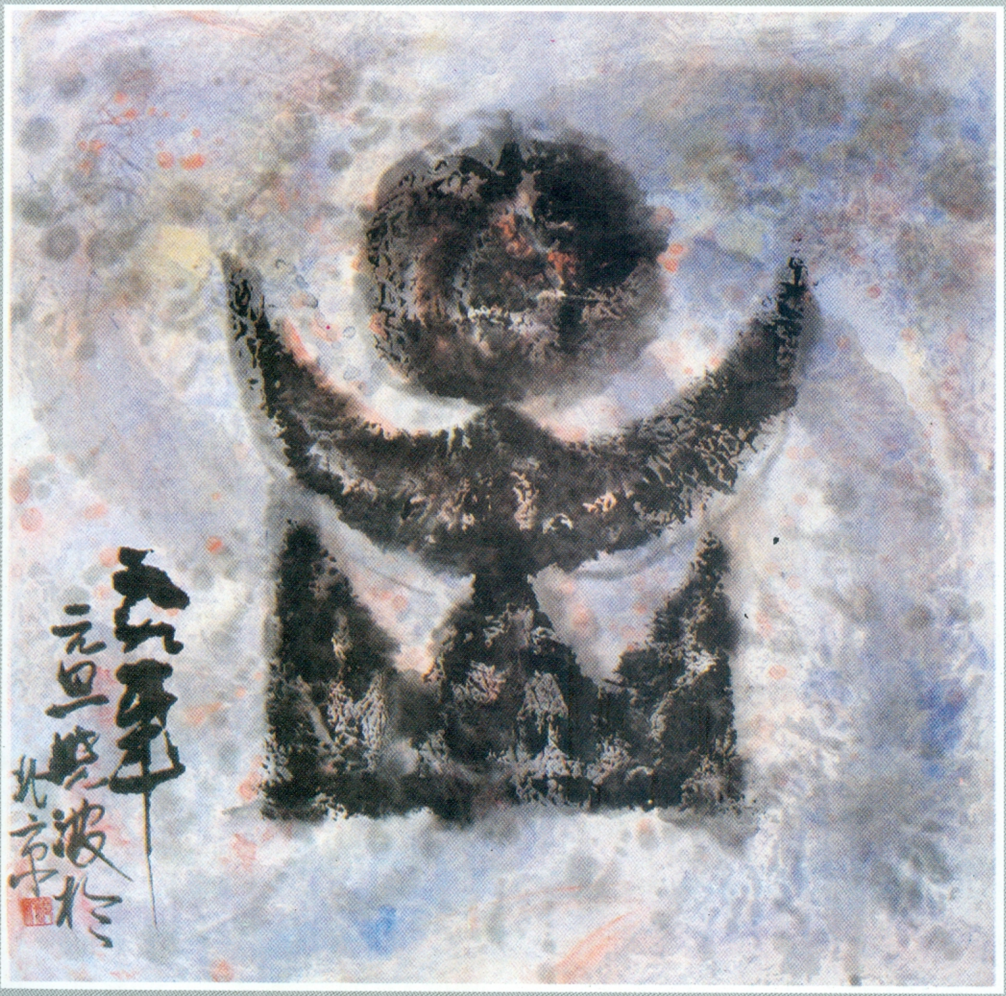
中国高等科学技术中心

CCAST—WL WORKSHOP SERIES: VOLUME 17

QUANTUM GROUPS and LOW DIMENSIONAL FIELD THEORY



edited by

Han—Ying Guo, Zhaoming Qiu, Ke Wu

and Zhong—yuan Zhu

Beijing March 12—April 10, 1992

Preface to the CCAST-World Laboratory Series

The China Center of Advanced Science and Technology (CCAST) was established in Beijing on October 17, 1986 to introduce important frontier areas of science to China, to foster their growth by providing a suitable environment, and to promote free exchange of scientific information between China and other nations.

An important component of CCAST's activities is the organization of domestic and international* symposium/workshops. Each academic year we hold about 15 domestic symposium/workshops which last an average of one month each. The subjects are carefully chosen to cover advanced areas that are of particular interest to Chinese scientists. About 20-60 participants, from senior scientists to graduate students, are selected on a nationwide basis for each program. During each workshop, these scientists hold daily seminars and work closely with each other.

Since 1990, CCAST has also sponsored a vigorous program for young Chinese scientists who have already made world-class contributions and are currently doing research abroad. They return to China to lecture at CCAST and to collaborate with their colleagues at home. In this way, they can bring to China their own expertise, and when they go back to their institutions abroad they will be able to circulate in turn the knowledge they have acquired in China.

China is at a pivotal point in her scientific development. She is gradually emerging as an important and dynamic force in shaping the advanced science and technology of the future. This series is part of this remarkable evolution. It records the effort, dedication, and sharing of knowledge by the Chinese scientists, at home and abroad.

T. D. Lee

*The CCAST international symposium/workshop series is published separately by Gordon and Breach Scientific Publishers

Contents

CCAST-92-04
ASIAM-92-07
ASITP-92-12
March, 1992

Bicovariant Differential Calculus on Quantum Group $GL_q(n)$ †

XIAODONG SUN SHIKUN WANG

CCAST (World laboratory),
P.O. Box 3730, Beijing 100080, China

and

Institute of Applied Mathematics,
Academia Sinica, Beijing 100080, China‡

Abstract

The de Rham complex of the quantum group $GL_q(n)$ is presented. And we show that the differential calculus on the quantum group $GL_q(n)$ given in this paper is bicovariant. The noncommutative differentail calculus on the quantum group $SL_q(n)$ is also discussed.

† Supported by National Natural Science Foundation of China and Open Foundation of Institute of Theoretical Physics, Academia Sinica.

‡ Mailing address.

§0. Introduction

Recently, much attention has been paid to the non-commutative differential calculus on quantum groups[1-5]. In this paper we will describe the exterior differential calculus on the quantum group $GL_q(n)$. Quantum group theories were developed and several different approaches to construct quantum groups were also introduced in papers [6-9]. In the first part of this paper, we will adopt Faddeev's approach to give the construction of the quantum group $GL_q(n)$ and a concrete subalgebra of the dual of the coordinate ring of $GL_q(n)$. The second part is mainly applied to discuss the first order differential calculus on the quantum group $GL_q(n)$, namely the construction of the exterior differential operator d and the first order differential bimodule. In the third part we will demonstrate in detail that the first order differential calculus given in section two is bicovariant. In the fourth part we will describe how to get the quantum de Rham complex of $GL_q(n)$. A general theory for bicovariant differential calculus on compact matrix pseudogroups was developed by Woronowicz [1]. And the discussions of noncommutative differential calculus on more general quantum groups and quantum spaces can be found in the papers [2]. In the third and fourth sections we mainly adopt Woronowicz's methods and some basic results that are true in Hopf algebras level for general quantum groups. In the last part we shortly remark how the quantum exterior differential calculus on the quantum group $GL_q(n)$ is induced to give the quantum de Rham complex on the quantum group $SL_q(n)$.

This paper is an extension of [10] for more general case $GL_q(n)$, most proofs in [10] are still valid in this paper. In this paper quantum groups are understood as the objects of the inverse category of the Hopf algebras with antipode, which are neither commutative, nor co-commutative. As to Hopf algebras, please see [11]. For simplicity, summation convention is used in the paper.

By the method provided in this paper, we can also give bicovariant differential calculus on quantum groups of B_n, C_n, D_n series and other types[12].

§1. Quantum group $GL_q(n)$

In this section, we will cite some results on the quantum group $GL_q(n)$ without proof, and give some explanation to the symbols applied in this paper.

Let

$$R_q = \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \chi \sum_{\substack{i,j=1 \\ i>j}}^{n} e_{ij} \otimes e_{ji}, \; q \in \mathbb{C}^*. \tag{1.1}$$

where $\chi = q - q^{-1}$, e_{ij} $(i, j = 1, 2, \cdots, n)$ is the element matrix of order n, entries of which are all zeros except that the one on i-th row j-th column is 1, and the symbol $\otimes$ means the tensor product of matrices. One easily checks that the matrix R_q is a solution of the quantum Yang-Baxter equation(QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.2}$$

with R_{ij} as $n^3 \times n^3$ matrices defined via

$$R_{12} = R_q \otimes E, \; R_{13} = (E \otimes P)R_{12}(E \otimes P), \; R_{23} = E \otimes R_q,$$

where E is the unit matrix of order n and P is the permutation matrix in $\mathbb{C}^n \otimes \mathbb{C}^n$. We can also write the matrix R_q in the form of submatrices, i.e.,

$$R_q = (r_{ij})_{1 \leq i,j \leq n}, \tag{1.3}$$

with

$$r_{ij} = \begin{cases} \chi e_{ji}, & i > j, \\ 0, & i < j, \\ E + (q-1)e_{ii}, & i = j, \end{cases} \tag{1.4}$$

Take n^2 elements t_{ij} $(i, j = 1, 2, \cdots, n)$ and arrange them into a matrix $T = (t_{ij})_{1 \leq i,j \leq n}$. Let $\mathbb{C}[T]$ denote the free associative algebra with unit 1 generated by the n^2 elements t_{ij} $(i, j = 1, 2, \cdots, n)$, and let $\{R_q T_1 T_2 - T_2 T_1 R_q\}$ be the two-sided ideal of $\mathbb{C}[T]$ generated by the relations $R_q T_1 T_2 - T_2 T_1 R_q$, where $T_1 = T \otimes E$, $T_2 = E \otimes T$. Then the quotient

$$\mathrm{Fun}(M_q(n)) = \mathbb{C}[T]/\{R_q T_1 T_2 - T_2 T_1 R_q\} \tag{1.5}$$

has the structure of a bialgebra with the $\mathbb{C}$-linear structure maps, the comultiplication Δ and the counit ε, fixed by the following values for the generators:

$$\Delta T = T \odot T, \tag{1.6}$$

$$\varepsilon(T) = E. \tag{1.7}$$

where the symbol $\odot$ means $\Delta t_{ij} = t_{ik} \otimes t_{kj}$. Both Δ and ε are algebra homomorphism. And the multiplication m on $\mathrm{Fun}(M_q(n))$ corresponds to the ordinary one of functions, i.e.,

$$m(x \otimes y) = xy, \; \forall x, y \in \mathrm{Fun}(M_q(n)),$$

and the unit map i is defined by

$$\begin{array}{rccl} i: & \mathbb{C} & \longrightarrow & \mathrm{Fun}(M_q(n)), \\ & \lambda & \longmapsto & \lambda \cdot 1. \end{array}$$

When $q = 1$, $\mathrm{Fun}(M_q(n))$ coincides with the commutative algebra $\mathrm{Fun}(M(n))$ of coordinate functions on the matrix algebra $M(n, \mathbb{C})$. So, we can regard $\mathrm{Fun}(M_q(n))$ as the deformation of $\mathrm{Fun}(M(n))$, or the algebra of coordinate functions on the quantum matrix algebra $M_q(n)$ of rank n associated with the matrix R_q.

Write S_n for the symmetric group on n letters and write $l(\sigma)$ for the length of $\sigma \in S_n$. Namely, $l(\sigma)$ is the minimal number of the terms required to express σ as a product of the simple transposition $(i, i+1)$. For the quantum matrix algebra $M_q(n)$, the quantum determinant can be defined as:

$$Det_q T = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1\sigma_1} t_{2\sigma_2} \cdots t_{n\sigma_n}. \tag{1.8}$$

The quantum determinant has the following properties

$$\Delta(Det_q T) = Det_q T \otimes Det_q T, \tag{1.9}$$

$$\varepsilon(Det_q T) = 1. \tag{1.10}$$

Remark 1. In what follows, we identify the element t_{ij} $(i, j = 1, 2, \cdots, n)$ and $Det_q T$ with their corresponding equivalent classes.

Definition 1.1

$$\text{Fun}(GL_q(n)) = \text{Fun}(M_q(n))[t]/\{tDet_q T - (Det_q T)t,\ tDet_q T - 1\}, \tag{1.11}$$

where t is a new generator and $\{tDet_q T - (Det_q T)t,\ tDet_q T - 1\}$ means the two-sided ideal of $\text{Fun}(M_q(n))[t]$ generated by the two relations $tDet_q T - (Det_q T)t$, $tDet_q T - 1$.

At this time, we naturally extend the structure maps m, i, Δ and ε of the bialgebra $\text{Fun}(M_q(n))$ to the quotient $\text{Fun}(GL_q(n))$ and require

$$\Delta(t) = t \otimes t,\ \varepsilon(t) = 1 \tag{1.12}$$

to make it also a bialgebra. Furthermore, the antipode S on $\text{Fun}(GL_q(n))$ can be uniquely determined by the requirement that $TS(T) = E \cdot \mathbf{1} = S(T)T$, its definition on the generators t_{ij} $(i, j = 1, 2, \cdots, n)$ and t is given by

$$S(t_{ij}) = (-q)^{i-j} t Det_q T_{ji}, \quad i, j = 1, 2, \cdots, n, \tag{1.13}$$

$$S(t) = Det_q T, \tag{1.14}$$

where T_{ij} denote the $(n-1) \times (n-1)$ generic matrix obtained by deleting row i and column j of the generated matrix $T = (t_{ij})_{1 \le i,j \le n}$. After introducing the antipode we obtain

Theorem 1.1 $\text{Fun}(GL_q(n))$ is a Hopf algebras with respect to m, i, Δ, ε and S.

$\text{Fun}^*(GL_q(n))$ denotes the dual of $\text{Fun}(GL_q(n))$. We now give two sets of linear functionals $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ and arrange them into two $n \times n$ matrices.

$$L^{\pm} = (l_{ij}^{\pm})_{1 \le i,j \le n}.$$

To describe $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ explicitly. We first define that the values of the linear functionals $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ on the generators t_{ij} $(i, j = 1, 2, \cdots, n)$ of $\text{Fun}(GL_q(n))$ are given by

$$l_{ij}^{\pm}(T) = \lambda_{\pm}^{\pm 1} r_{ij}^{\pm}, \quad 0 \ne \lambda_{\pm} \in \mathbb{C}, \tag{1.15}$$

$$l_{ij}^{\pm}(1) = \delta_{ij}, \quad i, j = 1, 2, \cdots, n, \tag{1.16}$$

where

$$r_{ij}^{+} = \begin{cases} \chi e_{ji}, & i < j, \\ E + (q-1)e_{ii}, & i = j, \\ 0, & i > j. \end{cases} \tag{1.17}$$

$$r_{ij}^{-} = \begin{cases} 0, & i < j, \\ E + (q^{-1}-1)e_{ii}, & i = j, \\ -\chi e_{ji}, & i > j. \end{cases} \tag{1.18}$$

If denote $R^{+} = (r_{ij}^{+})_{1\leq i,j\leq n}$, $R^{-} = (r_{ij}^{-})_{1\leq i,j\leq n}$, then $R^{+} = PR_qP$, $R^{-} = R_q^{-1}$, where P is the permutation matrix, and can be written in the form of submatrices as

$$P = (P_{ij})_{1\leq i,j\leq n} = (e_{ji})_{1\leq i,j\leq n}.$$

From the fact that the matrix R_q satisfies QYBE, it follows that

$$R_{12}^{+}R_{13}^{\pm}R_{23}^{\pm} = R_{23}^{\pm}R_{13}^{\pm}R_{12}^{+} \tag{1.19}$$

with $R_{ij}^{\pm}$ as $n^3 \times n^3$ matrices defined by

$$R_{12}^{\pm} = R^{\pm} \otimes E,\ R_{13}^{\pm} = (E\otimes P)R_{12}^{\pm}(E\otimes P),\ R_{23}^{\pm} = E \otimes R^{\pm}.$$

For arbitrary element of $\mathrm{Fun}(GL_q(n))$ the definition of $l_{ij}^{\pm}$ is given by the following induction,

$$l_{ij}^{\pm}(xy) = l_{ik}^{\pm}(x)l_{kj}^{\pm}(y),\quad \forall x,y \in \mathrm{Fun}(GL_q(n)). \tag{1.20}$$

Now what we need to do is to give the value of $l_{ij}^{\pm}$ $(i,j = 1,2,\cdots,n)$ on the generator t of $\mathrm{Fun}(GL_q(n))$. For this we rewrite (1.15), (1.16) and (1.20) in the form of submatrices as follows

$$< L^{\pm}, T >= (l_{ij}^{\pm}(T))_{1\leq i,j\leq n} = \lambda_{\pm}^{\pm 1}R^{\pm}, \tag{1.21}$$

$$< L^{\pm}, 1 >= E, \tag{1.22}$$

$$< L^{\pm}, xy >= (l_{ij}^{\pm}(xy))_{1\leq i,j\leq n} =< L^{\pm}, x >< L^{\pm}, y > . \tag{1.23}$$

Then the action of $l_{ij}^{\pm}$ $(i,j = 1,2,\cdots,n)$ on the generator t is

$$< L^{\pm}, t >= (l_{ij}^{\pm}(t))_{1\leq i,j\leq n} =< L^{\pm}, Det_qT >^{-1} . \tag{1.24}$$

In fact, we have

$$< L^{+}, t_{ij} >= \lambda_{+}r_{ij},$$

$$< L^{+}, Det_qT >=< L^{+}, \prod_{i=1}^{n} t_{ii} >= \lambda_{+}^{n}qE.$$

Thus

$$< L^{+}, t >= \lambda_{+}^{-n}q^{-1}E \tag{1.25}$$

holds. Similarly, we have

$$< L^{-}, t >= \lambda_{-}^{n}qE. \tag{1.26}$$

We can check that the action of the linear functionals $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ given in above way on the two-sided ideal generated by the relations $R_qT_1T_2 - T_2T_1R_qt$, $Det_qT-(Det_qT)t$, $tDet_qT-1$ is zero. This shows $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ is well defined on the Hopf algebra $\mathrm{Fun}(GL_q(n))$ and then the two sets of functionals $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ belong to $\mathrm{Fun}^*(GL_q(n))$ (also see Proposition 1.2 in [10]). Furthermore, with the comultiplication Δ of $\mathrm{Fun}(GL_q(n))$, the multiplication m^* among $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ can be introduced. Suppose ξ,η are two polynomials of $l_{ij}^{\pm}$. We define

$$m^*(\xi\otimes\eta)(x)=(\xi\eta)(x)=(\xi\otimes\eta)\Delta x, \qquad \forall x\in \mathrm{Fun}(GL_q(n)), \tag{1.27}$$

and introduce two new linear functionals $l^{\pm}$ by the following formulas

$$< l^{\pm},T >= l^{\pm}(T)=(l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm}(T))^{-1}, \tag{1.28}$$
$$< l^{\pm},t >= l^{\pm}(t)=(l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm}(t))^{-1}, \tag{1.29}$$
$$< l^{\pm},1 >= l^{\pm}(1)=1, \tag{1.30}$$
$$< l^{\pm},xy >= l^{\pm}(xy)=l^{\pm}(x)l^{\pm}(y),\ \forall x,y\in \mathrm{Fun}(GL_q(n)). \tag{1.31}$$

It is also easy to see that

$$l^{\pm}(\{R_qT_1T_2-T_2T_1R_q,\ tDet_qT-(Det_qT)t,\ tDet_qT-1\})=0.$$

$\mathrm{Fun}_0^*(GL_q(n))$ denotes the associative subalgebra of $\mathrm{Fun}^*(GL_q(n))$ generated by $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ and $l^{\pm}$ via the multiplication m^* in (1.27). Obviously, the unit of the algebra $\mathrm{Fun}_0^*(GL_q(n))$ is ε, i.e. the counit of $\mathrm{Fun}(GL_q(n))$. However, it should be pointed out that the $2(n^2+1)$ elements $l_{ij}^{\pm}$ $(i,j=1,2,\cdots,n)$ and $l^{\pm}$ are not free generators, which are subordinate to the communication relations given by the following two propositions proofs of which are due to (1.19) and definition of $L^{\pm}$ (also see Proposition 1.4 in [10]).

Proposition 1.1

$$R^+L_1^{\pm}L_2^{\pm}=L_2^{\pm}L_1^{\pm}R^+, \tag{1.32}$$
$$R^+L_1^{+}L_2^{-}=L_2^{-}L_1^{+}R^+, \tag{1.33}$$

where $L_1^{\pm}=L^{\pm}\otimes E$, $L_2^{\pm}=E\otimes L^{\pm}$.

Proposition 1.2

$$\text{(i)}\quad l^{\pm}\prod_{i=1}^{n}l_{ii}^{\pm}=\varepsilon, \tag{1.34}$$
$$\text{(ii)}\quad l^{+}L^{\pm}=L^{\pm}l^{+},\ l^{-}L^{\pm}=L^{\pm}l^{-}, \tag{1.35}$$
$$\text{(iii)}\quad l^{+}l^{-}=l^{-}l^{+}, \tag{1.36}$$
$$\text{(iv)}\quad l_{ij}^{+}=0,\ i>j,\ l_{ij}^{-}=0,\ i<j. \tag{1.37}$$

The homomorphisms Δ^*, ε^*, S^* on $\mathrm{Fun}_C^*(GL_q(n))$ are defined as

$$\Delta^*(L^{\pm})=L^{\pm}\odot L^{\pm}, \qquad \Delta^*(l^{\pm})=l^{\pm}\otimes l^{\pm},$$
$$\varepsilon^*(L^{\pm})=E, \qquad \varepsilon^*(l^{\pm})=1,$$
$$S^*(L^{\pm})=(-q)^{j-i}l^{\pm}(Det_{q^{-1}}L_{ji}^{\pm})_{1\le i,j\le n}, \qquad S^*(l^{\pm})=l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm},$$

where $L_{ij}^{\mp}$ is the submatrix of $L^{\pm}$ defined like T_{ij} in (1.13). We can check the compatibility of the maps Δ^*, ε^*, S^* and the relations in Propositions 1.1 and 1.2. Namely, the actions of Δ^*, ε^*, and S^* on the relations (1.32)-(1.37) are all zeros (as Proposition 1.5 and 1.6 of [10]). We can also see

$$S^*(L^{\pm})L^{\pm} = L^{\pm}S^*(L^{\pm}) = \varepsilon \cdot E. \tag{1.38}$$

Finally, we have

Theorem 1.2 $\text{Fun}_0^*(GL_q(n))$ is a Hopf subalgebra of $\text{Fun}^*(GL_q(n))$ with respect to m^*, Δ^*, ε^*, S^*.

§2. The first order differential calculus on $GL_q(n)$

Assume $\mathcal{A}$ is an associative algebra with unit. The first order differential calculus on $\mathcal{A}$, which is denoted by (Γ, δ), consists of a bi-module Γ of $\mathcal{A}$ and a linear operator δ satisfying

(i) Leibnitz rule

$$\delta(xy) = (\delta x)y + x\delta y, \qquad \forall x, y \in \mathcal{A}, \tag{2.1}$$

(ii) for arbitrary element ρ in Γ, there always exist some elements $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \cdots, N)$ in $\mathcal{A}$ such that

$$\rho = \sum_{k=1}^{N} x_k \delta y_k. \tag{2.2}$$

Now we regard $\text{Fun}(GL_q(n))$ as $\mathcal{A}$, and for simplicity, give it a special symbol Ω^0. To construct the one order differential calculus on quantum group $GL_q(n)$, what one first has to do is to determine a Ω^0-bimodule which is denoted by Ω^1. For this end, we introduce the convolution "$*$" on Ω^0. For $f \in \text{Fun}^*(GL_q(n))$, the convolution "$*$" from Ω^0 to Ω^0 is defined by

$$f * (x) = (id \otimes f)\Delta x, \quad x \in \text{Fun}(GL_q(n)), \tag{2.3}$$

where id is the identity operator on Ω^0. Furthermore, we introduce two sets of functionals on Ω^0 as follows:

$$\text{(i)} \quad \nabla_{ij} := \frac{1}{\chi}(S^*(l_{ik}^-)l_{kj}^+ - \delta_{ij}\varepsilon), \quad i, j = 1, 2, \cdots, n, \tag{2.4}$$

$$\text{(ii)} \quad \theta_{ijkl} := S^*(l_{ki}^-)l_{jl}^+, \quad i, j, k, l = 1, 2, \cdots, n. \tag{2.5}$$

For the operators $*$, ∇_{ij}, θ_{ijkl}, we have

Proposition 2.1 For $\forall x, y \in \Omega^0, i, j, k, l = 1, 2, \cdots, n$ the formulas

$$\text{(i)} \quad \nabla_{ij}(1) = 0, \quad \theta_{ijkl}(1) = \delta_{ik}\delta_{jl}, \tag{2.6}$$

$$\text{(ii)} \quad \Delta^*\nabla_{ij} = \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij}, \qquad \Delta^*\theta_{ijkl} = \theta_{ijuv} \otimes \theta_{uvkl}, \tag{2.7}$$

$$\text{(iii)} \quad \nabla_{ij} * (xy) = (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y), \qquad \theta_{ijkl} * (xy) = (\theta_{ijuv} * x)(\theta_{uvkl} * y) \tag{2.8}$$

hold.

Proof: Now we prove the first equation of (2.7). A directly calculation shows

$$\begin{aligned} \Delta^*\nabla_{ij} &= \tfrac{1}{\chi}\Delta^*(S^*(l_{ik}^-))\Delta^* l_{kj}^+ - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\ &= \tfrac{1}{\chi}S^*(l_{uk}^-)l_{kv}^+ \otimes S^*(l_{iu}^-)l_{vj}^+ - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\ &= (\nabla_{uv} + \tfrac{1}{\chi}\delta_{uv}\varepsilon) \otimes S^*(l_{iu}^-)l_{vj}^+ - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\ &= \nabla_{uv} \otimes \theta_{uvij} + \tfrac{1}{\chi}\varepsilon \otimes (\delta_{uv}S^*(l_{iu}^-)l_{vj}^+ - \delta_{ij}\varepsilon) \\ &= \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij}. \end{aligned}$$

Next we prove the first equation of (2.8). Let $\Delta x = x_{1,\alpha} \otimes x_{2,\alpha}$, $\Delta y = y_{1,\beta} \otimes y_{2,\beta}$. Since

$$\begin{aligned} \nabla_{ij} * (x \cdot y) &= (id \otimes \nabla_{ij})\Delta(xy) \\ &= (id \otimes \nabla_{ij})\Delta x \Delta y \\ &= x_{1,\alpha}y_{1,\beta}\nabla_{ij}(x_{2,\alpha}y_{2,\beta}) \\ &= x_{1,\alpha}y_{1,\beta}\Delta^*\nabla_{ij}(x_{2,\alpha} \otimes y_{2,\beta}), \end{aligned}$$

applying the first equation of (2.7), we have

$$\begin{aligned} \nabla_{ij} * (xy) &= x_{1,\alpha}y_{1,\beta}(\nabla_{uv}(x_{2,\alpha})\theta_{uvij}(y_{2,\beta}) + \varepsilon(x_{2,\beta})\nabla_{ij}(y_{2,\beta})) \\ &= x_{1,\alpha}\nabla_{uv}(x_{2,\alpha})y_{1,\beta}\theta_{uvij}(y_{2,\beta}) + x_{1,\alpha}\varepsilon(x_{2,\alpha})y_{1,\beta}\nabla_{ij}(y_{2,\beta}) \\ &= (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y). \end{aligned}$$

As for the remained formulae, we leave them to readers.

From (1.38) and (1.7) it follows that

$$< S^*(L^-)L^-, T >=< \varepsilon \cdot E, T >= (\delta_{ij}\varepsilon(T))_{1 \le i,j \le n} = E_{n^2},$$

where E_{n^2} is the unit matrix of order n^2. On the other hand, due to (1.6) one has

$$< S^*(L^-)L^-, T >=< S^*(L^-), T >< L^-, T >=< S^*(L^-), T > \lambda_-^{-1}R^-.$$

So we obtain

$$< S^*(L^-), T >= \lambda_- R_q,$$

i.e.

$$S^*(l_{ij}^-)(T) = \lambda_- r_{ij}. \tag{2.9}$$

Combining (1.4), (1.15), (1.17) with (2.9), and letting $r = \lambda_+\lambda_-$, we can get, if $i = j$,

$$\begin{aligned}(S(l_{ik}^-)l_{kj}^+)(T) &= S(l_{ik}^-)(T)l_{kj}^+(T) \\ &= r\sum_{k<i} r_{ik}r_{kj}^+ + rr_{ii}r_{ii}^+ \\ &= r\chi^2\sum_{k<i} e_{kk} + r(E_n + (q^2-1))e_{ii},\end{aligned}$$

and if $i \neq j$,

$$(S(l_{ik}^-)l_{kj}^+)(T) = r\chi e_{ji}.$$

Thus

$$\nabla_{ij}(T) = \begin{cases} re_{ji}, & i \neq j, \\ \frac{1}{\chi}[(r-1)E_n + r(q^2-1)e_{ij} + r\chi^2\sum_{k<i} e_{kk}], & i = j. \end{cases} \tag{2.10}$$

Similarly, it follows from (1.25) and (1.26) that

$$\nabla_{ij}(t) = \frac{1}{\chi}(\frac{1}{r^n q^2} - 1)\delta_{ij}. \tag{2.11}$$

If we arrange $\nabla_{ij}(t_{kl})$ as a matrix of $n \times n$ blocks,

$$\nabla(T) = (\nabla_{ij}(T))_{1\leq i,j\leq n},$$

where the submatrices are

$$\nabla_{ij}(T) = (\nabla_{ij}(t_{kl}))_{1\leq k,l\leq n},\ 1 \leq i,j \leq n.$$

Then

$$\begin{aligned}\nabla(T) &= \frac{1}{\chi}(S^*(L^-)L^+(T) - \varepsilon\cdot E(T)) \\ &= \frac{1}{\chi}(< S^*(L^-),T >< L^+,T > -E_{n^2}) \\ &= \frac{1}{\chi}(rR_qR^+ - E_{n^2}) \\ &= \frac{1}{\chi}(rR_qPR_qP - E_{n^2}).\end{aligned} \tag{2.12}$$

And

$$(\nabla_{ij}(t))_{1\leq i,j\leq n} = \frac{1}{\chi}(\frac{1}{r^n q^2} - 1)E. \tag{2.13}$$

Now we apply the matrices $\nabla(T)$ and $\frac{1}{\chi}(\frac{1}{r^n q^2} - 1)E$ to construct another matrix. Let

$$M(\lambda) = (M_{n(i-1)+j}^{n(k-1)+l})_{1\leq k,l,i,j\leq n} = (\nabla_{ij}(t_{kl}) + \lambda\delta_{kl}\nabla_{ij}(t))_{1\leq k,l,i,j\leq n}. \tag{2.14}$$

where λ is a complex parameter. Sometimes we also write the matrix $M(\lambda)$ as $M(\lambda) = (M_{ij}^{kl})_{1\leq k,l,i,j\leq n}$.

Proposition 2.2 If $r^n q^2 \neq 1$, for fixed q and r there always exists λ, s.t. the matrix $M(\lambda)$ is invertible.

Proof: If $i \neq j$, then one has

$$M^{n(k-1)+l}_{n(i-1)+j} = \frac{r}{\chi}\chi\delta_{k-j,i-l} = r\delta_{k-j,i-l}.$$

The above equation shows that there is only one non-zero element r in every row or column of $M(\lambda)$ except the rows $(n(k-1)+k)$ and columns $n(i-1)+i$ $(i,k = 1,2,\cdots,n)$. Hence, to determine whether or not the matrix $M(\lambda)$ is invertible we need only to consider the matrix $N(\lambda)$ of order n,

$$\begin{aligned} N(\lambda) &= (M^{n(k-1)+k}_{n(i-1)+i})_{1\le k,i\le n}, \\ M^{n(k-1)+k}_{n(i-1)+i} &= \nabla_{ii}(t_{kk}) + \frac{\lambda}{\chi}(\frac{1}{r^n q^2} - 1). \end{aligned}$$

In fact, the expression of the matrix $N(\lambda)$ is

$$N(\lambda) = \frac{1}{\chi}\begin{pmatrix} b & a & a & \cdots & a \\ c & b & a & \cdots & a \\ c & c & b & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & c & c & \cdots & b \end{pmatrix},$$

where

$$\begin{aligned} a &= r(\chi^2+1) - 1 + \lambda(\frac{1}{r^n q^2} - 1), \\ b &= rq^2 - 1 + \lambda(\frac{1}{r^n q^2} - 1), \\ c &= r - 1 + \lambda(\frac{1}{r^n q^2} - 1). \end{aligned}$$

Straightforward calculation gives

$$\begin{aligned} DetN(\lambda) &= \frac{(b-a)^{n-1}}{\chi^n}(b\frac{q^{2n}-1}{q^2-1} - q^2(b-a)\frac{q^{2n-2}-1}{q^2-1}) \\ &= \frac{r^{n-1}}{\chi q^{n-1}}\left(b\frac{q^{2n}-1}{q^2-1} - (q^{2n-2}-1)\right). \end{aligned}$$

So, when $r^n q^2 \neq 1$, for fixed q and r we have λ such that $DetN(\lambda) \neq 0$.

Now we are going to construct a Ω^0-bimodule Ω^1. Define dt_{ij} and dt the one order differentials of the generators t_{ij} $(i,j = 1,2,\cdots,n)$ and t of Ω^0. And let Ω^1 be the left Ω^0-module generated by the elements ω^{kl} $(k,l = 1,2,\cdots,n)$, satisfying the following conditions:

$$S(t_{im})dt_{mj} = \nabla_{kl}(t_{ij})\omega^{kl}, \tag{2.15}$$
$$Det_q T dt = \nabla_{kl}(t)\omega^{kl}. \tag{2.16}$$

The right multiplication in the left module Ω^1 is defined by

$$\omega^{ij}\cdot x = (\theta_{ijkl} * x)\omega^{kl}, \ \forall x \in \Omega^0, \tag{2.17}$$

so that Ω^1 is a bimodule of Ω^0. Due to the argument in the last section that θ_{ijkl} is a functional on Ω^0, the right multiplication is well defined. Namely,

$$\omega^{ij} \cdot x = (id \otimes \theta_{ijkl})\Delta x \omega^{kl}$$

is independent of the choice of the representation element x. Furthermore, by Proposition 2.1 this right multiplication is associable, i.e.

$$\omega^{ij}(xy) = (\omega^{ij}x)y, \quad \forall x, y \in \Omega^0.$$

And it is clear

$$\omega^{ij} \cdot \mathbf{1} = \omega^{ij}.$$

So Ω^1 is a Ω^0-bimodule.

Definition 2.1 The differential operation d from Ω^0 to Ω^1 is defined by

$$dx = \nabla_{ij} * (x)\omega^{ij}. \tag{2.18}$$

Theorem 2.1 $\{\Omega^1, d\}$ constructed above is the first order differential calculus on Ω^0.

Proof: We need only to check (2.1) and (2.2) hold. Combining (2.8) in Proposition 2.1 with (2.17) one directly verifies d is a differential operator satisfying Leibnitz rule (2.1). To verify $\{\Omega^1, d\}$ satisfies (2.2) we need to prove ω^{ij} $(i, j = 1, 2, \cdots, n)$, the generators of Ω^1, can be represented by the form $\sum_{k=1}^{N} x_k dy_k$, $x_k, y_k \in \Omega^0$. By (2.18), (2.15) and (2.16), we have

$$\begin{aligned} dt_{kl} &= (id \otimes \nabla_{ij})\Delta t_{kl}\omega^{ij}, \\ dt &= (id \otimes \nabla_{ij})\Delta t\omega^{ij}. \end{aligned}$$

Using the Proposition 2.2 we can find λ such that the matrix $M(\lambda)$ is nonsingular. So from

$$S(t_{km})dt_{ml} + \lambda\delta_{kl}Det_qTdt = (\nabla_{ij}(t_{kl}) + \lambda\delta_{kl}\nabla_{ij}(t))\omega^{ij},$$

one obtains

$$\omega^{ij} = M^{-1}(\lambda)^{n(i-1)+j}_{n(k-1)+l}[S(t_{km})dt_{ml} + \lambda\delta_{kl}Det_qTdt], \tag{2.19}$$

which implies that for $\{\Omega^1, d\}$, (2.2) holds.

It should be point out that the expression of ω^{ij} in (2.19) is independent of the parameter λ. This can be proved by a simple argument in linear algebra.

By (2.17), we also have the cross relation among dt_{ij}, dt and t_{ij}, t as follows,

$$\begin{aligned} dT \cdot z &= (\nabla_{ij} * T)\omega^{ij} z \\ &= (\nabla_{ij} * T)\theta_{ijkl} * (z)M^{-1}(\lambda)^{n(k-1)+l}_{n(\alpha-1)+\beta}(S(t_{\alpha\gamma})dt_{\gamma\beta} + \lambda\delta_{\alpha\beta}(Det_qT)dt), \\ dt \cdot z &= (\nabla_{ij} * t)\omega^{ij} z \\ &= (\nabla_{ij} * t)\theta_{ijkl} * (z)M^{-1}(\lambda)^{n(k-1)+l}_{n(\alpha-1)+\beta}(S(t_{\alpha\gamma})dt_{\gamma\beta} + \lambda\delta_{\alpha\beta}(Det_qT)dt), \end{aligned} \tag{2.20}$$

where $z = t_{uv}$ or t $(u, v = 1, 2, \cdots, n)$.

Remark 2. The case of $r^n q^2 = 1$ will be discussed in §5.

Proposition 2.3 Let $\mathbf{L} = S^*(L^-)L^+$. We have

$$R\mathbf{L}_1 R^+ \mathbf{L}_2 = \mathbf{L}_2 R \mathbf{L}_1 R^+$$

here $\mathbf{L}_1 = \mathbf{L} \otimes I$, $\mathbf{L}_2 = I \otimes \mathbf{L}$.

Proof: By Proposition 1.1 we easily obtain

$$\begin{aligned} & L_1^+ R^+ S^*(L_2^-) = S^*(L_2^-) R^+ L_1^+, \\ & RS^*(L_1^-)S^*(L_2^-) = S^*(L_2^-)S^*(L_1^-)R, \\ & R^+ L_1^+ L_2^+ = L_2^+ L_1^+ R^+, \\ & S^*(L_1^-)RL_2^+ = L_2^+ RS^*(L_1^-). \end{aligned}$$

Noticing $\mathbf{L}_1 = S^*(L_1^-)L_1^+$ and $\mathbf{L}_2 = S^*(L_2^-)L_2^+$, we have

$$\begin{aligned} R\mathbf{L}_1 R^+ \mathbf{L}_2 &= RS^*(L_1^-)L_1^+ R^+ S^*(L_2^-)L_2^+ \\ &= RS^*(L_1^-)S^*(L_2^-)R^+ L_1^+ L_2^+ \\ &= S^*(L_2^-)S^*(L_1^-)RL_2^+ L_1^+ R^+ \\ &= S^*(L_2^-)L_2^+ RS^*(L_1^-)L_1^+ R^+ \\ &= \mathbf{L}_2 R \mathbf{L}_1 R^+. \end{aligned}$$

Let

$$\begin{aligned} \mathbf{R}^{ij,kl}_{mn,uv} &=< \theta_{ijuv}, T_{mk}S(T_{ln}) >, \\ \mathbf{F}^{ij}_{kl,mn} &= \chi < \nabla_{mn}, T_{ki}S(T_{jl}) > . \end{aligned}$$

By Proposition 2.3, we obtain

Theorem 2.2 For $\mathbf{R}, \mathbf{F}$ defined in the above two formulae, there exist two sets of equations which are equivalent with each other:

$$\begin{aligned} \text{(i)}\quad & \nabla_k \nabla_l - \nabla_i \nabla_j \mathbf{R}^{ij}_{kl} = \nabla_m \mathbf{F}^m_{kl}, \\ & \nabla_k \theta_{ml} = \theta_{mi} \nabla_j \mathbf{R}^{ij}_{kl}, \\ & \mathbf{R}^{ij}_{kl} \theta_{ku} \theta_{lv} = \theta_{im} \theta_{jn} \mathbf{R}^{mn}_{uv}, \\ & \mathbf{F}^i_{jk} \theta_{ju} \theta_{kv} + \theta_{iu} \nabla_v = \nabla_m \theta_{in} \mathbf{R}^{mn}_{uv} + \theta_{in} \mathbf{F}^n_{uv}, \\ \text{(ii)}\quad & \mathbf{F}^w_{uk} \mathbf{F}^v_{wl} - \mathbf{F}^w_{ui} \mathbf{F}^v_{wj} \mathbf{R}^{ij}_{kl} = \mathbf{F}^v_{uw} \mathbf{F}^w_{kl}, \\ & \mathbf{F}^w_{uk} \mathbf{R}^{mv}_{wl} = \mathbf{R}^{mw}_{ui} \mathbf{F}^v_{wj} \mathbf{R}^{ij}_{kl}, \\ & \mathbf{R}^{ij}_{kl} \mathbf{R}^{lm}_{wv} \mathbf{R}^{kw}_{nu} = \mathbf{R}^{jm}_{wl} \mathbf{R}^{iw}_{nk} \mathbf{R}^{kl}_{uv}, \\ & \mathbf{R}^{ij}_{kl} \mathbf{F}^k_{uv} + \mathbf{F}^i_{mn} \mathbf{R}^{mj}_{kl} \mathbf{R}^{nk}_{uv} = \mathbf{F}^j_{km} \mathbf{R}^{ik}_{un} \mathbf{R}^{mn}_{lv} + \mathbf{R}^{ij}_{uw} \mathbf{F}^w_{lv}. \end{aligned}$$

Here, for simplicity, we use one index instead of two in above equations, for example i stands for ii', etc.

Proof: Here we only prove the first equations of (i) and (ii), the proofs of remain equations are similar and can be found in [5]. By Proposition 2.3, we have

$$\begin{aligned}
& \mathbf{L}_{cc'} R_{kc',dl'} \mathbf{L}_{dd'} R^{+}_{d'l',k'b'} = R_{kc,al} \mathbf{L}_{aa'} R^{+}_{a'l,k'b} \mathbf{L}_{bb'} \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} < S^*(l^-_{kd}), t_{c'l'} > \mathbf{L}_{dd'} < l^+_{d'k'}, t_{l'b'} > = < S^*(l^-_{ka}), t_{cl} > \mathbf{L}_{aa'} < l^+_{a'k'}, t_{lb} > \mathbf{L}_{bb'} \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} \mathbf{L}_{dd'} \theta_{dd'kk'}(t_{c'b'}) = \mathbf{L}_{aa'} \mathbf{L}_{bb'} \theta_{aa'kk'}(t_{cb}) \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} \mathbf{L}_{dd'} \theta_{dd'kk'}(t_{c'b'}) \theta_{kk'uu'}(S(t_{b'w})) = \mathbf{L}_{aa'} \mathbf{L}_{bb'} \theta_{aa'kk'}(t_{cb}) \theta_{kk'uu'}(S(t_{b'w})) \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} \mathbf{L}_{dd'} \theta_{dd'uu'}(t_{c'b'} S(t_{b'w})) = \mathbf{L}_{aa'} \mathbf{L}_{bb'} \theta_{aa'uu'}(t_{cb} S(t_{b'w})) \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} \mathbf{L}_{dd'} \delta^{dd'}_{uu'} \delta^{c'}_{w} = \mathbf{L}_{aa'} \mathbf{L}_{bb'} \mathbf{R}^{aa'bb'}_{cwuu'} \\
\Longleftrightarrow\ & \mathbf{L}_{cc'} \mathbf{L}_{dd'} = \mathbf{L}_{aa'} \mathbf{L}_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'} \\
\Longleftrightarrow\ & (\chi \nabla_{cc'} + \delta_{cc'} \varepsilon)(\chi \nabla_{dd'} + \delta_{dd'} \varepsilon) = (\chi \nabla_{aa'} + \delta_{aa'} \varepsilon)(\chi \nabla_{bb'} + \delta_{bb'} \varepsilon) \mathbf{R}^{aa'bb'}_{cc'dd'}.
\end{aligned}$$

Since

$$\begin{aligned}
\delta_{aa'} \mathbf{R}^{aa'bb'}_{cc'dd'} &= \delta_{aa'} \theta_{aa'dd'}(t_{cb} S(t_{b'c'})) \\
&= (\chi \nabla_{dd'} + \delta_{dd'} \varepsilon)(t_{cb} S(t_{b'c'})) \\
&= \mathbf{F}^{bb'}_{cc'dd'} + \delta_{dd'} \delta_{bc} \delta_{b'c'},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'} &= \delta_{bb'} \theta_{aa'dd'}(t_{cb} S(t_{b'c'})) \\
&= \delta_{ad} \delta_{a'd'} \delta_{cc'},
\end{aligned}$$

$$\delta_{aa'} \delta_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'} = \delta_{cc'} \delta_{dd'},$$

we have

$$\nabla_{cc'} \nabla_{dd'} - \nabla_{aa'} \nabla_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'} = \nabla_{ee'} \mathbf{F}^{ee'}_{cc'dd'}.$$

Therefore the first equation of (i) is proved, and by applying both sides of it to $t_{uv} S(t_{v'u'})$, we have on the left side

$$\begin{aligned}
& (\nabla_{cc'} \nabla_{dd'} - \nabla_{aa'} \nabla_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'})(t_{uv} S(t_{v'u'})) \\
=\ & (\nabla_{cc'} \otimes \nabla_{dd'} - \nabla_{aa'} \otimes \nabla_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'})(\Delta(t_{uv} S(t_{v'u'}))) \\
=\ & (\nabla_{cc'} \otimes \nabla_{dd'} - \nabla_{aa'} \otimes \nabla_{bb'} \mathbf{R}^{aa'bb'}_{cc'dd'})((t_{uw} S(t_{w'u'})) \otimes (t_{wv} S(t_{v'w'}))) \\
=\ & \mathbf{F}^{ww'}_{uu'cc'} \mathbf{F}^{vv'}_{ww'dd'} - \mathbf{F}^{ww'}_{uu'aa'} \mathbf{F}^{vv'}_{ww'bb'} \mathbf{R}^{aa'bb'}_{cc'dd'},
\end{aligned}$$

and on the right side

$$\nabla_{ee'}(t_{uv} S(t_{v'u'})) \mathbf{F}^{ee'}_{cc'dd'} = \mathbf{F}^{vv'}_{uu'ee'} \mathbf{F}^{ee'}_{cc'dd'},$$

i.e.

$$\mathbf{F}^{ww'}_{uu'cc'} \mathbf{F}^{vv'}_{ww'dd'} - \mathbf{F}^{ww'}_{uu'aa'} \mathbf{F}^{vv'}_{ww'bb'} \mathbf{R}^{aa'bb'}_{cc'dd'} = \mathbf{F}^{vv'}_{uu'ee'} \mathbf{F}^{ee'}_{cc'dd'}.$$

Therefore, the first equation of (ii) is true.

