s,i} = \gamma_s(-\sigma_s)^i/i!$. From Equation (28), it in turns results in the following exact structure function

$$\begin{aligned} \mathcal{F}\Big(a_{\frac{n}{k}}\Big) &= C - \sum_{s,t=0}^{k-1} \gamma_{s+t} \Bigg[\sum_{i=0}^{\infty} \frac{(-\sigma_{s+t})^i}{(i+1)!} \, B_{i+1}\left(\frac{t+n}{k}\right) \Bigg] \Delta_{n,s} \,, \\ &= C - \sum_{s,t=0}^{k-1} \gamma_{s+t} \Bigg[\frac{e^{-(\frac{t+n}{k})\,\sigma_{s+t}}}{e^{-\sigma_{s+t}} - 1} + \frac{1}{\sigma_{s+t}} \Bigg] \Delta_{n,s} \,, \end{aligned} \tag{33}$$

where to go from the first line to the second one, we use $B_0(x) = 1$ and a property of the generating function that defines the Bernoulli functions [36]. Now, according to Equation (17), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$, having the specific remainder (32), is written by

$$E_n^{(-)} = \sum_{t=0}^{k-1} \Bigg\{ \sum_{s=0}^{k-1} \gamma_{s+t} \Bigg[\frac{e^{-(\frac{t+n}{k})\,\sigma_{s+t}}}{e^{-\sigma_{s+t}} - 1} + \frac{1}{\sigma_{s+t}} \Bigg] \Delta_{n,s} - \gamma_t \Bigg[\frac{e^{-\frac{t}{k}\,\sigma_t}}{e^{-\sigma_t} - 1} + \frac{1}{\sigma_t} \Bigg] \Bigg\} \,, \tag{34}$$

where $n = 0, 1, 2, \cdots$.

Correctness of the exact expression (34) can be easily checked for the ordinary $k = 1$ shape invariant potentials. When $k = 1$, the grading indices are taken as $s = t = 0$. If we denote $\gamma = \gamma_0$ and $\sigma = \sigma_0$, Equation (34) then is reduced to the simple form

$$E_n^{(-)} = \gamma \, \frac{1 - e^{-n\sigma}}{1 - e^{-\sigma}} \,, \tag{35}$$

which yields $E_0^{(-)} = 0$, as is necessary by the requirement of unbroken SUSY. Further, we can compute the energy difference between two adjacent eigenstates from Equation (35). We find

$$\mathcal{R}(a_n) = E_{n+1}^{(-)} - E_n^{(-)} = \gamma \, e^{-n\sigma} \,. \tag{36}$$

Consistency of Equations (36) and (32) (when letting $k = 1$ in the latter one) is therefore evident. Note that, in the limiting case $\sigma \to 0$, Equation (35) becomes $E_n^{(-)} = n\gamma$, which is nothing but the energy spectrum of simple harmonic oscillators.

III-2. The scaling shape invariant potentials

We next present the algebraic properties of shape invariant potentials in k steps in the scaling class, in which the shape invariant parameters are related to others by the scaling transformation [9, 10]: $a_{m+1} = q a_m$. For more general cases, we will have $a_{m+\frac{s}{k}} = q^{\frac{s}{k}} a_m = q^{m+\frac{s}{k}} a_0$. Here, $m = 0, 1, 2, \cdots$, $s = 0, 1, \cdots, k-1$, a_0 is a constant, and the scaling parameter q is restricted to be in the interval $0 < q < 1$.

From Equations (19), the unified remainder in the scaling class takes the form

$$\mathcal{R}\Big(a_{m+\frac{s}{k}}\Big) = \sum_{i=0}^{\infty} \omega_{s,i} \Big(q^{\frac{s}{k}} a_m\Big)^i . \tag{37}$$

Comparing Equations (37) with (18), we obtain the relation between the expansion coefficients: $\alpha_{s,i} = \omega_{s,i}\, q^{i\frac{s}{k}}$. Likewise, if we denote $n = mk + s$, Equation (37) then becomes