In Theorem 2.2, the first set of equations is related to the Lie bracket of the generators of Lie algebra and the second set is a deformation of the Jacobian identity of the structure constants of the classical Lie group $GL(n)$.

§3. Bicovariant differential calculus on $GL_q(n)$

Definition 3.1 Suppose (Γ, δ) is the one order differential calculus on Hopf algebra $\mathcal{A}$. For arbitrary $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \cdots, N)$ satisfying $x_k \delta y_k = 0$, if $\Delta x_k(id \otimes \delta)\Delta y_k = 0$, then we call (Γ, δ) left-covariant; If $\Delta x_k(\delta \otimes id)\Delta y_k = 0$, then we call (Γ, δ) right-covariant; If (Γ, δ) is not only left-covariant but also right-covariant, then we call (Γ, δ) bicovariant.

Theorem 3.1 The differential calculus (Ω^1, d) on $GL_q(n)$ given in §2 is left-covariant.

Proof: According to the definition of left-covariant, we need only to prove that for arbitrary $x_k, y_k \in \Omega^0$ $(k = 1, 2, \cdots, N)$, if $x_k dy_k = 0$, then $\Delta x_k(id \otimes d)\Delta y_k = 0$.

Suppose $\Delta y_k = y_{1,\alpha}^{(k)} \otimes y_{2,\alpha}^{(k)}$. Then

$$\begin{aligned}
\Delta x_k(id \otimes d)\Delta y_k &= \Delta x_k(id \otimes id \otimes \nabla_{ij})(id \otimes \Delta)\Delta y_k \omega^{ij} \\
&= \Delta x_k(id \otimes id \otimes \nabla_{ij})(\Delta \otimes id)\Delta y_k \omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k(id \otimes id \otimes \nabla_{ij})((\Delta y_{1,\alpha}^{(k)}) \otimes y_{2,\alpha}^{(k)})\omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k(\Delta y_{1,\alpha}^{(k)} \nabla_{ij}(y_{2,\alpha}^{(k)}))\omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k \Delta(y_{1,\alpha}^{(k)} \nabla_{ij}(y_{2,\alpha}^{(k)}))\omega^{ij} \\
&= \Delta x_k \Delta(\nabla_{ij} * y_k)\omega^{ij} \\
&= \Delta(x_k(\nabla_{ij} * y_k))\omega^{ij}.
\end{aligned}$$

Since $0 = x_k dy_k = x_k(\nabla_{ij} * y_k)\omega^{ij}$, we have

$$x_k(\nabla_{ij} * y_k) = 0, \qquad \forall i, j = 1, 2, \cdots, n.$$

Thus

$$\Delta x_k(id \otimes d)\Delta y_k = 0.$$

Therefore, Theorem 3.1 holds.

Now we introduce the concept of ad-invariant. First let two linear mappings $r, s : \Omega^0 \otimes \Omega^0 \longrightarrow \Omega^0 \otimes \Omega^0$ be defined by the following formulas: for $\forall x, y \in \Omega^0$,

$$r(x \otimes y) = m^{\otimes}((x \otimes 1) \otimes \Delta y), \tag{3.1}$$

$$s(x \otimes y) = m^{\otimes}((1 \otimes x) \otimes \Delta y), \tag{3.2}$$

where $m^{\otimes}$ is the multiplication on $\Omega^0 \otimes \Omega^0$, i.e.

$$m^{\otimes}((x \otimes y) \otimes (z \otimes w)) = xz \otimes yw.$$

It can be proved that r, s are bijections, and (see [1])

$$r^{-1}(x \otimes y) = m^{\otimes}((x \otimes 1) \otimes (S \otimes id)\Delta y). \tag{3.3}$$

Definition 3.2 We call a linear subspace $\mathcal{B}$ of Ω^0 *ad*-invariant if

$$ad(\mathcal{B}) \subset \mathcal{B} \otimes \Omega^0$$

where the linear mapping $ad : \Omega^0 \longrightarrow \Omega^0 \otimes \Omega^0$ is defined by

$$ad(x) = s(r^{-1}(1 \otimes x)). \tag{3.4}$$

Proposition 3.1 Let $\mathcal{H} = ker\,\varepsilon \cap (\cap_{i,j=1}^{n} ker\nabla_{ij})$. Then $\mathcal{H}$ is a right ideal of Ω^0.

Proof: Assume $x \in \mathcal{H}$, i.e. the equations

$$\varepsilon(x) = 0, \quad \nabla_{ij}(x) = 0, \quad i, j = 1, 2, \cdots, n$$

hold. For $\forall y \in \Omega^0$ using (2.7) of Proposition 2.1, we have

$$\begin{aligned} \nabla_{ij}(xy) &= \Delta^*(\nabla_{ij})(x \otimes y) \\ &= \nabla_{uv}(x)\theta_{uvij}(y) + \varepsilon(x)\nabla_{ij}(y) \\ &= 0 \end{aligned}$$

and

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y) = 0.$$

So $xy \in \mathcal{H}$.

Now take such a parameter λ that the matrix $M(\lambda)$ is invertible and denote the dual basis of ∇_{kl} by

$$S_{kl} = M^{-1}(\lambda)_{ij}^{kl}(t_{ij} + \lambda\delta_{ij}t),$$

i.e. $\nabla_{ij}(S_{kl}) = \delta_{ik}\delta_{jl}$. For the operator *ad* we have

Proposition 3.2 The formula

$$\nabla_{uu'}(t_{ij}t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} = ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) \tag{3.5}$$

holds.

Proof: Due to Proposition 2.1 and the definition of the operator *ad*, the right side of (3.5) is

$$\begin{aligned} & ad[\nabla_{uu'}(t_{ab}t_{cd})M^{-1}(\lambda)_{vv'}^{uu'}(t_{vv'} + \lambda\delta_{vv'}t)] \\ = \; & \nabla_{uu'}(t_{ab}t_{cd})M^{-1}(\lambda)_{vv'}^{uu'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \; & [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) + \delta_{ab}\nabla_{uu'}(t_{cd})]M^{-1}(\lambda)_{vv'}^{uu'} \\ & \cdot(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \; & [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) + \delta_{ab}M(\lambda)_{uu'}^{cd} - \lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)] \\ & \cdot M^{-1}(\lambda)_{vv'}^{uu'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \; & [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) - \lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)]M^{-1}(\lambda)_{vv'}^{uu'} \\ & \cdot(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) + \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1). \end{aligned}$$

On the other hand, the left side of (3.5) is

$$\begin{aligned} & \nabla_{uu'}(t_{ij}t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = & [\nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl}) + \delta_{ij}\nabla_{uu'}(t_{kl})]S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = & \nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} + \delta_{ab}\nabla_{uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{ld} \\ = & I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld}, \\ I_2 &= \delta_{ab}\nabla_{uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{ld}. \end{aligned}$$

We also have

$$\begin{aligned} & \nabla_{ww'}(t_{ij})S(t_{ai})t_{jb} \\ = & [S^*(l^-_{wv})l^+_{vw'}(t_{ij}) - \delta_{ww'}\varepsilon(t_{ij})]S(t_{ai})t_{jb} \\ = & rS(t_{ai})R_{wi,vk}R^+_{vk,w'j}t_{jb} - \delta_{ww'}\delta_{ab}\mathbf{1} \\ = & rt_{wi}R_{ia,vk}R^+_{vk,jb}S(t_{jw'}) - \delta_{ww'}\delta_{ab}\mathbf{1} \\ = & [S^*(l^-_{iv})l^+_{vj}(t_{ab}) - \delta_{ij}\varepsilon(t_{ab})]t_{wi}S(t_{jw'}) \\ = & \nabla_{ij}(t_{ab})t_{wi}S(t_{jw'}), \end{aligned}$$

i.e.

$$\nabla_{ww'}(t_{ij})S(t_{ai})t_{jb} = \nabla_{ij}(t_{ab})t_{wi}S(t_{jw'}). \tag{3.6}$$

Here

$$\begin{aligned} (R_{wi,vk})_{1\le i,k\le n} &= r_{wv}, \\ (R^+_{vk,wj})_{1\le k,j\le n} &= r^+_{vw}. \end{aligned}$$

Applying (3.6) to I_1, we have

$$I_1 = \nabla_{ij}(t_{ab})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{wi}S(t_{jw'})t_{ld}.$$

Since

$$\begin{aligned} & \theta_{ww'uu'}(t_{kl})S(t_{ck})t_{wi}S(t_{jw'})t_{ld} \\ = & r(R_{uk,wm}S(t_{ck})t_{wi})(R^+_{w'm,u'l}S(t_{jw'})t_{ld}) \\ = & r(\delta_{u\alpha}S(t_{ck})R_{\alpha k,w\beta}t_{wi}\delta_{\beta m})(S(t_{jw'})\delta_{m\alpha}R^+_{w'\alpha,\beta l}\delta_{\beta u'}t_{ld}) \\ = & r(\delta_{u\alpha}S(T)r_{\alpha w}t_{wi}E)_{cm}(S(t_{jw'})Er^+_{w'\beta}\delta_{\beta u'}T)_{md} \\ = & r(E\otimes S(T)\cdot R_q\cdot T\otimes E)_{uc,im}(S(T)\otimes E\cdot R^+\cdot E\otimes T)_{jm,u'd} \\ = & r(S(T)_2RT_1)_{uc,im}(S(T)_1R^+T_2)_{jm,u'd} \\ = & r(T_1R_qS(T)_2)_{uc,im}(T_2R^+S(T)_1)_{jm,u'd} \\ = & rR_{kc,iw}t_{uk}S(t_{wm})R^+_{jw',ld}t_{mw'}S(t_{lu'}) \\ = & \theta_{ijkl}(t_{cd})t_{uk}S(t_{lu'}), \end{aligned}$$

in which we apply QYBE and

$$\begin{aligned} S(T_1) = S(T\otimes E) = S(T)\otimes E = S(T)_1, \\ S(T_2) = S(E\otimes T) = E\otimes S(T) = S(T)_2, \end{aligned}$$

we obtain

$$I_1 = \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})S_{uu'} \otimes t_{uk}S(t_{lu'}). \tag{3.7}$$

By (3.6), we have

$$(\nabla_{ww'}(t_{ij}) + \lambda\delta_{ij}\delta_{ww'}C_0)S(t_{ai})t_{jb} = (\nabla_{ij}(t_{ab}) + \lambda\delta_{ab}\delta_{ij}C_0)t_{wi}S(t_{jw'}),$$

where $\delta_{ij}C_0 = \nabla_{ij}(t)$, i.e.

$$M(\lambda)^{ij}_{ww'}S(t_{ai})t_{jb} = M(\lambda)^{ab}_{ij}t_{wi}S(t_{jw'}).$$

Therefore, one obtains

$$M^{-1}(\lambda)^{kl}_{ab}S(t_{ai})t_{jb} = M^{-1}(\lambda)^{ww'}_{ij}t_{wk}S(t_{lw'}),$$

or

$$M^{-1}(\lambda)^{uu'}_{vv'}t_{uk}S(t_{lu'}) = M^{-1}(\lambda)^{kl}_{ww'}S(t_{wv})t_{v'w'}. \tag{3.8}$$

Thus the equation (3.8) is applied to rewrite I_1 as

$$\begin{aligned} I_1 &= \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes t_{uk}S(t_{lu'}) \\ &= \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})M^{-1}(\lambda)^{kl}_{ww'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{wv})t_{v'w'} \\ &= \nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} + \lambda\delta_{kl}t) \otimes S(t_{vk})t_{lv'} \\ &= \nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1). \end{aligned}$$

Similarly, I_2 can be rewritten as

$$\begin{aligned} I_2 &= \delta_{ab}(\nabla_{uu'}(t_{kl}) + \lambda\delta_{kl}\nabla_{uu'}(t) - \lambda\delta_{kl}\nabla_{uu'}(t)) \\ &\quad \cdot M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{ck})t_{ld} \\ &= \delta_{ab}(t_{kl} + \lambda\delta_{kl}t) \otimes S(t_{ck})t_{ld} \\ &\quad -\lambda\delta_{ab}\delta_{kl}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{ck})t_{ld} \\ &= \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1) \\ &\quad -\lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes 1. \end{aligned}$$

By (3.8), we have

$$\begin{aligned} & \nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes 1 \\ =\ & \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes 1 \\ =\ & \delta_{kl}C_0M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes t_{uk}S(t_{lu'}) \\ =\ & \nabla_{kl}(t)M^{-1}(\lambda)^{kl}_{ww'}t_{vv'} \otimes S(t_{wv})t_{v'w'} \\ =\ & \nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}t_{kl} \otimes S(t_{vk})t_{lv'}. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} I_2 = \ & \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1) \\ & -\lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1). \end{aligned}$$

We complete the proof of the Proposition 3.2.

Theorem 3.2 The differential calculus (Ω^1, d) on $GL_q(n)$ given in §2 is right-covariant.

In Theorem 1.8 of [1], S.L. Woronowicz provided a theorem to decide whether a differential calculus is bicovariant. The theorem can also be said as "*Let* (Γ, δ) *be a left-covariant first order differential calculus. Then* (Γ, d) *is bicovariant if and only if* $\mathcal{H}$ *is ad-invariant.*" Based on this theorem, in order to prove the differential calculus on Ω^0 we have provided is bicovariant, it is sufficient to verify $\mathcal{H}$ is *ad*-invariant.

Proposition 3.3 Let $\mathcal{H} = ker\,\varepsilon \cap (\cap_{i,j=1}^{n} ker\nabla_{ij})$. Then $\mathcal{H}$ is *ad*-invariant.

Proof: It is easy to check that

$$\begin{aligned} &t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv} - C_{ijkl}\mathbf{1} \in \mathcal{H}, \\ &t - \nabla_{uv}(t)S_{uv} - \hat{C}\mathbf{1} \in \mathcal{H}, \end{aligned}$$

where

$$\begin{aligned} C_{ijkl} &= \varepsilon(t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv}), \\ \hat{C} &= \varepsilon(t - \nabla_{uv}(t)S_{uv}). \end{aligned}$$

Denote by $\widetilde{\mathcal{H}}$ the right ideal generated by $t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv} - C_{ijkl}\mathbf{1}$ $(i,j,k,l = 1,2,\cdots,n)$ and $t - \nabla_{uv}(t)S_{uv}\hat{C}\mathbf{1}$, and denote the set of the generators by Λ. Obviously, $\widetilde{\mathcal{H}} \subseteq \mathcal{H}$.

Now we define an equivalent relation in Ω^0, for $\xi, \eta \in \Omega^0$, we say ξ and η are equivalent or $\xi \sim \eta$ if $\xi - \eta \in \widetilde{\mathcal{H}}$.

An arbitrary element ρ of $\mathcal{H}$ can be represented by a polynomial of t_{ij} $(i,j = 1,2,\cdots,n)$ and t. By the definitions of the generators in Λ, we known that any two order polynomial of t_{ij} $(i,j = 1,2,\cdots,n)$ and one order polynomial of t is equivalent to a one order polynomial of S_{kl} $(k,l = 1,2,\cdots,n)$, so ρ is equivalent to $a_{kl}S_{kl}$, $a_{kl} \in \mathbb{C}$, i.e.,

$$\begin{aligned} 0 &= \varepsilon(\rho) = \varepsilon(a_{kl}S_{kl}), \\ 0 &= \nabla_{ij}(\rho) = \nabla_{ij}(a_{kl}S_{kl}), \quad i,j = 1,2,\cdots,n. \end{aligned}$$

Since S_{kl} $(k,l = 1,2,\cdots,n)$ is the dual basis of ∇_{ij} $(i,j = 1,2,\cdots,n)$, $a_{kl} = 0$, i.e. $\rho \in \widetilde{\mathcal{H}}$. So we have proved $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$, therefore $\mathcal{H} = \widetilde{\mathcal{H}}$.

Now we prove Λ is *ad*-invariant, i.e. $ad\Lambda \subset \Lambda \otimes \Omega^0$. By the definition of *ad*,

$$ad(t_{ij}) = t_{kl} \otimes S(t_{ik})t_{lj}, \tag{3.10}$$

$$ad(t_{ab}t_{cd}) = t_{ij}t_{kl} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld}, \tag{3.11}$$

$$ad(t) = t \otimes \mathbf{1}. \tag{3.12}$$

By the Proposition 3.2, we have

$$\begin{aligned} &C_{ijkl}\mathbf{1} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \;& \varepsilon(t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv})\mathbf{1} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \;& \delta_{ij}\delta_{kl}\mathbf{1} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} - (\varepsilon \otimes id)ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) \\ = \;& \delta_{ab}\delta_{cd}\mathbf{1} \otimes \mathbf{1} - \varepsilon(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'})\mathbf{1} \otimes \mathbf{1} \\ = \;& C_{abcd}\mathbf{1} \otimes \mathbf{1}\,. \end{aligned} \tag{3.13}$$

Therefore, by the Proposition 3.2 and (3.11),

$$\begin{aligned}
& ad(t_{ab}t_{cd} - \nabla_{uu'}(t_{ab}t_{cd})S_{uu'} - C_{abcd}\mathbf{1}) \\
= \; & t_{ij}t_{kl} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} - ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) - C_{abcd}\mathbf{1} \otimes \mathbf{1} \\
= \; & (t_{ij}t_{kl} - \nabla_{uu'}(t_{ij}t_{kl})S_{uu'} - C_{ijkl}\mathbf{1}) \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\
\in \; & \Lambda \otimes \Omega^0
\end{aligned}$$

holds. By (3.9) we have

$$\begin{aligned}
& ad(t - \nabla_{uu'}(t)S_{uu'} - \hat{C}\mathbf{1}) \\
= \; & ad(t - \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) - \hat{C}\mathbf{1}) \\
= \; & t \otimes \mathbf{1} - \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes \mathbf{1}) - \hat{C}\mathbf{1} \otimes \mathbf{1} \\
= \; & t \otimes \mathbf{1} - \nabla_{uu'}(t)C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} \otimes \mathbf{1} + \lambda\delta_{vv'}t \otimes \mathbf{1}) - \hat{C}\mathbf{1} \otimes \mathbf{1} \\
= \; & (t - \nabla_{uu'}(t)S_{uu'} - \hat{C}\mathbf{1}) \otimes \mathbf{1}.
\end{aligned}$$

Therefore we have proved

$$ad\Lambda \subset \Lambda \otimes \Omega^0.$$

Let ξ_i $(i = 1, 2, \cdots, n^4 + 1)$ be the $n^4 + 1$ generators in Λ. Take $\xi \in \Lambda$. For $\forall \eta \in \Omega^0$, one has

$$\begin{aligned}
& \Delta\xi = \xi_{1,\alpha} \otimes \xi_{2,\alpha}, \quad \Delta\eta = \eta_{1,\beta} \otimes \eta_{2,\beta}, \\
& \Delta(\xi\eta) = (\xi\eta)_{1,r} \otimes (\xi\eta)_{2,r} = \xi_{1,\alpha}\eta_{1,\beta} \otimes \xi_{2,\alpha}\eta_{2,\beta}.
\end{aligned}$$

Then

$$\begin{aligned}
ad(\xi) & = s(r^{-1}(\mathbf{1} \otimes \xi)) \\
& = s(m^{\otimes}((\mathbf{1} \otimes \mathbf{1}) \otimes (S \otimes id)\Delta\xi)) \\
& = s(S(\xi_{1,\alpha}) \otimes \xi_{2,\alpha}) \\
& = m^{\otimes}(\mathbf{1} \otimes S(\xi_{1,\alpha}) \otimes \Delta\xi_{2,\alpha}) \\
& = (\mathbf{1} \otimes S(\xi_{1,\alpha}))\Delta\xi_{2,\alpha}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
ad(\xi\eta) & = [\mathbf{1} \otimes S((\xi\eta)_{1,r})]\Delta((\xi\eta)_{2,r}) \\
& = [\mathbf{1} \otimes S(\xi_{1,\alpha}\eta_{1,\beta})]\Delta(\xi_{2,\alpha}\eta_{2,\beta}) \\
& = [\mathbf{1} \otimes S(\eta_{1,\beta})S(\xi_{1,\alpha})]\Delta(\xi_{2,\alpha})\Delta(\eta_{2,\beta}) \\
& = (\mathbf{1} \otimes S(\eta_{1,\beta}))ad(\xi)\Delta(\eta_{2,\beta}).
\end{aligned}$$

Since $ad\xi \in \Lambda \otimes \Omega^0$, one obtains

$$ad(\xi) = \sum_{i=1}^{n^4+1} \xi_i \otimes x_i, \quad x_i \in \Omega^0, \; i = 1, 2, \cdots, n^4 + 1 .$$

Thus

$$\begin{aligned}
ad(\xi\eta) & = (\mathbf{1} \otimes S(\eta_{1,\beta}))(\xi_i \otimes x_i)\Delta(\eta_{2,\beta}) \\
& = (\xi_i \otimes S(\eta_{1,\beta})x_i)\Delta(\eta_{2,\beta}).
\end{aligned}$$

Therefore $ad(\xi\eta) \in \mathcal{H} \otimes \Omega^0$. By the linear property of ad, we know

$$ad(\mathcal{H}) \subset \mathcal{H} \otimes \Omega^0,$$

i.e. $\mathcal{H}$ is *ad*-invariant.

Therefore, by Theorem 1.8 in [1], we have proved the differential calculus given in §2 is bicovariant.

§4. Quantum de Rham complex on $GL_q(n)$

Let Γ be a $\mathcal{A}$-bimodule consisting of all differential forms of one order on an associative algebra $\mathcal{A}$ with unit. Let $\Gamma^{\otimes n}$ be the n-fold tensor product of Γ. If $\mathcal{A}$ is commutative, for example the algebra consisting of all $\mathbb{C}^\infty$ functions on a smooth manifold, then the de Rham complex on $\mathcal{A}$ can be defined as follows.

$$\Gamma^\wedge = \Gamma^\otimes/N, \ \Gamma^\otimes = \oplus_{i=0}^{\infty}\Gamma^{\otimes i} \ , \tag{4.1}$$

where $\Gamma^0 = \mathcal{A}$, $\Gamma^{\otimes 1} = \Gamma$, and N is the two-sided ideal of $\Gamma^\otimes$ generated by the kernel of $\mathbf{1}-\sigma$, in which $\mathbf{1}$ is the identity operator on $\Gamma\otimes_{\mathcal{A}}\Gamma$ and σ is the automorphism given by the permutation on $\Gamma^{\otimes 2}$. As done in commutative geometry, in order to construct the high order differential calculus on the quantum group $GL_q(n)$, we should first decide a bimodule automorphism σ of $\Gamma\otimes_{\mathcal{A}}\Gamma$. For that reason we first introduce the concept of left-invariant and right-invariant 1-form.

Definition 4.1 Let $\Delta_\Omega : \Omega^1 \longrightarrow \Omega^0 \otimes \Omega^1$ be a linear mapping satisfying

(i) $\forall x \in \Omega^0, \omega \in \Omega^1$,

$$\Delta_\Omega(x\omega) = \Delta(x)\Delta_\Omega(\omega),$$
$$\Delta_\Omega(\omega x) = \Delta_\Omega(\omega)\Delta(x),$$

(ii) $(\Delta \otimes id)\Delta_\Omega = (id \otimes \Delta_\Omega)\Delta_\Omega$,

(iii) $(\varepsilon \otimes id)\Delta_\Omega = id$.

Then we call Δ_Ω the left action on Ω^1. If an element $\omega \in \Omega^1$ satisfying $\Delta_\Omega(\omega) = \mathbf{1}\otimes\omega$, then we call ω the left-invariant differential 1-form.

Definition 4.2 Let ${}_\Omega\Delta : \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^0$ be a linear mapping satisfying:

(i) $\forall x \in \Omega^0, \omega \in \Omega^1$,

$${}_\Omega\Delta(x\omega) = \Delta(x){}_\Omega\Delta(\omega),$$
$${}_\Omega\Delta(\omega x) = {}_\Omega\Delta(\omega)\Delta(x),$$

(ii) $(id \otimes \Delta){}_\Omega\Delta = ({}_\Omega\Delta \otimes id){}_\Omega\Delta$,

(iii) $(id \otimes \varepsilon){}_\Omega\Delta = id$.

Then we call ${}_\Omega\Delta$ the right action on Ω^1. If an element $\omega \in \Omega^1$ satisfying ${}_\Omega\Delta(\omega) = \omega\otimes\mathbf{1}$, then we call ω the right-invariant differential 1-form.

In general, the differential calculus on a Hopf algebra which only satisfies the conditions (2.1) and (2.2) can not always be provided with a left(right) action. But if the differential calculus is left(right)-covariant, the left(right) action on differential forms can be defined.

Proposition 4.1 For $\forall\omega \in \Omega^1$, $\omega = x_k dy_k$, the left action on ω is defined as

$$\Delta_\Omega(x_k dy_k) = \Delta x_k (id \otimes d)\Delta y_k.$$

and the right action on ω as

$$_\Omega\Delta(x_k dy_k) = \Delta x_k (d \otimes id)\Delta y_k.$$

For the proof of this proposition, please see the Proposition 1.2 and 1.3 in [1].
By this proposition, we have

$$\Delta_\Omega(Det_q T dt) = \Delta(Det_q T)(id \otimes d)\Delta t,$$
$$_\Omega\Delta(Det_q T dt) = \Delta(Det_q T)(d \otimes id)\Delta t.$$

Since

$$\Delta Det_q T = Det_q T \otimes Det_q T, \; \Delta t = t \otimes t,$$

we have

$$\Delta_\Omega(Det_q T dt) = 1 \otimes (Det_q T)dt, \tag{4.2}$$
$$_\Omega\Delta(Det_q T dt) = (Det_q T)dt \otimes 1. \tag{4.3}$$

Noticing

$$\Delta S(t_{ik}) = S(t_{mi}) \otimes S(t_{km}), \; i, k = 1, 2, \cdots, n, \tag{4.4}$$

we have

$$\Delta_\Omega(S(t_{ik})dt_{kj}) = 1 \otimes S(t_{ik})dt_{kj}, \tag{4.5}$$
$$_\Omega\Delta(S^{-1}(t_{ki})dt_{jk}) = S^{-1}(t_{ki})dt_{jk} \otimes 1. \tag{4.6}$$

Combining (4.2), (4.3), (4.5) with (4.6), we obtain the following proposition.

Proposition 4.2

(i) $Det_q T dt$ is left-invariant and right-invariant 1-form,
(ii) $S(t_{ik})dt_{kj}$, $(i, j = 1, 2, \cdots, n)$ are left-invariant 1-forms,
(iii) $S^{-1}(t_{ki})dt_{jk}$, $(i, j = 1, 2, \cdots, n)$ are right-invariant 1-forms.

By the proposition,

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl}[S(t_{km})dt_{ml} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{kl} Det_q T dt] \tag{4.7}$$

is left-invariant. It is easy to see

$$t_{mk} S^{-1}(t_{lm}) = \delta_{kl} 1. \tag{4.8}$$

By (4.8), we can rewrite (4.7) as

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl} S(t_{ku}) t_{vl} [S^{-1}(t_{wv})dt_{uw} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{uv} Det_q T dt], \tag{4.9}$$

Denote

$$\eta_{uv} = S^{-1}(t_{wv})dt_{uw} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{uv}Det_q T dt. \tag{4.10}$$

We have

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl}S(t_{ku})t_{vl}\eta_{uv}. \tag{4.11}$$

By Proposition 4.2, η_{uv} is right-invariant 1-form, and (4.10) shows η_{uv} $(u, v = 1, 2, \cdots, n)$ are also a group of right-invariant generators of Ω^1.

Now we define the bi-module automorphism $\sigma : \Omega^1 \otimes_{\Omega^0} \Omega^1 \longrightarrow \Omega^1 \otimes_{\Omega^0} \Omega^1$ by

$$\sigma(X^{mn}_{uv}\omega^{uv} \otimes \eta_{mn}) = X^{mn}_{uv}\eta_{mn} \otimes \omega^{uv}, \tag{4.12}$$

here $X^{mn}_{uv} \in \Omega^0$. It is easy to check σ satisfies the braid relation,

$$(id \otimes \sigma)(\sigma \otimes id)(id \otimes \sigma) = (\sigma \otimes id)(id \otimes \sigma)(\sigma \otimes id).$$

Obviously, $\omega^{ij} \otimes \omega^{uv}$ $(i, j, k, l = 1, 2, \cdots, n)$ is a group of generators of $\Omega^1 \otimes_{\Omega^0} \Omega^1$. By (4.10),

$$\begin{aligned} \sigma(\omega^{ij} \otimes \omega^{kl}) &= \sigma(\omega^{ij} \otimes M^{-1}(\lambda)^{kl}_{\alpha\beta}S(t_{\alpha u})t_{v\beta}\eta_{uv}) \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}(\theta_{ijuv} * (S(t_{\alpha\gamma})t_{\tau\beta}))\eta_{\gamma\tau} \otimes \omega^{uv} \end{aligned} \tag{4.13}$$

Applying (4.10) to (4.13), we obtain

$$\begin{aligned} \sigma(\omega^{ij} \otimes \omega^{kl}) &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})S(t_{\mu\gamma})t_{\tau w} \\ &\quad \cdot[S^{-1}(t_{m\tau})dt_{\gamma m} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{\tau\gamma}Det_q T dt] \otimes \omega^{uv} \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})[S(t_{\mu\gamma})dt_{\gamma w} \\ &\quad + \lambda(\frac{1}{r^n q^2} - 1)\delta_{\mu w}Det_q T dt] \otimes \omega^{uv} \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})M(\lambda)^{\mu w}_{mn}\omega^{mn} \otimes \omega^{uv}. \end{aligned} \tag{4.14}$$

Let

$$\begin{aligned} \sigma(\omega^{ij} \otimes \omega^{kl}) &= \tilde{\mathbf{R}}^{ijkl}_{mnuv}\omega^{mn} \otimes \omega^{uv}, \\ \mathbf{R}^{ijkl}_{mnuv} &= \theta_{ijuv}(t_{mk}S(t_{ln})). \end{aligned}$$

Proposition 4.3 $\mathbf{R}^{ijkl}_{mnuv} = \tilde{\mathbf{R}}^{ijkl}_{mnuv}$.

Proof: From (4.14), it follows that

$$\begin{aligned} \tilde{\mathbf{R}}^{ijkl}_{mnuv} &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})M(\lambda)^{\mu w}_{mn} \\ &= \theta_{ijuv}(M^{-1}(\lambda)^{kl}_{\alpha\beta}S(t_{\alpha\mu})t_{w\beta}M(\lambda)^{\mu w}_{mn}). \end{aligned}$$

Applying (3.8) to above equation, one obtains

$$\begin{aligned} \tilde{\mathbf{R}}^{ijkl}_{mnuv} &= \theta_{ijuv}(M^{-1}(\lambda)^{\alpha\beta}_{\mu w}S(t_{\alpha k})t_{l\beta}M(\lambda)^{\mu w}_{mn}) \\ &= \theta_{ijuv}(t_{mk}S(t_{ln})) \\ &= \mathbf{R}^{ijkl}_{mnuv}. \end{aligned} \tag{4.15}$$

Proposition 4.4 Let $\mathbf{R} = (\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda}) = (r^a_b)_{1\le a,b\le n^4}$, $a = n^3(i-1)+n^2(j-1)+n(k-1)+l$, $b = n^3(\alpha-1)+n^2(\beta-1)+n(\mu-1)+\lambda$. Then the minimal polynomial of $\mathbf{R}$ is $(\xi-1)(\xi+q^2)(\xi+q^{-2})$.

Proof: By (2.8) in Proposition 2.1,

$$\begin{aligned}\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= \theta_{ij\mu\lambda}(t_{\alpha k}S(t_{l\beta}))\\ &= \Delta^*\theta_{ij\mu\lambda}(t_{\alpha k}\otimes S(t_{l\beta}))\\ &= \theta_{ijab}(t_{\alpha k})\theta_{ab\mu\lambda}(S(t_{l\beta}))\\ &= S^*(l^-_{ai})\otimes l^+_{jb}(\Delta t_{\alpha k})S^*(l^-_{\mu a})\otimes l^+_{b\lambda}(\Delta S(t_{l\beta}))\cdot\\ &= S^*(l^-_{ai})(t_{\alpha c})l^+_{jb}(t_{ck})S^*(l^-_{\mu a})(S(t_{d\beta}))l^+_{b\lambda}(S(t_{ld})).\end{aligned}$$

By definition of $l^\pm_{ij}$ $(i,j=1,2,\cdots,n)$,

$$\begin{aligned} S^*(l^-_{ai})(t_{\alpha c}) &= \lambda_- R_{a\alpha,ic}, & l^+_{jb}(t_{ck}) &= \lambda_+ R^+_{jc,bk},\\ S^*(l^-_{\mu a})(S(t_{d\beta})) &= \lambda_-^{-1}R^{-1}_{\mu d,a\beta}, & l^+_{b\lambda}(S(t_{ld})) &= \lambda_+^{-1}(R^+)^{-1}_{bl,\lambda d}.\end{aligned} \tag{4.16}$$

These formulae give

$$\begin{aligned}\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= R^+_{jc,bk}R_{a\alpha,ic}(R^+)^{-1}_{bl,\lambda d}(R_q^{-1})_{\mu d,a\beta}\\ &= R_{cj,kb}(R^t_q)_{ic,a\alpha}(PR_q^{-1}P)_{bl,\lambda d}(R_q^{t_1})^{-1}_{ad,\mu\beta}\\ &= (PR_q^{t_1})_{jk,cb}(R^t_qP)_{ic,\alpha a}(PR_q^{-1})_{bl,d\lambda}(PR_q^{t_1})^{-1}_{ad,\beta\mu},\end{aligned} \tag{4.17}$$

where t is the transposition of the matrix, and t_1 is the permutation of the first and the third indexes. Write

$$\begin{aligned} (PR_q^{t_1})_{jk,cb} &= (PR_q^{t_1})^{jk}_{cb}, & (R^t_qP)_{ic,\alpha a} &= (R^t_qP)^{ic}_{\alpha a},\\ (PR_q^{-1})_{lb,\lambda d} &= (PR_q^{-1})^{lb}_{\lambda d}, & (PR_q^{t_1})^{-1}_{ad,\beta\mu} &= ((PR_q^{t_1})^{-1})^{ad}_{\beta\mu}.\end{aligned}$$

Then (4.17) can be rewritten as

$$\begin{aligned}\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= (\delta^i_a(PR_q^{t_1})^{jk}_{bc}\delta^l_d)((R^t_qP)^{ab}_{a'b'}\delta^c_{c'}\delta^d_{d'})\\ &\quad(\delta^{a'}_m\delta^{b'}_n(PR_q^{-1})^{c'd'}_{uv})(\delta^m_\alpha((PR_q^{t_1})^{-1})^{nu}_{\beta\mu}\delta^v_\lambda)\\ &= [(E\otimes PR_q^{t_1}\otimes E)\cdot(R^t_qP\otimes E\otimes E)\\ &\quad\cdot(E\otimes E\otimes PR_q^{-1})\cdot(E\otimes PR_q^{t_1}\otimes E)^{-1}]^{ijkl}_{\alpha\beta\mu\lambda}.\end{aligned} \tag{4.18}$$

Let $M_1 = R^t_qP\otimes E\otimes E$, $M_2 = E\otimes E\otimes PR_q^{-1}$. Obviously, $\mathbf{R}$ and M_1M_2 have the same minimal polynomial

$$(\xi-1)(\xi+q^2)(\xi+q^{-2}).$$

Therefore, we obtain

$$(E_{n^4}-\mathbf{R})(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4}) = 0,$$

where E_{n^4} is the unit matrix of order n^4.

Now we introduce the quantum de Rham complex on quantum group $GL_q(n)$. Denote

$$\Omega^{\otimes} = \oplus_{i=0}^{\infty}\Omega^{\otimes i},\ \Omega^{\otimes 0} = \Omega^0,\ \Omega^{\otimes 1} = \Omega^1.$$

Definition 4.3 The quantum de Rham complex on $GL_q(n)$ is defined as

$$\Omega^{\wedge} = \Omega^{\otimes}/\{\ker(1-\sigma)\}, \tag{4.19}$$

where $\{\ker(1-\sigma)\}$ is the two-sided ideal on $\Omega^{\otimes}$ generated by $\ker(1-\sigma)$, and $\ker(1-\sigma) = \{[(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4})]_{ijkl}^{\alpha\beta\mu\lambda}\omega^{ij}\otimes\omega^{kl}\ \alpha,\beta,\mu,\lambda = 1,2,\cdots,n\}$. The production in $\Omega^{\wedge}$ is denoted by $\wedge$.

Theorem 4.1 There exists a unique linear mapping

$$d:\Omega^{\wedge} \longrightarrow \Omega^{\wedge},$$

so that

(i) d is the derivation of order one, i.e. it maps differential forms of order n to ones of order $n+1$,

(ii) The definition of d on Ω^0 is given by (2.18),

(iii) $d(\xi\wedge\eta) = d\xi\wedge\eta + (-1)^{\deg\xi}\xi\wedge d\eta$, where $\deg\xi = n$ if ξ is a differential form of order n,

(iv) $d^2 = 0$.

The proof of Theorem 4.1 is similar to that of Theorem 4.1 in [1].

In fact, we can write (4.19) as

$$\Omega^{\wedge} = \mathrm{Fun}(GL_q(n))[\omega^{ij},1]/\{I_1,I_2\}, \tag{4.20}$$

where relation I_1 is given by

$$\omega^{ij}x - (\theta_{ijkl} * x)\omega^{kl},\ x = t_{mn}, t \tag{4.21}$$

and the relation I_2 is given by

$$[(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4})]_{ijkl}^{\alpha\beta\mu\lambda}\omega^{ij}\wedge\omega^{kl},\ \alpha,\beta,\mu,\lambda = 1,2,\cdots,n. \tag{4.22}$$

Additionally, we can also obtain the Maurer-Cartan equation by Theorem 4.1 and (2.19).

§5. Noncommutative differential calculus on quantum group $SL_q(n)$

Quantum group $SL_q(n)$ can be obtained by taking the quotient algebra

$$\mathrm{Fun}(M(n))/\{Det_qT - 1\}.$$

In fact, as for the Hopf algebra $\text{Fun}(GL_q(n))$, its generator t now equivalents to the unit 1, i.e.

$$t \equiv 1. \tag{5.1}$$

Thus, the algebra of coordinate functions $\text{Fun}(SL_q(n))$ is equivalent to $\text{Fun}(GL_q(n)) /\{Det_qT - 1\}$. Namely,

$$\text{Fun}(SL_q(n)) = \text{Fun}(M(n))/\{Det_qT - 1\} = \text{Fun}(GL_q(n))/\{Det_qT - 1\}\ . \tag{5.2}$$

To insure the linear functionals $l^{\pm}_{ij}$ $(i, j = 1, \cdots n)$ and $l^{\pm}$ given in §1 are well defined on $\text{Fun}(SL_q(n))$, from (1.25) and (1.26), we know that the following conditions must be satisfied,

$$\lambda^n_+ q = \lambda^n_- q = 1. \tag{5.3}$$

Therefore,

$$\nabla_{ij}(t) = \nabla_{ij}(1) = 0, \tag{5.4}$$

and by (1.15) (1.17) and (1.18) we have $l^+ = l^- = \varepsilon$. Or we can say, after the condition (5.2) is introduced, all of the equations in §1 still hold, and those related to t and $l^{\pm}$ become trivial. Obviously, $\text{Fun}(SL_q(n))$ and the corresponding algebra $\text{Fun}^*_0(SL_q(n))$ are Hopf algebras.

Now we discuss how to obtain the differential calculus on $SL_q(n)$ and its quantum de Rham complex from that of $GL_q(n)$.

Matrix $M(\lambda)$ plays a very important role in the discussions of the differential calculus on $GL_q(n)$. For quantum group $SL_q(n)$, we have only two extra conditions $Det_qT = t = 1$ and $\lambda^n_+ q = \lambda^n_- q = 1$. Thus,

$$M(\lambda) = (M^{n(k-1)+l}_{n(i-1)+j})_{1 \le k,l,i,j \le n} = (\nabla_{ij}(t_{kl}))_{1 \le k,l,i,j \le n}. \tag{5.5}$$

And the determinant of $N(\lambda)$ is

$$DetN(\lambda) = \frac{(1 - r^n)^{n-2}}{\chi^n r^{n^2 - n}}(1 - r^{n-1} - r^n + r^{2n} + r^{n^2+n-1} - r^{n^2+n}), \tag{5.6}$$

where $r^n q^2 = 1$. Therefore, except for finite isolated values of q, the matrix $M(\lambda)$ is invertible. When $M(\lambda)$ is invertible, we can add the conditions $Det_qT \equiv 1$ and $\lambda^n_+ q = \lambda^n_- q = 1$ to the differential calculus of $GL_q(n)$ to obtain that of $SL_q(n)$. The values of q that $M(\lambda)$ is not invertible are the 6th unit roots when $n = 2$, when $n \ge 3$, the discussions will be a bit more complicated, we will discuss the differential calculus of $SL_q(n)$ at the extra values of q elsewhere.

References

1. S. L. Woronowicz, Comm. Math. Phys. 122, 125(1989).

2. J. Wess and B. Zumino, Nucl. Phys. B(proc. Suppl.) 18B 302 (1990).
   B. Zumino, Mod. Phys. Lett. A6 1225 (1991), Preprint LBL-31432, UCB-PTH-62/91.
   A. Schirrmacher, J. Wess & B. Zumino, Z. Phys., C-Particles and Fields 49, 317-324(1991).

3. Yu. I. Manin, Notes on quantum groups and quantum de Rham complexes, Preprint (1991).

4. X. C. Song & L. Liao, New type covariant differential calculus on the quantum planes, Preprint ASITP-91-03, CCAST-91-04, January, 1991.

5. K. Wu & R. J. Zhang, Differential calculus on quantum matrix groups, Preprint ASITP-91-72, CCAST-91-69.

6. V. G. Drinfel'd, Quantum Groups, in Vol 1 of Proceedings of the International Congress of Mathematics, Berkeley, California, USA, 1986, Academic Press, 1987, 798-820.

7. N. Yu. Reshetikhin, L. A. Takhtadzhyan & L. D. Faddeev, Leningrad Math. J. Vol. 1, 193-224(1990).

8. Yu. I. Manin, Quantum groups and noncommutative geometry, CRM, Université de Montréal, 1988.

9. S. L. Woronowicz, Comm.Math.Phys. 111, 613-665(1987).

10. X. D. Sun, S. K. Wang & Yunxiao Wang, Bicovariant differential calculus on the two-parameter quantum group $GL_{p,q}(2)$, Preprint ASIAM-91-43, CCAST-91-62.

11. E. Abe, Hopf algebras, Cambridge Tracts Math., No. 74, Cambridge University Press, (1977).

12. X. D. Sun, S. K. Wang & K. Wu, Differential Calculus on Quantum Lorentz Group, Preprint CCAST-92-14, ASITP-92-18, ASIAM-92-11.

CCAST-92-027
ASIAM-92-016

Fourier Transform on Quantum Group $C(SU_q(2))$ [1]

JI-SHENG NA

CCAST (World Laboratory)
P.O. Box 3730, Beijing 100080, China

Institute of Applied Mathematics
Academia Sinica, Beijing, 100080, China

Abstract

This note sketches the finite-dimensional representations of quantum group $SL_q(2, C)$, in particular, its compact real form $SU_q(2)$, discusses the Fourier transform and the inversion formula on $SU_q(2)$, and generalizes to the case of its completion $C(SU_q(2))$.