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \varepsilon_{s,i}\, q^{\,i\frac{n}{k}}\Bigg] \Delta_{n,s} = \sum_{s=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \varepsilon_{s,i}\, e^{-\frac{n}{k}\,\sigma_i}\Bigg] \Delta_{n,s} \,, \tag{38}$$

where we have denoted $\varepsilon_{s,i} \equiv a_0^i\, \omega_{s,i} = \big(q^{-\frac{s}{k}} a_0\big)^i \alpha_{s,i}$ and $\sigma_i = -\ln q^{\,i}$. Obviously, the term that will be singled out in the index s summation satisfies the condition $n - s = 0 \bmod k$. The equivalent operator expression of Equation (38) can be established, in terms of the $\mathcal{N}$ and Π_s operators, as follows:

$$\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big) = \sum_{s=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \varepsilon_{s,i}\, q^{\,i(N_0-\mathcal{N})}\Bigg] \Pi_s \,. \tag{39}$$

Therefore, Equation (38) can be recovered with ease by acting with the remainder operator $\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big)$ (39) on the number eigenstate $\big|N_0 - \frac{n}{k}\big\rangle$.

It is noted that the remainder $\mathcal{R}\big(a_{\frac{n}{k}}\big)$ in the second equality of Equation (38) takes the standard exponential form, similar to that discussed in Equation (32). As a consequence, the corresponding algebraic properties can be immediately determined. By using Equation (33), we obtain the associated structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of the scaling shape invariant potentials in k steps as

$$\begin{aligned}
\mathcal{F}\Big(a_{\frac{n}{k}}\Big) &= C - \sum_{s,t=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \varepsilon_{s+t,i} \bigg(\frac{e^{-(\frac{t+n}{k})\,\sigma_i}}{e^{-\sigma_i}-1} + \frac{1}{\sigma_i}\bigg)\Bigg] \Delta_{n,s} \,, \\
&= C - \sum_{s,t=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \varepsilon_{s+t,i} \bigg(\frac{q^{\,i\,(\frac{t+n}{k})}}{q^{\,i}-1} - \frac{1}{\ln q^{\,i}}\bigg)\Bigg] \Delta_{n,s} \,,
\end{aligned} \tag{40}$$

where $n = 0, 1, 2, \cdots$, the convention $\varepsilon_{s+t,i} \equiv \varepsilon_{s+t \bmod k,i}$ is used, and C is a constant to yield the structure function positive definite. It is mentioned that the term inside the parenthesis on the second line reduces to $\frac{t+n}{k}$, in the limiting case $i \to 0$.

The structure function (40) together with the set of operators $\{\mathcal{N}, \Pi_s, \mathcal{A}, \mathcal{A}^\dagger\}$ in Equations (9) and (10) thus define the simplified algebra of the scaling shape invariance condition in k steps. Now, according to Equation (17), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in the scaling class will be ($n = 0, 1, 2, \cdots$)

$$E_n^{(-)} = \sum_{t=0}^{k-1} \sum_{i=0}^{\infty} \left\{ \sum_{s=0}^{k-1} \varepsilon_{s+t,i} \left[\frac{q^{i\,(\frac{t+n}{k})}}{q^i - 1} - \frac{1}{\ln q^i} \right] \Delta_{n,s} - \varepsilon_{t,i} \left[\frac{q^{i\,\frac{t}{k}}}{q^i - 1} - \frac{1}{\ln q^i} \right] \right\} . \tag{41}$$

Correctness of the exact energy expression (41) can be verified by comparing to that of the scaling shape invariant potentials in one step. When letting $k = 1$, the grading indices are taken to be $s = t = 0$. If we denote $\varepsilon_i = \varepsilon_{0,i}$, then Equation (41) reduces to the relatively simple form [9, 10]

$$E_n^{(-)} = \sum_{i=0}^{\infty} \varepsilon_i \, \frac{q^{i\,n} - 1}{q^i - 1} , \tag{42}$$

which yields $E_0^{(-)} = 0$, as is necessary by the requirement of unbroken SUSY. Finally, let us compute the energy difference between two adjacent eigenstates for Equation (42). We find that

$$\mathcal{R}(a_n) = E_{n+1}^{(-)} - E_n^{(-)} = \sum_{i=0}^{\infty} \varepsilon_i \, q^{i\,n} = \sum_{i=0}^{\infty} \omega_i \, a_n^i , \tag{43}$$

where $\varepsilon_i = \omega_i \, a_0^i$. Obviously, Equations (43) and (37) (by setting $s = 0$) are equivalent.

III-3. The 'exotic'-I shape invariant potentials

The detailed algebraic properties of the 'exotic'-I shape invariant potentials in k steps will be determined. In this class, the shape invariant parameters are related to others by the transformation [11]: $a_{m+1} = q \, a_m^p$, for $m = 0, 1, 2, \cdots$, $0 < q < 1$, and $p = 2, 3, ...$ Note that $p \neq 1$.

Before going into the details, let us redefine the relevant parameters to simplify the upcoming presentations. To that purpose, we denote $q \equiv \tilde{q}^{\,p-1}$ and $b_m \equiv \tilde{q} \, a_m$, so that the relationship between the shape invariant parameters turns into $b_{m+1} = b_m^p$ and $b_m = (b_0)^{p^m}$. In the general case of the shape invariance condition in k steps, we thus have $b_{m+\frac{s}{k}} = (b_0)^{p^{s/k}} b_m = (b_0)^{p^{m+s/k}}$, for $s = 0, 1, \cdots, k-1$. In terms of the newly defined shape invariant parameter b_m, the unified remainder (19) in the 'exotic'-I class would take the form

$$\mathcal{R}\Big(a_{m+\frac{s}{k}}\Big) = \sum_{i=0}^{\infty} \frac{\omega_{s,i}}{\tilde{q}^{\,i}} \Big(b_{m+\frac{s}{k}}\Big)^i = \sum_{i=0}^{\infty} \tilde{\alpha}_{s,i} \, b_m^i , \tag{44}$$

where we denote $\tilde{\alpha}_{s,i} \equiv \omega_{s,i} (b_0^i)^{p^{s/k}} / \tilde{q}^{\,i}$. In addition, when comparing Equations (44) and (18), we find that the relation between the coefficients is $\alpha_{s,i} = \tilde{q}^{\,i} \, \tilde{\alpha}_{s,i}$. In the same vein, if we write $n = mk + s$, Equation (44) then becomes

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1}\Bigg[\sum_{i=0}^{\infty}\frac{\omega_{s,i}}{\tilde{q}^{\,i}}\,(b_0^i)^{p^{n/k}}\Bigg]\Delta_{n,s} = \sum_{s=0}^{k-1}\Bigg[\sum_{i,j=0}^{\infty}\frac{\omega_{s,i}}{\tilde{q}^{\,i}}\frac{(\ln b_0^i)^j}{j!}\,e^{\frac{n}{k}\ln p^j}\Bigg]\Delta_{n,s}\,. \tag{45}$$

It is stressed that the term that survives in the index s summation above satisfies the condition $n - s = 0 \bmod k$.

Now, by denoting the new parameters $\varepsilon^{\,j}_{s,i} \equiv \omega_{s,i}(\ln b_0^i)^j/\tilde{q}^{\,i}j!$ and $\sigma_j \equiv \ln p^j$, we arrive at the expression

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1}\Bigg[\sum_{i,j=0}^{\infty}\varepsilon^{\,j}_{s,i}\,e^{\frac{n}{k}\,\sigma_j}\Bigg]\Delta_{n,s}\,. \tag{46}$$

Apparently, the remainder $\mathcal{R}\big(a_{\frac{n}{k}}\big)$ in Equation (46) again is of the standard exponential form, similar to that in Equation (32). As in the preceding class, the associated algebraic properties can be analogously determined. Besides, the equivalent operator expression $\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big)$ of Equation (46) are expressible as follows

$$\mathcal{R}\big(\boldsymbol{\alpha}(\mathcal{N})\big) = \sum_{s=0}^{k-1}\Bigg[\sum_{i,j=0}^{\infty}\varepsilon^{\,j}_{s,i}\,p^{j(N_0-\mathcal{N})}\Bigg]\Pi_s\,. \tag{47}$$

Here, Equation (46) can be reproduced from Equation (47) when the latter one acts on the number eigenstate $\big|N_0 - \frac{n}{k}\big\rangle$.