Since the theory of quantum groups was introduced systematically by Drinfeld [1], Fadeev et al. [3], Woronowicz [8] and others from different approaches, it has obtained remarkable advances both in physical and in mathematical aspects. Based on quantization, quantum groups are considered as the noncommutative deformation of functions on Lie groups, so they inherit some properties of the functions on groups in the noncommutative context. For example, the well-known Peter-Weyl theorem on compact groups has its counterpart in the case of compact quantum groups, such as $SU_q(2)$ [5][8]. It seems that many results in harmonic analysis on groups can be extended to "compact" (or "noncompact") quantum groups. First, for $SU_q(2)$ there are Fourier transform, Fourier expansion (inversion formula), which can be expressed explicitly in terms of matrix elements of its irreducible representations. Of course, $SU_q(2)$ is a (Hopf) algebra, its elements are finitely generated, and the equalities and relations appeared here are all algebraic. How to introduce the concept of limit (norm, topology) to the quantum groups is an interesting topic, and it is important for developing research on harmonic analysis on quantum groups and quantum homogeneous

[1]Supported by National Natural Science Foundation of China.

spaces, some examples can be found in [6] [7] [8]. The aim of this note is to generalize the result on Fourier transformation to $C(SU_q(2))$ which is the completion of $SU_q(2)$ in the norm of C^*-algebra.

1. Quantum group $SL_q(2,C)$ and its finite-dimensional representations

As usual, quantum group $SL_q(2,C)$ (or algebra of regular (or smooth) functions on it, or its coordinate ring $A(SL_q(2,C))$, we do not make difference between them in this note) means a noncommutative Hopf algebra generated by x, u, v, y over complex number field C:

$$A(SL_q(2,C)) = C[x,u,v,y]/\sim$$

where $\sim$ is the ideal generated by the following relations

$$ux = qxu,\ \ vx = qxv,\ \ yu = quy,\ \ yv = qvy,$$
$$uv = vu,\quad xy - q^{-1}uv = yx - quv = 1$$

for $q \in C\backslash 0$ and comultiplication Δ, counit ε and antipode s are defined on generators by

$$\Delta\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix} \otimes \begin{pmatrix} x & u \\ v & y \end{pmatrix},$$
$$\varepsilon\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$s\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix}$$

respectively. Let us describe some of its algebraic structure. First, as a vector space, $A(SL_q(2,C))$ has a basis consisted of

$$x^i u^j v^k,\quad u^j v^k y^l\quad (i,j,k \geq 0,\ l > 0)$$

On the other hand, there is a direct sum decomposition of vector spaces [5]:

$$A(SL_q(2,C)) = \oplus_{m,n\in Z}\ A[m,n]$$

where

$$A[m,n] = \{a \in A(SL_q(2,C)) | L_K(a) = t^m \otimes a,\quad R_K(a) = a \otimes t^n\}$$

$$L_K = (\pi \otimes id)\circ\Delta,\quad R_K = (id \otimes \pi)\circ\Delta$$

π is a canonical epimorphism of $A(SL_q(2,C))$ into its quantum subgroup $A(K) = C[t,t^{-1}]$ defined by

$$\pi\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},\quad \Delta(t) = t\otimes t,\ \varepsilon(t) = 1,\ s(t) = t^{-1}.$$

Here $A[0,0] = C[\zeta]$ $(\zeta = -q^{-1}uv)$, a polynomial ring of ζ, is an algebra of bi-invariants in $A(SL_q(2,C))$. For $m \equiv n \bmod 2$, $A[m,n]$ is a free left (or right) $C[\zeta]$-module of dimension one, otherwise $A[m,n] = 0$. For brevity, we also write $A, A(G)$ for $A(SL_q(2,C))$. Let's consider the (co-)representation of $SL_q(2,C)$, i.e., $A(G)$-comodule. Taking two C-vector spaces of $A(G)$:

$$V_l^L = \oplus_{i\in I_l} C\xi_i^{(l)}, \quad V_l^R = \oplus_{j\in I_l} C\eta_j^{(l)}$$

where index set $I_l = \{-l, -l+1, ..., l\}, l \in N/2$ (N: set of natural numbers)

$$\xi_i^{(l)} = \begin{bmatrix} 2l \\ l+i \end{bmatrix}_{q^2}^{1/2} x^{l-i} v^{l+i}, \quad \eta_j^{(l)} = \begin{bmatrix} 2l \\ l+j \end{bmatrix}_{q^2}^{1/2} x^{l-j} u^{l+j}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}, \quad (a;q)_m = \prod_{k=0}^{m-1} (1 - aq^k).$$

Obviously, V_l^L and V_l^R are left and right A-submodules respectively, which are called the spin l representations of $SL_q(2,C)$. Then we can write down

$$\Delta\xi_i^{(l)} = \sum_{j\in I_l} w_{ij}^{(l)} \otimes \xi_j^{(l)}, \quad i \in I_l,$$

and from the structure of comodule, it follows

$$\Delta w_{ij}^{(l)} = \sum_{k\in I_l} w_{ik}^{(l)} \otimes w_{kj}^{(l)}, \quad \varepsilon(w_{ij}^{(l)}) = \delta_{ij}$$

which can be written in the matrix form:

$$\Delta W_l = W_l \otimes W_l, \quad \varepsilon(W_l) = I, \quad W_l = (w_{ij}^{(l)})_{i,j\in I_l}.$$

$w_{ij}^{(l)}$ is called the matrix element of the representation, and

$$w_{ij}^{(l)} \in A[-2i, -2j].$$

In the same time, there is also an equality

$$\Delta\eta_j^{(l)} = \sum_{i\in I_l} \eta_i^{(l)} \otimes w_{ij}^{(l)}.$$

We have

Proposition. [5] For any $l \in N/2$, A-comodule $V_l^L (resp. V_l^R)$ is irreducible, and any finite-dimensional irreducible left- (resp. right-) comodule is isomorphic to $V_l^L (resp. V_l^R)$ for some $l \in N/2$.

Theorem.[5][8] $A(SL_q(2,C))$ has a direct sum decomposition into A-bicomodules

$$A(SL_q(2,C)) = \oplus_{l\in N/2} W_l$$

and

$$A[-2i,-2j] = \oplus_{l\in N/2}\, w_{ij}^{(l)}.$$

The matrix element $w_{ij}^{(l)}$ of the representation of quantum group $SL_q(2,C)$ can be explicitly expressed by the little q-Jacobi polynomials in ζ, hence a new interpretation of q-special functions in the representation theory of quantum groups is obtained [4][5][7]. Quantum group $SL_q(2,C)$ has another important property, i.e., it possesses bi-invariant linear functional (or the Haar measure as in the classical case). A linear functional $h: A \to C$ is called bi-invariant if it satisfies

$$(id \otimes h)\circ\Delta(a) = 1\bullet h(a), \quad (h\otimes id)\circ\Delta(a) = h(a)\bullet 1, \quad \forall a \in A.$$

It is unique, if taking normalization $h(1) = 1$. Such a functional h takes values as follows:

$$\begin{array}{l} h(a)\neq 0 \quad \forall a \in A[0,0], \\ h(a) = 0 \quad otherwise, \\ h(\zeta^n) = (1-q^2)/(1-q^{2(n+1)}) \quad n\in N, \\ h(s(a)) = h(a) \quad \forall a \in A. \end{array}$$

In particular,

$$\begin{array}{l} h(w_{00}^{(0)}) = h(1) = 1, \quad h(w_{00}^{(l)}) \in C[\zeta], \\ h(w_{ij}^{(l)}) = 0 \quad for\ (i,j)\neq(0,0). \end{array}$$

h has an explicit expression in terms of q-integral: for $f = f(\zeta)\in C[\zeta]$

$$h(f) = \int_0^1 f(\zeta)d_q^2\zeta = (1-q^2)\sum_{k=0}^{\infty} q^{2k} f(q^{2k}), \quad |q|<1$$

and for $|q|>1$ there is also a cooresponding formula.

2. Quantum group $SU_q(2)$ and the Fourier transform on it

According to Fadeev et al. [3], the real form of quantum group $SL_q(2,C)$ is a Hopf algebra which possesses the *-structure, that means, on $SL_q(2,C)$ is defined an operation (involution) *: $A\to A$ satisfying:

$$\begin{array}{c} (\lambda a)^* = \overline{\lambda}a^*, \quad (ab)^* = b^*a^*, \quad *^2 = id, \\ \Delta\circ * = (*\otimes *)\circ\Delta, \quad \varepsilon(a^*) = \overline{\varepsilon(a)}, \quad (*\circ s)^2 = id. \end{array}$$

There are three real forms for $SL_q(2,C)$, called $SU_q(2)$, $SU_q(1,1)$ and $SL_q(2,R)$, and the *-operation is defined as

$$\begin{pmatrix} x^* & v^* \\ u^* & y^* \end{pmatrix} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix}, \begin{pmatrix} y & qu \\ q^{-1}v & x \end{pmatrix}, \begin{pmatrix} y & q^{-1}v \\ qu & x \end{pmatrix}$$

respectively, and the value of q is: q real for $SU_q(2)$ and $SU_q(1,1)$, and $|q| = 1$ for $SL_q(2,R)$. Due to the property of *-operation, we have

Proposition. The spin l representation of $SL_q(2,C)$, as restricted on the "real" quantum groups mentioned above, gives rise to the finite- dimentional irreducible representation for them, respectively. But in these three cases, such a restriction provides a unitary representation only for $SU_q(2)$. Here the meaning of the unitarity of a representation may be stated in an obvious way, that is,

$$W_l W_l^* = W_l^* W_l = I$$

where $W_l = (w_{ij}^{(l)})_{i,j\in I_l}$, $W_l^* = (w_{ji}^{(l)*})_{i,j\in I_l}$, I is the $I_l \times I_l$ unit matrix, in other words, W_l is a unitary matrix: $W_l^* = W_l^{-1} = s(W_l)$.

As for the infinite-dimentional representation of quantum groups, since the concept of *limit/convergence/topology* should be introduced, so we don't discuss it here. Really, $SL_q(2,C)$ has infinite dimensional representation (infinite-dimensional comodule) and its restriction to $SU_q(1,1)$ gives rise to the unitary representation of the latter, see [6] for detail. Woronowicz has introduced $SU_q(2)$ from the angle of C^*-algebra and obtained many important results. $SU_q(2)$ is an analogue of compact group, it can be endowed with a metric in terms of bi-invariant functional h.

Proposition.[5] The Hermitian forms $<,>_R$ and $<,>_L$ on $A(SU_q(2))$ defined by

$$< a,b >_R = h(ab^*), \qquad < a,b >_L = h(a^*b)$$

are positive definite.

Hence, we can define the norm $||a||_h^2 =< a,a >_R = h(aa^*)$, $SU_q(2)$ is turned into a *metric/topological* space. We have the following Peter-Weyl theorem [5],[8].

Theorem. The $*$-Hopf algebra $A(SU_q(2))$ has an orthogonal decomposition

$$A(SU_q(2)) = \otimes_{l\in N/2} \quad W_l$$

with respect to $<,>_R$ $(or <,>_L)$, and the matrix elements $w_{ij}^{(l)}$ satisfy the following relations:

$$< w_{ij}^{(l)}, w_{st}^{(m)} >_R = < w_{ij}^{(l)}, w_{st}^{(m)} >_L = 0 \quad for \quad (l,i,j) \neq (m,s,t)$$
$$< w_{ij}^{(l)}, w_{ij}^{(l)} >_R = q^{2(l-j)}(1-q^2)/(1-q^{2(2l+1)})$$
$$< w_{ij}^{(l)}, w_{ij}^{(l)} >_L = q^{2(l+j)}(1-q^2)/(1-q^{2(2l+1)}).$$

Here, the quantum Schur lemma is available for the irreducible commodules. Since $w_{ij}^{(l)}$ can be expressed by q-Jacobi polynomials, P-W theorem produces the orthogonality relations of them. From P-W theorem we can establish the Fourier transform on $SU_q(2)$

$$\mathcal{F} : A = A(SU_q(2)) \to \widehat{A} := \oplus_{l\in N/2} Mat(I_l, C).$$

For $f \in A(SU_q(2))$, define the matrix-valued Fourier coefficients $\widehat{f}^{(l)} = (\widehat{f}_{ij}^{(l)})_{i,j\in I_l} \in Mat(I_l, C)$ as follows:

$$\widehat{f}^{(l)} = h(f.s(W_l)), \quad \mathcal{F}(f) = (\widehat{f})_{l\in N/2},$$

where W_l is the representation matrix of $SU_q(2)$ in V_l^L.

Theorem. (Fourier Inversion Formula) The Fourier transform $\mathcal{F}: A \to A$ is a C-isomorphism. The inversion formula is given by

$$f = \sum_{l \in N/2} [(q^{2l+1} - q^{-2l-1})/(q - q^{-1})] \sum_{i,j \in I_l} q^{2j} \widehat{f}_{ij}^{(l)} w_{ij}^{(l)}.$$

3. Fourier transform on quantum group $C(SU_q(2))$

According to Woronowicz, $C(SU_q(2))$, the algebra of continuous functions on $SU_q(2)$, is the completion of $A(SU_q(2))$ under the norm $\|\|_*$ of C^*-algebra, which he called the compact matrix pseudogroup. Norm $\|\|_*$ is defined by

$$\|a\|_* = \sup_{\pi \in \Pi} \|\pi(a)\|,$$

where Π is the set of all representations of C^*-algegra $SU_q(2)$, and $\|\pi(a)\|$ is the norm of operator. From the theory of C^*-algebras [2], $C(SU_q(2))$ is the C^*-algebra of type I, the comultiplication on $A(SU_q(2))$ can be extended continuously to $C(SU_q(2))$:

$$\Delta: \quad C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$$

and $C(SU_q(2))$ is turned into the C^*-bialgebra.

Proposition. The bi-invariant linear functional h on $A(SU_q(2))$ is continuously extended to $C(SU_q(2))$.

From [2], since $h(aa^*) \geq 0, \forall a \in A(SU_q(2))$, and $h(aa^*) = 0$ iff $a = 0$, then there exist some *-representation π, acting on the Hilbert space V with inner product $(,)_V$, and a fixed vector $\xi \in V$, such that

$$h(a) = (\pi(a)\xi, \xi)_V \qquad \forall a \in A(SU_q(2)).$$

Therefore

$$|h(a)| \leq |(\pi(a)\xi, \xi)_V| \leq \|\pi(a)\| \|\xi\|_V^2 \leq \|a\|_* \|\xi\|_V^2,$$

and since $\|a\|_*$ is continuous with respect to $a \in C(SU_q(2))$, then $h(a)$ is continuously extended to $C(SU_q(2))$ and is still the Haar measure.

Corollary. For $f \in C[\zeta] \bigcap C(SU_q(2)), \quad 0 < |q| < 1$ the Haar measure h has an expression:

$$h(f) = (1 - q^2) \sum_{k=0}^{\infty} q^{2k} f(q^{2k}).$$

It is obvious that for $a \in A(SU_q(2))$,

$$h(aa^*) \leq \|\pi(a)\pi(a^*)\| \|\xi\|_V^2 \leq \|a\|_*^2 \|\xi\|_V^2.$$

If $A(SU_q(2))$ is endowed with the norm $\|\|_h : \|a\|_h^2 = < a, a >_R = h(aa^*)$, then $\|\|_h$ can be extended to a norm of $C(SU_q(2))$. Of course, the inner product $<,>_R$ is extended, too.

Now there are two norms : $\|\|_*$ and $\|\|_h$ on $C(SU_q(2))$, the relation between them is included in the following

Proposition. $\|a\|_h \leq c_1\|a\|_*, \quad \forall a \in C(SU_q(2))$.

This means the norm $\|\|_*$ (hence its induced topology) is stronger than $\|\|_h$. At this point we can establish the Fourier transform and inversion formula for $C(SU_q(2))$, although formally they are the same as that for $A(SU_q(2))$, but now they are really related to the infinite summation.

Proposition. The set $w_{ij}^{(l)}, i, j \in I_l, l \in N/2$ forms a complete and orthogonal basis of $C(SU_q(2))$ with respect to the Haar measure h, and any element $a \in C(SU_q(2))$ has the unique expression of a series in $w_{ij}^{(l)}$.

We need only to prove the fact: if $\sum_l \sum_{i,j} c_{ij}^{(l)} w_{ij}^{(l)} = 0$, then all $c_{ij}^{(l)} = 0$. In fact, $<,>_R$ is continuous with respect to $\|\|_*$, it follows

$$\begin{aligned} 0 &=< 0, w_{st}^{(m)} >_R \\ &=< \textstyle\sum_l \sum_{i,j} c_{ij}^{(l)} w_{ij}^{(l)}, w_{st}^{(m)} >_R \\ &= \textstyle\sum_l \sum_{i,j} c_{ij}^{(l)} < w_{ij}^{(l)}, w_{st}^{(m)} >_R \\ &= q^{2(l-j)}(1-q^2)/(1-q^{2(2l+1)})c_{st}^{(m)}. \end{aligned}$$

then $c_{st}^{(m)} = 0 \qquad \forall m, s, t.$

The Fourier transform on $C(SU_q(2))$ is defined as

$$\begin{aligned} &\mathcal{F} : C(SU_q(2)) \to \widehat{C} = \oplus_{l \in N/2} Mat(I_l, C) \\ &\mathcal{F}(f) = (\widehat{f}^{(l)})_{l \in N/2}, \end{aligned}$$

where

$$\widehat{f}^{(l)} = h(f.(W_l)^*) = h(f.s(W_l)),$$

i.e.,

$$\widehat{f}_{ij}^{(l)} = h(f.s(w_{ij}^{(l)})).$$

Due to the above Proposition, as f is expressed in a series of $w_{ij}^{(l)}$, the coefficients are determined in a unique way, hence the Fourier inversion formula holds.

Theorem. For $(C(SU_q(2)), \|\|_*)$, holds the expansion

$$f = \sum_{l \in N/2} \sum_{i,j \in I_l} [(q^{2l+1} - q^{-2l-1})/(q - q^{-1})] q^{2j} \widehat{f}_{ij}^{(l)} w_{ij}^{(l)}.$$

The R.H.S. is called the Fourier series of f. Thus, it converges to f in the norm $\|\|_*$. Moreover, $\mathcal{F}$ is an isometry between $C(SU_q(2))$ and $\widehat{C}$, that means, we have

$$< f, g >_R = \sum_{l \in N/2} [(q^{2l+1} - q^{-2l-1})/(q - q^{-1})] \sum_{i,j \in I_l} q^{2j} < \widehat{f}_{ij}^{(l)}, \widehat{g}_{ij}^{(l)} >_R .$$

At last, let's introduce the convolution in $C(SU_q(2))$. For $a \in A(SU_q(2)), b \in C(SU_q(2))$ we can define another multiplication $\odot$, called convolution, as follows

$$a \odot b = (id \otimes \rho_a)(\Delta b)$$

where ρ_a is a linear functional on $C(SU_q(2))$ defined by

$$\rho_a(x) = h(x.s(a)), \qquad x \in C(SU_q(2)).$$

As well as

$$b \odot a = (\rho'_a \otimes id)(\Delta b),$$

where

$$\rho'_a(x) = h(x.s^{-1}(a)).$$

It is easy to verify that convolution $\odot$ satisfies the law of associativity, if the operation is reasonable. The following is an analogue of the convolution theorem for classical Fourier transform:

Theorem. If a or b belongs to $A(SU_q(2))$, then

$$\mathcal{F}(a \odot b) = \mathcal{F}(a).\mathcal{F}(b)$$

where the R.H.S. is the multiplication of matrices on C.

Using Fourier expansion and definitions, it is not difficult to verify the theorem. From the finiteness of the Fourier coefficients for $SU_q(2)$ and the inversion formula, we have

Corollary. $A(SU_q(2))$ is an two-sided ideal of $C(SU_q(2))$ with respect to the convolution.

This note is completed during the Workshop *on Quantum Groups and Low-Dimensional Field Theory* organized by the CCAST. The author would like to express his gratitude to the host for providing this good opportunity for academic exchange and excellent conditions for research work, and to Profs. Shikun Wang and Ke Wu for beneficial discussions.

References

1. V. G. Drinfeld, Quantum Groups, in Proc. ICM, Berkeley, CA USA, 1986, Academic Press, 1987, 798-820.

2. J. Dixmier, Les C^*-algebres et leurs representations, Paris: Gauthier-Villars, 1969.

3. L. D. Fadeev, N. Reshetikhin, L. A. Takhatajan, Quantization of Lie groups and Lie algebras, in "Algebraic Analysis", Academic Press, 1988, 129-140.

4. T. H. Koornwinder, Representations of twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials, Nederl. Akad. Wetensch. Proc. Ser. A 92(1989), 97-117.

5. T. Masuda et al., Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials, J. Funct. Anal., 99(1991), 357-386.

6. T. Masuda et al., Unitary representations of the quantum group $SU_q(1,1)$, Lett. Math. Phys., 19(1990), 187-194, 195-204.

7. L. L. Vaksman, Ya. Soibelman, Algebra of functions on the quantum group $SU(2)$, Funct. Anal. Appl., 22(1988), 170-181 (in Russian).

8. S. L. Woronowics, Compact matrix pseudogroups, Commun. Math. Phys., 111(1987), 613-665.

CCAST-92-14
ASITP-92-18
ASIAM-92-11
March, 1992

Differential Calculus on Quantum Lorentz Group*

XIAODONG SUN[1,2] SHIKUN WANG[1,2] KE WU[1,3]

[1]CCAST (World laboratory), P.O. Box 3730, Beijing 100080, China
[2]Institute of Applied Mathematics, Academia Sinica, Beijing 100080, China
[3]Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

Abstract

In this paper, we discuss the bicovariant differential calculus on quatum Lorentz group, and provide corresponding de Rham complex and Maurer-Cartan formulae.

*Supported in part by Natural Science Foundation of China and the Foundation of Institute of Theoretical Physics, Academia Sinica.

§0. Introduction

Quantum groups and quantum Lie algebras as deformed Lie groups and Lie algebras have been studied extensively following the pioneer papers of Drinfeld, Jimbo, Faddeev et al[1]. It is also very interesting to study the quantum groups from geometrical point of view[2], i.e. to find its relation with non-commutative geometry[3]. Although the bicovariant differential calculi on quantum groups have been provided by Woronowicz[4], their concrete constructions still attract much attention. So several groups have been working on this question on different subjects, for example, the differential calculi on quantum planes[5], on quantum groups[6,7,8] as well as on quantum groups with multi-deformed parameters[9], one can also find many results in the review paper of Zumino[10].

The construction of quantum Lorentz group was first given in [11] by Podlés and Woronowicz, and then discussed by Drabant et al.[12], Carow-Watamura et al.[13] and others[14]. Following the works of [11,12], the main purpose of this letter is to discuss the bicovariant differential calculus on quantum Lorentz group based on the method given in [7,8,9]. The results we obtained include the bicovariant first order and high order differential calculus on quantum Lorentz group, the quantum Maurer-Cartan formulae, i.e. the deformed de Rham complex on quantum Lorentz group as following sequence of bimodules of quantum Lorentz group

$$0 \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \xrightarrow{\mathrm{d}} \cdots. \qquad (0.1)$$

Summation convention will be used in this paper.

§1. Quantum Lorentz group: Ω^0

It is well known that the quantum matrix group $SL_q(2)$ is generated by four elements which can be written into a 2×2 matrix $T=(t_{ij})_{i,j=1,2}$ and also satisfy the following relations,

$$RT_1T_2 = T_2T_1R, \quad Det_qT = t_{11}t_{22} - qt_{12}t_{21} = 1, \qquad (1.1)$$

where

$$R = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \lambda = q - q^{-1},\ q \in I\!R\,.$$

For quantum Lorentz group $SL_q(2,\mathbb{C})$, we need to introduce t^*_{ij} $(i,j=1,2)$, the conjugates of elements t_{ij} $(i,j=1,2)$, and generators $t_{\bar{i}\bar{j}}$ $(i,j=1,2)$ as

$$\widehat{T} = (t_{\bar{i}\bar{j}})_{i,j=1,2}, \quad t_{\bar{i}\bar{j}} = (S(t_{ji}))^*,$$

where S is the antipode. With a few calculation, we have

$$R\widehat{T}_1\widehat{T}_2 = \widehat{T}_2\widehat{T}_1 R, \qquad Det_q\widehat{T} = t_{\bar{1}\bar{1}}t_{\bar{2}\bar{2}} - qt_{\bar{1}\bar{2}}t_{\bar{2}\bar{1}} = 1,$$

or

$$(R^+)^{-1}\widehat{T}_1\widehat{T}_2 = \widehat{T}_2\widehat{T}_1(R^+)^{-1}, \qquad Det_q\widehat{T} = t_{\bar{1}\bar{1}}t_{\bar{2}\bar{2}} - qt_{\bar{1}\bar{2}}t_{\bar{2}\bar{1}} = 1. \tag{1.2}$$

The commutation relations between t_{ij} and $t_{\bar{i}\bar{j}}$ are defined as[11]

$$R\widehat{T}_1 T_2 = T_2\widehat{T}_1 R. \tag{1.3}$$

Then we can simplely rewrite (1.1), (1.2), (1.3) as

$$\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R}, \tag{1.4}$$

where

$$\mathcal{T} = \begin{pmatrix} T & \\ & \widehat{T} \end{pmatrix},$$

$\mathcal{R} = (\mathcal{R}^{ab}_{cd})_{a,b,c,d=1,2,\bar{1},\bar{2}}$,

$$\mathcal{R}^{ij}_{kl} = R^{ij}_{kl},\ \mathcal{R}^{i\bar{j}}_{k\bar{l}} = ((R^+)^{-1})^{ij}_{kl},\ \mathcal{R}^{\bar{i}j}_{\bar{k}l} = R^{ij}_{kl},\ \mathcal{R}^{\bar{i}\bar{j}}_{\bar{k}\bar{l}} = ((R^+)^{-1})^{ij}_{kl},\ i,j,k,l = 1,2,$$

and other elements of $\mathcal{R}$ are zeros. It can be checked that $\mathcal{R}$ satisfies the Yang-Baxter equation.

Definition 1.1

$$Fun(SL_q(2,\mathbb{C})) := \mathbb{C}[t_{ij}, t_{\bar{i}\bar{j}}]/\{\mathcal{R}\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1\mathcal{R},\ Det_q T = 1,\ Det_q\widehat{T} = 1\}.$$

$SL_q(2,\mathbb{C})$ can be understood as the direct sum of two copies of $SL_q(2)$[11,12], so the comultiplication Δ, the counite ε and the antipode S on $SL_q(2,\mathbb{C})$ can be naturally induced from those operators on $SL_q(2)$. Therefore, $Fun(SL_q(2,\mathbb{C}))$ is a Hopf algebra. For simplicity we denote $Fun(SL_q(2,\mathbb{C}))$ by Ω^0 as the first bimodule of the de Rham complex in (0.1).

Now we can introduce two sets of linear functionals on Ω^0 by

$$\mathcal{L}^+ = \begin{pmatrix} L^+ & \\ & \widehat{L}^+ \end{pmatrix}, \qquad \mathcal{L}^- = \begin{pmatrix} L^- & \\ & \widehat{L}^- \end{pmatrix},$$

where $L^\pm = (l^\pm_{ij})_{i,j=1,2}$, $\widehat{L}^\pm = (l^\pm_{\bar{i}\bar{j}})_{i,j=1,2}$.

Definition 1.2

$$< l^\pm_{ab}, t_{cd} >= (\mathcal{R}^\pm)^{ac}_{bd}, \quad a,b,c,d = 1,2,\bar{1},\bar{2}$$

where $\mathcal{R}^+ = P\mathcal{R}P$, $\mathcal{R}^- = \mathcal{R}^{-1}$ and P is the permutation matrix[1][12].

Definition 1.2 can be also write down as

$$
\begin{array}{lll}
< l^{+}_{ij}, t_{kl} >= (R^{+})^{ik}_{jl}, & \text{i.e.} & < L^{+}, T >= R^{+}, \\
< l^{+}_{ij}, t_{\bar{k}\bar{l}} >= (R^{+})^{ik}_{jl}, & \text{i.e.} & < L^{+}, \widehat{T} >= R^{+}, \\
< l^{+}_{\bar{i}\bar{j}}, t_{kl} >= (R^{-})^{ik}_{jl}, & \text{i.e.} & < \widehat{L}^{+}, T >= R^{-}, \\
< l^{+}_{\bar{i}\bar{j}}, t_{\bar{k}\bar{l}} >= (R^{-})^{ik}_{jl}, & \text{i.e.} & < \widehat{L}^{+}, \widehat{T} >= R^{-}, \\
< l^{-}_{ij}, t_{kl} >= (R^{-})^{ik}_{jl}, & \text{i.e.} & < L^{-}, T >= R^{-}, \\
< l^{-}_{ij}, t_{\bar{k}\bar{l}} >= (R^{+})^{ik}_{jl}, & \text{i.e.} & < L^{-}, \widehat{T} >= R^{+}, \\
< l^{-}_{\bar{i}\bar{j}}, t_{kl} >= (R^{-})^{ik}_{jl}, & \text{i.e.} & < \widehat{L}^{-}, T >= R^{-}, \\
< l^{-}_{\bar{i}\bar{j}}, t_{\bar{k}\bar{l}} >= (R^{+})^{ik}_{jl}, & \text{i.e.} & < \widehat{L}^{-}, \widehat{T} >= R^{+},
\end{array}
$$

where $i, j, k, l = 1, 2$. From the above, we see l^{-}_{ij} and $l^{-}_{\bar{i}\bar{j}}$ have the same definition, so we may assume $l^{-}_{ij} = l^{-}_{\bar{i}\bar{j}}$.

By general theory of quantum groups, we know

$$
\mathcal{R}^{+}\mathcal{L}^{\pm}_{1}\mathcal{L}^{\pm}_{2} = \mathcal{L}^{\pm}_{2}\mathcal{L}^{\pm}_{1}\mathcal{R}^{+}, \qquad \mathcal{R}^{+}\mathcal{L}^{+}_{1}\mathcal{L}^{-}_{2} = \mathcal{L}^{-}_{2}\mathcal{L}^{+}_{1}\mathcal{R}^{+}. \tag{1.5}
$$

(1.5) can also be written as

$$
\begin{array}{ll}
R^{+}L^{\pm}_{1}L^{\pm}_{2} = L^{\pm}_{2}L^{\pm}_{1}R^{+}, & R^{+}L^{\pm}_{1}\widehat{L}^{\pm}_{2} = \widehat{L}^{\pm}_{2}L^{\pm}_{1}R^{+}, \\
R^{-}\widehat{L}^{\pm}_{1}L^{\pm}_{2} = L^{\pm}_{2}\widehat{L}^{\pm}_{1}R^{-}, & R^{-}\widehat{L}^{\pm}_{1}\widehat{L}^{\pm}_{2} = \widehat{L}^{\pm}_{2}\widehat{L}^{\pm}_{1}R^{-}, \\
R^{+}L^{+}_{1}L^{-}_{2} = L^{-}_{2}L^{+}_{1}R^{+}, & R^{+}L^{+}_{1}\widehat{L}^{-}_{2} = \widehat{L}^{-}_{2}L^{+}_{1}R^{+}, \\
R^{-}\widehat{L}^{+}_{1}L^{-}_{2} = L^{-}_{2}\widehat{L}^{+}_{1}R^{-}, & R^{-}\widehat{L}^{+}_{1}\widehat{L}^{-}_{2} = \widehat{L}^{-}_{2}\widehat{L}^{+}_{1}R^{-},
\end{array}
$$

Denote $Fun^{*}_{0}(SL_q(2,\mathbb{C}))$ the associate algebra generated by $L^{\pm}_{ij}$ and $L^{\pm}_{\bar{i}\bar{j}}$ $(i, j = 1, 2)$. The comultiplication Δ^{*}, the counite ε^{*} and the antipode S^{*} can also be defined on it. It is clear that $Fun^{*}_{0}(SL_q(2,\mathbb{C}))$ is a Hopf subalgebra of $Fun^{*}(SL_q(2,\mathbb{C}))$, the dual of Ω^{0}.

The main results in this section have already appeared in [12], here we use a different convention used in this letter.

§2. The first order differential calculus: Ω^1

Assume $\mathcal{A}$ is an associative algebra with unit. The first order differential calculus on $\mathcal{A}$, which is denoted by (Γ, δ), consists of a bi-module Γ of $\mathcal{A}$ and a linear operator δ satisfying

(i) Leibnitz rule

$$
\delta(xy) = (\delta x)y + x\delta y, \qquad \forall x, y \in \mathcal{A},
$$

(ii) for arbitrary element ρ in Γ, there always exist some elements $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \cdots, N)$ such that

$$
\rho = \sum_{k=1}^{N} x_k \delta y_k.
$$

Now we introduce the convolution "$*$" on Ω^0. For $f \in Fun^*(SL_q(2,\mathbb{C}))$, the convolution "$*$" from Ω^0 to Ω^0 is defined by

$$f*(x) = (id \otimes f)\Delta x, \; x \in \mathrm{Fun}(GL_q(n)), \tag{2.1}$$

where id is the identity operator on Ω^0. Furthermore, we introduce two sets of functionals on Ω^0 as follows:

$$\nabla_{ab} := \frac{1}{\lambda}(S^*(l^-_{ac})l^+_{cb} - \delta_{ab}\varepsilon),$$
$$\theta_{abcd} := S^*(l^-_{ca})l^+_{bd},$$

where $a,b,c,d = 1,2,\bar{1},\bar{2}$.

Proposition 2.1 For $\forall x, y \in \Omega^0$, $a,b,c,d,e,f = 1,2,\bar{1},\bar{2}$, we have

$$\begin{aligned}
&\text{(i)} && \nabla_{ab}(1) = 0, \quad \theta_{abcd}(1) = \delta_{ac}\delta_{bd},\\
&\text{(ii)} && \Delta^*\nabla_{ab} = \nabla_{ef} \otimes \theta_{efab} + \varepsilon \otimes \nabla_{ab},\\
& && \Delta^*\theta_{abcd} = \theta_{abef} \otimes \theta_{efcd},\\
&\text{(iii)} && \nabla_{ab} * (xy) = (\nabla_{ef} * x)(\theta_{efab} * y) + x(\nabla_{ab} * y),\\
& && \theta_{abcd} * (xy) = (\theta_{abef} * x)(\theta_{efcd} * y).
\end{aligned}$$

The proof of Proposition 2.1 can be found in [8]. Let Ω^1 be the left module generated by eight generators ω^{ij}, $\omega^{\bar{i}\bar{j}}$, $(i,j = 1,2)$, and define the right multiplication on Ω^1 by

$$\omega^{ab} \cdot x = (\theta_{abcd} * x)\omega^{cd}, \quad \forall x \in \Omega^0, \quad a,b,c,d = 1,2,\bar{1},\bar{2}.$$

Therefore, Ω^1 becomes a Ω^0-bimodule.

Definition 2.1

$$dx := (\nabla_{ab} * x)\omega^{ab}, \quad \forall x \in \Omega^0, a,b = 1,2,\bar{1},\bar{2}. \tag{2.2}$$

It is easy to check

$$d(xy) = (dx)y + xdy, \quad \forall x, y \in \Omega^0$$

To prove (Ω^1, d) is a differential calculus on Ω^0, we must show that each of ω^{ij}, $\omega^{\bar{i}\bar{j}}$, $(i,j = 1,2)$ can be represented by

$$\sum_{k=1}^{N} x_k dy_k, \quad x_k, y_k \in \Omega^0.$$

We have for $i,j,k,l,u,v = 1,2$,

$$\begin{aligned}
\nabla_{ij}(t_{\bar{k}\bar{l}}) &= \tfrac{1}{\lambda}(S^*(l^-_{iu})l^+_{uj} - \delta_{ij}\varepsilon)(t_{\bar{k}\bar{l}})\\
&= \tfrac{1}{\lambda}[< S^*(l^-_{iu}), t_{\bar{k}\bar{v}} >< l^+_{uj}, t_{\bar{v}\bar{l}} > -\delta_{ij}\delta_{kl}]\\
&= \tfrac{1}{\lambda}[((R^+)^{-1})^{ik}_{uv}(R^+)^{uv}_{jl} - \delta_{ij}\delta_{kl}]\\
&= 0.
\end{aligned}$$

Similarly, we have $\nabla_{\bar{i}\bar{j}}(t_{kl}) = 0$. Therefore, we have for $i, j, k, u, v = 1, 2$,

$$S(t_{ik})dt_{kj} = \nabla_{uv}(t_{ij})\omega^{uv},$$
$$S(t_{\bar{i}\bar{k}})dt_{\bar{k}\bar{j}} = \nabla_{\bar{u}\bar{v}}(t_{\bar{i}\bar{j}})\omega^{\bar{u}\bar{v}}.$$

Now we can treat the representation of ω^{ij} and $\omega^{\bar{i}\bar{j}}$, $(i, j = 1, 2)$ seperately, therefore we have

$$\begin{aligned}
\omega^1 &= [q^3(t_{22}dt_{11} - q^{-1}t_{12}dt_{21}) - (q^3 - q - 1)(-qt_{21}dt_{12} + t_{11}dt_{22})]/(q^3 - 1),\\
\omega^2 &= q(-qt_{21}dt_{11} + t_{11}dt_{21}),\\
\omega^3 &= q(t_{22}dt_{12} - q^{-1}t_{12}dt_{22}),\\
\omega^4 &= [q^2(t_{22}dt_{11} - q^{-1}t_{12}dt_{21}) + q^3(-qt_{21}dt_{12} + t_{11}dt_{22})]/(q^3 - 1),\\
\omega^{\bar{1}} &= [(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{1}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{1}}) + q(-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{2}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{2}})]/(q^3 - 1),\\
\omega^{\bar{2}} &= -q^{-1}(-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{1}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{1}}),\\
\omega^{\bar{3}} &= -q^{-1}(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{2}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{2}}),\\
\omega^{\bar{4}} &= [(q^3 + q^2 - 1)(t_{\bar{2}\bar{2}}dt_{\bar{1}\bar{1}} - q^{-1}t_{\bar{1}\bar{2}}dt_{\bar{2}\bar{1}}) + (-qt_{\bar{2}\bar{1}}dt_{\bar{1}\bar{2}} + t_{\bar{1}\bar{1}}dt_{\bar{2}\bar{2}})]/(q^3 - 1).
\end{aligned}$$

By result of [8], the differential calculus provided above on quantum Lorentz group is bicovariant.

Let $\widetilde{\mathcal{L}} = S^*(\mathcal{L}^-)\mathcal{L}^+$, i.e. $\mathbf{L} = S^*(L^-)L^+$, $\widehat{\mathbf{L}} = S^*(\widehat{L}^-)\widehat{L}^+$,

$$\widetilde{\mathcal{L}} = \begin{pmatrix} \mathbf{L} & \\ & \widehat{\mathbf{L}} \end{pmatrix},$$

we have

$$\mathcal{R}\widetilde{\mathcal{L}}_1\mathcal{R}^+\widetilde{\mathcal{L}}_2 = \widetilde{\mathcal{L}}_2\mathcal{R}\widetilde{\mathcal{L}}_1\mathcal{R}^+. \tag{2.3}$$

We can also write (2.3) as

$$\begin{aligned}
&R\mathbf{L}_1R^+\mathbf{L}_2 = \mathbf{L}_2R\mathbf{L}_1R^+, && (R^+)^{-1}\mathbf{L}_1R^+\widehat{\mathbf{L}}_2 = \widehat{\mathbf{L}}_2(R^+)^{-1}\mathbf{L}_1R^+,\\
&R\widehat{\mathbf{L}}_1R^-\mathbf{L}_2 = \mathbf{L}_2R\widehat{\mathbf{L}}_1R^-, && (R^+)^{-1}\widehat{\mathbf{L}}_1R^-\widehat{\mathbf{L}}_2 = \widehat{\mathbf{L}}_2(R^+)^{-1}\widehat{\mathbf{L}}_1R^-.
\end{aligned}$$

Since

$$\mathbf{L}_{ij} = (\lambda\nabla_{ij} + \delta_{ij}\varepsilon), \qquad \widehat{\mathbf{L}}_{\bar{i}\bar{j}} = (\lambda\nabla_{\bar{i}\bar{j}} + \delta_{ij}\varepsilon),$$

by the results of [7], we have the commutation relations of derivatives ∇_{ab} $(a, b = 1, 2, \bar{1}, \bar{2})$ on quantum Lorentz group

$$\nabla_{cc'}\nabla_{dd'} - \nabla_{aa'}\nabla_{bb'}\mathbf{R}^{aa'bb'}_{cc'dd'} = \nabla_{ee'}\mathbf{F}^{ee'}_{cc'dd'}, \tag{2.4}$$

where

$$\begin{aligned}
\mathbf{R}^{aa',bb'}_{cc',dd'} &= < \theta_{aa'dd'}, T_{cb}S(T_{b'c'}) >,\\
\mathbf{F}^{aa'}_{bb',cc'} &= \lambda < \nabla_{cc'}, T_{ca}S(T_{a'c'}) >,
\end{aligned}$$

$a, a', b, b', c, c', d, d', e, e' = 1, 2, \bar{1}, \bar{2}$, and $\mathbf{R}$ is a 256×256 matrix satisfying Y-B equation. (2.4) give the commutation relations of ∇_{ab} $(a, b = 1, 2, \bar{1}, \bar{2})$ which are shown in Table 1.

Table 1

(i)
$$
\begin{aligned}
&\nabla_2\nabla_1 - q^2\nabla_1\nabla_2 = (q^2-1)\nabla_2,\\
&\nabla_3\nabla_1 - q^{-2}\nabla_1\nabla_2 = (q^{-2}-1)\nabla_3,\\
&\nabla_4\nabla_1 - \nabla_1\nabla_4 = 0,\\
&\nabla_3\nabla_2 - \nabla_2\nabla_3 + (1-q^{-2})\nabla_4\nabla_1 - (1-q^{-2})\nabla_1\nabla_1 = (1-q^{-2})(\nabla_1 - \nabla_4),\\
&\nabla_4\nabla_2 - \nabla_2\nabla_4 + (q^{-2}-1)\nabla_1\nabla_2 = (1-q^{-2})\nabla_2,\\
&\nabla_4\nabla_3 - \nabla_3\nabla_4 + q^{-2}(1-q^{-2})\nabla_1\nabla_3 = -q^{-2}(1-q^{-2})\nabla_3,
\end{aligned}
$$

(ii)
$$
\begin{aligned}
&\nabla_{\bar 2}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 2} + (q^2-1)\nabla_{\bar 2}\nabla_{\bar 4} = -(q^2-1)\nabla_{\bar 2},\\
&\nabla_{\bar 3}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 3} - (q^2-1)\nabla_{\bar 4}\nabla_{\bar 3} = (q^2-1)\nabla_{\bar 3},\\
&\nabla_{\bar 4}\nabla_{\bar 1} - \nabla_{\bar 1}\nabla_{\bar 4} = 0,\\
&\nabla_{\bar 3}\nabla_{\bar 2} - \nabla_{\bar 2}\nabla_{\bar 3} + (q^2-1)\nabla_{\bar 1}\nabla_{\bar 4} - (q^2-1)\nabla_{\bar 4}\nabla_{\bar 4} = -(q^2-1)(\nabla_{\bar 1} - \nabla_{\bar 4}),\\
&\nabla_{\bar 4}\nabla_{\bar 2} - q^{-2}\nabla_{\bar 2}\nabla_{\bar 4} = (q^{-2}-1)\nabla_{\bar 2},\\
&\nabla_{\bar 4}\nabla_{\bar 3} - q^{2}\nabla_{\bar 3}\nabla_{\bar 4} = (q^{2}-1)\nabla_{\bar 3},
\end{aligned}
$$

(iii)
$$
\begin{aligned}
&\nabla_{\bar 1}\nabla_1 = \nabla_1\nabla_{\bar 1} + q^2(q^2-1)\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 1}\nabla_2 = -(q^2-1)\nabla_1\nabla_{\bar 2} + \nabla_2\nabla_{\bar 1} + (q^2-1)\nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 1}\nabla_3 = \nabla_3\nabla_{\bar 1},\\
&\nabla_{\bar 1}\nabla_4 = -(q^2-1)\nabla_3\nabla_{\bar 2} + \nabla_4\nabla_{\bar 1},\\
&\nabla_{\bar 2}\nabla_1 = \nabla_1\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_2 = q^{-2}\nabla_2\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_3 = q^{2}\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 2}\nabla_4 = \nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 3}\nabla_1 = \nabla_1\nabla_{\bar 3} - (q^2-1)\nabla_3\nabla_{\bar 1} + (q^2-1)\nabla_3\nabla_{\bar 4},\\
&\nabla_{\bar 3}\nabla_2 = (q^2-1)\nabla_1\nabla_{\bar 1} - (q^2-1)\nabla_1\nabla_{\bar 4} + q^2\nabla_2\nabla_{\bar 3}\\
&\qquad\qquad +(q^4-1)(1-q^{-2})\nabla_3\nabla_{\bar 2} - (q^2-1)\nabla_4\nabla_{\bar 1} + (q^2-1)\nabla_4\nabla_{\bar 4},\\
&\nabla_{\bar 3}\nabla_3 = q^{-2}\nabla_3\nabla_{\bar 3},\\
&\nabla_{\bar 3}\nabla_4 = (1-q^{-2})\nabla_3\nabla_{\bar 1} - (1-q^{-2})\nabla_3\nabla_{\bar 4} + \nabla_4\nabla_{\bar 3},\\
&\nabla_{\bar 4}\nabla_1 = \nabla_1\nabla_{\bar 4} - (q^2-1)\nabla_3\nabla_{\bar 2},\\
&\nabla_{\bar 4}\nabla_2 = (1-q^{-2})\nabla_1\nabla_{\bar 2} + \nabla_2\nabla_{\bar 4} - (1-q^{-2})\nabla_4\nabla_{\bar 2},\\
&\nabla_{\bar 4}\nabla_3 = \nabla_3\nabla_{\bar 4},\\
&\nabla_{\bar 4}\nabla_4 = (1-q^{-2})\nabla_3\nabla_{\bar 2} + \nabla_4\nabla_{\bar 4}.
\end{aligned}
$$

For simplicity, in Table 1 and following two tables we write "$1,2,3,4,\bar 1,\bar 2,\bar 3,\bar 4$" instead of "$11,12,21,22,\bar 1\bar 1,\bar 1\bar 2,\bar 2\bar 1,\bar 2\bar 2$" respectively.

By result of [7], we also have the deformed Jacobian identities satified by ∇_{ab} and θ_{abcd} together with (2.4)

$$
\begin{aligned}
&\nabla_{aa'}\theta_{bb'cc'} = \theta_{bb'dd'}\nabla_{ee'}\mathbf{R}^{dd'ee'}_{aa'cc'},\\
&\mathbf{R}^{aa'bb'}_{ee'ff'}\theta_{ee'cc'}\theta_{ff'dd'} = \theta_{aa'ee'}\theta_{bb'ff'}\mathbf{R}^{ee'ff'}_{cc'dd'},\\
&\mathbf{F}^{aa'}_{dd'ee'}\theta_{dd'bb'}\theta_{ee'cc'} + \theta_{aa'bb'}\nabla_{cc'} = \nabla_{dd'}\theta_{aa'ee'}\mathbf{R}^{dd'ee'}_{bb'cc'} + \theta_{aa'dd'}\mathbf{F}^{dd'}_{bb'cc'}.
\end{aligned}
$$

§3. High order differential calculus: $\Omega^\wedge$

As Woronowitz pointed out in [4], the high order differential calculus $\Omega^\wedge$ is constructed as follows

$$\Omega^\wedge = \Omega^\otimes / N, \quad \Omega^k = \Omega^{\otimes k}/N_k,$$

where

$$\Omega^\otimes = \oplus_{n=0}^{\infty}\Omega^{\otimes n}, \quad \Omega^\otimes = \oplus_{n=0}^{k}\Omega^{\otimes n}, \quad \Omega^{\otimes 0} = \Omega^0, \quad \Omega^{\otimes 1} = \Omega^1,$$

N and N_k are the two-sided ideal generated by $ker(1-\sigma)$ in $\Omega^\otimes$ and $\Omega^{\otimes k}$ respectively, σ is an automorphism on $\Omega^1 \otimes \Omega^1$, compatible with left and right group actions, and also satisfies the braid relation.

By the result of [8], the automorphism σ on $\Omega^1 \otimes \Omega^1$ can be constructed as

$$\begin{aligned}&\sigma(X_{aa'bb'}\omega^{aa'} \otimes \omega^{bb'}) = X_{aa'bb'}\mathbf{R}^{aa'bb'}_{cc'dd'}(\omega^{cc'} \otimes \omega^{dd'}),\\ &\forall X_{aa'bb'} \in \Omega^0, \ a, a', b, b' = 1, 2, \bar{1}, \bar{2}.\end{aligned}$$

Therefore, we obtain $\Omega^\wedge$, the external algebra of quantum Lorentz group and Ω^k, the bimodules in (0.1) of de Rham complex of quantum Lorentz group. By some calculations, we obtain the minimal polynomial of $\mathbf{R}$ as following,

$$(t-1)(t+1)(t+q^2)(t+q^{-2}),$$

so that N is generated by following elements

$$[(\mathbf{R}+1)(\mathbf{R}+q^2)(\mathbf{R}+q^{-2})]^{aa'bb'}_{cc'dd'}(\omega^{cc'} \otimes \omega^{dd'}), \qquad a, a', b, b', c, c', d, d' = 1, 2, \bar{1}, \bar{2}.$$

Therefore, we obtain an equivalent relation on $\Omega^1 \otimes \Omega^1$ which are shown in Table 2.

Table 2

(i) $\omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 = \omega^4 \wedge \omega^4 = 0,$
$\omega^3 \wedge \omega^2 = -\omega^2 \wedge \omega^3, \quad \omega^4 \wedge \omega^2 = -\omega^2 \wedge \omega^4,$
$\omega^4 \wedge \omega^3 = -\omega^3 \wedge \omega^4,$
$\omega^2 \wedge \omega^1 = -q^{-2}\omega^1 \wedge \omega^2 - q^{-2}(q^{-2}-1)\omega^2 \wedge \omega^4,$
$\omega^3 \wedge \omega^1 = -q^2\omega^1 \wedge \omega^3 + (q^{-2}-1)\omega^3 \wedge \omega^4,$
$\omega^4 \wedge \omega^1 = -\omega^1 \wedge \omega^4 + (q^{-2}-1)\omega^2 \wedge \omega^3,$
$\omega^1 \wedge \omega^1 = (1-q^{-2})\omega^2 \wedge \omega^3,$

(ii) $\omega^{\bar 1} \wedge \omega^{\bar 1} = \omega^{\bar 2} \wedge \omega^{\bar 2} = \omega^{\bar 3} \wedge \omega^{\bar 3} = 0,$
$\omega^{\bar 2} \wedge \omega^{\bar 1} = -\omega^{\bar 1} \wedge \omega^{\bar 2}, \quad \omega^{\bar 3} \wedge \omega^{\bar 1} = -\omega^{\bar 1} \wedge \omega^{\bar 3},$
$\omega^{\bar 3} \wedge \omega^{\bar 2} = -\omega^{\bar 2} \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^{\bar 1} = -\omega^{\bar 1} \wedge \omega^{\bar 4} - (q^2-1)\omega^{\bar 2} \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^{\bar 2} = -q^2\omega^{\bar 2} \wedge \omega^{\bar 4} - q^2(q^2-1)\omega^{\bar 1} \wedge \omega^{\bar 2},$
$\omega^{\bar 4} \wedge \omega^{\bar 3} = -q^{-2}\omega^{\bar 3} \wedge \omega^{\bar 4} + (q^2-1)\omega^{\bar 1} \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^{\bar 4} = (q^2-1)\omega^{\bar 2} \wedge \omega^{\bar 3},$

(iii) $\omega^{\bar 1} \wedge \omega^1 = -\omega^1 \wedge \omega^{\bar 1} + (1-q^{-2})\omega^2 \wedge \omega^{\bar 3},$
$\omega^{\bar 1} \wedge \omega^2 = -\omega^2 \wedge \omega^{\bar 1},$
$\omega^{\bar 1} \wedge \omega^3 = -\omega^3 \wedge \omega^{\bar 1} - (q^2-1)\omega^1 \wedge \omega^{\bar 3} + (1-q^{-2})\omega^4 \wedge \omega^{\bar 3},$
$\omega^{\bar 1} \wedge \omega^4 = -\omega^4 \wedge \omega^{\bar 1} - (1-q^{-2})\omega^2 \wedge \omega^{\bar 3},$
$\omega^{\bar 2} \wedge \omega^1 = -\omega^1 \wedge \omega^{\bar 2} - (q^2-1)\omega^2 \wedge \omega^{\bar 1} + (1-q^{-2})\omega^2 \wedge \omega^{\bar 4},$
$\omega^{\bar 2} \wedge \omega^2 = -q^2\omega^2 \wedge \omega^{\bar 2},$
$\omega^{\bar 2} \wedge \omega^3 = (q^2-1)\omega^1 \wedge \omega^{\bar 1} - (1-q^{-2})\omega^1 \wedge \omega^{\bar 4} - (1-q^{-4})(q^2-1)\omega^2 \wedge \omega^{\bar 3}$
$\qquad -q^{-2}\omega^3 \wedge \omega^{\bar 2} + (q^{-2}-1)\omega^4 \wedge \omega^{\bar 1} + q^{-4}(q^2-1)\omega^4 \wedge \omega^{\bar 4},$
$\omega^{\bar 2} \wedge \omega^4 = (q^2-1)\omega^2 \wedge \omega^{\bar 1} + (q^{-2}-1)\omega^2 \wedge \omega^{\bar 4} - \omega^4 \wedge \omega^{\bar 2},$
$\omega^{\bar 3} \wedge \omega^1 = -\omega^1 \wedge \omega^{\bar 3},$
$\omega^{\bar 3} \wedge \omega^2 = -q^{-2}\omega^2 \wedge \omega^{\bar 3},$
$\omega^{\bar 3} \wedge \omega^3 = -q^2\omega^3 \wedge \omega^{\bar 3},$
$\omega^{\bar 3} \wedge \omega^4 = -\omega^4 \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^1 = -\omega^1 \wedge \omega^{\bar 4} + (q^{-2}-1)\omega^2 \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^2 = -\omega^2 \wedge \omega^{\bar 4},$
$\omega^{\bar 4} \wedge \omega^3 = (q^2-1)\omega^1 \wedge \omega^{\bar 3} - \omega^3 \wedge \omega^{\bar 4} + (q^{-2}-1)\omega^4 \wedge \omega^{\bar 3},$
$\omega^{\bar 4} \wedge \omega^4 = (1-q^{-2})\omega^2 \wedge \omega^{\bar 3} - \omega^4 \wedge \omega^{\bar 4}.$

We can also obtain quantum Maurer-Cartan formulae which are shown in Table 3.

Table 3

$$
\begin{aligned}
d\omega^1 &= q^{-1}\omega^2 \wedge \omega^3, \\
d\omega^2 &= q^{-1}\omega^1 \wedge \omega^2 + q^{-3}\omega^2 \wedge \omega^4, \\
d\omega^3 &= -q\omega^1 \wedge \omega^3 - q^{-1}\omega^3 \wedge \omega^4, \\
d\omega^4 &= -q^{-1}\omega^2 \wedge \omega^3, \\
d\omega^{\bar{1}} &= -q\omega^{\bar{2}} \wedge \omega^{\bar{3}}, \\
d\omega^{\bar{2}} &= -q^3\omega^{\bar{1}} \wedge \omega^{\bar{2}} - q\omega^{\bar{2}} \wedge \omega^{\bar{4}}, \\
d\omega^{\bar{3}} &= q\omega^{\bar{1}} \wedge \omega^{\bar{3}} + q^{-1}\omega^{\bar{3}} \wedge \omega^{\bar{4}}, \\
d\omega^{\bar{4}} &= q\omega^2 \wedge \omega^3.
\end{aligned}
$$

Based on the quantum Maurer-Cartan formulae, it is possible to construct the de Rham cohomology on quantum Lorentz group which will appear in a seperate paper.

Remark After the present paper was finished, we received a very nice preprint titled "Vector Fields on Complex Quantum Groups" by C. Chryssomalakos, B. Drabant, M. Schlieker, W. Weich and B. Zumino. Although the vector fields related to bicovariant differential calculus on quantum groups including quantum Lorentz group is discussed in this paper, the main results of our present paper on quantum de Rham complex of quantum Lorentz group are still useful.

References

1. V.G. Drinfeld, Proc. Int. Congr. Math., Berkeley (1986) 798.
   M. Jimbo, Lett. Math. Phy. **10** (1985) 63.
   L. Faddeev, N. Reshetikhin and L. Takhtajan, Leningrad Math. J. 1 (1990) 193.

2. S.L. Woronowicz, Commum. Math. Phys. **111** (1987) 613.

3. Yu.I. Manin, "Quantum groups and non-commutative geometry", Centre de Recherches Math., Univ. de Montreal (1989).

4. S.L. Woronowicz, Commun. Math. Phys. **122** (1989) 125.

5. J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) 18B (1990) 302.

6. D. Bernard, Prog. Theor. Phys. Suppl. **102** (1990) 49.
   B. Jurčo, Lett. Math. Phys. **22** (1991) 177.
   U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, Commun. Math. Phys. **142** (1991) 605.

7. K. Wu & R. J. Zhang, preprint ASITP-91-72, CCAST-91-69, to appear in Commun. Theor. Phys.

8. X. D. Sun, S. K. Wang, preprint CCAST-92-04, ASIAM-92-07, ASITP-92-12.

9. X. D. Sun, S. K. Wang & Y. X. Wang, preprint ASIAM-91-43, CCAST-91-62.

10. B. Zumino, preprint LBL-31432.

11. P. Podlés and S.L. Woronowicz, Commun. Math. Phys. **130** (1990) 381.

12. B. Drabant, M. Schlieker, W. Weich, B. Zumino, preprint MPI-PTh/91-75, LMU-TPW 1991-5.

13. U. Carow-Watamura, M. Schlieker, M. Scholl & S. Watamura, Z. Phys. **C48** (1990) 159, and Int. J. Mod. Phys. **A6** (1991) 3081.

14. X.C. Song, preprint ASITP-91-66, ASITP-92-07.
    Z. Xu, preprint (1991).
    W.B. Schmidke, J. Wess & B. Zumino, Z. Phys. **C52** (1991) 471.
    O. Ogievetsky, W.B. Schmidke, J. Wess & B. Zumino, MPI-Ph/91-51 (1991).

AS-ITP-92-07
March, 1992

The Covariant Generators of the Quantum Transformation Groups on the Spinor Spaces [1]

Xing-Chang SONG

CCAST(World Laboratory)
P.O.BOX 8730, Beijing 100080

Department of Physics[2]
Peking University, Beijing 100871

Institute of Theoretical Physics
Academia Sinica, Beijing 100080

Abstract

The covariant formalism of the generators for the quantum group $SL_q(2, C)$ on the quantum spinor space is constructed manifestly, very similar to the ones on the quantum group itself proposed by Woronowicz through his 4 D_+ calculus. Also constructed is the set of eight generators for the quantum Lorentz group on the bispinor space. In the limit $q \longrightarrow 1$, these generators reduce to those of the left and the right $SL(2, C)$ plus two corresponding Casimirs.

[1]Work supported in part by the National Natural Science Foundation of China and the Doctoral Programme Foundation Institution of High Education.

[2]Communication address.

I Introduction

The importance of the idea of the quantum groups [1] is now more and more extensively understood by most of physicists. This is due to its close connection to the Yang-Baxter equation [2] which plays a deep role in various physical problems. In our opinion, as for the classical group, for more direct physical application, the view considering the quantum group as the "quantum" symmetry of some basic physical objects, or "quantum" space, is more attractive.

The noncommutative differential calculus over the quantum group itself and the generators of the quantum group have been established by Woronowicz [1]. A general construction of quantum group as linear transformations upon the quantum plane has been suggested by Manin[3]. And the covariant differential calculus on the quantum plane was developed by Wess and Zumino [4] and generalized to the more general quantum spaces including the quantum orthogonal plane and symplectic plane[5], and more recently, to the quantum Minkowski space[6].

In this paper we would like to give the explicit covariant form of the generators of quantum groups $SL_q(2,C)$ and $SO_q(3,1)$ on these quantum spaces. The main tools are the consistent covariant differential calculus on these spaces[4,5] and the projection operator method developed in Ref[5-7]. We start with the linear representation of the $SL_q(2,c)$ on the spinor space in Section II. With the help of the differential calculus on the spinor space, we construct the differential realization of the covariant generators explicitly in Section III, just as the ones for the ordinary $SL(2,c)$ group. We have a set of four generators satisfying the relations similar to those given by Woronowicz [8] in considering the 4 D_+ calculus on quantum group $SL_q(2,C)$ itself and by Wu & Zhang recently [12] in developing RTF method [1] to discuss the differential calculus on quantum matrix groups. In $q \longrightarrow 1$ limit three of them reduce to the generators of classical $SL(2,C)$ and the fourth is connected with the Casimir operator(total angular momentum)[9].Then we turn to discuss the counterpart set of generators in conjugate spinor (dotted spinor) space in Section IV. Combining these two sets we get the total eight generators of quantum Lorentz group $SO_q(3,1)$ in the bispinor space. The action of these generators on the spinors as well as the 4-vectors is presented in Section V.

II. Quantum Spinors and $\check{R}$ Matrix

We start from two-dimensional q-spinor $u^\alpha = (u^1, u^2) = (u, v)$ with its components obeying the q-deformed commutation relation

$$uv = q\ vu\ . \tag{2.1}$$

This relation is preserved under the transformation of the q-spinor (The summation over the repeated indeces is understood throughout this paper)

$$u^\alpha \longrightarrow u'^\alpha = M^\alpha{}_\beta u^\beta\ , \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2}$$

if M is a $GL_q(2)$ matrix with its entries satisfying the definition relations

$$\begin{array}{lll} ab = q\ ba & ac = q\ ca & ad - da = (q - q^{-1})bc \\ bc = cb & bd = q\ db & cd = q\ dc \end{array} \tag{2.3}$$

and commuting with the components of the spinor, *i.e.*, $au = ua$ etc. The relations (2.1) and (2.3) can be put into the following form

$$u^\alpha u^\beta = q^{-1}\ \check{R}^{\alpha\beta}{}_{\gamma\delta} u^\gamma u^\delta\ , \tag{2.4a}$$

$$\check{R}^{\alpha\beta}{}_{\gamma\delta} M^\gamma{}_{\gamma'} M^\delta{}_{\delta'} = M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} \check{R}^{\alpha'\beta'}{}_{\gamma'\delta'}\ , \quad (\check{R}_{12} M_1 M_2 = M_1 M_2 \check{R}_{12}) \tag{2.4b}$$

by introducing the numerical $\check{R}_{12}$ matrix associated with $GL_q(2)$

$$\check{R}(q) = \left(\check{R}^{\alpha\beta}{}_{\gamma\delta}\right) = \begin{pmatrix} q & & & \\ & q - q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \tag{2.5}$$

which satisfies the Yang-Baxter equation (in the braid form)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{2.6a}$$

and the reduction relation

$$(\check{R} - q)(\check{R} + q^{-1}) = 0\ . \tag{2.6b}$$

The left-acting as well as the right-acting eigenvalue equations can be written as

$$\check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ t_m(q)^{\gamma\delta} = q\ t_m(q)^{\alpha\beta}\ , \quad \check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ s(q)^{\gamma\delta} = -q^{-1} s(q)^{\alpha\beta}\ , \tag{2.7a}$$

$$\bar{t}^m(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = q\ \bar{t}^m(q)_{\gamma\delta}\ , \quad \bar{s}(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = -q^{-1}\bar{s}(q)_{\gamma\delta}\ . \tag{2.7b}$$

Now since the matrix $\check{R}$ is symmetric, the components of $\bar{t}^m(q)$ and $\bar{s}(q)$ may be taken the same as those of their left-acting counterparts $t_m(q)$ and $s(q)$, namely[3]

$$\begin{aligned} t_+(q)^{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_3(q)^{\alpha\beta} = \begin{pmatrix} 0 & -q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} [2]^{-1/2}\ , \\ t_-(q)^{\alpha\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad s(q)^{\alpha\beta} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} [2]^{-1/2}, \end{aligned} \tag{2.8}$$

where the q-number is defined as $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. The q-analogue of the Levi-Civita symbols $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are related to the singlet eigenvectors

$$\epsilon(q)_{\alpha\beta} = -[2]^{1/2}\bar{s}(q)_{\alpha\beta}\ , \quad \epsilon(q)^{\alpha\beta} = [2]^{1/2}s(q)^{\alpha\beta} \tag{2.9a}$$

and are normalized in the way such that

$$\epsilon(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma} = \delta_\alpha{}^\gamma\ , \quad \epsilon(q)^{\alpha\beta}\epsilon(q)_{\beta\gamma} = \delta^\alpha{}_\gamma\ . \tag{2.9b}$$

while

$$\epsilon(q)^{\alpha\gamma}\ \epsilon(q)_{\beta\gamma} = -\xi^{-1}(q)^\alpha{}_\beta\ , \quad \epsilon(q)^{\gamma\alpha}\ \epsilon(q)_{\gamma\beta} = -\xi(q)^\alpha{}_\beta \tag{2.9c}$$

where

$$\xi(q)^\alpha{}_\beta = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} = q^{2c(\alpha)}\delta^\alpha{}_\beta \tag{2.9d}$$

with the "charge" of indices α defined as $c(1) = 1/2$, $c(2) = -1/2$.

As is used in Ref[6], $t_m(q)$ and $s(q)$ are grouped together to form the matrix-valued four vectors

$$t_\mu(q) = (t_0(q),\ \ t_m(q))\ , \quad t_0(q)^{\alpha\beta} = q\ s(q)^{\alpha\beta}\ ; \tag{2.10a}$$

$$\bar{t}^\mu(q) = (\bar{t}^0(q),\ \ \bar{t}^m(q))\ , \quad \bar{t}^0(q)_{\alpha\beta} = q^{-1}\ \bar{s}(q)_{\alpha\beta}\ . \tag{2.10b}$$

It is easy to check that $t_\mu(q)$ and $\bar{t}^\mu(q)$ satisfy the following orthonomality condition

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\nu(q)_{\alpha\beta} = \delta_\mu{}^\nu\ , \tag{2.11a}$$

the completeness relation

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\mu(q)_{\gamma\delta} = \delta^\alpha{}_\gamma\delta^\beta{}_\delta \equiv E^{\alpha\beta}{}_{\gamma\delta}\ , \tag{2.11b}$$

[3]The convention we adopted here is different from that in Ref[6] by exchanging + and −.

and the symmetric relations

$$\bar{t}_m(q)_{\alpha\beta} = \bar{t}_m(q^{-1})_{\beta\alpha} , \quad \bar{s}_m(q)_{\alpha\beta} = -\bar{s}_m(q^{-1})_{\beta\alpha} . \tag{2.11c}$$

from which follow the important relations

$$\check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\, t_\mu(q)^{\gamma\delta} = q\, t_\mu(q^{-1})^{\beta\alpha} , \quad \bar{t}^\mu(q)_{\gamma\delta}\, \check{R}^{-1}(q)^{\gamma\delta}{}_{\alpha\beta} = q^{-1}\, \bar{t}^\mu(q^{-1})_{\beta\alpha} . \tag{2.12}$$

The projection operators for the triplet and the singlet can be defined as

$$Q^{(2)}(q)^{\alpha\beta}{}_{\gamma\delta} = t_m(q)^{\alpha\beta}\bar{t}^m(q)_{\gamma\delta} , \quad Q^{(1)}(q)^{\alpha\beta}{}_{\gamma\delta} = s(q)^{\alpha\beta}\bar{s}(q)_{\gamma\delta} \tag{2.13a}$$

respectively, with the properties

$$Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(j)} , \quad Q^{(1)} + Q^{(2)} = E , \tag{2.13b}$$

$\check{R}$ matrix and other relevant matrices can be expressed as the linear combination of Q's, i.e.,

$$\check{R}(q) = \lambda_2 Q^{(2)}(q) + \lambda_1 Q^{(1)}(q) = qQ^{(2)} - q^{-1}Q^{(1)} . \tag{2.14}$$

Conversely, the projectors can be re-expressed in terms of $\check{R}$:

$$Q^{(2)} = \frac{\check{R} - \lambda_1 E}{\lambda_2 - \lambda_1} , \quad Q^{(1)} = \frac{\check{R} - \lambda_2 E}{\lambda_1 - \lambda_2} . \tag{2.15}$$

From these relations and the Yang-Baxter relation (2.4b) and the Yang-Baxter equation (2.6a) we see immediately that

$$Q^{(i)}_{12} M_1 M_2 = M_1 M_2 Q^{(i)}_{12} , \tag{2.16a}$$

$$Q^{(i)}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}Q^{(i)}_{23} , \quad \check{R}_{12}\check{R}_{23}Q^{(i)}_{12} = Q^{(i)}_{23}\check{R}_{12}\check{R}_{23} . \tag{2.16b}$$

The preservation of the q-commutation relation (2.1) comes from the fact that $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are the eigenvectors of $M \otimes M$with $Det_q M$ being the associated eigenvalue:

$$M^\alpha{}_\gamma M^\beta{}_\delta \epsilon(q)^{\gamma\delta} = Det_q M\ \epsilon(q)^{\alpha\beta} , \quad \epsilon(q)_{\alpha\beta} M^\alpha{}_\gamma M^\beta{}_\delta = Det_q M\ \epsilon(q)_{\gamma\delta} \tag{2.17}$$

where $Det_q M = ad - qbc$ is the center of algebra generated by a, b, c and d. For M with $Det_q M = 1$ we say M is an $SL_q(2, C)$ matrix. In this case we see immediately from (2.17) that

$$M^\alpha{}_\gamma \epsilon(q)^{\gamma\delta} M^\beta{}_\delta \epsilon(q)_{\beta\alpha'} = \delta^\alpha{}_{\alpha'} \tag{2.18a}$$

$$\epsilon(q)_{\beta'\alpha}M^{\alpha}{}_{\gamma}\epsilon(q)^{\gamma\delta}M^{\beta}{}_{\delta} = \delta_{\beta'}{}^{\beta} \tag{2.18b}$$

This implies that

$$\epsilon(q)^{\gamma\delta}M^{\beta}{}_{\delta}\epsilon(q)_{\beta\alpha} = M^{-1\gamma}{}_{\alpha} \tag{2.19a}$$

$$\epsilon(q)_{\beta\alpha}M^{\alpha}{}_{\gamma}\epsilon(q)^{\gamma\delta} = (M^t)^{-1}{}_{\beta}{}^{\delta} \ , \tag{2.19b}$$

where M^t is the matrix transpose of M. Now since $\epsilon(q)_{\beta\alpha}(= -\epsilon(q^{-1})_{\alpha\beta} \neq -\epsilon(q)_{\alpha\beta})$ is not ordinary antisymmetric with respect to α and β, $(M^{-1})^t \neq (M^t)^{-1}$. This fact tells us that starting from the basic spinor u^{α}, we can build two different types of lower index spinors. The one is

$${}_{\alpha}u \equiv u^{\beta}\ \epsilon(q)_{\beta\alpha} \longrightarrow {}_{\gamma}u\ M^{-1\gamma}{}_{\alpha} \tag{2.20a}$$

and the other

$$u_{\alpha} \equiv \epsilon(q)_{\alpha\beta}\ u^{\beta} \longrightarrow (M^t)^{-1}{}_{\alpha}{}^{\gamma}\ u_{\gamma} \tag{2.20b}$$

So two types of invariants can be formed

$${}_{\alpha}u\ u^{\alpha} \longrightarrow {}_{\gamma}u\ M^{-1\gamma}{}_{\alpha}\ M^{\alpha}{}_{\beta}\ u^{\beta} = {}_{\beta}u\ u^{\beta} \ , \tag{2.21a}$$

$$u^{\alpha}\ u_{\alpha} \longrightarrow u^{\beta}\ M^{t}{}_{\beta}{}^{\alpha}\ (M^t)^{-1}{}_{\alpha}{}^{\gamma}\ u_{\gamma} = u^{\beta}\ u_{\beta} \ . \tag{2.21b}$$

${}_{\alpha}u$ and u_{β} defined in (2.20) are the same covariant spinor in different form. They are related by ${}_{\alpha}u = -u_{\beta}\ \xi^{\beta}{}_{\alpha}$. The (left) derivative spinor

$$\partial_{\alpha} = \frac{\partial}{\partial u^{\alpha}} \ , \quad (\partial_{\alpha}\ u^{\beta}) = \delta_{\alpha}{}^{\beta}$$

should transform as

$$\partial_{\alpha} \longrightarrow (M^t)^{-1}{}_{\alpha}{}^{\gamma}\ \partial_{\gamma} \ . \tag{2.22a}$$

If we introduce the right derivative ${}_{\alpha}\tilde{\partial}$, $(u^{\beta}\ {}_{\alpha}\tilde{\partial}) = \delta^{\beta}{}_{\alpha}$ then it should transform as

$${}_{\alpha}\tilde{\partial} \longrightarrow {}_{\gamma}\tilde{\partial}\ M^{-1\gamma}{}_{\alpha} \ . \tag{2.22b}$$

The correctness of these definitions can be seen from

$$\delta_{\alpha}{}^{\beta} = \partial_{\alpha}\ u^{\beta} \longrightarrow (M^t)^{-1}{}_{\alpha}{}^{\gamma}\ \partial_{\gamma}\ u^{\delta}\ M^{t}{}_{\delta}{}^{\beta} = (M^t)^{-1}{}_{\alpha}{}^{\gamma}\ \delta_{\gamma}{}^{\delta}\ M^{t}{}_{\delta}{}^{\beta} = \delta_{\alpha}{}^{\beta} \ ,$$

$$\delta^{\beta}{}_{\alpha} = (u^{\beta}\ {}_{\alpha}\tilde{\partial}) \longrightarrow M^{\beta}{}_{\delta}\ u^{\delta}\ {}_{\gamma}\tilde{\partial}\ M^{-1\gamma}{}_{\alpha} = M^{\beta}{}_{\delta}\ \delta^{\delta}{}_{\gamma}\ M^{-1\gamma}{}_{\alpha} = \delta^{\beta}{}_{\alpha} \ .$$

Therefore the derivative spinors with upper index, which transform just as the basic spinor u^{α} does, are

$$\epsilon(q)^{\alpha\gamma}\ \partial_{\gamma} \equiv \partial^{\alpha} \longrightarrow M^{\alpha}{}_{\beta}\ \partial^{\beta} \ , \quad {}_{\gamma}\tilde{\partial}\ \epsilon(q)^{\gamma\alpha} \equiv \tilde{\partial}^{\alpha} \longrightarrow M^{\alpha}{}_{\beta}\ \tilde{\partial}^{\beta} \ . \tag{2.23}$$

The components of the derivative spinor ∂_α obey a q-commutation relation similar to (2.1)

$$\partial_2 \partial_1 = q\ \partial_1 \partial_2 \qquad (2.24a)$$

which can be re-written as

$$\partial_\beta\ \partial_\alpha = q^{-1}\ \partial_\delta\ \partial_\gamma\ \check{R}(q)^{\gamma\delta}{}_{\alpha\beta}. \qquad (2.24b)$$

For consistent differential calculus, we also need the relations between coordinates and derivatives. The result was first given by Wess and Zumino[4]:

$$\partial_\alpha\ u^\beta = \delta_\alpha{}^\beta + q^{-1}\ \check{R}^{-1}(q)^{\beta\alpha'}{}_{\alpha\beta'}\ u^{\beta'}\ \partial_{\alpha'}\ . \qquad (2.25)$$

Of the two possible choises we choose $C = q^{-1}\check{R}^{-1}$ for definiteness. It can be easily checked that the relations (2.4a), (2.24b) and (2.25) are covariant with respect to the transformation in (2.2) and (2.22)

The conjugate spinor $\bar{u}_{\dot\alpha} \equiv (u^\alpha)^*$ transforms[6] according to the hermitian conjugate (complex transpose) of the quantum matrix M:

$$\bar{u}_{\dot\alpha} \longrightarrow \bar{u}_{\dot\beta}\ M^{+\dot\beta}{}_{\dot\alpha}\ . \qquad (2.26a)$$

Then raising the dotted index, we see

$$\bar{u}^{\dot\alpha} \equiv \bar{u}_{\dot\beta}\ \epsilon(q)^{\dot\beta\dot\alpha} \longrightarrow \bar{M}^{\dot\alpha}{}_{\dot\gamma}\ \bar{u}^{\dot\gamma} \qquad (2.26b)$$

from a similar relation for $\bar{M}$ as in (2.18). Quantum matrix $\bar{M} = (M^+)^{-1}$ satisfies the Yang-Baxter relation similar to (2.4b),

$$\check{R}^{\dot\alpha\dot\beta}{}_{\dot\gamma\dot\delta}\ \bar{M}^{\dot\gamma}{}_{\dot\gamma'}\ \bar{M}^{\dot\delta}{}_{\dot\delta'} = \bar{M}^{\dot\alpha}{}_{\dot\alpha'}\ \bar{M}^{\dot\beta}{}_{\dot\beta'}\check{R}^{\dot\alpha'\dot\beta'}{}_{\dot\gamma'\dot\delta'} \qquad (2.27a)$$

and an additional relation[9] with M:

$$\check{R}^{\dot\alpha\beta}{}_{\alpha\dot\beta}\ M^\alpha{}_\gamma\ \bar{M}^{\dot\beta}{}_{\dot\delta} = \bar{M}^{\dot\alpha}{}_{\dot\gamma}\ M^\beta{}_\delta\ \check{R}^{\dot\gamma\delta}{}_{\gamma\dot\delta}\ . \qquad (2.27b)$$

The dotted derivative spinor

$$\bar\partial^{\dot\alpha} = \frac{\partial}{\partial \bar{u}_{\dot\alpha}}\ , \quad (\bar\partial^{\dot\alpha}\bar{u}_{\dot\beta}) = \delta^{\dot\alpha}{}_{\dot\beta}$$

transforms just as $\bar{u}^{\dot\alpha}$ does in (2.26a). Now by considering q real and taking the complex conjugate of Eq(2.1) or (2.4a) one obtain that

$$\bar{u}_{\dot 2}\ \bar{u}_{\dot 1} = q\ \bar{u}_{\dot 1}\ \bar{u}_{\dot 2} \quad or \quad \bar{u}_{\dot\beta}\ \bar{u}_{\dot\alpha} = q^{-1}\bar{u}_{\dot\delta}\ \bar{u}_{\dot\gamma}\ \check{R}^{\dot\gamma\dot\delta}{}_{\dot\alpha\dot\beta}\ . \qquad (2.28a)$$

Other commutation relations can be obtained by a similar Wess-Zunmino method

$$\bar{\partial}^{\dot{\alpha}}\ \bar{\partial}^{\dot{\beta}} = q^{-1}\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}}\ \bar{\partial}^{\dot{\gamma}}\ \bar{\partial}^{\dot{\delta}}\ , \quad \bar{\partial}^{\dot{\alpha}}\ \bar{u}_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} + q^{-1}\check{R}^{-1}(q)^{\dot{\alpha}\dot{\beta}'}{}_{\dot{\beta}\dot{\alpha}'}\ \bar{u}^{\dot{\beta}'}\ \bar{\partial}^{\dot{\alpha}'}\ . \tag{2.28b}$$

In the following discussion we need the relations in upper index form: we list them together as follows

$$u^{\alpha}\ u^{\beta} = q^{-1}\check{R}^{\alpha\beta}{}_{\gamma\delta}\ u^{\gamma}\ u^{\delta}\ , \quad \partial^{\alpha}\ \partial^{\beta} = q^{-1}\check{R}^{\alpha\beta}{}_{\gamma\delta}\ \partial^{\gamma}\ \partial^{\delta}\ ; \tag{2.29a}$$

$$\partial^{\alpha}\ u^{\beta} = \epsilon(q)^{\alpha\beta} + q^{-2}\check{R}^{\alpha\beta}{}_{\gamma\delta}\ u^{\gamma}\ \partial^{\delta}\ ; \tag{2.29b}$$

$$\bar{u}^{\dot{\alpha}}\ \bar{u}^{\dot{\beta}} = q^{-1}\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}}\ \bar{u}^{\dot{\gamma}}\ \bar{u}^{\dot{\delta}}\ , \quad \bar{\partial}^{\dot{\alpha}}\ \bar{\partial}^{\dot{\beta}} = q^{-1}\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}}\ \bar{\partial}^{\dot{\gamma}}\ \bar{\partial}^{\dot{\delta}}\ ; \tag{2.30a}$$

$$\bar{\partial}^{\dot{\alpha}}\ \bar{u}^{\dot{\beta}} = \epsilon(q)^{\dot{\alpha}\dot{\beta}} + q^{-2}\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}}\ \bar{u}^{\dot{\gamma}}\ \bar{\partial}^{\dot{\delta}}\ . \tag{2.30b}$$

In discussing the bispinor comprising both u^{α} and $\bar{u}_{\dot{\alpha}}$, we also need the cross commutation relations between undotted quantity with dotted one:

$$u^{\alpha}\ \bar{u}^{\dot{\alpha}} = \check{R}^{\alpha\dot{\alpha}}{}_{\dot{\beta}\beta}\ \bar{u}^{\dot{\beta}}\ u^{\beta}\ , \tag{2.31a}$$

$$\partial^{\alpha}\ \bar{u}^{\dot{\alpha}} = q^{-1}\check{R}^{\alpha\dot{\alpha}}{}_{\dot{\beta}\beta}\ \bar{u}^{\dot{\beta}}\ \partial^{\beta}\ , \quad u^{\alpha}\ \bar{\partial}^{\dot{\alpha}} = q^{-1}\check{R}^{\alpha\dot{\alpha}}{}_{\dot{\beta}\beta}\ \bar{\partial}^{\dot{\beta}}\ u^{\beta}\ , \tag{2.31b}$$

$$\partial^{\alpha}\ \bar{\partial}^{\dot{\alpha}} = \check{R}^{\alpha\dot{\alpha}}{}_{\dot{\beta}\beta}\ \bar{\partial}^{\dot{\beta}}\ \partial^{\beta}\ . \tag{2.31c}$$

The consistency of all these relations can be directly checked by considering the triplet product of operators chosen from (u^{α}, ∂^{β}, $\bar{u}^{\dot{\alpha}}$, $\bar{\partial}^{\dot{\beta}}$) and altering the order in two different ways.

Starting from the given 4×4 $\check{R}$ matrix (2.5) we can obtain different higher dimensional $\check{R}$ matrices by using different "fusion": for example

$$\check{R}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = \check{R}^{\alpha\gamma}{}_{\alpha'\gamma'}\ \check{R}^{\beta\delta}{}_{\beta'\delta'}(= P_{23}\check{R}_{12}\check{R}_{34}P_{23}) \tag{2.32a}$$

is the $\check{R}$ matrix corresponding to the quantum group $SO_q(4)$,

$$\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = q^{-1}\check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{-1}(q)^{b'c'}{}_{\beta'\gamma'}(= q^{-1}\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}^{-1}) \tag{2.32b}$$

is the $\check{R}$ matrix for the quantum Lorentz group $SO_q(3,1)$ [6,10],

$$\check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = q^{-2}\check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{b'c'}{}_{\beta'\gamma'}(= q^{-2}\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}) \tag{2.32c}$$

represents a reducible $\check{R}$ matrix[7,11], and

$$|\check{R}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = \check{R}(q)^{\beta\gamma}_{bc}\check{R}(q)^{\alpha b}_{\alpha'b'}\check{R}^{-1}(q)^{c\delta}_{c'\delta'}\check{R}^{-1}(q)^{b'c'}_{\beta'\gamma'} = \check{R}_{23}\check{R}_{12}\check{R}_{34}^{-1}\check{R}_{23}^{-1} \tag{2.32d}$$

is another reducible $\check{R}$ matrix. The reduction comes from the repeated use of (2.16b) giving

$$\begin{aligned} Q_{12}^{(i)}\ \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}\ \check{R}_{23} &= \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}\ \check{R}_{23}\ Q_{34}^{(i)} \\ Q_{34}^{(j)}\ \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}\ \check{R}_{23} &= \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}\ \check{R}_{23}\ Q_{12}^{(j)} \\ Q_{12}^{(i)}\ \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}^{-1}\ \check{R}_{23}^{-1} &= \check{R}_{23}\ \check{R}_{12}\ \check{R}_{34}^{-1}\ \check{R}_{23}^{-1}\ Q_{34}^{(i)} \end{aligned} \tag{2.33}$$

Then by multiplying $\bar{t}^{\mu}{}_{\alpha\beta}$, $\bar{t}^{\nu}_{\gamma\delta}$, $t_{\kappa}^{\alpha'\beta'}$ and $t_{\lambda}^{\gamma'\delta'}$, Eq(2.32c) leads to

$$\begin{aligned} \check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda} &\equiv \bar{t}^{\mu}(q)_{\alpha\beta}\ \bar{t}^{\nu}(q)_{\gamma\delta}\ \check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}\ t_{\kappa}(q)^{\alpha'\beta'}\ t_{\lambda}(q)^{\gamma'\delta'} \\ &= \check{\mathbf{R}}^{00}{}_{00} \oplus \check{\mathbf{R}}^{m0}{}_{0l} \oplus \check{\mathbf{R}}^{0n}{}_{k0} \oplus \check{\mathbf{R}}^{mn}{}_{kl}\ . \end{aligned} \tag{2.34}$$

Here $\check{\mathbf{R}}^{mn}{}_{kl}$ is a $\check{R}$ matrix associated with $SO_{q^2}(3)$, corresponding quantum matrix being[11]

$$D^{m}{}_{k} = \bar{t}^{m}(q)_{\alpha\gamma} M^{\alpha}{}_{\beta} M^{\gamma}{}_{\delta} t_{k}(q)^{\beta\delta}\ . \tag{2.35}$$

The same procedure gives

$$\begin{aligned} |\check{R}^{\mu\nu}{}_{\kappa\lambda} &\equiv \bar{t}^{\mu}_{12}\bar{t}^{\nu}_{34}|\check{R}_{(12)(34)}t_{\kappa}^{12}t_{\lambda}^{34} \\ &= |\check{R}^{0\nu}{}_{\kappa 0} \oplus |\check{R}^{m\nu}{}_{\kappa l} \end{aligned} \tag{2.36}$$

where $|\check{R}^{m\nu}{}_{\kappa l}$ is the $\check{R}$ matrix between the triplet (1,0) and quartet $(\frac{1}{2},\frac{1}{2})$. Similarly, from Eq(2.32b) we obtain

$$\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} \equiv \bar{t}^{\mu}(q)_{\alpha\beta}\ \bar{t}^{\nu}(q)_{\gamma\delta}\ \check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}\ t_{\kappa}(q)^{\alpha'\beta'}\ t_{\lambda}(q)^{\gamma'\delta'} \tag{2.37}$$

which is indeed the $\check{R}$ matrix for quantum Lorentz group with its singlet eigenvector

$$g_{\mu\nu} = (g_{+-}, g_{33}, g_{00}, g_{-+}) = (-q^{-1}, -1, 1, -q) \tag{2.38}$$

being identified as the q-deformed Lorentz metric[6].