The corresponding structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of the 'exotic'-I shape invariant potentials in k steps, which has the unified remainder of the form (47), is consequently determined, via Equation (33), as

$$\begin{aligned}\mathcal{F}\Big(a_{\frac{n}{k}}\Big) &= C - \sum_{s,t=0}^{k-1}\Bigg[\sum_{i,j=0}^{\infty}\varepsilon^{\,j}_{s+t,i}\bigg(\frac{e^{(\frac{t+n}{k})\,\sigma_j}}{e^{\sigma_j}-1}+\frac{1}{\sigma_j}\bigg)\Bigg]\Delta_{n,s}\,,\\ &= C - \sum_{s,t=0}^{k-1}\Bigg[\sum_{i,j=0}^{\infty}\varepsilon^{\,j}_{s+t,i}\bigg(\frac{p^{j(\frac{t+n}{k})}}{p^j-1}+\frac{1}{\ln p^j}\bigg)\Bigg]\Delta_{n,s}\,,\end{aligned} \tag{48}$$

where $n = 0, 1, 2, \cdots$, the convention $\varepsilon^{\,j}_{s+t,i} \equiv \varepsilon^{\,j}_{s+t \bmod k,i}$ is used, and C is a constant to yield the structure function positive definite. It is mentioned that the term inside the parenthesis on the second line reduces to $\frac{t+n}{k}$, in the limit $j \to 0$.

We mention that the structure function (48) and the operators $\{\mathcal{N}, \Pi_s, \mathcal{A}, \mathcal{A}^\dagger\}$ in Equations (9) and (10) altogether define the simplified algebra of the 'exotic'-I shape invariance condition in arbitrary k steps. Now, according to Equation (17), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in this class is given by ($n = 0, 1, 2, \cdots$)

$$E_n^{(-)} = \sum_{t=0}^{k-1}\sum_{i,j=0}^{\infty}\Bigg\{\sum_{s=0}^{k-1}\varepsilon^{\,j}_{s+t,i}\bigg[\frac{p^{j(\frac{t+n}{k})}}{p^j-1}+\frac{1}{\ln p^j}\bigg]\Delta_{n,s} - \varepsilon^{\,j}_{t,i}\bigg[\frac{p^{j\frac{t}{k}}}{p^j-1}+\frac{1}{\ln p^j}\bigg]\Bigg\}\,. \tag{49}$$

Correctness of the exact energy expression (49) can be checked for the ordinary 'exotic'-I shape invariant potentials. When letting $k = 1$, the grading indices are taken to be $s = t = 0$. By denoting $\varepsilon_i^j = \varepsilon_{0,i}^j$ in Equation (49), we obtain [11]

$$E_n^{(-)} = \sum_{i,j=0}^{\infty} \varepsilon_i^j \, \frac{p^{j\,n} - 1}{p^j - 1} = \sum_{m=0}^{n-1} \sum_{i=0}^{\infty} \frac{\omega_i}{\tilde{q}^{\,i}} \, (b_0^i)^{p^m} , \tag{50}$$

which yields $E_0^{(-)} = 0$, as is a must for unbroken SUSY. Note that $\sum_{m=0}^{-1} = 0$ and $\varepsilon_i^j \equiv \omega_i (\ln b_0^i)^j / \tilde{q}^{\,i} j!$. Finally, let us compute the energy difference between two adjacent eigenstates from Equation (50). It is found that

$$\mathcal{R}(a_n) = E_{n+1}^{(-)} - E_n^{(-)} = \sum_{i=0}^{\infty} \frac{\omega_i}{\tilde{q}^{\,i}} \, (b_n)^i = \sum_{i=0}^{\infty} \frac{\omega_i}{q^{\,i/p-1}} \Big(q^{\frac{1}{p-1}} a_0 \Big)^{i\,p^n} , \tag{51}$$

where $\omega_i = \omega_{0,i}$. Consistency of Equations (51) and (44) (by setting $s = 0$) is manifest.

III-4. The 'exotic'-II shape invariant potentials

In the literature, the fourth known class is the 'exotic'-II shape invariant potentials [11], in which the shape invariant parameters are related by the relation $a_1 = qa_0/(1 + ra_0)$, where $0 < q < 1$, a_0 a constant, and $ra_0 \ll 1$, so that we can expand the denominator $(1 + ra_0)^{-1}$ in powers of a_0. By recursively using this relation, we arrive for the general a_m at

$$a_m = \frac{q^m a_0}{1 + ra_0 \Big(\frac{1-q^m}{1-q} \Big)} \equiv \frac{\rho \, q^m}{1 + \delta \, q^m} , \tag{52}$$

where in the second equality the parameters ρ and δ are denoted, respectively, as

$$\rho = \frac{(q-1)a_0}{q - 1 - ra_0} , \qquad \delta = \frac{ra_0}{q - 1 - ra_0} . \tag{53}$$

Note that $|\delta| < 1$ for $0 < q < 1$, such that the denominator of a_m in the second equality of Equation (52) can be further expanded in powers of q^m.

In the present 'exotic'-II class, the unified remainder (19) of the shape invariance condition in k steps still takes the standard form $\mathcal{R}\big(a_{m+\frac{s}{k}}\big) = \sum_{i=0}^{\infty} \omega_{s,i} \big(a_{m+\frac{s}{k}}\big)^i$. Therefore, if we write $n = mk + s$, this remainder becomes

$$\mathcal{R}\Big(a_{\frac{n}{k}}\Big) = \sum_{s=0}^{k-1} \Bigg[\sum_{i=0}^{\infty} \omega_{s,i} \rho^i \sum_{j=0}^{\infty} \binom{-i}{j} \delta^j q^{(i+j)\frac{n}{k}} \Bigg] \Delta_{n,s} = \sum_{s=0}^{k-1} \Bigg[\sum_{i,j=0}^{\infty} \varepsilon_{s,i}^j \, e^{\frac{n}{k}\,\sigma_{i+j}} \Bigg] \Delta_{n,s} . \tag{54}$$

Here, in the first equality we have expanded the denominator $(1 + \delta\, q^{n/k})^{-i}$ of $(a_{\frac{n}{k}})^i$ with the help of the second expression of Equation (52) and $\binom{-i}{j}$ is the usual binomial coefficient. While in the second equality we set the new parameters $\varepsilon_{s,i}^j \equiv \binom{-i}{j} \omega_{s,i} \rho^i \delta^j$ and $\sigma_{i+j} \equiv \ln q^{\,i+j}$

to simplify the notations. It is once again noted that the term that survives in the index s summation above satisfies the condition: $n - s = 0 \bmod k$.

The remainder $\mathcal{R}(a_{\frac{n}{k}})$ in Equation (54) is of the expected exponential form, similar to that in Equation (32). As a result, the associated algebraic properties can similarly be determined. In addition, the equivalent operator expression $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ of Equation (54) can be represented by

$$\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1} \Big[\sum_{i,j=0}^{\infty} \varepsilon_{s,i}^{j} \, q^{(i+j)(N_0-\mathcal{N})} \Big] \Pi_s \, . \tag{55}$$

Hence, Equation (54) can be easily reproduced from Equation (55).

For the unified remainder (55), we determine the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of the 'exotic'-II shape invariant potentials in k steps, via Equation (33), as

$$\begin{aligned} \mathcal{F}\Big(a_{\frac{n}{k}}\Big) &= C - \sum_{s,t=0}^{k-1} \Big[\sum_{i,j=0}^{\infty} \varepsilon_{s+t,i}^{j} \Big(\frac{e^{(\frac{t+n}{k})\,\sigma_{i+j}}}{e^{\sigma_{i+j}} - 1} + \frac{1}{\sigma_{i+j}} \Big) \Big] \Delta_{n,s} \, , \\ &= C - \sum_{s,t=0}^{k-1} \Big[\sum_{i,j=0}^{\infty} \varepsilon_{s+t,i}^{j} \Big(\frac{q^{(i+j)(\frac{t+n}{k})}}{q^{i+j} - 1} + \frac{1}{\ln q^{i+j}} \Big) \Big] \Delta_{n,s} \, , \end{aligned} \tag{56}$$

where $n = 0, 1, 2, \cdots$, the convention $\varepsilon_{s+t,i}^{j} \equiv \varepsilon_{s+t \bmod k,i}^{j}$ is used, and C is a constant to yield the structure function positive definite. It is mentioned that the term inside the parenthesis on the second line reduces to $\frac{t+n}{k}$, in the limit $i + j \to 0$.