III. $SL_q(2,C)$ Generators on Spinor Space

Now we are ready to construct the set of generators on the spinor space. As for ordinary angular momentum operators in the classical spinor space[9], we consider the combination operators

$$L^{\alpha\beta} = u^{\alpha}\ \partial^{\beta} = u^{\alpha}\ \epsilon(q)^{\beta\gamma}\ \partial_{\gamma}\ . \tag{3.1}$$

Then by using the relations (2.29) we obtain through a straightforward but tedious derivation

$$\begin{aligned} & L^{\alpha\beta}\, L^{\gamma\delta} - q^{-1}\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}\, L^{\alpha'\beta'}\, L^{\gamma'\delta'} \\ & = q\, \check{R}^{-1}(q)^{\alpha\beta}{}_{\alpha'\beta'}\, \epsilon(q)^{\beta'\gamma'}\, \check{R}^{-1}(q)^{\gamma\delta}{}_{\gamma'\delta'}\, \check{R}^{\alpha'\delta'}{}_{\xi\eta}\, L^{\xi\eta} + \epsilon(q)^{\beta\gamma}\, L^{\alpha\delta} \end{aligned} \tag{3.2}$$

where $\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}$ is the $\check{R}$ matrix defined in Eq(2.32b) which can be transferred to $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ as in Eq(2.35). As is discussed in Ref[6], the 16×16 $\check{R}$ matrix $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ has three distinctive eigenvalues: the single one $\lambda_0(q) = q^{-3}$, the sixfold one $\lambda_1(q) = -q^{-1}$ and ninefold one $\lambda_2(q) = q$.

$$\begin{aligned} \bar{v}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} &= \lambda_0(q)\bar{v}(q)_{\kappa\lambda}\ , & \check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}v(q)^{\kappa\lambda} &= \lambda_0(q)v(q)^{\mu\nu}\ , \\ \bar{u}^{ms}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} &= \lambda_1(q)\bar{u}^{ms}(q)_{\kappa\lambda}\ , & \check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}u_{ms}(q)^{\kappa\lambda} &= \lambda_1(q)u_{ms}(q)^{\mu\nu}\ , \\ \bar{w}^{mn}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} &= \lambda_2(q)\bar{w}^{mn}(q)_{\kappa\lambda}\ , & \check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}w_{ms}(q)^{\kappa\lambda} &= \lambda_2(q)w_{ms}(q)^{\mu\nu}\ , \end{aligned} \tag{3.3}$$

where for $\bar{u}(q)$, $m = +, 3, -$ and $s = \pm$, for $\bar{w}(q)$ $(m,n) = (2,0)$, $(1,\pm 1)$, $(0,\pm 2)$, $(0,0)$, $(\bar{1},\pm 1)$ and $(\bar{2},0)$. These eigenvectors are normalized in such a way that

$$\begin{aligned} \bar{u}^{mr}(q)_{\mu\nu}\ u_{ls}(q)^{\mu\nu} &= \delta^{m}{}_{l}\ \delta^{r}{}_{s}\ , \\ \bar{v}(q)_{\mu\nu}\ v(q)^{\mu\nu} &= 1\ , \\ \bar{u}^{mr}(q)_{\mu\nu}\ v(q)^{\mu\nu} = \bar{v}(q)_{\mu\nu}\ u_{ls}(q)^{\mu\nu} &= 0 \end{aligned} \tag{3.4}$$

and so forth. To write down the explicit form of $\check{\mathcal{R}}$, we order the Lorentz index $\mu = (+,3,0,-)$ and define a "charge" for each index: $c(+) = +1$, $c(-) = -1$, $c(3)=c(0)=0$. Then $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ is "charge" conservative $c(\mu)+c(\nu) = c(\kappa)+c(\lambda) = m$, and breaks into the block diagonal form according to the total "charge" m.

$$\check{\mathcal{R}}(q) = S_1^{(2)} \oplus S_4^{(1)} \oplus S_6^{(0)} \oplus S_4^{(\bar{1})} \oplus S_1^{(\bar{2})} \tag{3.5}$$

where $S_d^{(m)}$ is a $d\times d$ matrix with total "charge" m. The following standard order for the indeces pair (μ,ν) or (κ,λ) is adopted throughout this paper:

$$\begin{aligned} &(\mu,\nu) = (++) \quad for \quad m = 2, \\ &(\mu,\nu) = (+3,+0,3+,0+) \quad for \quad m = 1, \\ &(\mu,\nu) = (+-,33,30,03,00,-+) \quad for \quad m = 0, \\ &(\mu,\nu) = (3-,0-,-3,-0) \quad for \quad m = \bar{1}, \\ &(\mu,\nu) = (--) \quad for \quad m = \bar{2}. \end{aligned} \tag{3.6}$$

Then the singlet eigenvector(with total "charge" zero)

$$\bar{v}(q)_{\mu\nu} = (-q^{-1}, -1, 0, 0, 1, -q)[2]^{-1} \ , \tag{3.7}$$

is proportional to the Lorentz metric $g(q)_{\mu\nu}$. And the sextet eigenvectors are chosen as

$$\begin{aligned}
\bar{u}^{1+}(q)_{\mu\nu} &= (-q^{-1}, q^{-1}, q, -q)[2]^{-1}, \\
\bar{u}^{1-}(q)_{\mu\nu} &= (-q^{-1}, -q, q, q^{-1})[2]^{-1} \ ; \\
\bar{u}^{3+}(q)_{\mu\nu} &= (1, q - q^{-1}, q^{-1}, -q, 0, -1))[2]^{-1}, \\
\bar{u}^{3-}(q)_{\mu\nu} &= (1, q - q^{-1}, -q, q^{-1}, 0, -1)[2]^{-1} \ ; \\
\bar{u}^{\bar{1}+}(q)_{\mu\nu} &= (-q^{-1}, -q, q, q^{-1})[2]^{-1}, \\
\bar{u}^{\bar{1}-}(q)_{\mu\nu} &= (-q^{-1}, q^{-1}, q, -q)[2]^{-1} \ .
\end{aligned} \tag{3.8}$$

Then by changing the bispinor index (α, β) into 4-vector index μ and defining

$$L^{\mu} = \bar{t}^{\mu}(q)_{\alpha\beta} L^{\alpha\beta} \tag{3.9}$$

we obtain

$$\begin{aligned}
&L^{\mu}L^{\nu} - q^{-1}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}L^{\kappa}L^{\lambda} = f^{\mu\nu}{}_{\rho}L^{\rho} \\
&= [2]^{1/2}[u_{l+}(q)^{\mu\nu} + u_{l-}(q)^{\mu\nu}]L^{l} + [2]^{1/2}(q - q^{-1})v(q)^{\mu\nu}L^{0}
\end{aligned} \tag{3.10}$$

which can be rewritten in the form

$$\begin{aligned}
\bar{u}^{l+}(q)_{\mu\nu}L^{\mu}L^{\nu}(1 + q^{-2}) &= [2]^{1/2}L^{l} \ , \\
\bar{u}^{l-}(q)_{\mu\nu}L^{\mu}L^{\nu}(1 + q^{-2}) &= [2]^{1/2}L^{l} \ , \\
\bar{v}(q)_{\mu\nu}L^{\mu}L^{\nu}(1 - q^{-4}) &= [2]^{1/2}(q - q^{-1})L^{0} \ .
\end{aligned} \tag{3.11}$$

Or more explicitly

$$\begin{aligned}
&[L^{0}, L^{m}] = 0 \ , \\
&q(L^{3} - L^{0})L^{+} - q^{-1}L^{+}(L^{3} - L^{0}) = q[2]^{1/2}L^{+} \ , \\
&qL^{-}(L^{3} - L^{0}) - q^{-1}(L^{3} - L^{0})L^{-} = q[2]^{1/2}L^{-} \ , \\
&L^{+}L^{-} - L^{-}L^{+} + (q - q^{-1})L^{3}(L^{3} - L^{0}) = q[2]^{1/2}L^{3} \ , \\
&-q^{-1}L^{+}L^{-} - L^{3}L^{3} + L^{0}L^{0} - qL^{-}L^{+} = q^{2}[2]^{1/2}L^{0} \ .
\end{aligned} \tag{3.12}$$

This set of relations are similar to those given in [8] from the 4 D_+ differential calculus on quantum group $SU_q(2)$ itself [8]. So the operators L^μ defined in (3.1) and (3.9) are indeed the derivative realization of the $SL_q(2,C)$ generators on the spinor space.

In the limit $q \longrightarrow 1$, it is easily seen that

$$L^+ \sim u^1\partial_2 \ , \quad L^- \sim u^2\partial_1 \ ,$$
$$L^3 \sim (u^1\partial_1 - u^2\partial_2)/\sqrt{2} \ , \quad L^0 \sim (u^1\partial_1 + u^2\partial_2)/\sqrt{2} \ . \tag{3.13}$$

This means that L^m are the generators of $SL(2,C)$ and L^0 an operator related to the Casimir operator L^2.

IV. Generators on Conjugate Space

On the conjugate spinor space, the corresponding generators can be defined in a similar way. Consider

$$L^{\dot\alpha\dot\beta} = \bar{u}^{\dot\alpha}\bar{\partial}^{\dot\alpha} \ , \tag{4.1}$$

then we see

$$\begin{aligned} &\bar{L}^{\dot\alpha\dot\beta}\bar{L}^{\dot\gamma\dot\delta} - q^{-1}\check{R}^{\dot\alpha\dot\beta,\dot\gamma\dot\delta}{}_{\dot\alpha'\dot\beta',\dot\gamma'\dot\delta'}\bar{L}^{\dot\alpha'\dot\beta'}\bar{L}^{\dot\gamma'\dot\delta'} \\ &= q\ \check{R}^{-1}(q)^{\dot\alpha\dot\beta}{}_{\dot\alpha'\dot\beta'}\epsilon(q)^{\dot\beta'\dot\gamma'}\check{R}^{-1}(q)^{\dot\gamma\dot\delta}{}_{\dot\gamma'\dot\delta'}\check{R}^{\dot\alpha'\dot\delta'}{}_{\dot\xi\dot\eta}\bar{L}^{\dot\xi\dot\eta} + \epsilon(q)^{\dot\beta\dot\gamma}\bar{L}^{\dot\alpha\dot\delta} \ . \end{aligned} \tag{4.2}$$

In deriving (4.2) use has been made of the relations in the upper index form (2.30) which are completely similar to the relations with undotted indices, (2.29) Then by introducing

$$\bar{L}^\mu = \bar{t}^\mu(q^{-1})_{\dot\beta\dot\alpha}\ L^{\dot\alpha\dot\beta} \ , \tag{4.3}$$

we have relations similar to those in (3.10)

$$\begin{aligned} &\bar{L}^\mu\bar{L}^\nu - q^{-1}\check{R}^{\mu\nu}{}_{\kappa\lambda}\bar{L}^\kappa\bar{L}^\lambda = f^{\mu\nu}{}_\rho\bar{L}^\rho \\ &= [2]^{1/2}(u_{l+}(q)^{\mu\nu} + u_{l-}(q)^{\mu\nu})\bar{L}^l + [2]^{1/2}(q - q^{-1})v(q)^{\mu\nu}\bar{L}^0 \ . \end{aligned} \tag{4.4}$$

where we have used the relation (3.12). Then it follows that

$$
\begin{aligned}
&[\bar{L}^0, \bar{L}^m] = 0\ , \\
&q(\bar{L}^3 - \bar{L}^0)\bar{L}^+ - q^{-1}\bar{L}^+(\bar{L}^3 - \bar{L}^0) = q[2]^{1/2}\bar{L}^+\ , \\
&q\bar{L}^-(\bar{L}^3 - \bar{L}^0) - q^{-1}(\bar{L}^3 - \bar{L}^0)\bar{L}^- = q[2]^{1/2}\bar{L}^-\ , \\
&\bar{L}^+\bar{L}^- - \bar{L}^-\bar{L}^+ + (q - q^{-1})\bar{L}^3(\bar{L}^3 - \bar{L}^0) = q[2]^{1/2}\bar{L}^3\ , \\
&-q^{-1}\bar{L}^+\bar{L}^- - \bar{L}^3\bar{L}^3 + \bar{L}^0\bar{L}^0 - q\bar{L}^-\bar{L}^+ = q^2[2]^{1/2}\bar{L}^0\ .
\end{aligned}
\tag{4.5}
$$

Now by making use of the relations between one undotted object with dotted one (2.31), we see immediately that

$$
\begin{aligned}
u^\alpha \partial^\beta \bar{u}^{\dot\alpha} \bar\partial^{\dot\beta} &= u^\alpha\, q^{-1} \check{R}^{\beta\dot\alpha}{}_{\dot\alpha'\beta'} \bar{u}^{\dot\alpha'} \partial^{\beta'} \bar\partial^{\dot\beta'} \\
&= q^{-1}\ \check{R}^{\beta\dot\alpha}{}_{\dot\alpha'\beta'}\ \check{R}^{\alpha\dot\alpha'}{}_{\dot\alpha''\alpha'}\ \bar{u}^{\dot\alpha''}\ u^{\alpha'}\ \check{R}^{\beta'\dot\beta}{}_{\dot\beta''\beta''}\ \bar\partial^{\dot\beta'}\ \partial^{\beta''} \\
&= q^{-2}\ \check{R}^{\beta\dot\alpha}{}_{\dot\alpha'\beta'}\ \check{R}^{\alpha\dot\alpha'}{}_{\dot\alpha''\alpha'}\ \check{R}^{\beta'\dot\beta}{}_{\dot\beta'\beta''}\ \check{R}^{\alpha'\dot\beta'}{}_{\dot\beta''\alpha''}\ \bar{u}^{\dot\alpha''}\ \bar\partial^{\dot\beta''}\ u^{\alpha''} \partial^{\beta''}\ .
\end{aligned}
\tag{4.6}
$$

This gives

$$
L^{\alpha\beta}\ \bar{L}^{\dot\alpha\dot\beta} = \check{\mathbf{R}}(q)^{\alpha\beta,\dot\alpha\dot\beta}{}_{\dot\alpha''\dot\beta'',\alpha''\beta''}\ \bar{L}^{\dot\alpha''\dot\beta''}\ L^{\alpha''\beta''}
\tag{4.7}
$$

which means that $L^{\alpha\beta}$ is $\check{R}$ commuting with $\bar{L}^{\dot\alpha\dot\beta}$. Here $\check{\mathbf{R}}$ is the $\check{R}$ matrix given in Eq(2.32c), which can be transferred to the vector index form as in (2.34)

$$
\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda} = \check{\mathbf{R}}^{00}{}_{00} \oplus \check{\mathbf{R}}^{m0}{}_{0l} \oplus \check{\mathbf{R}}^{0n}{}_{k0} \oplus \check{\mathbf{R}}^{mn}{}_{kl}
\tag{4.8}
$$

with $\check{\mathbf{R}}^{00}{}_{00} = 1$, $\check{\mathbf{R}}^{m0}{}_{0l} = \delta^m{}_l$, $\check{\mathbf{R}}^{0n}{}_{k0} = \delta^n{}_k$, and $\check{\mathbf{R}}^{mn}{}_{kl}$ being an $SO_{q^2}(3)$ $\check{R}$ matrix.

$$
\begin{array}{l}
(\check{\mathbf{R}}^{mn}{}_{kl}) = q^2 \oplus \begin{pmatrix} \theta & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} (1-q^{-2})\theta & -q^{-1}\theta & q^{-2} \\ -q^{-1}\theta & 1 & 0 \\ q^{-2} & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \theta & 1 \\ 1 & 0 \end{pmatrix} \oplus q^2 \\
kl:\quad ++ \quad +3 \quad 3+ \quad +- \quad 33 \quad -+ \quad 3- \quad -3 \quad --
\end{array}
\tag{4.9}
$$

where θ stands for $q^2 - q^{-2}$. This leads to

$$
[L^0, \bar{L}^0] = [L^n, \bar{L}^0] = [L^0, \bar{L}^m] = 0
\tag{4.10a}
$$

and

$$
L^m \bar{L}^n = \check{\mathbf{R}}^{mn}{}_{kl}\ \bar{L}^k L^l\ .
\tag{4.10b}
$$

The latter relation can be written out more explicitly as follows

$$\begin{aligned}
&L^+\bar{L}^+ = q^2\bar{L}^+L^+ \ , \\
&L^+\bar{L}^3 = \bar{L}^3L^+ \ , \\
&L^+\bar{L}^- = q^{-2}\bar{L}^-L^+ \ , \\
&L^3\bar{L}^+ = \bar{L}^+L^3 + \theta\ \bar{L}^3L^+ \ , \\
&L^3\bar{L}^3 = \bar{L}^3L^3 - q^{-1}\theta\bar{L}^-L^+ \ , \\
&L^3\bar{L}^- = \bar{L}^-L^3 \ , \\
&L^-\bar{L}^+ = q^{-2}\bar{L}^+L^- - q^{-1}\theta\bar{L}^3L^3 + (1-q^{-2})\theta\bar{L}^-L^+ \ , \\
&L^-\bar{L}^3 = \bar{L}^3L^-\theta\bar{L}^-L^3 \ , \\
&L^-\bar{L}^- = q^2\ \bar{L}^-L^- \ .
\end{aligned} \tag{4.11}$$

The relations (3.2), (4.2) and (4.6), or equivalently (3.12), (4.5), (4.10) and (4.11) complete our cross commutation relations for quantum Lorentz algebra. Among total eight generators $(L^\mu, \bar{L}^\nu)$, two of them, L^0 and $\bar{L}^0$, are centers which must be added to complete the algebra.

In the limit $q \longrightarrow 1$, two of these relations bocome the definiton of L^0 and $\bar{L}_0$ which are decoupling from the other six generators and these six generators fall into two commuting sets of angular momentum operators. This is just the case for classical Lorentz algebra.

V. The Action of Generators

Now we are in the position to give the explicit results when the generators are acting upon the spinors and 4-vectors. The cross commutation relations given in Section II are enough to give all the following results.

The action of generators on the basic spinor u^γ gives

$$(u^\alpha\partial^\beta)u^\gamma = u^\alpha\ \epsilon(q)^{\beta\gamma} + q^{-3}\check{R}^{\beta\gamma}{}_{\gamma'\beta'}\check{R}^{\alpha\gamma'}{}_{\delta\alpha'}\ u^\delta\ (u^{\alpha'}\ \partial^{\beta'}) \ . \tag{5.1}$$

This yields

$$L^0\ u^\gamma = -q^{-1}[2]^{-1/2}u^\gamma + q^{-2}u^\gamma\ L^0 \ , \tag{5.2a}$$

$$L^m u^\gamma = \bar{t}^m(q)_{\alpha\beta}\, u^\alpha\, \epsilon(q)^{\beta\gamma} + q^{-2}\check{R}^{\gamma m}{}_{l\delta}\, u^\delta\, L^l \tag{5.2b}$$

where $\check{R}(q)^{\gamma m}{}_{l\delta}$ is the $\check{R}$ matrix between spin 1 and spin 1/2 . If we set

$$R(q)^{m\gamma}{}_{l\delta} \equiv \check{R}(q)^{\gamma m}{}_{l\delta} ,$$

then

$$R(q)^{m\gamma}{}_{l\delta} = \begin{pmatrix} q & & & & & \\ & q^{-1} & \omega & & & \\ & 0 & 1 & & & \\ & & & 1 & -\omega & \\ & & & 0 & q^{-1} & \\ & & & & & q \end{pmatrix} \tag{5.3}$$

where $\omega = (q^{-1} - q)q^{-1/2}[2]^{1/2}$, and the indeces pair (m, γ) or (l, δ) is ordered $(+,1)$, $(+,2)$, $(3,1)$, $(3,2)$, $(-,1)$ and $(-,2)$. Then (5.2b) can be written out more explicitly

$$\begin{aligned}
L^+u^1 &= q^{-1}u^1L^+, \\
L^+u^2 &= q^{-1/2}u^1 - (q - q^{-1})q^{-5/2}[2]^{1/2}u^1L^3 + q^{-3}u^2L^+, \\
L^3u^1 &= q[2]^{-1/2}u^1 + q^{-2}u^1L^3, \\
L^3u^2 &= -q^{-1}[2]^{-1/2}u^2 + (q - q^{-1})q^{-5/2}[2]^{1/2}u^1L^- + q^{-2}u^2L^3, \\
L^-u^1 &= q^{1/2}u^2 + q^{-3}u^1L^-, \\
L^-u^2 &= q^{-1}u^2L^- .
\end{aligned} \tag{5.2c}$$

The similar results can be obtained when the generators are acting upon the conjugate spinor $\bar{u}_{\dot\gamma}$. Indeed we have

$$(u^\alpha\partial^\beta)\bar{u}_{\dot\gamma} = q\; \check{R}^{-1}(q)^{\dot\delta\alpha}{}_{\alpha'\dot\gamma'}\check{R}^{-1}(q)^{\dot\gamma'\beta}{}_{\beta'\dot\gamma}\bar{u}_{\dot\delta}(u^{\alpha'}\partial^{\beta'}) . \tag{5.4}$$

This gives

$$L^0\bar{u}_{\dot\gamma} = \bar{u}_{\dot\gamma}L^0 , \tag{5.5a}$$

$$L^m\bar{u}_{\dot\gamma} = \check{R}^{-1}(q)^{m\dot\delta}{}_{\dot\gamma l}\bar{u}_{\dot\delta}L^l . \tag{5.5b}$$

Here $\check{R}^{-1}(q)^{m\dot{\delta}}{}_{\dot{\gamma}l}$ is the inverse of the $\check{R}$ matrix appearing in (5.2b). And (5.5b) becomes

$$\begin{aligned}
&L^{+}\bar{u}_{\dot{1}} = q^{-1}\bar{u}_{\dot{1}}L^{+} + (q-q^{-1})q^{1/2}[2]^{1/2}\bar{u}_{\dot{2}}L^{3}, \\
&L^{+}\bar{u}_{\dot{2}} = q\ \bar{u}_{\dot{2}}L^{+}, \\
&L^{3}\bar{u}_{\dot{1}} = \bar{u}_{\dot{1}}L^{3} - (q-q^{-1})q^{1/2}[2]^{1/2}\bar{u}_{\dot{2}}L^{-}, \\
&L^{3}\bar{u}_{\dot{2}} = \bar{u}_{\dot{2}}L^{3}, \\
&L^{-}\bar{u}_{\dot{1}} = q\ \bar{u}_{\dot{1}}L^{-}, \\
&L^{-}\bar{u}_{\dot{2}} = q^{-1}\bar{u}_{\dot{2}}L^{-}\ .
\end{aligned} \tag{5.5c}$$

Similarly we have

$$(\bar{u}^{\dot{\alpha}}\bar{\partial}^{\dot{\beta}})u^{\gamma} = q\ \check{R}^{-1}(q)^{\beta\gamma}{}_{\gamma'\dot{\beta}'}\check{R}^{-1}(q)^{\dot{\alpha}\gamma'}{}_{\delta\dot{\alpha}'}u^{\delta}(\bar{u}^{\dot{\alpha}'}\bar{\partial}^{\dot{\beta}'}) \tag{5.6}$$

This gives

$$\bar{L}^{0}\ u^{\gamma} = u^{\gamma}\ \bar{L}^{0}\ , \tag{5.7a}$$

$$\bar{L}^{m}\ u^{\gamma} = \check{\bar{R}}^{-1}(q)^{m\gamma}{}_{\delta l}u^{\delta}\ \bar{L}^{l}\ , \tag{5.7b}$$

where $\check{\bar{R}}^{-1}(q)^{m\gamma}{}_{\delta l}$ is another $\check{R}$ matrix between spin 1 and spin 1/2 with $\check{\bar{R}}^{-1}(q)^{m\gamma}{}_{\delta l}$ $=\check{R}^{-1}(q)^{l\delta}{}_{\gamma m}$. Then (5.7a) becomes

$$\begin{aligned}
&\bar{L}^{+}\ u^{1} = q^{-1}u^{1}\ \bar{L}^{+}\ , \\
&\bar{L}^{+}\ u^{2} = q\ u^{2}\bar{L}^{+}\ , \\
&\bar{L}^{3}\ u^{1} = -\omega q\ u^{2}\bar{L}^{+} + u^{1}\bar{L}^{3}\ , \\
&\bar{L}^{3}\ u^{2} = u^{2}\ \bar{L}^{3}\ , \\
&\bar{L}^{-}\ u^{1} = \omega q\ u^{2}\bar{L}^{3} + q\ u^{1}\ \bar{L}^{-}\ , \\
&\bar{L}^{-}\ u^{2} = q^{-1}\ u^{2}\ \bar{L}^{-}\ .
\end{aligned} \tag{5.7c}$$

When $L^{\dot{\alpha}\dot{\beta}}$ is acting upon $\bar{u}_{\dot{\gamma}}$ we have

$$(\bar{u}^{\dot{\alpha}}\ \bar{\partial}^{\dot{\beta}})\ \bar{u}_{\dot{\gamma}} = \bar{u}^{\dot{\alpha}}\ \delta^{\dot{\beta}}{}_{\dot{\gamma}} + q^{-1}\ \check{R}^{-1}(q)^{\dot{\gamma}'\dot{\beta}}{}_{\dot{\beta}'\dot{\gamma}}\ \check{R}^{-1}(q)^{\dot{\delta}\dot{\alpha}}{}_{\dot{\alpha}'\dot{\gamma}'}\ \bar{u}_{\dot{\gamma}}\ (\bar{u}^{\dot{\alpha}'}\ \bar{\partial}^{\dot{\beta}'})\ . \tag{5.8}$$

This gives

$$\bar{L}^{0}\ \bar{u}_{\dot{\gamma}} = -q\ [2]^{-1/2}\bar{u}_{\dot{\gamma}} + q^{-2}\ \bar{u}_{\dot{\gamma}}\bar{L}^{0}\ , \tag{5.9a}$$

$$\bar{L}^m \; \bar{u}_{\dot\gamma} = -\bar{u}_{\dot\beta}\epsilon(q)^{\dot\beta\dot\alpha} \; \bar{t}^m(q)_{\dot\alpha\dot\gamma} + q^{-2}\check{R}^{-1}(q)^{m\dot\delta}{}_{\dot\gamma l} \; \bar{u}_{\dot\delta}\bar{L}^l \;, \tag{5.9b}$$

where $\check{R}^{-1}(q)^{m\dot\delta}{}_{\dot\gamma l}$ is the same matrix appearing in (5.5b). Then it follows that

$$\begin{aligned}
&\bar{L}^+ \; \bar{u}_{\dot 1} = q^{1/2}\bar{u}_{\dot 2} + q^{-3}\bar{u}_{\dot 1}L^+ - \omega q^{-1}\bar{u}_{\dot 2}\bar{L}^3 \;, \\
&\bar{L}^+\bar{u}_{\dot 2} = q^{-1}\bar{u}_{\dot 2}\bar{L}^+ \;, \\
&\bar{L}^3 \; \bar{u}_{\dot 1} = q^{-1}[2]^{-1/2}\bar{u}_{\dot 1} + q^{-2}\bar{u}_{\dot 1}L^3 + \omega q^{-1}\bar{u}_{\dot 2}\bar{L}^- \;, \\
&\bar{L}^3 \; \bar{u}_{\dot 2} = -q \; [2]^{-1/2}\bar{u}_{\dot 2} + q^{-2}\bar{u}_{\dot 2}L^3 \;, \\
&\bar{L}^-\bar{u}_{\dot 1} = q^{-1}\bar{u}_{\dot 1}\bar{L}^- \;, \\
&\bar{L}^- \; \bar{u}_{\dot 2} = q^{1/2}\bar{u}_{\dot 1} + q^{-3}\bar{u}_{\dot 2}L^- \;.
\end{aligned} \tag{5.9c}$$

Then when we consider the coordinates 4-vector as product of a basic spinor u^γ with a conjugate spinor $\bar{w}_{\dot\gamma}$ transformed in the same way as $\bar{u}_{\dot\gamma}$:

$$X^\gamma{}_{\dot\gamma} = \tau_\mu{}^\gamma{}_{\dot\gamma}x^\mu \sim u^\gamma\bar{w}_{\dot\gamma} \;. \tag{5.10}$$

We can easily obtain that

$$\begin{aligned}
&(u^\alpha\partial^\beta)(u^\gamma\bar{w}_{\dot\gamma}) \\
&= u^\alpha\bar{w}_{\dot\gamma}\epsilon(q)^{\beta\gamma} + q^{-2}\check{R}^{\beta\gamma}{}_{\gamma'\beta'}\check{R}^{\alpha\gamma'}{}_{\delta\alpha'}u^\delta\bar{w}_{\dot\delta}u^{\alpha''}\partial^{\beta''}\check{R}^{-1}(q)^{\dot\delta\alpha'}{}_{\alpha''\dot\gamma'}\check{R}^{-1}(q)^{\dot\gamma'\beta'}{}_{\beta''\dot\gamma} \;.
\end{aligned} \tag{5.11}$$

It follows that

$$\begin{aligned}
&L^\mu x^\nu = C^{\mu\nu}(q) \;\; {}_\kappa x^\kappa + q^{-2}\check{\mathbf{R}}(q)^{\mu\nu}{}_{\kappa\lambda}x^\kappa L^\lambda \;, \\
&C(q)^{\mu\nu}{}_\sigma = \bar{t}^\mu(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma}\bar{t}^\nu(q)_{\gamma\dot\gamma}t_\sigma(q)^{\alpha\dot\gamma} \;.
\end{aligned} \tag{5.12}$$

where $\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda}$ is reducible

$$\check{\mathbf{R}}^{\mu\nu}_{\kappa\lambda} = \check{\mathbf{R}}^{00}_{00} \oplus \check{\mathbf{R}}^{m0}_{0l} \oplus \check{\mathbf{R}}^{0n}_{k0} \oplus \check{\mathbf{R}}^{mn}_{kl} \;,$$

and

$$\begin{aligned}
&C(q)^{0\nu}{}_\sigma = C(q)^{\nu 0}{}_\sigma = -q^{-1}[2]^{-1/2}\delta^\nu{}_\sigma \;, \\
&C(q)^{mn}_0 = -q[2]^{-1/2}\mathbf{g}(q^2)^{mn} \;, \\
&C(q)^{mn}_k = \tfrac{[4]^{1/2}}{[2]}\mathbf{u}_k(q^2)^{mn} \;,
\end{aligned} \tag{5.13}$$

where $\mathbf{g}(q^2)^{mn}$ and $\mathbf{u}_s(q^2)^{mn}$ are respectively the singlet and triplet eigenvectors [11] of the $\check{\mathbf{R}}^{mn}{}_{kl}$. Then Eq(5.12) is equivalent to

$$
\begin{aligned}
L^0 x^0 &= -q^{-1}[2]^{-1/2}x^0 + q^{-2}x^0 L^0 \ , \\
L^m x^0 &= -q^{-1}[2]^{-1/2}x^m + q^{-2}x^0 L^m \ , \\
L^0 x^n &= -q^{-1}[2]^{-1/2}x^n + q^{-2}x^n L^0 \ , \\
L^m x^n &= -q\ [2]^{-1/2}\mathbf{g}(q^2)^{mn}x^0 + \tfrac{[4]^{1/2}}{[2]}\mathbf{u}_s(q^2)^{mn}x^s + q^{-2}\check{\mathbf{R}}^{mn}{}_{kl}x^k L^l \ .
\end{aligned}
\tag{5.14}
$$

Similarly we have

$$
\begin{aligned}
\bar{L}^{\dot\alpha\dot\beta}\ u^\gamma\ \bar{w}_{\dot\gamma} = q\check{R}^{-1}(q)^{\dot\beta\gamma}{}_{\gamma'\dot\beta'}\check{R}^{-1}(q)^{\dot\alpha\gamma'}{}_{\delta\dot\alpha'}u^\delta\ \bar{w}_{\dot\delta}\{\epsilon(q)^{\dot\delta\dot\alpha'}\ \delta^{\dot\beta'}{}_{\dot\gamma} \\
+q^{-1}\ \check{R}^{-1}(q)^{\dot\gamma'\dot\beta'}{}_{\dot\beta''\dot\gamma}\check{R}^{-1}(q)^{\dot\delta\dot\alpha'}{}_{\dot\alpha''\dot\gamma'}\bar{L}^{\dot\alpha''\dot\beta''}\}
\end{aligned}
\tag{5.15}
$$

from which we obtain a tedious derivation that

$$
\bar{L}^\mu\ x^\nu = -C(q^{-1})^{\nu\mu}{}_\sigma\ x^\sigma + q^{-2}\ |\check{R}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda}x^\kappa\ \bar{L}^\lambda \ , \tag{5.16}
$$

where $|\check{R}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda}$ is the $\check{R}$ matrix in (2.32d) which is reducible in the sense

$$
\begin{aligned}
|\check{R}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda} &= |\check{R}^{-1}(q)^{0\nu}{}_{\kappa 0} \oplus |\check{R}^{-1}(q)^{m\nu}{}_{\kappa l} \ , \\
|\check{R}^{-1}(q)^{0\nu}{}_{\kappa 0} &= \delta^\nu{}_\kappa \ , \\
|\check{R}^{-1}(q)^{m\nu}{}_{\kappa l} &= \bar{t}^\nu(q)_{\gamma\dot\gamma}\check{\bar{R}}^{-1}(q)^{m\gamma}{}_{\delta j}\check{R}(q)^{\dot\gamma j}{}_{l\dot\delta}t_\kappa(q)^{\delta\dot\delta} \ .
\end{aligned}
\tag{5.17}
$$

The latter $|\check{R}^{-1}(q)^{m\nu}{}_{\kappa l}$, representing the scattering between the triplet (m,l=+,3,−) and the quartet (ν, κ =+,3,0,−), can be constructed by $\check{\bar{R}}(q)$ appearing in (5.7b) and $\check{R}(q)$ appearing in (5.2b)

$$
\begin{aligned}
&|\check{R}^{-1}(q)^{m0}{}_{0l} = (q^2 - 1 + q^{-2})\delta^m{}_l \ , \\
&|\check{R}^{-1}(q)^{m0}{}_{kl} = (q - q^{-1})q^{-2}\sqrt{q^2 + q^{-2}}\,\bar{\mathbf{u}}^m(q^2)_{kl} \ , \\
&|\check{R}^{-1}(q)^{mn}{}_{0l} = (q - q^{-1})q^2\sqrt{q^2 + q^{-2}}\,\mathbf{u}_l(q^2)^{mn} \ , \\
&(|\check{R}^{mn}{}_{kl}) = 1 \oplus \begin{pmatrix} dq^{-1} & 1 \\ 1 & -dq \end{pmatrix} \oplus \begin{pmatrix} 0 & -d & 1 \\ -d & 3 - q^2 - q^{-2} & d \\ 1 & d & 0 \end{pmatrix} \oplus \begin{pmatrix} dq^{-1} & 1 \\ 1 & -dq \end{pmatrix} \oplus 1 \\
&\quad kl: \quad ++ \quad +3 \quad 3+ \quad +- \quad 33 \quad -+ \quad 3- \quad -3 \quad --
\end{aligned}
\tag{5.18}
$$

with $d = q - q^{-1}$ in this equation. Therefore, Eq(5.16) gives

$$
\begin{aligned}
\bar{L}^0\, x^\nu &= q\,[2]^{-1/2} x^\nu + q^{-2}\, x^\nu\, \bar{L}^0\,, \\
\bar{L}^m\, x^0 &= q[2]^{-1/2} x^m + q^{-2}\{(q^2 - 1 + q^{-2}) x^0\, \bar{L}^m \\
&\qquad + d q^{-2}\sqrt{q^2 + q^{-2}}|\bar{u}^m(q^2)_{kl}\, x^k\, \bar{L}^l\}\,, \\
\bar{L}^m\, x^n &= q^{-1}[2]^{-1/2}\mathbf{g}(q^2)^{mn} + [2]^{-1/2}\sqrt{q^2 + q^{-2}}\mathbf{u}_k(q^2)^{mn} x^k \\
&\qquad - d\sqrt{q^2 + q^{-2}}\mathbf{u}_l(q^2)^{mn} x^0\, \bar{L}^l - q^{-2}|\check{R}^{-1}(q)^{mn}{}_{kl} x^k\, \bar{L}^l\,.
\end{aligned}
\tag{5.19}
$$

In the limit $q \longrightarrow 1$, we obtain from (5.14) and (5.19) that

$$
\begin{aligned}
&[L^+, x^+] = 0, && [\bar{L}^+, x^+] = 0, \\
&[L^+, x^3] = -\tfrac{1}{\sqrt{2}} x^+, && [\bar{L}^+, x^3] = -\tfrac{1}{\sqrt{2}} x^+, \\
&[L^+, x^-] = \tfrac{1}{\sqrt{2}}(x^3 - x^0), && [\bar{L}^+, x^-] = \tfrac{1}{\sqrt{2}}(x^3 + x^0), \\
&[L^3, x^+] = \tfrac{1}{\sqrt{2}} x^+, && [\bar{L}^3, x^+] = \tfrac{1}{\sqrt{2}} x^+, \\
&[L^3, x^3] = -\tfrac{1}{\sqrt{2}} x^0, && [\bar{L}^3, x^3] = \tfrac{1}{\sqrt{2}} x^0, \\
&[L^3, x^-] = -\tfrac{1}{\sqrt{2}} x^-, && [\bar{L}^3, x^-] = -\tfrac{1}{\sqrt{2}} x^-, \\
&[L^-, x^+] = -\tfrac{1}{\sqrt{2}}(x^3 + x^0), && [\bar{L}^-, x^+] = -\tfrac{1}{\sqrt{2}}(x^3 - x^0), \\
&[L^-, x^3] = \tfrac{1}{\sqrt{2}} x^-, && [\bar{L}^-, x^3] = \tfrac{1}{\sqrt{2}} x^-, \\
&[L^-, x^-] = 0, && [\bar{L}^-, x^-] = 0, \\
&[L^m, x^0] = -x^m/\sqrt{2}, && [\bar{L}^m, x^0] = x^m/\sqrt{2}.
\end{aligned}
\tag{5.20}
$$

It implies that $J^m = \frac{1}{\sqrt{2}}(L^m + \bar{L}^m)$ are the rotation generators while $K^m = \frac{1}{\sqrt{2}}(\bar{L}^m - L^m)$ the boost generators. In the present stage we have not considered the reality of the coordinate vector x^μ. It will be discussed in a separate paper.

Note Added

After completing this manuscript we saw a paper by W.B. Schmidke et al (Z.Phys. **C** 52(1991)471) in which the ansatz-consistency method is used to give the generators of the quantum Lorentz group acting upon spinors and 4-vectors similar to those in Section V. We believe that their results will be equivalent to ours if they used an ansatz corresponding to the 4 D_+ differential calculus rather than the 3 D calculus they adopted. This is also the reason why their results were less compact and less explicitly covariant.

References

[1] V.G. Drinfeld, Quantum groups, Proccedings of ICM, 1986, Vol.1,798; M. Jimbo, Lett. Math. Phys. 10(1986)63, 11(1986)247, Commun. Math. Phys. 102(1986)537; N.Yu. Reshetikhin, LOMI preprint E-4-87, E-17-87 (1987); N.Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Leningrad Math. J. 1(1990)193; S.L. Woronowicz, Publ. RIMS, Kyoto Univ., Vol.23, (1987)117, Commun. Math. Phys. 111(1987)613.

[2] C.N. Yang, Phys. Rev. Lett. 19(1967)1312; R.J. Baxter, Exactly Solved Models, Academic Press, London, 1982.

[3] Yu.I. Manin, Quantum groups and non-commutative geometry, Les publications du Cetre de Recherches Mathematiques. Universite de Montreal, 1988.

[4] J. Wess and B. Zumino, Nucl. Phys. B(Proc. Suppl.) 18B(1990)302.

[5] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys.C49(1991)439; X.C. Song and L. Liao, J. Phys. A25(1992)623.

[6] X.C. Song, Covariant Differential Calculus on Quantum Minkowski Space and The q-Analogue of Dirac Equation, Preprint CCAST-91-63, Beijing.

[7] X.C. Song, forthcoming paper.

[8] P. Podles, Lett. Math. Phys. 18(1989)107; S.L. Woronowicz, Commun. Math. Phys.122(1989)125.

[9] See for example, A.R. Edmonds, Angular Momentum in Quantum Mechanics, Sec. 2.6, Princeton University Press, 1957.

[10] P. Podles and S.L. Woronowicz, Commun. Math. Phys. 130(1990)381. U. Carow-Watamura, M. Schlicker, M. Scholl and S. Watamura, Z. Phys.C48(1991)159.

[11] X.C. Song, Spinor Analysis for Quantum Group $SU_q(2)$, preprint CCAST-91-28, Beijing, to appear in J.Phys. A.

[12] K. Wu & R. Zhang, Differential Calculus on Quantum Matrix Lie Groups, ASITP-91-72, CCAST-91-69, Beijing.

CCAST 92-10
ASITP 92-19
TUIMP-TH 92/46
April 1992

Duality in Quantum Minkowski Space-time

Zhan XU

CCAST (World Lab), P. O. Box 8730, Beijing 100080
and
Institute of Theoretical Physics, Academia Sinica, Beijing 100080
and
Department of Physics and Institute of Modern Physics,
Tsinghua University, Beijing 100084 [1]

ABSTRACT

The quantum Minkowski space-time has real structure and this seems to be contradictive to the differential calculus in it. Dual differentiations are introduced to solve the problem here. And this duality can be extended to differential calculus in any C^*-algebra.

[1]Mailing address.

The standard model of elementary particles has, as known, both great success and obvious imperfection, among which the Higgs mechanism responsible for the breakdown of unified gauge symmetry of electroweak interaction is the most puzzling feature. Several ideas based on the non-commutative geometry have been proposed [1–4] to account for the appearance of the non-vanishing vacuum expectation value of Higgs scalar fields without contradiction to the gauge invariance of the theory. In fact these efforts are extensions of the active research in mathematical physics on quantum groups, Hopf algebra, quantum space, Yang-Baxter equations and non-commutative geometry in the recent decade [5–10].

As known the Lorentz invariance describes the fundamental structure of space-time, the Minkowski space, in the world at our reach and the conventional relativistic quantum field theory constructed in this space is the foundation of current theories for particles and their interactions. But it is not clear that whether the relativistic quantum field theory is applicable when the scale of physics was down to the sub-microscopic one [11]. In other words should we assume new structure of space-time for the sub-microscopic world? In the view of quantum field theory space-time "coordinate" are but a set of parameters characterizing the degrees of freedom of the fields, which in special relativity forms a manifold. Mathematics, however, supports physics with more spaces than manifold. Could physics survive in, say, algebraic space-time? More concretely, as well known, the Lorentz group $SO(3,1)$ is locally isomorphic to the group $SL(2,C)$, and the latter has its quantum counterpart $SL_q(2,C)$. Therefore it is interesting to consider $SL_q(2,C)$ as a new symmetry of space-time.

Motivated by this idea, research on quantum space-time is carrying on. The structure of deformed Lorentz group and algebra have been described [11–16,21,22]. From the mathematical point of view the quantum group $SL_q(2,C)$ is interesting because it is non-compact and yet no general theory for non-compact quantum group is available. P. Podleś and W. L. Woronowicz [11] introduced $SL_q(2,C)$, which they called quantum Lorentz group directly, by the cartesian product of quantum group $SU_q(2)(\sim SO_q(3))$ and its Pontryagin dual. But from the physical point of view the non-compactness may be less important. U. Carow-Watamura *et al.* [12,13] constructed the quantum space-time by means of a method analogous to the the twister theory, which is more physical. W. B. Schmidke *et al.* [15] did it similarly. And many authors in different institutions [12,13,21,22] worked out the differential calculus in quantum Minkowski space-time. At this point, however, there is still a very important problem to be solved. That is the contradiction between the $*$-conjugation operation and the exterior differentiation operation in this "real" quantum space.