We mention again that the structure function (56) and the operators $\{\mathcal{N}, \Pi_s, \mathcal{A}, \mathcal{A}^\dagger\}$ in Equations (9) and (10) altogether define the simplified algebra of the 'exotic'-II shape invariance condition in arbitrary k steps. According to Equation (17), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in the present class will be $(n = 0, 1, 2, \cdots)$

$$E_n^{(-)} = \sum_{t=0}^{k-1} \sum_{i,j=0}^{\infty} \Big\{ \sum_{s=0}^{k-1} \varepsilon_{s+t,i}^{j} \Big[\frac{q^{(i+j)(\frac{t+n}{k})}}{q^{i+j} - 1} + \frac{1}{\ln q^{i+j}} \Big] \Delta_{n,s} - \varepsilon_{t,i}^{j} \Big[\frac{q^{(i+j)\frac{t}{k}}}{q^{i+j} - 1} + \frac{1}{\ln q^{i+j}} \Big] \Big\} . \tag{57}$$

Correctness of the exact expression (57) can be checked for the ordinary 'exotic'-II shape invariant potentials. When letting $k = 1$, the grading indices are chosen as $s = t = 0$. By denoting $\varepsilon_i^j = \varepsilon_{0,i}^j$ in Equation (57) and letting $\omega_i = \omega_{0,i}$, we obtain [11]

$$\begin{aligned} E_n^{(-)} &= \sum_{i,j=0}^{\infty} \varepsilon_i^j \, \frac{q^{(i+j)n} - 1}{q^{i+j} - 1} = \sum_{m=0}^{n-1} \Big[\sum_{i=0}^{\infty} \omega_i \rho^i \sum_{j=0}^{\infty} \binom{-i}{j} \delta^j q^{(i+j)m} \Big] \\ &= \sum_{m=0}^{n-1} \Big[\sum_{i=0}^{\infty} \omega_i \Big(\frac{\rho\, q^m}{1 + \delta q^m} \Big)^i \Big] , \end{aligned} \tag{58}$$

which yields $E_0^{(-)} = 0$, as is necessary for unbroken SUSY. Here, by definition $\sum_{m=0}^{-1} = 0$.

Finally, we report the energy difference between two adjacent eigenstates. From equation (58), we obtain

$$\mathcal{R}(a_n) \;=\; E_{n+1}^{(-)} - E_n^{(-)} \;=\; \sum_{i=0}^{\infty} \omega_i \left(\frac{\rho\, q^n}{1 + \delta q^n} \right)^i . \tag{59}$$

Consistency of Equations (59) and (54) (by setting $s = 0$) is evident.

IV. CONCLUSIONS

In the literature of SUSY QM, it is known that the shape invariance condition (in one step) leads immediately to all popular, analytically solvable potentials, which can be further classified into four distinct classes, namely, the translational, the scaling, the 'exotic'-I, and the 'exotic'-II classes, depending on how the shape invariant parameters are related. It is also known that all the shape invariant potentials (in one step) possess the so-called potential algebra; thus they can be studied by algebraic methods.

In this paper, we present the general algebraic properties for the four known classes of shape invariant potentials that are extended from the ordinary one step to arbitrary k steps, based on the formalism of the Z_k-graded deformed oscillator algebra developed in [24]. For each class, we first discuss the respective relationship between the neighboring k-step shape invariant parameters, $a_{m+\frac{s}{k}}$. We also assume that the Z_k-graded unified remainder $\mathcal{R}\big(a_{\frac{n}{k}}\big)$ is analytic, admitting a power series expansion about $a_{\frac{n}{k}}$. Then according to the remainder-structure-function relation, we are able to construct the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ in a systematic way, which together with the operators $\mathcal{N}$, Π_s, $\mathcal{A}$, and $\mathcal{A}^\dagger$ define, for each class, the simplified shape invariant potential algebra in k steps. Furthermore, the energy spectrum $E_n^{(-)}$ for the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ can be straightforwardly determined. Finally, correctness of these obtained k-step results are verified by comparing to those of the ordinary shape invariant potentials in one step.

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