At first a method to construct differential calculus in quantum space was suggested by J. Wess and B. Zumino [17]. They introduced the Cartan exterior differentiation d satisfying the following axioms,

$$\begin{cases} \mathrm{d}^2 = 0, & \text{(Cartan rule)} \\ \mathrm{d}(fg) = (\mathrm{d}f)g + (-1)^{P(f)} f(\mathrm{d}g), & \text{(Leibniz rule)} \end{cases} \tag{1}$$

where $P(f)$ is the order of element f in the algebra, and defined the partial differentiation

∂_μ by

$$d = \xi^\mu \partial_\mu, \qquad \text{(summed over } \mu\text{)} \tag{2}$$

in which

$$\xi^\mu \equiv dx^\mu, \tag{3}$$

x^μ is the "coordinate" of the space. Then the calculus is determined by requirement of consistency. The differential calculus in quantum Minkowski space, according to this method, is [21,22]

$$\begin{cases} (\mathcal{P}_A)^{\mu\nu}{}_{\kappa\lambda} x^\kappa x^\lambda = 0, \\ (\mathcal{P}_S)^{\mu\nu}{}_{\kappa\lambda} \xi^\kappa \xi^\lambda = 0, \\ (\mathcal{P}_1)^{\mu\nu}{}_{\kappa\lambda} \xi^\kappa \xi^\lambda = 0, \\ x^\mu \xi^\nu = q \hat{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} \xi^\kappa x^\lambda, \\ \partial_\mu x^\nu = \delta^\nu_\mu + q \hat{\mathcal{R}}^{\nu\kappa}{}_{\mu\lambda} x^\lambda \partial_\kappa, \\ \partial_\mu \xi^\nu = q^{-1} (\hat{\mathcal{R}}^{-1})^{\nu\kappa}{}_{\mu\lambda} \xi^\lambda \partial_\kappa, \\ \partial_\mu \partial_\nu (\mathcal{P}_A)^{\nu\mu}{}_{\lambda\kappa} = 0, \end{cases} \tag{4}$$

where $\mathcal{P}_i (i = 1, A, S)$ are the projective operators for singlet, antisymmetric and symmetric multiplets, respectively, and $\hat{\mathcal{R}}$ is the $\hat{R}$ matrix of the vector representation of quantum group $SL_q(2, C)$ (see Appendix for details). The explicit commutation relations for coordinate x^μ, under some proper choice of basis ($\mu = 0, +, 3, -$), are

$$\begin{cases} x^0 x^+ - x^+ x^0 = x^0 x^3 - x^3 x^0 = x^0 x^- - x^- x^0 = 0, \\ q x^+ x^3 - q^{-1} x^3 x^+ = \omega x^0 x^+, \\ q x^3 x^- - q^{-1} x^- x^3 = \omega x^0 x^-, \\ x^+ x^- - x^- x^+ = \omega (x^3 - x^0) x^3. \end{cases} \tag{5}$$

in which

$$\omega = q - q^{-1}.$$

This differential calculus is covariant under the quantum Lorentz transformation

$$x'^\mu = L^\mu{}_\nu x^\nu, \tag{6}$$

where the quantum Lorentz matrix $L^\mu{}_\nu$ satisfies the Yang-Baxter relation

$$\hat{\mathcal{R}}_{12} L_1 L_2 = L_1 L_2 \hat{\mathcal{R}}_{12}, \tag{7}$$

and the $\hat{\mathcal{R}}$ matrix itself satisfies the Yang-Baxter equation

$$\hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} = \hat{\mathcal{R}}_{23} \hat{\mathcal{R}}_{12} \hat{\mathcal{R}}_{23}. \tag{8}$$

Furthermore U. Carow-Watamura *et al.* [18] pointed that when there exists a metric in the quantum space, the algebra in the space becomes Birman-Wenzl-Murakami algebra

[19,20], in which the $\hat{R}$ matrix has three different eigenvalues and hence satisfies a cubic algebraic equation, and this is just the case of quantum Minkowski space. In the above basis the invariant metric of quantum Minkowski space is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & -1 & 0 \\ 0 & -q^{-1} & 0 & 0 \end{pmatrix}, \qquad (\mu, \nu = 0, +, 3, -) \tag{9}$$

that is,

$$\begin{aligned} J &= g_{\mu\nu} x^\mu x^\nu \\ &= x^0 x^0 - q x^+ x^- - x^3 x^3 - q^{-1} x^- x^+ \\ &= \text{invariant center of the algebra.} \end{aligned} \tag{10}$$

By means of the metric the subscript of partial differentiation can be transformed to superscript,

$$\partial^\mu = g^{\mu\nu} \partial_\nu, \tag{11}$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ satisfying

$$g^{\mu\lambda} g_{\lambda\nu} = g_{\nu\lambda} g^{\lambda\mu} = \delta^\mu_\nu, \tag{12}$$

and in fact

$$g^{\mu\nu} = g_{\mu\nu}. \tag{13}$$

Then the last three commutation relations in Eq.(4) reduce to

$$\begin{cases} \partial^\mu x^\nu = g^{\mu\nu} + q(\hat{\mathcal{R}}^{-1})^{\mu\nu}{}_{\kappa\lambda} x^\kappa \partial^\lambda, \\ \partial^\mu \xi^\nu = q^{-1} \hat{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} \xi^\kappa \partial^\lambda, \\ (\mathcal{P}_A)^{\mu\nu}{}_{\kappa\lambda} \partial^\kappa \partial^\lambda = 0, \end{cases} \tag{14}$$

because $g^{\mu\nu}$ and $\hat{\mathcal{R}}$ obey the relations

$$\begin{cases} g^{\mu\rho} \hat{\mathcal{R}}^{\nu\kappa}{}_{\rho\lambda} = (\hat{\mathcal{R}}^{-1})^{\mu\nu}{}_{\lambda\rho} g^{\rho\kappa}, \\ g^{\mu\rho} (\hat{\mathcal{R}}^{-1})^{\nu\kappa}{}_{\rho\lambda} = \hat{\mathcal{R}}^{\mu\nu}{}_{\lambda\rho} g^{\rho\kappa}, \\ g_{\mu\rho} \hat{\mathcal{R}}^{\rho\lambda}{}_{\nu\kappa} = (\hat{\mathcal{R}}^{-1})^{\lambda\rho}{}_{\mu\nu} g_{\rho\kappa}, \\ g_{\mu\rho} (\hat{\mathcal{R}}^{-1})^{\rho\lambda}{}_{\nu\kappa} = \hat{\mathcal{R}}^{\lambda\rho}{}_{\mu\nu} g_{\rho\kappa}. \end{cases} \tag{15}$$

Therefore we are able to write the set of commutation relations in compact tensor form as

$$\begin{cases} (\mathcal{P}_A)_{12}x_1x_2 = 0, \\ (\mathcal{P}_S)_{12}\xi_1\xi_2 = 0, \\ (\mathcal{P}_1)_{12}\xi_1\xi_2 = 0, \\ x_1\xi_2 = q\hat{\mathcal{R}}_{12}\xi_1x_2, \\ \partial_1x_2 = (G^{-1})_{12} + q(\hat{\mathcal{R}}^{-1})_{12}x_1\partial_2, \\ \partial_1\xi_2 = q^{-1}\hat{\mathcal{R}}_{12}\xi_1\partial_2, \\ (\mathcal{P}_A)_{12}\partial_1\partial_2 = 0, \end{cases} \tag{16}$$

where

$$x = (x^\mu), \qquad \xi = (\xi^\mu), \qquad \partial = (\partial^\mu), \qquad G^{-1} = (g^{\mu\nu}),$$

and the covariance of these commutation relations under quantum Lorentz transformation becomes apparent because the transformation is

$$\begin{Bmatrix} x'^\mu \\ \xi'^\mu \\ \partial'^\mu \end{Bmatrix} = L^\mu{}_\nu \begin{Bmatrix} x^\mu \\ \xi^\mu \\ \partial^\mu \end{Bmatrix}, \tag{17}$$

and we have Eq.(7) and relation

$$L^\mu{}_\kappa L^\nu{}_\lambda g^{\kappa\lambda} = g^{\mu\nu}. \tag{18}$$

Now the problem is that in the quantum group $SL_q(2, C)$ and its representation spaces regarded as C^*- algebras, the $*$-conjugation has been introduced, which obeys the fundamental axioms

$$\begin{cases} (f^*)^* = f, & \text{(idempotence)} \\ (fg)^* = g^*f^*, & \text{(algebraic antihomomorphism)} \end{cases} \tag{19}$$

for any elements f and g in the algebra, and quantum Minkowski space is a "real" representation of the quantum group $SL_q(2, C)$. This means that although the coordinate x^μ itself is not real, i. e.

$$(x^\mu)^* \neq x^\mu,$$

there still exists a relation for x^μ and its conjugation,

$$(x^\mu)^* = C^\mu{}_\nu x^\nu, \tag{20}$$

where the matrix $C = (C^\mu{}_\nu)$ obeys

$$C^2 = I, \tag{21}$$

and this conjugation relation keeps invariant under quantum Lorentz transformation, because we have for quantum Lorentz matrix $L^{\mu}{}_{\nu}$,

$$(L^*)^{\mu}{}_{\nu} = C^{\mu}{}_{\kappa} L^{\kappa}{}_{\lambda} C^{\lambda}{}_{\nu}. \tag{22}$$

We call Eq.(20) and (22) the reality condition of coordinates x and quantum Lorentz matrix L, respectively. The explicit form of the conjugation matrix C is

$$C^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \qquad (\mu, \nu = 0, +, 3, -) \tag{23}$$

that is

$$(x^0)^* = x^0, \quad (x^+)^* = qx^-, \quad (x^3)^* = x^3, \quad (x^-)^* = q^{-1}x^+. \tag{24}$$

Therefore in quantum space two kinds of fundamental operations, d and $*$, have been defined. But it is not difficult to see that they must be non-commutative, i.e.

$$(\mathrm{d}f)^* \neq \mathrm{d}(f^*),$$

because the Leibniz rule for exterior differentiation and the algebraic anti-homomorphism for $*$-conjugation contradict to their commutativity, and indeed the commutation relations Eq.(16) are not consistent if the commutativity holds.

The solution to this contradiction is to introduce the dual (or, in some sense, conjugation) of the differentiation d. Let us redenote the previous differential as left one,

$$\mathrm{d}f \equiv \overrightarrow{\mathrm{d}} f, \tag{25}$$

which obeys the rules

$$\begin{cases} \overrightarrow{\mathrm{d}}^2 = 0, \\ \overrightarrow{\mathrm{d}}(fg) = (\overrightarrow{\mathrm{d}} f)g + (-1)^{P(f)} f(\overrightarrow{\mathrm{d}} g), \end{cases} \tag{26}$$

and define its dual as right differential $\overleftarrow{\mathrm{d}}$ which obeys

$$\begin{cases} \overleftarrow{\mathrm{d}}^2 = 0, \\ (fg)\overleftarrow{\mathrm{d}} = f(g\overleftarrow{\mathrm{d}}) + (-1)^{P(g)}(f\overleftarrow{\mathrm{d}})g, \end{cases} \tag{27}$$

and simultaneously assume they cooperate with $*$-conjugation according to the rule

$$\begin{cases} (\overrightarrow{\mathrm{d}} f)^* = (f^*)\overleftarrow{\mathrm{d}}, \\ (f\overleftarrow{\mathrm{d}})^* = \overrightarrow{\mathrm{d}}(f^*). \end{cases} \tag{28}$$

For mixed application of the two differentiations we assume

$$(\overrightarrow{\mathrm{d}} f)\overleftarrow{\mathrm{d}} = \overrightarrow{\mathrm{d}}(f\overleftarrow{\mathrm{d}}) = \overrightarrow{\mathrm{d}} f \overleftarrow{\mathrm{d}} = 0. \tag{29}$$

Now it is easy to prove that the fundamental conjectures Eqs.(19,26,27,28,29) are consistent.

The differentiations $\overrightarrow{d}$ and $\overleftarrow{d}$ should be invariant operations under transformation,

$$\begin{cases} \overrightarrow{d}' = \overrightarrow{d}, \\ \overleftarrow{d}' = \overleftarrow{d}, \end{cases} \tag{30}$$

therefore if we write

$$\begin{cases} \overrightarrow{d} = \xi^\mu \overrightarrow{\partial}_\mu, & \xi^\mu \equiv \overrightarrow{d} x^\mu \\ \overleftarrow{d} = \overleftarrow{\partial}_\mu \eta^\mu, & \eta^\mu \equiv x^\mu \overleftarrow{d} \end{cases} \tag{31}$$

the partial differentiations $\overrightarrow{\partial}_\mu$ and $\overleftarrow{\partial}_\mu$ will transform differently,

$$\begin{cases} \overrightarrow{\partial}'_\mu = \overrightarrow{\partial}_\nu \, (\tilde{L}^{-1})^\nu{}_\mu, \\ \overleftarrow{\partial}'_\mu = \overleftarrow{\partial}_\nu \, (L^{-1})^\nu{}_\mu, \end{cases} \tag{32}$$

in which L^{-1} is the usual inverse matrix of L,

$$(L^{-1})^\mu{}_\kappa L^\kappa{}_\nu = L^\mu{}_\kappa (L^{-1})^\kappa{}_\nu = \delta^\mu_\nu, \tag{33}$$

but $\tilde{L}^{-1}$ is the opposite-ordered inverse matrix of L,

$$(\tilde{L}^{-1})^\kappa{}_\nu L^\mu{}_\kappa = L^\kappa{}_\nu (\tilde{L}^{-1})^\mu{}_\kappa = \delta^\mu_\nu. \tag{34}$$

$\tilde{L}^{-1}$ is different from L^{-1} because we are now working in non-commutative geometry. Also raising of the subscripts of $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ are different,

$$\begin{cases} \overrightarrow{\partial}^\mu = g^{\mu\nu} \, \overrightarrow{\partial}_\nu, \\ \overleftarrow{\partial}^\mu = \overleftarrow{\partial}_\nu \, g^{\nu\mu}, \end{cases} \tag{35}$$

(notice that $g_{\mu\nu} \neq g_{\nu\mu}$), but they both transform like vectors,

$$\begin{Bmatrix} \overrightarrow{\partial}^\mu \\ \overleftarrow{\partial}^\mu \end{Bmatrix} = L^\mu{}_\nu \begin{Bmatrix} \overrightarrow{\partial}^\nu \\ \overleftarrow{\partial}^\nu \end{Bmatrix}. \tag{36}$$

Quite parallel to the case of left differentiation, we have for the right differentiation the following calculus,

$$\begin{cases} (\mathcal{P}_S)_{12}\eta_1\eta_2 = 0, \\ (\mathcal{P}_1)_{12}\eta_1\eta_2 = 0, \\ \eta_1 x_2 = q\hat{\mathcal{R}}_{12} x_1 \eta_2, \\ x_1 \, \overleftarrow{\partial}_2 = (G^{-1})_{12} + q(\hat{\mathcal{R}}^{-1})_{12} \, \overleftarrow{\partial}_1 \, x_2, \\ \eta_1 \, \overleftarrow{\partial}_2 = q^{-1}\hat{\mathcal{R}}_{12} \, \overleftarrow{\partial}_1 \, \eta_2, \\ (\mathcal{P}_A)_{12} \, \overleftarrow{\partial}_1 \overleftarrow{\partial}_2 = 0, \end{cases} \tag{37}$$

where

$$\eta = (\eta^\mu), \qquad \overleftarrow{\partial} = (\overleftarrow{\partial}^\mu).$$

As to the mixed commutation relations, by requirement of consistency and conjecture Eq.(29) we have

$$\begin{cases} \xi_1\eta_2 = \eta_1\xi_2, \\ (\mathcal{P}_S)_{12}\xi_1\eta_2 = 0, \\ (\mathcal{P}_1)_{12}\xi_1\eta_2 = 0, \\ \overrightarrow{\partial}_1\overleftarrow{\partial}_2 = q^{-1}(\hat{\mathcal{R}}^{-1})_{12}\overleftarrow{\partial}_1\overrightarrow{\partial}_2, \\ \overrightarrow{\partial}_1\eta_2 = q(\hat{\mathcal{R}}^{-1})_{12}\eta_1\overrightarrow{\partial}_2, \\ \xi_1\overleftarrow{\partial}_2 = q(\hat{\mathcal{R}}^{-1})_{12}\overleftarrow{\partial}_1\xi_2. \end{cases} \tag{38}$$

From Eq.(20) and (28) we see that ξ^μ and η^μ are not themselves real but dual with one another,

$$\begin{cases} (\xi^\mu)^* = C^\mu{}_\nu\eta^\nu, \\ (\eta^\mu)^* = C^\mu{}_\nu\xi^\nu, \end{cases} \tag{39}$$

and $\overrightarrow{\partial}^\mu$ and $\overleftarrow{\partial}^\mu$ are similar,

$$\begin{cases} (\overrightarrow{\partial}^\mu)^* = C^\mu{}_\nu\overleftarrow{\partial}^\nu, \\ (\overleftarrow{\partial}^\mu)^* = C^\mu{}_\nu\overrightarrow{\partial}^\nu. \end{cases} \tag{40}$$

It is now worth collecting the commutation relations in Eqs.(16,37,38) for coordinate, left and right differentials as well as left and right partial differentiations in one set. That is

$$\left\{\begin{array}{l}(\mathcal{P}_A)_{12}x_1x_2=0,\\ (\mathcal{P}_S)_{12}\xi_1\xi_2=(\mathcal{P}_S)_{12}\eta_1\eta_2=(\mathcal{P}_S)_{12}\xi_1\eta_2=0,\\ (\mathcal{P}_1)_{12}\xi_1\xi_2=(\mathcal{P}_1)_{12}\eta_1\eta_2=(\mathcal{P}_1)_{12}\xi_1\eta_2=0,\\ \xi_1\eta_2=\eta_1\xi_2,\\ x_1\xi_2=q\hat{\mathcal{R}}_{12}\xi_1x_2,\\ \eta_1x_2=q\hat{\mathcal{R}}_{12}x_1\eta_2,\\ \overrightarrow{\partial}_1\,x_2=(G^{-1})_{12}+q(\hat{\mathcal{R}}^{-1})_{12}x_1\,\overrightarrow{\partial}_2,\\ x_1\,\overleftarrow{\partial}_2=(G^{-1})_{12}+q(\hat{\mathcal{R}}^{-1})_{12}\,\overleftarrow{\partial}_1\,x_2,\\ \overrightarrow{\partial}_1\,\xi_2=q^{-1}\hat{\mathcal{R}}_{12}\xi_1\,\overrightarrow{\partial}_2,\\ \eta_1\,\overleftarrow{\partial}_2=q^{-1}\hat{\mathcal{R}}_{12}\,\overleftarrow{\partial}_1\,\eta_2,\\ \overrightarrow{\partial}_1\,\eta_2=q(\hat{\mathcal{R}}^{-1})_{12}\eta_1\,\overrightarrow{\partial}_2,\\ \xi_1\,\overleftarrow{\partial}_2=q(\hat{\mathcal{R}}^{-1})_{12}\,\overleftarrow{\partial}_1\,\xi_2,\\ (\mathcal{P}_A)_{12}\,\overrightarrow{\partial}_1\overrightarrow{\partial}_2=0,\\ (\mathcal{P}_A)_{12}\,\overleftarrow{\partial}_1\overleftarrow{\partial}_2=0,\\ \overrightarrow{\partial}_1\overleftarrow{\partial}_2=q^{-1}(\hat{\mathcal{R}}^{-1})_{12}\,\overleftarrow{\partial}_1\overrightarrow{\partial}_2\,.\end{array}\right. \tag{41}$$

And it is easy to check this set of relations is self-consistent under $*$-conjugation operation Eqs.(20,39,40), because we have relations for conjugation matrix, metric, projective operators and $\hat{\mathcal{R}}$ matrix as follows,

$$\left\{\begin{array}{l}C^\mu{}_\kappa C^\nu{}_\lambda g^{\kappa\lambda}=g^{\mu\nu},\\ g_{\mu\nu}C^\mu{}_\kappa C^\nu{}_\lambda=g_{\kappa\lambda},\end{array}\right. \tag{42}$$

and

$$C^\mu{}_\tau C^\nu{}_\rho(\mathcal{P}_i)^{\rho\tau}{}_{\kappa\lambda}=(\mathcal{P}_i)^{\mu\nu}{}_{\rho\tau}C^\tau{}_\kappa C^\rho{}_\lambda,\qquad i=1,A,S \tag{43}$$

and Eq.(43) holds also for the matrices $\hat{\mathcal{R}}$, $\hat{\mathcal{R}}^{-1}$ and E, the 16×16 unit matrix.

In conclusion, the differential calculus defined in quantum Minkowski space-time should satisfy three conditions. First, it is covariant under quantum Lorentz transformation. Second, the commutation relations are consistent. Third, it is compatible with the reality of the space. And the introduction of dual differentials is a necessary step to meet those requirements. More generally this duality can be extended to differential calculus in any non-commutative C^*-algebra.

Appendix

Here we collect some basic definitions of quantum group $SL_q(2,C)$ and quantum Minkowski space.

The decomposition of the $\hat{R}$ matrix into projective operators for quantum group $SL_q(2,C)$ is

$$\hat{R} = qP_S - q^{-1}P_A, \tag{44}$$

where $(P_A)^{ab}{}_{cd}$ is the anti-symmetric (singlet) one and $(P_S)^{ab}{}_{cd}$ is the symmetric (triplet) one, and $a,b,c,d = 1,2$. They satisfy the conditions

$$\begin{cases} P_A^2 = P_A, \quad P_S^2 = P_S, \\ P_A P_S = 0, \\ P_A + P_S = I. \quad\quad I : 4\times 4 \text{ unit matrix} \end{cases} \tag{45}$$

These operators are constructed from the eigenvectors of $\hat{R}$ matrix [21], which are

$$\begin{cases} \bar{t}^{\,0}{}_{ab} = t^{ab}{}_0 = [2]^{-1/2}\begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}, \\ \bar{t}^{\,+}{}_{ab} = t^{ab}{}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \bar{t}^{\,3}{}_{ab} = t^{ab}{}_3 = [2]^{-1/2}\begin{pmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}, \\ \bar{t}^{\,-}{}_{ab} = t^{ab}{}_- = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \tag{46}$$

($a,b = 1,2;\ [2] = q + q^{-1}$), and satisfy

$$\begin{cases} \hat{R}^{ab}{}_{cd} t^{cd}{}_0 = -q^{-1} t^{ab}{}_0, \\ \hat{R}^{ab}{}_{cd} t^{cd}{}_m = q t^{ab}{}_m, \quad\quad m = +,3,- \end{cases} \tag{47}$$

$$\begin{cases} \bar{t}^{\,0}{}_{ab} \hat{R}^{ab}{}_{cd} = -q^{-1} \bar{t}^{\,0}{}_{cd}, \\ \bar{t}^{\,m}{}_{ab} \hat{R}^{ab}{}_{cd} = q \bar{t}^{\,m}{}_{cd}, \quad\quad m = +,3,- \end{cases} \tag{48}$$

and

$$\begin{cases} \bar{t}^{\,\mu}{}_{ab} t^{ab}{}_\nu = \delta^\mu_\nu, \\ t^{ab}{}_\mu \bar{t}^{\,\mu}{}_{cd} = \delta^a_c \delta^b_d, \end{cases} \tag{49}$$

where $\mu,\nu = 0,+,3,-$. Then

$$\begin{cases} (P_A)^{ab}{}_{cd} = t^{ab}{}_0 \bar{t}^{\,0}{}_{cd}, \\ (P_S)^{ab}{}_{cd} = t^{ab}{}_m \bar{t}^{\,m}{}_{cd}. \quad \text{(summed over } m) \end{cases} \tag{50}$$

For bi-spinor representation of $SL_q(2,C)$ the $\hat{\mathbf{R}}$ matrix is [19]

$$\hat{\mathbf{R}}^{a\dot{a}b\dot{b}}{}_{c\dot{c}d\dot{d}} = q^{-1}\hat{R}^{\dot{a}b}{}_{e\dot{e}}\hat{R}^{ae}{}_{cf}\hat{R}^{\dot{e}\dot{b}}{}_{\dot{f}\dot{d}}(\hat{R}^{-1})^{f\dot{f}}{}_{\dot{c}d}, \tag{51}$$

or, in compact tensor notation,

$$\hat{\mathbf{R}}_{(12)(34)} = q^{-1}\hat{R}_{23}\hat{R}_{12}\hat{R}_{34}\hat{R}_{23}^{-1}. \tag{52}$$

Its projective operators are

$$\begin{cases} (\mathbf{P}_1)_{(12)(34)} = \hat{R}_{23}(P_A)_{12}(P_A)_{34}\hat{R}_{23}^{-1}, \\ (\mathbf{P}_A)_{(12)(34)} = \hat{R}_{23}[(P_A)_{12}(P_S)_{34} + (P_S)_{12}(P_A)_{34}]\hat{R}_{23}^{-1}, \\ (\mathbf{P}_S)_{(12)(34)} = \hat{R}_{23}(P_S)_{12}(P_S)_{34}\hat{R}_{23}^{-1}, \end{cases} \tag{53}$$

where $\mathbf{P}_1$, $\mathbf{P}_A$, $\mathbf{P}_S$ are the singlet, anti-symmetric (hexet) and symmetric (nonet) projective operators, respectively.

Then by virtue of the eigenvectors of $\hat{R}$ matrix we can transit to the vector representation. Taking

$$\begin{cases} K^0{}_{ab} = \bar{t}\,^0{}_{ab}, \\ K^m{}_{ab} = q\bar{t}\,^m{}_{ab}, \qquad m = +,3,- \end{cases} \tag{54}$$

and their inverse matrices

$$\begin{cases} (K^{-1})^{ab}{}_0 = t^{ab}{}_0, \\ (K^{-1})^{ab}{}_m = q^{-1}t^{ab}{}_m, \qquad m = +,3,- \end{cases} \tag{55}$$

we put

$$\hat{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} = K^\mu{}_{a\dot{a}}K^\nu{}_{b\dot{b}}\hat{\mathbf{R}}^{a\dot{a}b\dot{b}}{}_{c\dot{c}d\dot{d}}(K^{-1})^{c\dot{c}}{}_\kappa(K^{-1})^{d\dot{d}}{}_\lambda, \tag{56}$$

and similarly

$$(\mathcal{P}_i)^{\mu\nu}{}_{\kappa\lambda} = K^\mu{}_{a\dot{a}}K^\nu{}_{b\dot{b}}(\mathbf{P}_i)^{a\dot{a}b\dot{b}}{}_{c\dot{c}d\dot{d}}(K^{-1})^{c\dot{c}}{}_\kappa(K^{-1})^{d\dot{d}}{}_\lambda, \tag{57}$$

where $i = 1, A, S$. Those projective operators satisfy the conditions

$$\begin{cases} (\mathcal{P}_i)^2 = \mathcal{P}_i, & i = 1, A, S \\ \mathcal{P}_i\mathcal{P}_j = 0, & i \neq j = 1, A, S \\ \mathcal{P}_1 + \mathcal{P}_A + \mathcal{P}_S = E, & E: 16 \times 16 \text{ unit matrix} \end{cases} \tag{58}$$

and the decomposition of $\hat{\mathcal{R}}$ matrix is

$$\hat{\mathcal{R}} = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{-3}\mathcal{P}_1, \tag{59}$$

therefore it satisfies the equation

$$(\hat{\mathcal{R}} - q)(\hat{\mathcal{R}} + q^{-1})(\hat{\mathcal{R}} - q^{-3}) = 0. \tag{60}$$

Especially the singlet projective operator $\mathcal{P}_1$ is

$$(\mathcal{P}_1)^{\mu\nu}{}_{\kappa\lambda} = [2]^{-2} g^{\mu\nu} g_{\kappa\lambda}, \tag{61}$$

where $g_{\mu\nu}$ and $g^{\mu\nu}$ are metric and its inverse, respectively.

The Lorentz transformation matrix is

$$L^{\mu}{}_{\nu} = K^{\mu}{}_{a\dot{a}} M^{a}{}_{b} \bar{M}^{\dot{a}}{}_{\dot{b}} (K^{-1})^{b\dot{b}}{}_{\nu}, \tag{62}$$

where $M^a{}_b \in SL_q(2, C)$,

$$\bar{M} = (M^{\dagger})^{-1}, \tag{63}$$

and they obey the commutation relations

$$\begin{cases} \hat{R}_{12} M_1 M_2 = M_1 M_2 \hat{R}_{12}, \\ \hat{R}_{12} \bar{M}_1 \bar{M}_2 = \bar{M}_1 \bar{M}_2 \hat{R}_{12}, \\ \hat{R}_{12} M_1 \bar{M}_2 = \bar{M}_1 M_2 \hat{R}_{12}. \end{cases} \tag{64}$$

The auther is grateful to CCAST (World Lab), Beijing and ITP, Academia Sinica, Beijing and National Natural Science Foundation of China for their supports in this work which is completed in the " Workshop on Quantum Group and Field Theory in Low Dimensions" held on March 12 to April 10, 1992 and to Professors H. Y. Guo, X. C. Song, K. Wu and Dr. L. Liao for their helpful discussions.

REFERENCES

1. A. Connes and J. Lott, *Nucl. Phys.* (Proc. Suppl.)**18B** (1990)29.
2. M. Dubios-Violette, R. Kerner and J. Madore, *J. Math. Phys.* **31**(2)(1990)323.
3. D. Kastler, CPT-90/P2456 (France, Nov. 1990).
4. B. S. Balakrishna, F. Gürsey and K. C. Wali, *Phys. Lett.* **B254**(1991)430.
5. V. G. Drinfel'd, in "Proceedings ICM, Berkeley, 1986", pp.798–820.
6. L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, in " Algebraic Analysis", Academic Press, 1988, pp.129–140.
7. M. Jimbo, *Lett. Math. Phys.* **10**(1985)63; *Comm. Math. Phys.* **102**(1986)537.
8. S. L. Woronowicz, Publ. RIMS, Kyoto Univ. **23**(1987)117; *Comm. Math. Phys.* **111**(1987)613; **122**(1989)125.

9. Yu. I. Manin, "Quantum Groups and Non-commutative Geometry", Université de Montrél preprint, CRM-1561(1988); *Ann. Inst. Fourier*, **37**(1987)191; *Comm. Math. Phys.* **123**(1989)163.

10. S. L. Woronowicz and P. Podleś, Warsaw Univ. preprint (1989); Mittag-Leffler Ins. report 20(1988/89).

11. P. Podleś and S. L. Woronowicz, *Comm. Math. Phys.* **130**(1990)381.

12. U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, *Z. Phys.* **C48**(1990)159.

13. U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, *Int. J. Mod. Phys.* **A6**(1991)3081.

14. A. Schirrmacher, MPI-PTh/91-77 (Oct. 1991).

15. W. B. Schmidke, J. Wess and B. Zumino, MPI-Ph/91-15 (March 1991).

16. O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, MPI-Ph/91-51 (July 1991).

17. J. Wess and B. Zumino, *Nucl. Phys.* (Proc. Suppl.)**18B** (1990)302.

18. U. Carow-Watamura, M. Schlieker and S. Watamura, *Z. Phys.* **C49**(1991)439.

19. J. Birman and H. Wenzl, "Braids, Link Polynomials and a New Algebra", Columbia University Press, New York, 1989.

20. J. Murakami, *Osaka J. Math.* **24**(1987)745.

21. X. C. Song, CCAST 91-63 (Beijing, Nov. 1991), to appear in *Z. Phys.*C.

22. Z. Xu, IC/91/366 (ICTP Miramare-Trieste, Nov. 1991).

BIHEP-TH-92-21
CCAST-92-18

Generalized Jaynes-Cummings model with intensity-dependent coupling interacting with quantum group-theoretic coherent state

Zhe Chang

CCAST (World Laboratory)
P. O. Box 8730, Beijing 100080, China

Institute of High Energy Physics, Academia Sinica
P.O.Box 918(4), Beijing 100039, China [1]

Abstract

We present a generalized Jaynes-Cummings model with intensity-dependent coupling interacting with quantum group-theoretic coherent state. The deformed multiboson operators are used to give the Holstein-Primakoff realization of the $SU_q(1,1)$ quantum group. And the field operators of the JC Hamiltonian are identified as the elements of the $SU_q(1,1)$ quantum group. The $SU_q(2)$ and $SU_q(1,1)$ quantum group-theoretic coherent states are introduced and the squeezing properties of these quantum group-theoretic coherent states are investigated. The revivals of radiation squeezing are obtained for any values of initial squeezing.

[1]Mailing address.

I. INTRODUCTION

The Jaynes-Cummings model (JCM) [1] gives an idealized description of the interaction of matter with electromagnetic radiation by a simple Hamiltonian of a two-level atom coupled to a single bosonic mode. The time-evolution of the variances of the field quadratures for a squeezed vacuum state, described as an $SU(1,1)$ coherent state, interacting with a two-photon generalization of the JCM has been studied recently [2]. The revivals of radiation squeezing [3] have been found for any values of initial squeezing. Because the dynamics predicted by the JCM has been supported by the Rydberg maser experiments [4], there are intensive interests in the theoretical generalizations of the JCM [2,5,6]. It is of interest to mention that the recent developments of multiboson Holstein-Primakoff (HP) squeezed states [7] and quantum group symmetry [8—13] provide us a possibility to construct a more general JCM.

In this paper we present a generalized JCM with intensity-dependent coupling interacting with quantum group-theoretic coherent state. We take the JC Hamiltonian with intensity-dependent coupling constant. The field operators of this Hamiltonian are identified as the elements of $SU_q(1,1)$ quantum group in HP realization. We rely on quantum group-theoretic approach to define a new set of highly nontrivial generalized squeezed states which in suitable limits reproduce both the usual squeezed states and the HP squeezed states. From the quantum group-theoretic point of view these generalized squeezed states are connected with both $SU_q(2)$ and $SU_q(1,1)$ in their HP realizations. We study the time evolution of the field quadratures and show the revivals of squeezing in such a generalized JCM for any values of the initial squeezing.

This paper is organized as following: In Section II, we present a generalized realization of quantum Weyl-Heisenberg algebra. The deformed multiboson operators are introduced. In Section III the generalized JCM is discussed. Section IV is devoted to the definition and analysis of $SU_q(2)$ quantum group-theoretic coherent states. In Section V the time evolution of the generalized JCM with intensity-dependent coupling interacting with the $SU_q(2)$ coherent states is shown. Section VI contains the definition and analysis of $SU_q(1,1)$ quantum group-theoretic coherent states. In Section VII we present the time evolution of the generalized JCM with intensity-dependent coupling interacting with the $SU_q(1,1)$ coherent states. A summary and some further discussion are given in Section VIII.

II. DEFORMED MULTIBOSON OPERATORS AND QUANTUM WEYL-HEISENBERG ALGEBRA

The q-deformed oscillator are intensively investigated in the field of quantum group recently [14,10,11]. Its annihilation and creation operators a_q, $a_q^\dagger$ are connected with the annihilation and creation operators a, $a^\dagger$ of the usual harmonic oscillator in the following way

$$a_q = \sqrt{\frac{[N+1]}{N+1}}a, \qquad a_q^+ = a^+\sqrt{\frac{[N+1]}{N+1}}\,, \tag{1}$$

where $N = a^+a$ and $[x] = \dfrac{x^q - x^{-q}}{q - q^{-1}}$ (throughout this paper we limit us in the case of $q \in \mathcal{R}$). The deformed oscillator satisfies the quantum Weyl-Heisenberg algebra $H_q(4)$

$$\begin{aligned} [a_q, a_q^+] &= [N+1] - [N]\,, \\ [N, a_q] &= -a_q\,, \qquad [N, a_q^+] = a_q^+\,. \end{aligned} \tag{2}$$

The quantum Weyl-Heisenberg group is the simplest quantum group and it plays important rule in the study of the quantum group theory and its applications. Now, we give a more general realization of the quantum Weyl-Héisenberg group, the deformed k-boson realization. It should be shown that the above given realization of the quantum Weyl-Heisenberg algebra is a special case ($k = 1$) of the general form. First of all, we introduce the non-linear transformations

$$
\begin{aligned}
A^{\dagger}_{(k,q)} &\equiv \left(\frac{[N-k]!}{[N]!}\left[\left\langle\frac{N}{k}\right\rangle\right]\right)^{\frac{1}{2}}(a^{\dagger}_q)^k ,\\
A_{(k,q)} &\equiv a^k_q\left(\left[\left\langle\frac{N}{k}\right\rangle\right]\frac{[N-k]!}{[N]!}\right)^{\frac{1}{2}} ,\\
N_{(k,q)} &\equiv A^{\dagger}_{(k,q)}A_{(k,q)} ,
\end{aligned}
\tag{3}
$$

where k is positive integer, $[n]! = [n][n-1]\cdots[2][1]$ and $< x >$ is defined as the greatest integer less than or equal to x, functions of the number operator $N = a^{\dagger}a$ are only evaluated on eigenstates of N, and assume the value of the functions of the respective eigenvalues. It is not difficult to check that the above defined the deformed k-boson operators $A_{(k,q)}$ and $A^{\dagger}_{(k,q)}$ satisfy the quantum Weyl-Heisenberg group $H_q(4)$

$$
\begin{aligned}
\left[A_{(k,q)}, A^{\dagger}_{(k,q)}\right] &= \left[\left\langle\frac{N}{k}+1\right\rangle\right] - \left[\left\langle\frac{N}{k}\right\rangle\right] ,\\
\left[\left\langle\frac{N}{k}\right\rangle, A^{\dagger}_{(k,q)}\right] &= A^{\dagger}_{(k,q)} ,\\
\left[\left\langle\frac{N}{k}\right\rangle, A_{(k,q)}\right] &= -A_{(k,q)} .
\end{aligned}
\tag{4}
$$

The Hopf operations: co-multiplication, antipode and counit can be defined explicitly. The non-linear transformations $\{F_{(k)},\ k \text{ is integer}\}$, $F_{(k)} :\ a^{\dagger}_q \to F_{(k)}(a^{\dagger}_q) \equiv A^{\dagger}_{(k,q)}$ form a semigroup. If we envisage a situation in which we wish to compute general moments of quantities such as

$$
X_{(k',q)} \equiv \frac{1}{\sqrt{2}}\left(A_{(k',q)} + A^{\dagger}_{(k',q)}\right) , \quad P_{(k',q)} \equiv \frac{1}{\sqrt{2}}\left(A_{(k',q)} - A^{\dagger}_{(k',q)}\right) , \tag{5}
$$

in eigenstates of the number operator $N_{(k,q)}$, associated with generalized k' and k deformed bosons, respectively, then we are lead to consider the expectation $< km|(A^{\dagger}_{(k',q)})^u(A_{(k',q)})^v|km' >$, where $k,\ k',\ u,\ v,\ m,\ m'$ are positive integers. It is a straightforward excise to evaluate this expectation

$$< km|(A^\dagger_{(k',q)})^u(A_{(k',q)})^v|km' >= \left(\frac{\left[\left\langle \frac{km}{k'} \right\rangle\right]! \left[\left\langle \frac{km'}{k'} \right\rangle\right]!}{\left[\left\langle \frac{km}{k'} - u \right\rangle\right]! \left[\left\langle \frac{km'}{k'} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \delta_{m,m'+t} \;, \tag{6}$$

where we have defined $t \equiv \frac{k'}{k}(u-v)$. It should be noticed that when $u = v$, then $t = 0$ and the expectation (6) always has nonzero values (for $m = m'$). When $u \neq v$, (6) vanishes unless t is an integer. The expectation (6) depends only on k' and k through their ratio $r = \frac{k'}{k}$. Here r is the positive rational fraction of the fractional transformation $F_{(r)}$. We may equate $< km|(A^\dagger_{(k',q)})^u(A_{(k',q)})^v|km' >$ formally to an expectation involving fractional particles

$$< km|(A^\dagger_{(k',q)})^u(A_{(k',q)})^v|km' >=< m|(A^\dagger_{(r,q)})^u(A_{(r,q)})^v|m' > \;. \tag{7}$$

Notice the properties of the nonlinear transformation $F_{(k)}$

$$F_{(1)}(a^\dagger_q) = a^\dagger_q \;, \tag{8}$$

and

$$F_{(k)} \circ F_{(k')}(a^\dagger_q) = F_{(kk')}(a^\dagger_q) \;, \tag{9}$$

if we define the inverse transformation $F^{-1}_{(k)}$ as

$$F^{-1}_{(k)} \circ F_{(k)}(a^\dagger_q) = a^\dagger_q = F_{(1)}(a^\dagger_q) \;, \tag{10}$$

we may equate $F^{-1}_{(k)} = F_{(\frac{1}{k})}$. Similarly we have

$$F^{-1}_{(k)} \circ F_{(k')} = F_{(\frac{k'}{k})} = F_{(r)} \;, \tag{11}$$

where $r \equiv \frac{k'}{k}$ is a positive rational number. Now we have extended the semigroup of the nonlinear transformation $F_{(k)}$ to an Abelian group $\{F_{(k)} : \text{rational } k > 0\}$.

III. THE GENERALIZED JAYNES-CUMMINGS MODEL

The Hamiltonian of the system is of the form

$$H = \hbar\omega\sigma^z + \hbar\Omega a^\dagger a + g\left(\sigma^+ A_{(k,q)}\sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]} + \sigma^-\sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]}A^\dagger_{(k,q)}\right) . \tag{12}$$

We begin by separate the Hamiltonian into two commutative parts

$$H = H_1 + H_2 \ , \tag{13}$$

where

$$\begin{aligned} H_1 &= \hbar\Omega a^\dagger a + \hbar\Omega\sigma^z \ , \\ H_2 &= \hbar\Delta\Omega\sigma^z + g\left(\sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]}A^\dagger_{(k,q)}\sigma^- + A_{(k,q)}\sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]}\sigma^+\right) \ , \end{aligned} \tag{14}$$

with $\Delta\Omega = \omega - \Omega$. The HP realization of the $SU_q(1,1)$ quantum algebra are identified as

$$K_+^{(\frac{1}{2})} = \sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]}A^\dagger_{(k,q)} \ , \quad K_-^{(\frac{1}{2})} = A_{(k,q)}\sqrt{\left[\left\langle\frac{N}{k}\right\rangle\right]} \ , \quad K_0^{(\frac{1}{2})} = \left\langle\frac{N}{k}\right\rangle + \frac{1}{2} \ , \tag{15}$$

with the commutation relations

$$\left[K_0^{(\frac{1}{2})}, K_\pm^{(\frac{1}{2})}\right] = \pm K_\pm^{(\frac{1}{2})}, \qquad \left[K_-^{(\frac{1}{2})}, K_+^{(\frac{1}{2})}\right] = \left[2K_0^{(\frac{1}{2})}\right] . \tag{16}$$

And then the interacting Hamiltonian can be rewritten as

$$H_2 = \hbar\Delta\Omega\sigma^z + g\left(K_+^{(\frac{1}{2})}\sigma^- + K_-^{(\frac{1}{2})}\sigma^+\right) . \tag{17}$$

Hence the development operator factors

$$\begin{aligned} U(t,0) &= \exp\left(-iHt/\hbar\right) \\ &= \exp\left(-iH_1t/\hbar\right)\cdot\exp\left(-iH_2t/\hbar\right) . \end{aligned} \tag{18}$$

The first part is easily diagonalized

$$
\begin{aligned}
U_1 &= \exp\left(-iH_1 t/\hbar\right) \\
&= \exp\left(-i\Omega a^+ a t\right) \begin{pmatrix} \exp\left(-i\Omega t/2\right) & 0 \\ 0 & \exp\left(i\Omega t/2\right) \end{pmatrix} .
\end{aligned}
\tag{19}
$$

The second one equals

$$
\begin{aligned}
U_2 &= \exp\left(-iH_2 t/\hbar\right) \\
&= \begin{pmatrix} \cos\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) & -i\Lambda \dfrac{\sin\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}} K_-^{(\frac{1}{2})} \\ -i\Lambda K_+^{(\frac{1}{2})} \dfrac{\sin\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}} & \cos\left(t\sqrt{\Lambda^2 K_+^{(\frac{1}{2})} K_-^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) \end{pmatrix} \\
&+ \begin{pmatrix} -i\frac{1}{2}\Delta\Omega \dfrac{\sin\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}} & 0 \\ 0 & i\frac{1}{2}\Delta\Omega \dfrac{\sin\left(t\sqrt{\Lambda^2 K_+^{(\frac{1}{2})} K_-^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2 K_+^{(\frac{1}{2})} K_-^{(\frac{1}{2})} + \left(\frac{1}{2}\Delta\Omega\right)^2}} \end{pmatrix} ,
\end{aligned}
\tag{20}
$$

with $\Lambda = \frac{g}{\hbar}$. In the case of resonance, i.e., $\Delta\Omega = 0$,

$$
U_2 = \begin{pmatrix} \cos\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})}}\right) & -i\Lambda \dfrac{\sin\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})}}\right)}{\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})}}} K_-^{(\frac{1}{2})} \\ -i\Lambda K_+^{(\frac{1}{2})} \dfrac{\sin\left(t\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})}}\right)}{\sqrt{\Lambda^2 K_-^{(\frac{1}{2})} K_+^{(\frac{1}{2})}}} & \cos\left(t\sqrt{\Lambda^2 K_+^{(\frac{1}{2})} K_-^{(\frac{1}{2})}}\right) \end{pmatrix} .
\tag{21}
$$

IV. $SU_q(2)$ QUANTUM GROUP-THEORETIC COHERENT STATES

The HP relations form a realization of the quantum algebra $SU_q(2)$

$$\begin{aligned} K_+^{(\sigma)} &= A^\dagger_{(k,q)}\left[2\sigma - \left\langle\frac{N}{k}\right\rangle\right]^{\frac{1}{2}} , \\ K_-^{(\sigma)} &= \left[2\sigma - \left\langle\frac{N}{k}\right\rangle\right]^{\frac{1}{2}} A_{(k,q)} , \\ K_3^{(\sigma)} &= \left\langle\frac{N}{k}\right\rangle - \sigma . \end{aligned} \tag{22}$$

The generators $K_\pm^{(\sigma)}$ and $K_3^{(\sigma)}$ satisfy the well known quantum algebra relations

$$[K_+^{(\sigma)}, K_-^{(\sigma)}] = [2K_3^{(\sigma)}] , \quad [K_3^{(\sigma)}, K_\pm^{(\sigma)}] = \pm K_\pm^{(\sigma)} . \tag{23}$$

The $(2\sigma+1)$-dimensional representations of the quantum algebra $SU_q(2)$ is spanned by the states $|0>$, $|k>$, $\cdots$, $|2\sigma k>$, where the normalized bosonic states $|n>$ is defined as

$$|n> = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0> . \tag{24}$$

The set of coherent states for a quantum algebra G can be defined using a unitary irreducible representation of the quantum algebra, choosing a fixed vector $|\omega>$ in the representation space, and acting on it by the whole algebra. It turns out that the coherent states are labeled by means of the left cosets of the quantum algebra G with respect to the subalgebra leaving $|\omega>$ invariant up to a phase factor. For the quantum algebra $SU_q(2)$ the coherent states within the $(2\sigma+1)$ dimensional representation is given by the formula

$$|\sigma, k, \xi> \equiv \frac{1}{D}\exp\left(\xi K_+^{(\sigma)}\right)|0> , \tag{25}$$

where $\xi \in C$ and D is the renormalization coefficient. Expanding the exponential we obtained

$$
\begin{aligned}
|\sigma, k, \xi > &= \frac{1}{D} \sum_{l=0}^{2\sigma} \frac{\xi^l}{l!} (K_+^{(\sigma)})^l |0> \\
&= \frac{1}{D} \sum_{l=0}^{2\sigma} \frac{\xi^l}{l!} \left(\frac{[2\sigma]![l]!}{[2\sigma - l]!} \right)^{\frac{1}{2}} |kl> \ .
\end{aligned}
\tag{26}
$$

This is the general expression for the quantum group-theoretic coherent states of $SU_q(2)$. The inner product for the coherent state is

$$
< \sigma, k, \xi | \sigma, k, \xi > = \frac{1}{D^2} \sum_{l=0}^{2\sigma} \frac{|\xi|^{2l}}{(l!)^2} \frac{[2\sigma]![l]!}{[2\sigma - l]!} \ . \tag{27}
$$

It follows that

$$
D = \frac{1}{\sqrt{\sum_{l=0}^{2\sigma} \frac{|\xi|^{2l}}{(l!)^2} \frac{[2\sigma]![l]!}{[2\sigma - l]!}}} \ . \tag{28}
$$

All required moments may be obtained from evaluation of the expectation

$$
\begin{aligned}
&< \sigma, k, \xi | \left(A^\dagger_{(k',q)}\right)^u \left(A_{(k',q)}\right)^v |\sigma, k, \xi > \\
&= \frac{1}{D^2} \sum_{m,m'=0}^{2\sigma} \frac{\bar{\xi}^m \xi^{m'}}{m! m'!} < km| \left(A^\dagger_{(k',q)}\right)^u \left(A_{(k',q)}\right)^v |km'> \\
&\qquad \times \left(\frac{[2\sigma]![m]!}{[2\sigma - m]!} \frac{[2\sigma]![m']!}{[2\sigma - m']!} \right)^{\frac{1}{2}} \\
&= \frac{1}{D^2} \bar{\xi}^t \sum_{m=0}^{2\sigma - t} \frac{|\xi|^{2m}}{(m+t)! m!} \left(\frac{\left[\left\langle \frac{m+t}{r} \right\rangle\right]! \left[\left\langle \frac{m}{r} \right\rangle\right]!}{\left[\left\langle \frac{m+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \\
&\qquad \times \left(\frac{[2\sigma]![m+t]!}{[2\sigma - m - t]!} \frac{[2\sigma]![m]!}{[2\sigma - m]!} \right)^{\frac{1}{2}} \ .
\end{aligned}
\tag{29}
$$

To evaluate $\Delta X_{(k',q)}$, $\Delta P_{(k',q)}$ we use the expressions

$$
\begin{aligned}
\left(\Delta X_{(k',q)}\right)^2 &= \frac{\left[\left\langle\frac{N}{k'}\right\rangle+1\right]-\left[\left\langle\frac{N}{k'}\right\rangle\right]}{2}+\left\langle A^\dagger_{(k',q)}A_{(k',q)}\right\rangle-\left|< A^\dagger_{(k',q)} >\right|^2 \\
&\quad +\mathrm{Re}\left(\left\langle\left(A^\dagger_{(k',q)}\right)^2\right\rangle-\left\langle A^\dagger_{(k',q)}\right\rangle^2\right) , \\
\left(\Delta P_{(k',q)}\right)^2 &= \frac{\left[\left\langle\frac{N}{k'}+1\right\rangle\right]-\left[\left\langle\frac{N}{k'}\right\rangle\right]}{2}+\left\langle A^\dagger_{(k',q)}A_{(k',q)}\right\rangle-\left|\left\langle A^\dagger_{(k',q)}\right\rangle\right|^2 \\
&\quad -\mathrm{Re}\left(\left\langle\left(A^\dagger_{(k',q)}\right)^2\right\rangle-\left\langle A^\dagger_{(k',q)}\right\rangle^2\right) ,
\end{aligned}
\tag{30}
$$

where all expectations are taken with respect to the quantum group-theoretic coherent states of $SU_q(2)$, $|\sigma, k, \xi>$.

From the general form of expectations we can write down the necessary moments for evaluating of the variances $\Delta X_{(k',q)}$ and $\Delta P_{(k',q)}$ easily

$$
\begin{aligned}
< A^\dagger_{(k',q)} > &= \frac{1}{D^2}\bar{\xi}^r\sum_{m=0}^{2\sigma-r}\frac{|\xi|^{2m}}{(m+r)!m!}\left(\frac{[2\sigma]![m+r]!}{[2\sigma-m-r]!}\frac{[2\sigma]![m]!}{[2\sigma-m]!}\right)^{\frac{1}{2}} \\
&\quad \times\sqrt{\left[\left\langle\frac{m}{r}+1\right\rangle\right]} \qquad (\text{if } r = \text{integer} \leq 2\sigma) , \\
&= 0 \quad (\text{otherwise}) ; \\
< A^\dagger_{(k',q)}A_{(k',q)} > &= \frac{1}{D^2}\sum_{m=0}^{2\sigma}\frac{|\xi|^{2m}}{(m!)^2}\frac{[2\sigma]![m]!}{[2\sigma-m]!}\left[\left\langle\frac{m}{r}\right\rangle\right] , \\
< (A^\dagger_{(k',q)})^2 > &= \frac{1}{D^2}\bar{\xi}^{2r}\sum_{m=0}^{2\sigma-2r}\frac{|\xi|^{2m}}{(m+2r)!m!}\left(\frac{[2\sigma]!}{[2\sigma-m-2r+1]![m+2r]!}\frac{[2\sigma]!}{[2\sigma-m+1]![m]!}\right) \\
&\quad \times\sqrt{\left[\left\langle\frac{m}{r}+2\right\rangle\right]\left[\left\langle\frac{m}{r}+1\right\rangle\right]} \quad (\text{if } 2r = \text{integer} \leq 2\sigma) , \\
&= 0 \quad (\text{otherwise}) .
\end{aligned}
\tag{31}
$$

V. TIME EVOLUTION OF THE GENERALIZED JCM INTERACTING WITH $SU_q(2)$ COHERENT STATE

We assume the initial state of the field to be an $SU_q(2)$ quantum group-theoretic coherent state. If the atom is supposed to be in the ground state $|- >$ at the initial moment ($t = 0$), then the initial state vector $|\psi(t=0)>$ of the system can be written as

$$\begin{aligned} |\psi(t=0)> &= |\sigma, k, \xi> \otimes |-> \\ &= \frac{1}{D} \sum_{l=0}^{2\sigma} \frac{\xi^l}{l!} \left(\frac{[2\sigma]![l]!}{[2\sigma - l]!} \right)^{\frac{1}{2}} |-, kl> . \end{aligned} \tag{32}$$

From the time evolution operator U given in Eqs. (19) and (20), we can write down the state vector $|\psi(t)>$

$$\begin{aligned} |\psi(t)> &= \frac{1}{D} \sum_{l=0}^{2\sigma} \exp\left(-i\left(-\frac{1}{2}+kl\right)\Omega t\right) \frac{\xi^l}{l!} \left(\frac{[2\sigma]![l]!}{[2\sigma - l]!}\right)^{\frac{1}{2}} \\ &\times \left(\frac{-i\Lambda[l]}{\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}} \sin\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) |+, k(l-1)> \right. \\ &+ \left. \left(\cos\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) + \frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}} \sin\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)\right) \right. \end{aligned} \tag{33}$$

To see how the system under consideration evolves we calculate first of all the atomic population inversion

$$\begin{aligned} <\sigma^z(t)> &= \frac{1}{D^2} \sum_{l=0}^{2\sigma} \frac{|\xi|^{2l}}{(l!)^2} \frac{[2\sigma]![l]!}{[2\sigma - l]!} \\ &\times \left(\frac{-\Lambda^2[l]^2}{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2} \sin^2\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) |+, k(l-1)> \right. \\ &+ \left. \left(\cos\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) + \frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}} \sin\left(t\sqrt{\Lambda^2[l]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) \right. \right. \end{aligned} \tag{34}$$

In the case of resonance, i.e., $\omega = \Omega$

$$< \sigma^z(t) >= \frac{1}{D^2}\sum_{l=0}^{2\sigma}\frac{|\xi|^{2l}}{(l!)^2}\frac{[2\sigma]![l]!}{[2\sigma-l]!}\cos\left(2\Lambda t[l]\right) \ . \tag{35}$$

The time evolution of the general expectation is of the form

$$\begin{aligned}
&\left\langle \left(A^{\dagger}_{(k',q)}(t)\right)^{u}\left(A_{(k',q)}(t)\right)^{v}\right\rangle \\
&= \frac{1}{D^2}\bar{\xi}^t\sum_{m=0}^{2\sigma-t}\frac{|\xi|^{2m}}{(m+t)!m!}\left(\frac{[2\sigma]![m+t]!}{[2\sigma-m-t]!}\frac{[2\sigma]![m]!}{[2\sigma-m]!}\right)^{\frac{1}{2}} \\
&\quad\times\left(-\left(\frac{\left[\left\langle\frac{m-1+t}{r}\right\rangle\right]!\left[\left\langle\frac{m-1}{r}\right\rangle\right]!}{\left[\left\langle\frac{m-1+t}{r}-u\right\rangle\right]!\left[\left\langle\frac{m-1}{r}-v\right\rangle\right]!}\right)^{\frac{1}{2}}\right. \\
&\quad\quad\times\frac{\Lambda^2[m+t][m]\sin\left(t\sqrt{\Lambda^2[m+t]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)\sin\left(t\sqrt{\Lambda^2[m]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2[m+t]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\sqrt{\Lambda^2[m]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}} \\
&\quad\quad+\left(\frac{\left[\left\langle\frac{m+t}{r}\right\rangle\right]!\left[\left\langle\frac{m}{r}\right\rangle\right]!}{\left[\left\langle\frac{m+t}{r}-u\right\rangle\right]!\left[\left\langle\frac{m}{r}-v\right\rangle\right]!}\right)^{\frac{1}{2}} \\
&\quad\times\left(\cos\left(t\sqrt{\Lambda^2[m+t]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)+\frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[m+t]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}}\sin\left(t\sqrt{\Lambda^2[m+t]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)\right) \\
&\quad\left.\times\left(\cos\left(t\sqrt{\Lambda^2[m]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)+\frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[m]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}}\sin\left(t\sqrt{\Lambda^2[m]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)\right)\right) \ .
\end{aligned} \tag{36}$$

In the case of resonance, i.e., $\omega = \Omega$, we have

$$\left\langle \left(A^{\dagger}_{(k',q)}(t)\right)^{u} \left(A_{(k',q)}(t)\right)^{v} \right\rangle$$

$$= \frac{1}{D^2}\bar{\xi}^{t} \sum_{m=0}^{2\sigma-t} \frac{|\xi|^{2m}}{(m+t)!m!} \left(\frac{[2\sigma]![m+t]!}{[2\sigma-m-t]!} \frac{[2\sigma]![m]!}{[2\sigma-m]!} \right)^{\frac{1}{2}}$$

$$\times \left(- \left(\frac{\left[\left\langle \frac{m-1+t}{r} \right\rangle\right]! \left[\left\langle \frac{m-1}{r} \right\rangle\right]!}{\left[\left\langle \frac{m-1+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m-1}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \sin\left(t\Lambda[m+t]\right) \sin\left(t\Lambda[m]\right) \right.$$

$$\left. + \left(\frac{\left[\left\langle \frac{m+t}{r} \right\rangle\right]! \left[\left\langle \frac{m}{r} \right\rangle\right]!}{\left[\left\langle \frac{m+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \cos\left(t\Lambda[m+t]\right) \cos\left(t\Lambda[m]\right) \right) . \tag{37}$$

From the above evolution formulas, we can easily obtain the evolutions of necessary moments for evaluating of the variances $\Delta X_{(k',q)}$ and $\Delta P_{(k',q)}$

$$\left\langle A^{\dagger}_{(k',q)}(t) \right\rangle = \frac{1}{D^2}\bar{\xi}^{r} \sum_{m=0}^{2\sigma-r} \frac{|\xi|^{2m}}{(m+r)!m!} \left(\frac{[2\sigma]![m+r]!}{[2\sigma-m-r]!} \frac{[2\sigma]![m]!}{[2\sigma-m]!} \right)^{\frac{1}{2}}$$

$$\times \left(- \left(\left[\left\langle \frac{m-1}{r} + 1 \right\rangle\right] \right)^{\frac{1}{2}} \sin\left(t\Lambda[m+r]\right) \sin\left(t\Lambda[m]\right) \right. \tag{38}$$

$$\left. + \left(\left[\left\langle \frac{m}{r} + 1 \right\rangle\right] \right)^{\frac{1}{2}} \cos\left(t\Lambda[m+r]\right) \cos\left(t\Lambda[m]\right) \right)$$

$$(\text{if } r = \text{integer} \le 2\sigma) ,$$

$$= 0 \quad (\text{otherwise}) ;$$

$$\left\langle A^{\dagger}_{(k',q)}(t) A_{(k',q)}(t) \right\rangle = \frac{1}{D^2} \sum_{m=0}^{2\sigma} \frac{|\xi|^{2m}}{(m!)^2} \frac{[2\sigma]![m]!}{[2\sigma-m]!}$$

$$\times \left(- \left[\left\langle \frac{m-1}{r} \right\rangle\right] \sin^2\left(t\Lambda[m]\right) + \left[\left\langle \frac{m}{r} \right\rangle\right] \cos^2\left(t\Lambda[m]\right) \right) ; \tag{39}$$

$$
\begin{aligned}
\left\langle \left(A^{\dagger}_{(k',q)}(t) \right)^2 \right\rangle &= \frac{1}{D^2} \bar{\xi}^{2r} \sum_{m=0}^{2\sigma-2r} \frac{|\xi|^{2m}}{(m+2r)!m!} \left(\frac{[2\sigma]![m+2r]!}{[2\sigma-m-2r]!} \frac{[2\sigma]![m]!}{[2\sigma-m]!} \right)^{\frac{1}{2}} \\
&\times \left(- \left(\left[\left\langle \frac{m-1}{r} + 2 \right\rangle \right] \left[\left\langle \frac{m-1}{r} + 1 \right\rangle \right] \right)^{\frac{1}{2}} \sin(t\Lambda[m+r]) \sin(t\Lambda[m]) \right. \\
&\left. + \left(\left[\left\langle \frac{m}{r} + 1 \right\rangle \right] \left[\left\langle \frac{m}{r} + 2 \right\rangle \right] \right)^{\frac{1}{2}} \cos(t\Lambda[m+r]) \cos(t\Lambda[m]) \right) \\
&\qquad \text{(if } 2r = \text{integer} \le 2\sigma) , \\
&= 0 \quad \text{(otherwise)} .
\end{aligned} \tag{40}
$$

VI. $SU_q(1,1)$ QUANTUM GROUP-THEORETIC COHERENT STATES

Using the deformed k-boson operators, we can realize the quantum group $SU_q(1,1)$ by the following HP relations

$$
\begin{aligned}
K_+^{(\sigma)} &= \left[2\sigma - 1 + \left\langle \frac{N}{k} \right\rangle \right]^{\frac{1}{2}} A^{\dagger}_{(k,q)} , \\
K_-^{(\sigma)} &= A_{(k,q)} \left[2\sigma - 1 + \left\langle \frac{N}{k} \right\rangle \right]^{\frac{1}{2}} , \\
K_3^{(\sigma)} &= \sigma + \left\langle \frac{N}{k} \right\rangle .
\end{aligned} \tag{41}
$$

The operators $K_\pm^{(\sigma)}$ and $K_3^{(\sigma)}$ satisfy the commutation relations of quantum group $SU_q(1,1)$

$$
[K_3^{(\sigma)}, K_\pm^{(\sigma)}] = \pm K_\pm^{(\sigma)} , \qquad [K_+^{(\sigma)}, K_-^{(\sigma)}] = -[2K_3^{(\sigma)}] . \tag{42}
$$

The representation of the $SU_1(1,1)$ quantum algebra is spanned by the states $|0>$, $|k>$, $|2k>$, $\cdots$. Note that the representation labeled by σ are now infinite dimensional.

The quantum group-theoretic coherent states of $SU_q(1,1)$ are defined as

$$|\sigma, k, \alpha> \equiv \frac{1}{Q} \exp\left(\alpha K_+^{(\sigma)}\right) |0> , \tag{43}$$

where $\alpha \in C$ and Q is the renormalization coefficient. Expanding the exponential we obtained

$$\begin{aligned} |\sigma, k, \alpha> &= \frac{1}{Q} \sum_{l=0}^{\infty} \frac{\alpha^l}{l!} (K_+^{(\sigma)})^l |0> \\ &= \frac{1}{Q} \sum_{l=0}^{\infty} \frac{\alpha^l}{l!} \left(\frac{[2\sigma + l - 1]![l]!}{[2\sigma - 1]!} \right)^{\frac{1}{2}} |kl> . \end{aligned} \tag{44}$$

This is the general expression for the $SU_q(1,1)$ quantum group-theoretic coherent states. The inner product for the coherent state is

$$< \sigma, k, \alpha | \sigma, k, \alpha > = \frac{1}{Q^2} \sum_{l=0}^{\infty} \frac{|\alpha|^{2l}}{(l!)^2} \frac{[2\sigma + l - 1]![l]!}{[2\sigma - 1]!} . \tag{45}$$

It follows that

$$Q = \frac{1}{\sqrt{\sum_{l=0}^{\infty} \frac{|\alpha|^{2l}}{(l!)^2} \frac{[2\sigma + l - 1]![l]!}{[2\sigma - 1]!}}} . \tag{46}$$

All required moments may be obtained from evaluation of the general expectation $< \sigma, k, \alpha | \left(A^\dagger_{(k',q)}\right)^u \left(A_{(k',q)}\right)^v |\sigma, k, \alpha >$

$$\begin{aligned} &= \frac{1}{Q^2} \sum_{m,m'=0}^{\infty} \frac{\bar{\alpha}^m \alpha^{m'}}{m! m'!} < km | \left(A^\dagger_{(k',q)}\right)^u \left(A_{(k',q)}\right)^v |km' > \\ &\quad \times \left(\frac{[2\sigma + m - 1]![m]!}{[2\sigma - 1]!} \frac{[2\sigma + m' - 1]![m']!}{[2\sigma - 1]} \right)^{\frac{1}{2}} \\ &= \frac{1}{Q^2} \bar{\alpha}^t \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m+t)! m!} \left(\frac{\left[\left\langle \frac{m+t}{r} \right\rangle\right]! \left[\left\langle \frac{m}{r} \right\rangle\right]!}{\left[\left\langle \frac{m+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \\ &\quad \times \left(\frac{[2\sigma + m + t - 1]![m+t]!}{[2\sigma - 1]!} \frac{[2\sigma + m - 1]![m]!}{[2\sigma - 1]!} \right)^{\frac{1}{2}} . \end{aligned} \tag{47}$$

From the general form of expectations we can obtain the moments which is necessary to evaluate the variances $\Delta X_{(k',q)}$ and $\Delta P_{(k',q)}$

$$
\begin{aligned}
< A^{\dagger}_{(k',q)} > &= \frac{1}{Q^2}\bar{\alpha}^r \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m+r)!m!} \left(\frac{[2\sigma+m+r-1]![m+r]!}{[2\sigma-1]!} \frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!} \right)^{\frac{1}{2}} \\
&\quad \times \sqrt{\left[\left\langle \frac{m}{r}+1 \right\rangle\right]} \qquad \text{(if } r = \text{integer)} \,, \\
&= 0 \quad \text{(otherwise)} \,; \\
< A^{\dagger}_{(k',q)} A_{(k',q)} > &= \frac{1}{Q^2} \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m!)^2} \frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!} \left[\left\langle \frac{m}{r} \right\rangle\right] \,, \\
< (A^{\dagger}_{(k',q)})^2 > &= \frac{1}{Q^2}\bar{\alpha}^{2r} \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m+2r)!m!} \left(\frac{[2\sigma+m+2r-1]![m+2r]!}{[2\sigma-1]!} \frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!} \right)^{\frac{1}{2}} \\
&\quad \times \sqrt{\left[\left\langle \frac{m}{r}+2 \right\rangle\right] \left[\left\langle \frac{m}{r}+1 \right\rangle\right]} \qquad \text{(if } 2r = \text{integer)} \,, \\
&= 0 \quad \text{(otherwise)} \,.
\end{aligned}
\tag{48}
$$

VII. TIME EVOLUTION OF THE GENERALIZED JCM INTERACTING WITH $SU_q(1,1)$ COHERENT STATES

As in the case of $SU_q(2)$, we assume the initial state of the field to be an $SU_q(1,1)$ quantum group-theoretic coherent state. If the atom is supposed to be in the ground state $|->$ at the initial moment ($t=0$), then the initial state vector $|\psi(t=0)>$ of the system can be written as

$$
\begin{aligned}
|\psi(t=0)> &= |\sigma, k, \alpha> \otimes |-> \\
&= \frac{1}{Q}\sum_{l=0}^{\infty} \frac{\alpha^l}{l!} \left(\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!} \right)^{\frac{1}{2}} |-, kl> \,.
\end{aligned}
\tag{49}
$$

From the time evolution operator U given in Eqs. (19) and (20), we can write down the state vector $|\psi(t)>$

$$
\begin{aligned}
|\psi(t)> &= \frac{1}{Q}\sum_{l=0}^{\infty}\exp\left(-i\left(-\frac{1}{2}+ki\right)\Omega t\right)\frac{\alpha^l}{l!}\left(\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!}\right)^{\frac{1}{2}} \\
&\times\left(\frac{-i\Lambda[l]}{\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}}\sin\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)|+,k(l-1)>\right. \\
&+\left.\left(\cos\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)+\frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}}\sin\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)\right)|-,kl>\right).
\end{aligned}
\tag{50}
$$

To see how the system under consideration evolves we calculate first of all the atomic population inversion

$$
\begin{aligned}
<\sigma^z(t)> &= \frac{1}{Q^2}\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l!)^2}\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!} \\
&\times\left(\frac{-\Lambda^2[l]^2}{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\sin^2\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)|+,k(l-1)>\right. \\
&+\left.\left(\cos\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)+\frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}}\sin\left(t\sqrt{\Lambda^2[l]^2+\left(\frac{1}{2}\Delta\Omega\right)^2}\right)\right)^2|-,kl>\right)
\end{aligned}
\tag{51}
$$

In the case of resonance, i.e., $\omega=\Omega$

$$
<\sigma^z(t)>=\frac{1}{Q^2}\sum_{l=0}^{\infty}\frac{|\alpha|^{2l}}{(l!)^2}\frac{[2\sigma+l-1]![l]!}{[2\sigma-1]!}\cos(2\Lambda t[l]) \tag{52}
$$

As in the case of $SU_q(2)$, to show the time evolutions of the squeezing properties of the interacting system, we should calculate the expectations $<A^{\dagger}_{(k',q)}(t)>$, $<A^{\dagger}_{(k',q)}(t)A_{(k',q)}(t)>$ and $<(A^{\dagger}_{(k',q)}(t))^2>$. The time evolution of the general expectation is of the form

$$\left\langle \left(A^{\dagger}_{(k',q)}(t)\right)^{u} \left(A_{(k',q)}(t)\right)^{v} \right\rangle$$

$$= \frac{1}{Q^2}\bar{\alpha}^t \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m+t)!m!} \left(\frac{[2\sigma+m+t-1]![m+t]!}{[2\sigma-1]!} \frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!} \right)^{\frac{1}{2}}$$

$$\times \left(- \left(\frac{\left[\left\langle \frac{m-1+t}{r} \right\rangle\right]! \left[\left\langle \frac{m-1}{r} \right\rangle\right]!}{\left[\left\langle \frac{m-1+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m-1}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}} \right.$$

$$\times \frac{\Lambda^2[m+t][m] \sin\left(t\sqrt{\Lambda^2[m+t]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) \sin\left(t\sqrt{\Lambda^2[m]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right)}{\sqrt{\Lambda^2[m+t]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\sqrt{\Lambda^2[m]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}}$$

$$+ \left(\frac{\left[\left\langle \frac{m+t}{r} \right\rangle\right]! \left[\left\langle \frac{m}{r} \right\rangle\right]!}{\left[\left\langle \frac{m+t}{r} - u \right\rangle\right]! \left[\left\langle \frac{m}{r} - v \right\rangle\right]!} \right)^{\frac{1}{2}}$$

$$\times \left(\cos\left(t\sqrt{\Lambda^2[m+t]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) + \frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[m+t]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}} \sin\left(t\sqrt{\Lambda^2[m+t]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) \right)$$

$$\left. \times \left(\cos\left(t\sqrt{\Lambda^2[m]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) + \frac{\frac{1}{2}\Delta\Omega}{\sqrt{\Lambda^2[m]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}} \sin\left(t\sqrt{\Lambda^2[m]^2 + \left(\frac{1}{2}\Delta\Omega\right)^2}\right) \right) \right) . \tag{53}$$

In the case of resonance, i.e., $\omega = \Omega$, we have

$$\left\langle \left(A^{\dagger}_{(k',q)}(t)\right)^{u}\left(A_{(k',q)}(t)\right)^{v}\right\rangle$$

$$= \frac{1}{Q^2}\bar{\alpha}^{t}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m+t)!m!}\left(\frac{[2\sigma+m+t-1]![m+t]!}{[2\sigma-1]!}\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right)^{\frac{1}{2}}$$

$$\times\left(-\left(\frac{\left[\left\langle\frac{m-1+t}{r}\right\rangle\right]!\left[\left\langle\frac{m-1}{r}\right\rangle\right]!}{\left[\left\langle\frac{m-1+t}{r}-u\right\rangle\right]!\left[\left\langle\frac{m-1}{r}-v\right\rangle\right]!}\right)^{\frac{1}{2}}\sin\left(t\Lambda[m+t]\right)\sin\left(t\Lambda[m]\right)\right.$$

$$\left.+\left(\frac{\left[\left\langle\frac{m+t}{r}\right\rangle\right]!\left[\left\langle\frac{m}{r}\right\rangle\right]!}{\left[\left\langle\frac{m+t}{r}-u\right\rangle\right]!\left[\left\langle\frac{m}{r}-v\right\rangle\right]!}\right)^{\frac{1}{2}}\cos\left(t\Lambda[m+t]\right)\cos\left(t\Lambda[m]\right)\right) . \qquad (54)$$

From the above evolution formula, we can easily obtain the evolutions of necessary moments for evaluating of the variances $\Delta X_{(k',q)}$ and $\Delta P_{(k',q)}$

$$\left\langle A^{\dagger}_{(k',q)}(t)\right\rangle = \frac{1}{Q^2}\bar{\alpha}^{r}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m+r)!m!}\left(\frac{[2\sigma+m+r-1]![m+r]!}{[2\sigma-1]!}\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}\right)^{\frac{1}{2}}$$

$$\times\left(-\left(\left[\left\langle\frac{m-1}{r}+1\right\rangle\right]\right)^{\frac{1}{2}}\sin\left(t\Lambda[m+r]\right)\sin\left(t\Lambda[m]\right)\right.$$

$$\left.+\left(\left[\left\langle\frac{m}{r}+1\right\rangle\right]\right)^{\frac{1}{2}}\cos\left(t\Lambda[m+r]\right)\cos\left(t\Lambda[m]\right)\right)$$

$$(\text{if } r = \text{integer}) ,$$

$$= 0 \quad (\text{otherwise}) ; \qquad (55)$$

$$\left\langle A^{\dagger}_{(k',q)}(t)A_{(k',q)}(t)\right\rangle = \frac{1}{Q^2}\sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{(m!)^2}\frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!}$$

$$\times\left(-\left[\left\langle\frac{m-1}{r}\right\rangle\right]\sin^2\left(t\Lambda[m]\right)+\left[\left\langle\frac{m}{r}\right\rangle\right]\cos^2\left(t\Lambda[m]\right)\right) ; \qquad (56)$$

$$\left\langle \left(A^{\dagger}_{(k',q)}(t)\right)^2 \right\rangle = \frac{1}{Q^2}\bar{\alpha}^{2r} \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{(m+2r)!m!} \left(\frac{[2\sigma+m+2r-1]![m+2r]!}{[2\sigma-1]!} \frac{[2\sigma+m-1]![m]!}{[2\sigma-1]!} \right)^{\frac{1}{2}}$$

$$\times \left(- \left(\left[\left\langle \frac{m-1}{r} + 2 \right\rangle \right] \left[\left\langle \frac{m-1}{r} + 1 \right\rangle \right] \right)^{\frac{1}{2}} \sin\left(t\Lambda[m+r]\right) \sin\left(t\Lambda[m]\right) \right.$$

$$\left. + \left(\left[\left\langle \frac{m}{r} + 1 \right\rangle \right] \left[\left\langle \frac{m}{r} + 2 \right\rangle \right] \right)^{\frac{1}{2}} \cos\left(t\Lambda[m+r]\right) \cos\left(t\Lambda[m]\right) \right)$$

$$(\text{if } 2r = \text{integer}) \,,$$

$$= 0 \quad (\text{otherwise}) \,. \tag{57}$$

VIII. CONCLUSIONS AND REMARKS

In this paper we have present a generalized JCM with intensity-dependent coupling interacting with quantum group-theoretic coherent state. We have taken the JC Hamiltonian with intensity-dependent coupling constant. The field operators of this Hamiltonian were identified as the elements of $SU_q(1,1)$ quantum group in HP realization. We rely on quantum group-theoretic approach defined a new set of highly nontrivial generalized squeezed states. From the quantum group-theoretic point of view these generalized squeezed states are connected with both $SU_q(2)$ and $SU_q(1,1)$ in their HP realizations. To see clearly the squeezing properties of these generalized squeezed states, we can investigate the limiting form of some results presented in the previous sections. For the $SU_q(2)$ case, in the limit in which $\sigma \to \infty$ such that $\rho = \sqrt{2\sigma}\xi$ remain finite, we have

$$K_+^{(\sigma)} \to \sqrt{2\sigma} A^{\dagger}_{(k',q)} \,, \qquad K_-^{(\sigma)} \to \sqrt{2\sigma} A_{(k',q)} \,; \tag{58}$$

and the $SU_q(2)$ coherent states reduce to the generalized Biedenharn-Macfarlane coherent states (BMCS) [14,15]. This result is well known in the case $k = 1$ corresponding to the standard generalized Holstein-Primakoff coherent states reducing to the standard BMCS. Retaining terms up to order $\frac{1}{\sigma}$, we can obtain in the special case $q = 1$ and $k = 1$,

$$\begin{aligned} (\Delta X)^2 &\simeq \frac{1}{2} - \frac{\rho^2}{4\sigma} , \\ (\Delta P)^2 &\simeq \frac{1}{2} + \frac{\rho^2}{4\sigma} . \end{aligned} \tag{59}$$

In the case $q = 1$ and $k = 2$,

$$\begin{aligned} (\Delta X)^2 &\simeq \frac{1}{2} + 2\rho^2 + \sqrt{2}F_1(\rho^2) \\ &\quad -\frac{1}{\sigma}\left(\frac{1}{2\sqrt{2}}\left(F_3(\rho^2) + F_5(\rho^2) - \rho^4 F_1(\rho^2)\right) + \rho^4\right) , \\ (\Delta P)^2 &\simeq \frac{1}{2} + 2\rho^2 - \sqrt{2}F_1(\rho^2) \\ &\quad +\frac{1}{\sigma}\left(\frac{1}{2\sqrt{2}}\left(F_3(\rho^2) + F_5(\rho^2) - \rho^4 F_1(\rho^2)\right) - \rho^4\right) , \end{aligned} \tag{60}$$

where $F_n(x) = x^{\frac{n}{2}} e^{-x} \sum_{l=0}^{\infty} \frac{x^l}{l!}\sqrt{2l+n}$.

For the $SU_q(1,1)$ case, we may get two interesting limits: (1) $\sigma \to \infty$ with $\left\langle A^{\dagger}_{(k,q)} A_{(k,q)}\right\rangle$ finite. (2) σ finite with $\left\langle A^{\dagger}_{(k,q)} A_{(k,q)}\right\rangle \to \infty$. The limit of the first kind, if taken in such a way that $\rho = |\alpha|\sqrt{2\sigma}$ remain finite, results in

$$|\sigma, k, \alpha > \longrightarrow \frac{1}{Q} e^{\rho A^{\dagger}_{(k,q)}} |0> ,$$

which is again the generalized BMCS. The results for $(\Delta X)^2$ and $(\Delta P)^2$ both for $k = 1$ and $k = 2$ ($q = 1$) can be obtained from the expressions given in Eqs. (59) and (60) by reversing the sign of the coefficients of $\frac{1}{\sigma}$. The limit of the second kind requires that $\alpha \to 1$. For $k = 2$ we obtain in the limit

$$K_{+}^{(\sigma)} \longrightarrow \frac{1}{2}\left(a_q^{\dagger}\right)^2 ,$$

so that $|\sigma, k, \alpha >$ becomes a particular deformed oscillator squeezed state [15]. We have studied the time evolution of the field quadratures and the atomic population inversion.

In the case $q = 1$ ($SU_q(1,1)$), the expression for the atomic population inversion exhibits the exact periodicity. When $q \neq 1$ the periodic revivals of the generalized JCM are destroyed increasingly for large values of the deformation parameter q.

ACKNOWLEDGEMENTS

This work is supported in part by the National Natural Science Foundation of China.

References

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

F. W. Cummings, Phys. Rev. **140**. A1051 (1965).

[2] B. Buck and C. V. Sukumar, Phys. Lett. A**81**, 132 (1981).

V. Buzek, Phys. Rev. A**39**, 3196 (1989).

C. C. Gerry, Phys. Rev. A**37**, 2683 (1988).

[3] For a review of the current experiments for the squeezed light see the special issues of J. Mod. Opt. **34**, (6/7) (1987) and J. Opt. Soc. Am. B4, (10) (1987).

[4] D. Meschede, H. Waltner and G. Muller, Phys. Rev. Lett. **54**, 551 (1985).

G. Rempe, H. Walther and N. Klein, Phys. Rev. Lett. **58**, 353 (1987).

[5] B. Buck and C. V. Sukumar, Phys. Lett. A**83**, 211 (1981).

S. Singh, Phys. Rev. A**25**, 3206 (1982).

C. Buzano, M. G. Rasseti and M. L. Rastello, Phys. Rev. Lett. **62**, 137 (1989).

[6] M. Chaichian, D. Ellinas and P. Kulish, Phys. Rev. Lett. **65**, 980 (1990).

Z. Chang, *"Dynamics of the generalized photonic fields interacting with atoms"*, preprint BIHEP-TH-92-1, Phys. Rev A (to be published).

[7] G. D'Ariano, M. Rasetti and M. Vadacchino. Phys. Rev. D**32**, 1034 (1985).

J. Katriel, A.I. Solomon, G. D'Ariano and M. Rasseti, Phys. Rev. D**34**, 2332 (1986).

J. Katriel, M. Rasseti and A. I. Solomon, Phys. Rev. D**35**, 1248 (1987), Phys. Rev. D**35**, 2601 (1987).

[8] V. G. Drinfel'd, *"Quantum Groups"*, in Proceedings of the International Congress of Mathematics (Berkeley), ed, A. M. Gleason, P798 (Am. Math. Soc. ,Providence, R. I., 1986).

M. Jimbo, Lett. Math. Phys. **10**, 63 (1985), Comm. Math. Phys. **102**, 537 (1986).

[9] Yu. Manin, Ann. Inst. Fourier, **37**, 191 (1987).

S. Woronowicz, Publ. RIMS-Kyoto **23**,117 (1987).

E. Witten, Nucl. Phys. B330, 285 (1990).

M. Rosso, Comm. Math. Phys. **117**, 581 (1988).

G. Lustig, Contemp. Math. **82**, 59 (1989).

[10] E. Floratos, Phys. Lett. B**233**, 395 (1989).

T. Hayashi, Commun. Math. Phys. **127**, 129 (1990).

T. L. Curtright and C. K. Zachos, Phys. Lett. B**243**, 237 (1990).

T. L. Curtright, G. I. Ghandour and C. K. Zachos, J. Math. Phys. **32**, 171 (1991).

A. Schirmacher, J. Wess and B. Zumino, Z. Phys. C**49**, 317 (1991).

[11] Z. Chang, W. Chen and H. Y. Guo, J. Phys. A**23**, 1427 (1990).

Z. Chang, W. Chen, H. Y. Guo and H. Yan, J. Phys. A**23**, 5371 (1990); J. Phys. A**24**, 1427 (1991).

Z. Chang, J. X. Wang and H. Yan, J. Math. Phys. **32**, 3241 (1991).

[12] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).

R. J. Baxter, "*Exactly Solved Models in Statistical Mechanics*" (Academic Press, New York, 1982).

[13] V. Pasquier and H. Saleur, Nucl. Phys. B**330**, 523 (1990).

D. Levy, Phys. Rev. Lett. **64**, 499 (1990), Phys. Rev. Lett. **67**, 1971 (1991).

L. Alvarez-Gaume, C. Gomez and G. Sierra, Phys. Lett. B**220**, 142 (1989).

C. Gomez and G. Sierra, Phys. Lett. B**240**, 149 (1990).

Z. Chang and H. Yan, Phys. Rev. A**43**, 6043 (1991), Phys. Rev. A**44**, 7405, (1991).

[14] L. C. Biedenharn, J. Phys. A**22**, L873 (1989).

A. J. Macfarlane, J. Phys. A**22**, 4581 (1989).

Y. J. Ng. J. Phys. A**23**, 1023 (1990).

X. C. Song, J. Phys. A**23**, L821 (1990).

[15] E. Celeghini, M. Rasetti and G. Vitiello, Phys. Rev. Lett. **66**, 2056 (1991).

Z. Chang, "*Minimum Uncertainty States, Two-mode Device and Quantum Group Symmetry*", BIHEP-TH-92-3.

CCAST-92-17
ASITP-92-22
ZIMP-92-01

A Deformation of Quantum Mechanics[1]

YOU-QUAN LI†‡ ZHENG-MAO SHENG‡§

† *CCAST (World laboratory)*
P.O. Box 8730, Beijing 100080
[2]‡ *Zhejiang Institute of Modern Physics*
Zhejiang University, Hangzhou 310027
§ *Department of Physics*
Hangzhou University, Hangzhou 310028
People's Republic of China

ABSTRACT

On an one-dimensional lattice without uniform interval, the Hermitian conjugation of q-differential operator are discussed. Then a deformation of quantum mechanics in one dimension is presented. As an application, the harmonic oscillator are discussed. The energy spectrum and the eigenfunctions are solved to depend on an arbitrary deformation function. The deformed coherent states are also discussed. It is found that the completeness relation of coherent states holds for the case of q-coherent states, i.e. the deformation of Heisenberg-Weyl algebra is q-analogue one, a Hopf algebra.

PACS: 03.65.Fd 02.20.+b 03.65.Ge

---

[1]Work supported in part by the NNSF of China through ASITP and CCAST (World Laboratory).
[2]mailing address

I. Introduction

Recently, much attention has been paid to the studies of called quantum group in both aspects of physics and mathematics. Quantum groups [1] are the dual category of Hopf algebra which is neither commutative nor co-commutative. Most of the well studied concrete examples of quantum groups are deformations of the universal enveloping algebra of semisimple Lie algebra[2-5]. These mathematical structure, algebraic structure, arise in quantum inverse scattering theory[6] and statistical mechanics[7]. They may be thought of as matrix group in which the elements themselves are not commutative but obeying a set of bilinear product relations[4,8].

Dozens of works are devoted to the study of $U_q(Sl(2))$ etc.[9] via the called q-deformation of bosonic realization, which is a q-analogue of Schwinger technique in quantum mechanics[10]. It is known in the work of Manin[11] and Woronowicz[12] and a further development of Wess and Zumino[13] that quantum groups provide a concrete example of non-commutative differential geometry. A connection between quantum group and Lie-admissible Q-algebra was realized in the work of Janussis[14].

As one of the attempts to explore the physical significant of quantum groups, a q-extension of one-dimensional harmonic oscillator in Schrödinger picture is given in the work of Minahan[15]. We try to establish a deformation of quantum mechanics so that the standard quantum mechanics is its limit case, and quantum group symmetry is contained in it. To do so is not only to explore the meaning of quantum group in the contents of quantum mechanics, but to provide possibilities of nonperturbation explanations of some perturbation corrections[16] as well. Present paper reports one step toward the above goal. In next section, we briefly illustrate some notations and derive some useful formulae of q-differential calculus. In section 3, we discuss Hermitian conjugation and establish a deformation of one-dimensional stationary Schrödinger equation. In section 4, we discuss the Harmonic Oscillators. The energy spectrum and eigenfunctions are obtained to depend on a function, which involves concrete deformations. In section 5, we discuss deformations of coherent states, especially discuss a q-coherent states and the q-coherent states representation, an analogue of Bargman space representation. Finally, we give some conclusions and discussion.

II. The q-differential integral calculus

The lattice formulation of quantum field theories allows the nonperturbation calculation of bound state mass and decay amplitude. In the usual lattice approach to Schrödinger equation either using finite difference[17] or using finite-element[18] method, the lattice spacing is the same everywhere, i.e. the coordinates of lattice points are arithmetic sequence. However, it is known that eigenfunctions of bound state always descend properly as the coordinate goes to infinite such that they are square integrable. i.e. L^2-functions in mathematical terminology. By means of the basic idea of Lebesgue integral, we may consider a lattice with a constant lattice spacing no more. The lattice spacing should increases as the coordinates of lattice points does. Evidently, the simplest case is such a case that the coordinates of lattice points take as geometric sequence. This can be regarded as a lattice deformation from arithmetic lattice to geometric one (q-lattice) i.e.

$$\{x_n = na + x_0 \mid n \in Z\} \to \{x_n' = q^{2n} x_0' \mid n \in Z\} \tag{2.1}$$

which is realized by exponential map $x_n' = e^{x_n}$ ($q := e^{a/2}$).

On the basis of the above consideration, we will reintroduce the following definitions and formulations. For $x \in \mathcal{R}$, let $\mathcal{F}$ denotes the set of all complex functions on $\mathcal{R}$, i.e.

$$\mathcal{F} = \{f \mid f(x) \in \mathcal{C}\} = Fun(\mathcal{R}, \mathcal{C})$$

composition is defined by

$$(f \circ g)(x) = f(x)g(x) \quad \forall f, g \in \mathcal{F}.$$

Now we introduce a dilation operator $\hat{q}$

$$\hat{q} : \mathcal{F} \quad \longrightarrow \quad \mathcal{F}$$

defined by

$$(\hat{q}f)(x) = f(qx) \quad f \in \mathcal{F} \tag{2.2}$$

obviously $(\hat{q})^{-1} = (\hat{q^{-1}})$. Then a q-difference can be defined by

$$d_q f := \hat{q}f - \hat{q}^{-1}f \tag{2.3}$$

And q-difference quotient operator is given by

$$\frac{d_q}{d_q id_{\mathcal{R}}} = \frac{\hat{q} - \hat{q}^{-1}}{(q - q^{-1})id_{\mathcal{R}}} \tag{2.4a}$$

or explicitly

$$\frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} \tag{2.4b}$$

which is invariant under $q \rightarrow q^{-1}$, and recovers the usual definition of function derivation when $q \rightarrow 1$. The q-analogue of Lebnitz rule is easily obtained from definition (2.4)

$$\begin{aligned} \frac{d_q}{d_q x}(f \circ g) &= \frac{d_q f}{d_q x} \circ \hat{q}g + (\hat{q}^{-1}f) \circ \frac{d_q g}{d_q x} \\ &= \frac{d_q f}{d_q x} \circ \hat{q}^{-1}g + (\hat{q}f) \circ \frac{d_q g}{d_q x} \end{aligned} \tag{2.5}$$

If let $f(x) = x$, (2.5) gives an operator relation

$$\frac{d_q}{d_q x} x - q^{-1} x \frac{d_q}{d_q x} = \hat{q} \tag{2.6}$$

similarly we have another useful operator relation

$$\hat{q} \frac{d_q}{d_q x} - q^{-1} \frac{d_q}{d_q x} \hat{q} = 0 \tag{2.7}$$

One may define a q-integral as the inverse of q-difference quotient, denoted by

$$\int f(x) d_q x = F(x) + C \tag{2.8}$$

where $\frac{d_q F(x)}{d_q x} = f(x)$. Then one can show that the summation along a q-lattice can be calculated from a q-analogue of Newton-Lebnitz formula

$$\sum_{l=i}^{f} f(q^{2l} x)(q - q^{-1}) q^{2l} x = \int_{x_i}^{x_f} f(x) d_q x = F(x_f) - F(x_i) \tag{2.9}$$

where $x_f = q^f x$, $x_i = q^i x$ (if $x_f x_i < 0$, the summation should be divided into two parts, i.e. x_i to 0 and 0 to x_f). Evidently, the following identity holds

$$\int_{-\infty}^{+\infty} f(x) d_q x = \int_{-\infty}^{+\infty} q^l \hat{q}^l f(x) d_q x \tag{2.10}$$

The inverse of Jackson[19] q-integral (2.4) was first used to study the relation between rational conformal feild theories and quantum groups in [20]. In [21] one can find some discussions about q-integration rules.

III. Hermitian conjugation and q-Schrödinger equation

In this section, we will try to establish q-Schrödinger equation in coordinates representation. We define inner product as

$$< \psi \mid \varphi >= \int_{-\infty}^{+\infty} \psi(x)^* \varphi(x) d_q x \tag{3.1}$$

We consider the case that wave functions are continuous at origin and vanished at infinite i.e.

$$\begin{cases} \psi(0^+) &= \psi(0^-) \\ \psi(\infty) &= 0 \end{cases} \tag{3.2}$$

Using (2.5) and (2.10), we obtain from

$$\int_{-\infty}^{+\infty} \frac{d_q}{d_q x}(\psi(x)^* \varphi(x)) d_q x = 0$$

that

$$[(q\hat{q})^{-1} \frac{d_q}{d_q x}]^\dagger = -(q\hat{q}) \frac{d_q}{d_q x}. \tag{3.3}$$

Similarly we have from (2.10) that

$$\hat{q}^\dagger = q^{-1} \hat{q}^{-1} \tag{3.4}$$

Then we obtain from (3.3), (3.4) and (2.6) that

$$(\frac{d_q}{d_q x})^\dagger = -\frac{d_q}{d_q x} \tag{3.5}$$

Thus a deformation of time-independent Schrödinger equation with positive definite energy spectrum can be defined by the following Hamiltonian

$$H_q = -\frac{d_q^2}{d_q x^2} + V(x) \tag{3.6}$$

Where V(x) stands for potential. Obviously, it recovers the standard quantum mechanics when $q \rightarrow 1$.

IV. Harmonic oscillator

Let us consider a harmonic oscillator, its potential is $V(x) = x^2$, then Hamiltonian reads

$$H_q = -\frac{d_q^2}{d_q x^2} + x^2 \tag{4.1}$$

It can be written as

$$H_q = \frac{1}{2}(a_q^+ a_q + a_q a_q^+) \tag{4.2a}$$

where

$$a_q = x + \frac{d_q}{d_q x}$$

$$a_q^+ = (a_q)^\dagger = x - \frac{d_q}{d_q x} \tag{4.2b}$$

It is easy to check that a_q and a_q^+ are energy decrease and increase operator no more i.e. $[\ H_q, a_q\ ] \neq -a_q$. However, we can introduce a new Hermitian operator N_q such that

$$[\ N_q, a_q\ ] = -a_q,$$

$$[\ N_q, a_q^+\ ] = a_q^+. \tag{4.3}$$

of cause the latter is an immediate consequence of the former of (4.3) due to $N_q^\dagger = N_q$. Obviously $[\ N_q, H_q\ ] = 0$, then an eigenstate of N_q is also an eigenstate of H_q. If we assume $[\ a_q, a_q^+\ ] = X$, from Jacobi identity

$$[\ [\ N_q,\ a_q\ ],\ a_q^+\ ] + [\ [\ a_q,\ a_q^+\ ],\ N_q\ ] + [\ [\ a_q^+,\ N_q\ ],\ a_q\ ] = 0$$

we have

$$[\ [\ a_q,\ a_q^+\ ],\ N_q\ ] = 0 \tag{4.4}$$

This shows that $X = \mu(N_q)$ i.e. X may be any function of N_q, then

$$[\ a_q,\ a_q^+\ ] = \mu(N_q) \tag{4.5}$$

In order to recover standard quantum mechanics, the function must go to unit as the deformation parameter q goes to 1. It is known that $\mu(N_q) = [\ N_q + 1\ ] - [\ N_q\ ]$ where $[\ x\ ] := (q^x - q^{-x})/(q - q^{-1})$ is the case indicated by Biedenharn in the study of quantum group[9]. For a given function μ, chosen in accordance with experiment results, the Hamiltonian (4.2) becomes

$$H_q = a_q^+ a_q + \frac{1}{2}\mu(N_q) \tag{4.6}$$

Commutator relations (4.3) and (4.5) is the defining relations of a deformed Heisenberg-Weyl algebra. From those defining relations, one can find that a_q^+ and a_q are creation and annihilation operators of eigenvalues of N_q, a quantum number operator. i.e.

$$\begin{aligned} N_q \mid n > &= n \mid n > \\ a_q^+ \mid n > &= (\sum_{i=0}^{n} \mu(i))^{1/2} \mid n+1 > \\ a_q \mid n > &= (\sum_{i=0}^{n-1} \mu(i))^{1/2} \mid n-1 > \end{aligned} \tag{4.7}$$

Then eigenvalues of Hamiltonian (4.6) is easily calculated

$$E_q(n) = \sum_{i=0}^{n-1} \mu(i) + \frac{1}{2}\mu(n) \tag{4.8}$$

The whole eigenstates $\{ \mid n > \mid n = 0,\ 1,\ 2, \cdots \infty\}$ span a Fork space. In terms of vacuum state $\mid 0 >$ (i.e. ground state) the normalized eigenstates in Fork representation are expressed as

$$\mid n > = \frac{(a_q^+)^n}{[\ \prod_{j=1}^{n} (\sum_{i=0}^{j-1} \mu(i)\ )\ ]^{1/2}} \mid 0 > \tag{4.9}$$

Then the eigenfunctions in coordinate representation can be derived from (4.9) without much difficulty. First we consider the vacuum state $\mid 0 >$ which satisfies

$$a_q \mid 0 > = 0 \tag{4.10}$$

Using the expression of a_q in coordinate representation (4.2b), we have the following q-differential equation

$$(x + \frac{d_q}{d_q x})\psi_0 = 0 \tag{4.11}$$

where $\psi_0 :=< x \mid 0 >$ Solving (4.11), We have the eigenfunction of ground state

$$\psi_0(x) = \frac{1}{(\pi)^{1/4}} exp_{q^2}(-x^2/[2]) \tag{4.12}$$

where $exp_q x := \sum_{n=0}^{\infty} \frac{x^n}{[\ n]!}$. Then we obtain eigenfunctions of excited states

$$\begin{aligned} \psi_n(x) &:= < x \mid n > \\ &= \frac{1}{[\ \sqrt{\pi} \prod_{j=1}^{n} (\sum_{i=0}^{j-1} \mu(i))\ ]^{1/2}} (x - \frac{d_q}{d_q x})^n exp_{q^2}(-x^2/[2]) \end{aligned} \tag{4.13}$$

V. Coherent states

We now observe the spectrum problem of q-annihilation operator a_q. The eigenstates of a_q

$$a_q \mid \alpha >:= \alpha \mid \alpha > \tag{5.1}$$

is a deformation of usual coherent states[22]. In Fork representation, (5.1) is easily solved by using (4.7)

$$\begin{aligned} \mid \alpha > &= [\ e\tilde{x}p_\mu(-|\alpha|^2)\ ]^{1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{[\ \prod_{j=1}^{n} (\sum_{i=0}^{j-1} \mu(i)\ )\ ]^{1/2}} \mid n > \\ &= [\ e\tilde{x}p_\mu(-|\alpha|^2)\ ]^{1/2} e\tilde{x}p_\mu(\ \alpha a_q^+) \mid 0 > \end{aligned} \tag{5.2}$$

Where α takes any value in complex plane and $e\tilde{x}p_\mu x$ stands for a deformed exponential function.

$$e\tilde{x}p_\mu x := \sum_{n=0}^{\infty} \frac{x^n}{\prod_{j=1}^{n} (\sum_{i=0}^{j-1} \mu(i))}$$

Obviously, which is just $exp_q x$ appeared in section 4 when $\mu(i) = [\ i+1\ ] - [\ i\ ]$. For $\mu(i) = 1$, it is the usual exponential function and then (5.2) recovers the usual coherent states in quantum mechanics

As we have known the expression of coherent states in Fork representation and the transformation function (4.13) from Fork space to coordinate space, we can easily obtain its expression in coordinate representation i.e. wave functions of eigenstates of a_q

$$\phi_\alpha(x) = \sum_{n=0}^{\infty} < x \mid n >< n \mid \alpha >$$

$$= [\ e\tilde{x}p_\mu(-|\alpha|^2)\ ]^{1/2} e\tilde{x}p_\mu[\ \alpha(x - \frac{d_q}{d_q x})]\ exp_{q^2}(-x^2/[2]) \qquad (5.3)$$

The probability distribution of a deformed coherent state in Fork representation is

$$|< n \mid \alpha > |^2 = e\tilde{x}p_\mu(-|\alpha|^2) \frac{(|\alpha|^2)^n}{\prod_{j=1}^{n-1}(\sum_{i=0}^{j}\mu(i))} \qquad (5.4)$$

this is a deformation of Poison distribution. The deformed coherent states is also not orthogonal to each other due to

$$< \beta \mid \alpha >= [\ e\tilde{x}p_\mu(-|\alpha|^2) e\tilde{x}p_\mu(-|\beta|^2)\ ]^{1/2} e\tilde{x}p_\mu \alpha\beta^* \qquad (5.5)$$

The completeness relation for the deformed coherent states is shown to hold only in the case $\mu(x) = [\ x+1\ ] - [\ x\ ]$, i.e. q-coherent states (see Appendix).

$$\int | \alpha >< \alpha | \frac{d_q^2\alpha}{\pi} = 1 \quad for \quad \mu(x) = [\ x+1\ ] - [\ x\ ] \qquad (5.6)$$

This is an interesting consequence. In this case, (4.7) becomes

$$\begin{aligned} N_q | n > &= N_q | n > \\ a_q^+ | n > &= (\ [\ n+1\ ]\ )^{1/2} | n > \\ a_q | n > &= (\ [\ n\ ]\ )^{1/2} | n > \end{aligned} \qquad (5.7)$$

On the basis of the completeness relation (5.6), we can expand a n-quantum state in terms of q-coherent states

$$| n >= \int | \alpha >< \alpha | n > \frac{d_q^2\alpha}{\pi}$$

$$= \int [\ exp_q(-|\alpha|^2)\ ]^{1/2} \frac{\bar{\alpha}^n}{(\ [\ n\ ]!\ )^{1/2}} \frac{d_q^2\alpha}{\pi} | \alpha > \qquad (5.8)$$

where $\bar{\alpha}$ stands for α^*. Substituting (5.9) into (5.8), we obtain that

$$a_q^+ \bar{\alpha}^n = \bar{\alpha}^{n+1}$$

$$a_q \bar{\alpha}^n = [\ n\ ]\bar{\alpha}^{n-1} \tag{5.9}$$

Then we immediately have an expression of creation and annihilation operator in coherent states representation. i.e.

$$a_q^+ = \bar{\alpha}$$

$$a_q = \frac{d_q}{d_q \bar{\alpha}} \tag{5.10}$$

Any state of a harmonic oscillator must possess the following expansion in q-Fork space

$$|\ \psi\ > = \sum_{n=o}^{\infty} c_n |\ n\ > \tag{5.11}$$

where $\sum |c_n|^2 = 1$. In order to expand the arbitrary state in terms of q-coherent states, we must use the completeness relation which has been used in deriving (5.9). Substituting (5.8) into (5.11), we obtain the following expansion in q-coherent representation

$$|\ \psi\ > = \int \sum_{n=0}^{\infty} c_n \frac{\bar{\alpha}^n}{(\ [n]\ )^{1/2}} [\ exp_q(-|\alpha|^2)\ ]^{1/2} \frac{d_q^2 \alpha}{\pi} |\ \alpha\ > \tag{5.12}$$

Obviously the amplitude distribution function in this representation is not an entire function

$$<\ \alpha\ |\ \psi\ > = \chi(\bar{\alpha})[\ exp_q(-\bar{\alpha}\alpha)\ ]^{1/2} \tag{5.13}$$

where $\chi(\bar{\alpha})$ is an (anti-)analytical function on the complex α-plane and is defined by the expansion coefficients $\{c_n\}$ of the state $|\ \psi\ >$ in Fork space, i.e.

$$\chi(\beta) = \sum_{n=0}^{\infty} c_n \frac{\beta^n}{(\ [n]!\ )^{1/2}} \tag{5.14}$$

There is apparently a one-to-one correspondence between the entire function (5.15) and the state in Fork space (5.12). The Hilbert space of such functions $\chi(\beta)$ is the known Bargman space[24], in which the inner product of two vectors φ and χ is defined by

$$< \varphi \mid \chi >= \int [\ \varphi(\bar{\alpha})\ ]^{*}\chi(\bar{\alpha})exp_q(-|\alpha|^2)\frac{d_q^2\alpha}{\pi} \qquad (5.15)$$

This definition can be easily derived via q-coherent state representation i.e. by using (5.12) and (5.14).

VI. Conclusion and discussion

In above we have attempted to establish a deformation of quantum mechanics. We considered in fact a discrete quantum mechanics in one dimension, in which the intervals are not uniform. Instead, the intervals are divided by a geometric sequents. The Hermitian conjugation of q-differential operator (strictly speak quotient of q-difference) are discussion and then one-dimensional positive definite stationary Schrödinger equation is set up.

For the case of Harmonic oscillator, we have solved the energy spectrum and the eigenfunctions by means of operator method. Owning to the constraints of Jacobi identity, the oscillator algebra may contain an arbitrary function of q-quantum number operator N_q only. In order to recover usual quantum mechanics, this function is only obliged to unit when the deformation parameter goes to unit. So the eigenvalues and eigenfunctions of the Hamiltonian contain an deformation function, which can be chosen according to experiment results.

Furthermore, we discussed the coherent states for the deformed Heisenberg-Weyl algebra. Certainly the deformed coherent states also contain the deformation function. However the coherent states satisfy the completeness relation only for a special deformation function. This is just the case of the known q-analogue of Heisenberg-Weyl algebra, a Hopf algebra. Other potential cases and three dimensional case is now in discussion. Different from the noncommutative geometry approach to deformations of quantum mechanics[25], it is also worthwhile to notice the connections between quantum group and discrete quantum mechanics.

Appendix

From the definition (2.4), one can easily find

$$\frac{d_q}{d_q x} x^n = [n] x^{n-1} \tag{a}$$

$$\frac{d_q}{d_q x} exp_q x = exp_q x \tag{b}$$

$$\hat{q} x^n = q^n x^n \tag{c}$$

$$[\ n\ ]_{q^m} = [\ mn\ ]/[m] \tag{d}$$

$$\frac{d_q}{d_q x} f(x^m) = [m] x^{m-1} \frac{d_{q^m}}{d_{q^m}(x^m)} f(x^m) \tag{e}$$

The following formula of integration by part is a direct consequence of (2.5)

$$\int_{x_i}^{x_f} (\hat{q} f) d_q g = f g |_{x_i}^{x_f} - \int_{x_i}^{x_f} (\hat{q}^{-1} g) d_q f \tag{f}$$

The q-analogue of Γ-function is defined by

$$\Gamma_q(\rho) := \int_0^\infty x^{\rho-1} exp_q(-x) d_q x$$

$$\overset{\text{or}}{=} \frac{[2]}{2} \int_{-\infty}^{+\infty} x^{2\rho-1} exp_q(-x^2) d_q x \tag{g}$$

Using (a-d), one can show that

$$\Gamma_q(\rho+1) = [\ \rho\ ] \Gamma_q(\rho)$$

$$\Gamma_q(n+1) = [\ n\ ]! \qquad (h)$$

The completeness relation (5.7) is shown in the following

$$\int |\ \alpha >< \alpha\ | d_q^2\alpha$$
$$= \sum_{m,n} \frac{|\ \alpha >< \alpha\ |}{(\ [n]![m]!\ )^{1/2}} \int_0^\infty d_q|\alpha| exp_q(-|\alpha|_?^2) |\alpha|^{n+m+1} \int_0^{2\pi} d\phi e^{\ i(n-m)\phi}$$
$$= \pi \sum_n |\ n >< n\ | = \pi \qquad (i)$$

Acknowledgements

The authors would like to thank Prof R Wang and Dr J M Shen for interesting discussions. Li would like to thank Profs H Y Guo, Z M Qiu, K Wu and Z Y Zhu for their hospitality and discussions, and Dr H Yan for useful comment.

References

1. V G Drinfeld, 1986 *Proc. ICM Berkeley* 798; *Zap. Nauch. Sem. LOMI* **155** 18.

2. P Kulish, N Reshetikhin, *Zap. Nauch. Sem. LOMI* **101**(1981)101.
   E Sklyanin, *Func. Anal. Appl.* **16**(1982)263; **17**(1983)273.

3. M. Jimbo, *Lett. Math. Phys.* **10**(1985)63; *Lett. Math. Phys.* **11**(1986)247; *Comm. Math. Phys.* **102**(1986)537.

4. N. Reshetikhin, 1987 preprints LOMI E-4-87; E-17-87.
   N Reshetikhin, L Takhtajan, L Faddeev, *Leningrad Math. J.* **1**(1990)193.
   preprint LOMI E-14-87.

5. G Lusztig, *J. Amer. Math. Soc.* **3**(1990)257.

6. L Faddeev, *Recent Advances in Field Theory and Statistical Mechanics* Les Houches Session *XXX IX* ed J-B Zuber and R Stora (Amsterdam: Elsevier) p261 and reference therein.

7. R J Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academia Press, 1982).

8. Y Manin, *Ann. Inst. Fourier* **37**(1987)191.

9. L C Biedenharn, *J. Phys.* **A22**(1989)L873.
C P Su, H C Fu, *J. Phys.* **A22**(1989)L983.
A J MacFarlane, *J. Phys.* **A22**(1989)4581.
M Chaichian, P Kulish, *Phys. Lett.* **B234**(1990)72.
Y J Ng, *J. Phys.* **A23**(1990)1023.
Z Chang, W Chen, H Y Guo, H Yan, *J. Phys.* **A23** (1990)5371.
X C Song, *J. Phys.* **A23**(1990)L821.

10. J Schwinger, *On angular momentum* ed L C Biedenharn and V Dam (New York: Academic 1965).

11. Y Manin, Quantum Groups and non-commutation geometry, *Montreal Univ. report* CRM-156 1988; *Commun. Math. Phys.* **123**(1989)167.

12. S Woronowicz, *Commun. Math. Phys.* **111**(1988)613; **122**(1989)125.

13. J Wess, B Zumino, Covariant differential calculus on the quantum hyperplane preprint 1990 CERN-TH-90-5697.
B Zumino, *Mod. Phys. Lett.* **A6**(1991)1225.

14. A Janussis, *Hadron. J.* 1991 to appear.
A Janussis, G Brodimas, R Mignani, *J. Phys.* **A24**(1991)L77.

15. J Minahan, *Mod. Phys. Lett.* **A5**(1990)2625.

16. for example M Mizushima, *The Theory of Rotating Diatomic Molecules* (John Wiley and Sons, New York 1975)
Z Chang, H Y Guo, H Yan, 1990 preprint CCAST-90-54.

17. J Jurkiewicz, J Wosiek, *Nucl. Phys.* **B145**(1978)416; 445.

18. C Bender, D Sharp, *Phys. Rev. Lett.* **51**(1983)1815.
A Common, F Ebrahimi, S Hafez, *J. Phys.* **A22** (1989)3229.

19. F Jackson, *Q. J. Pure. Appl. Math.* **41**(1970)193.

20. L Alvarez-Gaume, C Gomer, G Sierra, *Nucl. Phys.* B319(1989)155.

21. D Bernard, A Leclair, *Phys. Lett.* **B227**(1989)417.
H Ruegg, *J. Math. Phys.* **31**(1990)1085.
A J Bracken, D S McAnally, R B Zhang, M D Gould, *J. Phys.* A24 (1991)1379.

22. R J Glauber, *Phys. Rev.* **131**(1963)2766.

23. B Jurco, *Lett. Math. Phys.* **21**(1991)51.

24. V Bargman, *Commun. Pure. Appl. Math.* **14**(1961)187.

25. X C Song, L Liao, *J. Phys.* **A25**(1992)623.

CCAST-92-20
ZIMP-1992-02

Is There Any Hopf Algebra Structure in CZ Algebra?

KANG LI

CCAST(World Laboratory),P.O.Box 8730,
Beijing 100080,P.R.China
and Zhejiang Institute of Modern Physics, Zhejiang University,
Hangzhou 310027,P.R.China[1]
And
Physics Department,Yan'an University,
Yan'an 716000,P.R.China.

ABSTRACT

After a brief review of CZ type deformation of virasoro algebra, I want to search out the Hopf algebra structure in CZ algebra, unfortunately I find out that there is no usual Hopf algebra structure in CZ algebra, in this sense, CZ algebra is not a quantum group. I discuss also the CILPP type deformation of virasoro algebra, this deformation is equivalent to CZ algebra under special cases. I give also some comments on HMNS type deformation of the virasoro algebra in the last section of the paper.

[1] Mailing address

1. Introduction

In recent two years, the quantum group theory becomes one of the most interesting subjects both in theoretic physics and mathematics. Originally, quantum group emerged in the context of the quantum Yang-Baxter equation as well as the quantum inverse scattering method [1-2], later on, it is widely used in physics, for example, it is proved that quantum group has deep relation with rational conformal field theory [3-5] as well as integral physics models [6]. General speak, quantum group is the q deformation (or q anologue) of Lie algebra, this deformation keep the Hopf algebra structure, when $q \to 1$, the deformed algebra (quantum group) reduce to the original Lie algebra. So, in some sense, we can say that to study a quantum group is to study the Hopf algebra in it.

As we know that virasoro algebra is a infinite dimensional Lie algebra, it plays a important role in conformal field theory. Recently, many paper have been devoted to discuss the q deformations of virasoro algebra [7-13], there are many different deformations. But, as I know, there are only three different kind of deformations. They are Curtright and Zachose's deformation (we call it CZ algebra)[7], and Chaichian, Isaev, Lukierski, Popowicz and Presnajder's deformation (we call it CILPP deformation)[11], as well as Hiro-oka, Matsui, Naito and Saito' deformation (we call it HMNS deformation)[13]. In the first two type deformations, the authors did not give the Hopf algebra structure in the deformed algebra. In the third type deformation, although the authors found out the Hopf algebra structure, but we can see from the discussion in the last section of the paper that HMNS deformation can hardly be considered as a deformation of virasoro algebra in common sense, in fact, it is a new infinite dimsional quantum algebra which is infinite times great than the virasoro algebra. CILPP deformation of virasoro algebra is equivalent to CZ algebra under some special cases, but in the general cases, the CILPP deformation changes the structure constant of virasoro algebra into operator, so I doubt that whether it can be treated as a deformation of virasoro algebra. So, up to now, a hopeful candidate for the q deformation of virasoro algebra is CZ algebra. But, as I said above, we do not find out the Hopf algebra structure in CZ algebra, this is the main topics will be discussed in this paper.

In the sections 2 and 3 of the paper, I recall the CZ algebra and CILPP deformations respectively; in section 4, I discuss the problem of Hopf algebra structure in CZ algebra, there are some comments given in the last section of the paper.

2. CZ Algebra and It's Central Extension

The q deformation of virasoro algebra has been first proposed by Curtright and Zachos. The original deformed algebra (CZ algebra) has the form:

$$[L_n, L_m]_{(q^{m-n}, q^{n-m})} = [n-m]L_{n+m}. \tag{2.1}$$

where

$$[A, B]_{(p,q)} = pAB - qBA, \tag{2.2}$$

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{2.3}$$

and CZ's operators realization of L_n's are give by

$$L_n = z^{-n}(q^{2z\partial} - 1)/(q - q^{-1}) = z^{-n}q^{z\partial}[z\partial]. \tag{2.4}$$

The (p,q)-commutators, unlike the usual commutators, satisfy the following deformed relations:

$$[A, B]_{(p,q)} = -[B, A]_{(q,p)}, \qquad (2.5)_a$$

$$[A + B, C]_{(p,q)} = [A, C]_{(p,q)} + [B, C]_{(p,q)}, \qquad (2.5)_b$$

$$[AB, C]_{(p,q)} = A[B, C]_{(p,r)} + [A, C]_{(r,q)}B, \qquad (2.5)_c$$

$$[A, [B, C]_{(q_1, q_1^{-1})}]_{(\frac{q_3}{q_2}, \frac{q_2}{q_3})} + [B, [C, A]_{(q_2, q_2^{-1})}]_{(\frac{q_1}{q_3}, \frac{q_3}{q_1})}$$
$$+ [C, [A, B]_{(q_3, q_3^{-1})}]_{(\frac{q_2}{q_1}, \frac{q_1}{q_2})} = 0. \qquad (2.5)_d$$

The last relation above is the deformed Jacobi relation, in CZ algebra which has the form:

$$[L_n, [L_m, L_l]_{(q^{l-m}, q^{m-l})}]_{(q^{m+l-2n}, q^{2n-m-l})} + cycl.perms. = 0. \tag{2.6}$$

The l.h.s. of (2.6) trivilly vanishes without any constraint to $L'_n s$. If L_n's satisfy the CZ algebraic relation (2.1), then the deformed Jacobi identity (2.6) becomes

$$[m-l][L_n, L_{m+l}]_{(q^{m+l-2n}, q^{2n-m-l})} + cycl.perms. = 0. \tag{2.7}$$

If we note the identity:

$$(q^n + q^{-n})[m-l][n-m-l] + cycl.perms. = 0, \tag{2.8}$$

the relation (2.7) becomes

$$[m-l][L_n, L_{m+l}]_{(q^{m+l}, q^{-m-l})} + cycl.perms. = 0. \tag{2.9}$$

This is just the braid Jacobi constraint condition given by Sato et.al [8]. Combination of (2.6) and (2.9), we get

$$(q^n + q^{-n})[L_n, [L_m, L_l]_{(q^{l-m}, q^{m-l})}]_{(q^{l+m-n}, q^{n-m-l})} + cycl.perms. = 0. \tag{2.10}$$

Now let us discuss the central extension of CZ algebra. We assume that the central extension of CZ algebra has the following form

$$[L_n, L_m]_{(q^{m-n}, q^{n-m})} = [n-m]L_{m+n} + \hat{C}(n, m). \tag{2.11}$$

We assume also that formula (2.10) is valid in the central extension case, then the substitution of (2.11) to (2.10) we get

$$(q^n + q^{-n})[L_n, [m-l]L_{m+l} + \hat{C}(m, l)]_{(q^{l+m-n}, q^{n-m-l})} + cycl.perms. = 0. \tag{2.12}$$

(2.12) gives

$$(q^n + q^{-n})[L_n, \hat{C}(m, n)]_{(q^{l+m-n}, q^{n-m-l})} + cycl.perms. = 0, \tag{2.13}$$

and

$$[m-l](q^n+q^{-n})\hat{C}(n,m+l)+cycl.perms.=0. \tag{2.14}$$

Considering the constraints

$$\hat{C}(n,0)=\hat{C}(1,-1)=0, \tag{2.15}$$

we can uniquely solve out $\hat{C}(n,m)$ from (2.13) and (2.14)

$$\hat{C}(n,m)=\hat{C}\cdot c(n,m)\delta_{m+n,0}, \tag{2.16}$$

where

$$\hat{C}=(1+(q-q^{-1})L_0)^2, \tag{2.17$_a$}$$

$$c(n,m)=\frac{c}{q^n+q^{-n}}\cdot\frac{[n][n-1][n+1]}{[3]!}, \tag{2.17$_b$}$$

and

$$[x]=[x][x-1]\cdots[1],\quad c=(q^2+q^{-2})C(2,-2) \tag{2.18}$$

So the CZ algebra with central extension has the form

$$[L_n,L_m]_{(q^{m-n},q^{n-m})}=[n-m]L_{n+m}$$
$$+(1+(q-q^{-1})L_0)^2\frac{[n]}{[2n]}\cdot\frac{c[n+1][n][n-1]}{[3]!}\delta_{m+n,0}. \tag{2.19}$$

3. The CILPP Deformation of Virasoro Algebra

The CILPP deformation related to the conformal dimension [11]. Under a conformal transformation, a primary field $\phi(z)$ with conformal dimension Δ becomes

$$\phi(z)\to(f'(z))^{\Delta}\phi(f(z)). \tag{3.1}$$

If

$$f(z)=z+\epsilon(z), \tag{3.2}$$

then

$$\delta\phi(z)=\epsilon(z)^{1-\Delta}\partial\cdot(\epsilon^{\Delta}\phi). \tag{3.3}$$

Now let $\epsilon = z^{n+1}$, then (3.3) becomes

$$\begin{aligned}\delta_n\phi(z) &= (z^{n+1}\partial + \Delta(n+1)z^n)\phi(z) \\ &= (z\partial + \Delta(n+1) - n)z^n\phi(z) \\ &= l_n\phi(z),\end{aligned} \tag{3.4}$$

$$l_n = (z\partial + \Delta(n+1) - n)z^n. \tag{3.5}$$

It is easy to check that l_n satisfy the relation of centraless virasoro algebra, namely:

$$[l_n, l_m] = (n-m)l_{m+n}. \tag{3.6}$$

Similar to (3.3), we can define the deformation of $\delta\phi(z)$ as

$$\delta^q(\phi(z)) = (\epsilon(z))^{1-\Delta} D_q(\epsilon(z)^\Delta\phi(z)), \tag{3.7}$$

where

$$D_q = \frac{1}{z}\frac{q^{z\partial} - q^{-z\partial}}{q - q^{-1}}. \tag{3.8}$$

Let also $\epsilon(z) = z^{n+1}$, the equation (3.7) becomes

$$\delta^q_n\phi(z) = L^{(\Delta)}_n(q)\phi(z)$$

$$= [z\partial + \Delta(n+1) - n]z^n\phi(z), \tag{3.9}$$

and

$$L^{(\Delta)}_n = [z\partial + \Delta(n+1) - n]z^n. \tag{3.10}$$

Formula (3.10) is just the operator realization of virasoro algebra given by CILPP, this realization depends on the conformal dimension Δ.

Now let us discuss the CILPP realization. The (p,q) commutators of $L^{(\Delta)}_n$'s are given by

$$[L^{(\Delta)}_n, L^{(\Delta)}_m]_{(x,y)} = \frac{1}{q-q^{-1}}(q^N(xq^{-n} - yq^{-m}) - q^{-N}(xq^n - yq^m))L^{(\Delta)}_{m+n}, \tag{3.11}$$

where N satisfies

$$[N] \equiv L^{(\Delta)}_0. \tag{3.12}$$

If the equation (3.11) hold, x and y must satisfy

$$x(q^{\Delta(n-m)+m}+q^{\Delta(m-n)-m}-q^{-\Delta(m+n)+m}-q^{\Delta(m+n)-m}$$

$$=y(q^{\Delta(n-m)-n}+q^{\Delta(m-n)+n}-q^{-\Delta(m+n)+n}-q^{\Delta(m+n)-n}). \tag{3.13}$$

We can see that when $\Delta=0,1$, the equation (3.13) holds for any values of x and y, but then $\Delta\neq 0.1$, the equation (3.13) holds only for the following values of x and y

$$x=[n(\Delta-1)][\Delta m], \qquad (3.14)_a$$

$$y=[m(\Delta-1)][\Delta n]. \qquad (3.14)_b$$

When $\Delta=0,1$, because there are no constraints on x and y, we can chose suitable values of x and y so that the structure constants in (3.11) become [m-n]. But for $\Delta\neq 0,1$, x and y are give by equations (3.14), there is no feedom for chosing other values of x and y, this makes the structure constants of $L_n^{(\Delta)}$ become operators, this, indeed, contradict to the virasoro algebra. For the case $\Delta\neq 0,1$, (3.11) can be written as

$$[L_n^{(\Delta)},L_m^{(\Delta)}]_{(S_{nm},S_{mn})}=[m-n]L_{m+n}^{(\Delta)}, \tag{3.15}$$

where

$$S_{mn}=\frac{q^{m-n}-q^{n-m}}{q^N(X(nm)q^{-n}-q^{-m})-q^{-N}(X(nm)q^{n}-q^{m})}, \qquad (3.16)_a$$

and

$$X(nm)=\frac{[n(\Delta-1)][\Delta m]}{[m(\Delta-1)][\Delta n]}. \qquad (3.16)_b$$

Though the structure constants become classical numbers, the deformation parameters S_{nm} change into operators, this can hardly be considered as a deformation of the algebra.

From the discussion above we see that the meaningful cases of CILPP deformation corespond to $\Delta=0,1$. In this case, because there are no constraints on x and y, so we can chose $xq^{-n}-yq^{-m}=0$, and $x=1$, then (3.11) becomes

$$[L_n^{(\Delta)},L_m^{(\Delta)}]_{(1,q^{m-n})}=[m-n]q^{-N_\Delta+m}L_{n+m}^{(\Delta)},(\Delta=0,1). \tag{3.17}$$

If we redefine

$$L_n = q^{N_\Delta} L_n^{(\Delta)} \quad (\Delta = 0, 1), \tag{3.18}$$

then we have

$$[L_n, L_m]_{(q^{n-m}, q^{m-n})} = [m - n] L_{m+n}. \tag{3.19}$$

This is equivalent to CZ algebra.

4. Is There Any Hopf Algebra Structure in CZ Algebra?

In this section, I will discuss the problem of whether existing hopf algebra structure in CZ algebra.

The hopf algebra is a associated algebra with unity. Let A be the hopf algebra, then there must exist three operations in A: coproduct Δ, antipode γ, and counity ϵ. They are defined as

$$\begin{aligned} \Delta : \quad & A \rightarrow \quad A \otimes A, \\ \gamma : \quad & A \rightarrow \quad A, \\ \epsilon : \quad & A \rightarrow \quad C. \end{aligned} \tag{4.1}$$

where C is a complex field. The three operations satisfy the following axiomes. For $\forall: \quad a, b \in A$ we have

$$\begin{aligned} i): \quad & (id \otimes \Delta)\Delta(a) && = (\Delta \otimes id)\Delta(a), \\ ii): \quad & m(id \otimes \gamma)\Delta(a) && = m(\gamma \otimes id)\Delta(a) = \epsilon(a) \cdot 1, \\ iii): \quad & (\epsilon \otimes id)\Delta(a) && = (id \otimes \epsilon)\Delta(a) = a. \end{aligned} \tag{4.2}$$

Where m is

$$m: \quad A \otimes A \rightarrow A. \tag{4.3}$$

For example $m(a \otimes b) = a \cdot b$.

If CZ algebra has the hopf algebra structure, then we can define Δ, γ, ϵ operations which satisfy (4.2), and the coproduct keep the relation (2.1), namely:

$$[\Delta L_n, \Delta L_m]_{(q^{m-n}, q^{n-m})} = [n - m] \Delta L_{m+n}. \tag{4.4}$$

If we want to search wether a algebra has the hopf algebra structure, it is important for us to search out a coproduct operation which keeps the original algebraic relations. usually, (4.2) can be satisfied easily.

At following, I want to discuss wether CZ algebra has the hopf algebra structure. First of all I want to search out a coproduct which satisfies equation (4.4).

I assume that the coproduct for generators of CZ algebra has the following general form:

$$\Delta L_n = \hat{f}(n,q) \otimes L_n + L_n \otimes \hat{g}(n,q), \tag{4.5}$$

where $\hat{f}(n,q)$ and $\hat{g}(n,q)$ satisfy

$$[\hat{f}(n,q), \hat{f}(m,q)] = [\hat{g}(n,q), \hat{g}(m,q)] = 0. \tag{4.6}$$

The correspondent antipode and counity are given by

$$\begin{aligned} \gamma(\hat{f}(n,q)) &= \hat{f}^{-1}(n,q), \\ \gamma(\hat{g}(n,q)) &= \hat{g}^{-1}(n,q), \\ \gamma(L_n) &= -\hat{f}^{-1} L_n \hat{g}^{-1}, \\ \epsilon(\hat{f}(n,q)) &= \epsilon(\hat{g}(n,q)) = 1, \\ \epsilon(L_n) &= 0. \end{aligned} \tag{4.7}$$

It is easy to check that the Δ, γ, ϵ defined by (4.5) and (4.7) satisfy the hopf algebra relations in (4.2). The main task following is searching $\hat{f}(n,q)$ and $\hat{g}(n,q)$ that satisfy the equation (4.4).

Inserting (4.5) into (4.4) we obtain

$$\begin{aligned} l.h.s. = {} & q^{m-n} \hat{f}(n,q)\hat{f}(m,q) \otimes L_n L_m - q^{n-m} \hat{f}(m,q)\hat{f}(n,q) \otimes L_m L_n \\ & + q^{m-n} \hat{f}(n,q) L_m \otimes L_n \hat{g}(m,q) - q^{n-m} L_m \hat{f}(n,q) \otimes \hat{g}(m,q) L_n \\ & + q^{m-n} L_n \hat{f}(m,q) \otimes \hat{g}(n,q) L_m - q^{n-m} \hat{f}(m,q) L_n \otimes L_m \hat{g}(n,q) \\ & + q^{m-n} L_n L_m \otimes \hat{g}(n,q)\hat{g}(m,q) - q^{n-m} L_m L_n \otimes \hat{g}(m,q)\hat{g}(n,q), \end{aligned} \tag{4.8$_a$}$$

$$r.h.s. = [n-m]\{\hat{f}(m+n,q) \otimes L_{n+m} + L_{n+m} \otimes \hat{g}(n+m,q)\}. \tag{4.8$_b$}$$

From (4.8) and (4.6) we find that $\hat{f}(n,q)$ and $\hat{g}(n,q)$ must satisfy the following relations.

$$\begin{aligned} \hat{f}(n,q)\hat{f}(m,q) &= \hat{f}(m+n,q), \\ \hat{g}(n,q)\hat{g}(m,q) &= \hat{g}(m+n,q), \end{aligned} \tag{4.9$_a$}$$

and

$$\begin{aligned} \hat{f}(n,q)L_m &= q^{-2m}K_{nm}L_m\hat{f}(n,q), \\ L_n\hat{g}(m,q) &= q^{2n}K_{mn}^{-1}\hat{g}(m,q)L_n, \end{aligned} \tag{4.9$_b$}$$

where K'_{mn}s are constants that depend on the m,n, and satisfy

$$K_{nm}K_{mn}^{-1} = 1. \tag{4.10}$$

From $(4.9)_{a,b}$ we find that $\hat{f}(n,q)$ and $\hat{g}(n,q)$ satisfy the same equations, so we only discuss $\hat{f}(n,q)$ in the following, discussion for $\hat{g}(n,q)$ is just the same.

From $(4.9)_{a,b}$ we have

$$\begin{aligned} \hat{f}(n_1,q)\hat{f}(n_2,q)L_m &\overset{(4.9)_a}{=} \hat{f}(n_1+n_2,q)L_m \\ &\overset{(4.9)_b}{=} q^{-2m}K_{n_1+n_2,m}L_m\hat{f}(n_1+n_2,q), \end{aligned}$$

on the other hand

$$\begin{aligned} \hat{f}(n_1,q)\hat{f}(n_2,q)L_m &\overset{(4.9)_b}{=} q^{-4m}K_{n_2,m}K_{n_1,m}L_m\hat{f}(n_1,q)\hat{f}(n_2,q) \\ &\overset{(4.9)_a}{=} q^{-4m}K_{n_2,m}K_{n_1,m}L_m\hat{f}(n_1+n_2,q). \end{aligned}$$

Compare the two equations above we obtain

$$K_{n_2,m}\cdot K_{n_1,m} = q^{2m}K_{n_1+n_2,m}. \tag{4.11}$$

The solution to (4.11) is

$$K_{n,m} = q^{\alpha(m)\cdot n+2m}. \tag{4.12}$$

From (4.10) we know that

$$K_{n,m} = K_{m,n}. \tag{4.13}$$

(4.12) and (4.13) give

$$\alpha(m) = \alpha m + 2, \tag{4.14}$$

where α is a constant. So the explicit form of $K_{n,m}$ is

$$K_{n,m} = q^{\alpha\cdot m\cdot n+2(n+m)}, \tag{4.15}$$

and the explicit form of $(4.9)_b$ is

$$\hat{f}(n,q)L_m = q^{\alpha mn+2n}L_m\hat{f}(n,q). \tag{4.16}$$

Multiplying both side of CZ algebra relation (2.1) with $\hat{f}(l,q)$ from righ t, one gets

$$(q^{m-n}L_nL_m - q^{n-m}L_mL_n)\hat{f}(l,q) = [n-m]L_{m+n}\hat{f}(l,q). \tag{4.17}$$

Using (4.16), we can take $\hat{f}(l,q)$ in (4.17) form right to the left, then multiplying $\hat{f}^{-1}(l,q)$ to both side on the left, we obtain

$$[L_n, L_m]_{(q^{m-n},q^{n-m})} = q^{2l}[n-m]L_{m+n}. \tag{4.18}$$

The surplus factor q^{2l} appears, which contradict to the CZ algebraic relation (2.1). In other words, $\hat{f}(n,q)$ which satisfies (4.16) does not exist, namely, the coproduct given by (4.5) does not exist.

If we chose

$$\Delta L_n = \hat{f}(n,q)\otimes L_n + L_n\otimes\hat{g}(n,q) + \hat{\beta}(n,q)1\otimes 1. \tag{4.20}$$

I find that any choice of $\hat{\beta}(n,q)$ can not cancel the contradiction said above. So there is no usual hopf algebra structure ((4.5) and (4.7)) in CZ algebra.

5. Conclusion Remark

To my knowledge, there are three kinds of deformations of virasoro algebra. They are CZ deformation, CILPP deformation and HMNS deformation. From the discussion in section 3, we know that the meaningful cases of CILPP deformation is $\Delta = 0, 1$, for general Δ cases, because it change the structure constants of the virasoro algebra, so it can not be treated as the deformation of virasoro algebra. The main problem in HMNS deformation is that one generator L_n changes into infinite generators under the deformation. Usually deformation does not change the number of generators of original algebra. Although HMNS found out the hopf algebra structure in their deformed algebra, the hopf algebra structure in correct q-deformation of virasoro algebra is still absent

After all, I think CZ algebra a best candidate for the q-deformation of the virasoro algebra by now. Unfortunately I find that there is no usually hopf algebra structure in CZ algebra. Does CZ algebra is unique deformed virasoro algebra which without hopf algebra structure? Or are there any other reasonable deformation of virasoro algebra which have the hopf algebra structure? These are all open problems.

Acknowledgements

Part of the work was accomplished during my stay in CCAST. I would like to thank professor H.Y.Guo and professor K.Wu for their guidance and help. I would also like to thank W.F.Chen, B.L.Wang, and Z.N. Hu for their useful discussion.

references

[1] V.G.Drinfeld, Sov.Math.Dokl. 32(1985)254

Quantum Groups, Proc.of ICM Berkeley, (1986)798

[2] M.Jimbo, Lett.Math.Phys. 10(1985)63; 11(1986)247;

Commun.Math.Phys. 102(1986)573

[3] L.Alvarez-Gaume,C.Gomez and G.Sierra, Phys.Lett.B220(1989)142;

Nucl.Phys. B319(1989)155

[4] B.Y.Hou,K.Li and P.Wang, J.Phys.A23(1990)3431

[5] B.Y.Hou,K.Li and P.Wang, High Ener.Nucl.Phys. 14(1990)711

[6] V.Pasquier and H.Saleurs, Nucl.Phys. B330(1990)523

[7] T.Curtright and C.Zachos, Phys.Lett.B243(1990)237

[8] N.Aizawa and H.Sato, Phys.lett. B256(1991)185

[9] H.Sato, YITP/K-930(1991)

[10] M.Chaichian, P.P.Kulish and J.Lukierski,Phys.Lett.B237(1990)401

[11] M.Chaichian, A.P.Isaev,J.Lukierski,Z.Popowicz and P.Presnajder, Phys.Lett. B262(1991)32

[12] M.chaichian,Z.Popowicz and P.presnajder,Phys.lett.B249(1990)63

[13] H.Hiro-oka, O.Matsui,T.Naito and S.Saito,TMUP-HEL-9004(1990)

[14] C.Kassel, ULP-IRMA-Preprint,August, 1991.

$\mathcal{ASITP}$-92-20
$\mathcal{CCAST}$-92-11
$\mathcal{ZIMP}$-92-05
March 1992

The symmetries of $Z_n \times Z_n$ Belavin model [1]

Rui-Hong Yue†§ and Yi-Xin Chen †‡

†CCAST (World Laboratory)
P.O.Box 8730, Beijing 100080, China

§Institute of Theoretical Physics, Academia Sinica
P.O.Box 2735, Beijing 100080, China [2]

‡Institute of Modern Physics, Zhejiang University
Hangzhou 310027, China[3]

Abstract

It is shown that there exist some new symmetries in $Z_n \times Z_n$ Belavin model. These symmetric properties can be used to construct the new exactly solvable statistical model with nontrivial boundary terms.

[1]Work supported in part by the National Natural Science Foundation of China and CCAST (World Laboratory).

[2]Mailing address.

[3]Permanent address.

1. Introduction

The quantum Yang-Baxter equation(QYBE) was discovered by Yang [1] and Baxter [2]. The solutions of QYBE have been classified as rational, trigonometric and elliptic according to their dependence of spectral parameters [3,4]. A host of investigation revealed that QYBE has played an important role in quantum field theories [5-7] and integrable statistical models [8-11]. Much attention was recently paid to the systems on a finite interval with independent boundary conditions on each end proposed by Sklyanian [12], Mezincescu and Nepomechie [13]. For a given trigonometric solution of QYBE based on the classical Lie algebra except $A_n(n > 1)$, it has been shown that the system with special boundary condition has the quantum group symmetry [12,13]. The trigonometric solution of the QYBE is a limit case of elliptic case. Minimal models in conformal field theory are closely related with the critical states of the elliptic RSOS [8]. The trigonometric limit of the Boltzmann weights for the elliptic case are identical with the fusion and braiding matrices in the minimal conformal field theory, which are the Racah coefficients of quantum group [5-7]. The Belavin's Z_n symmetric solution of QYBE is related with $A_{n-1}^{(1)}$ algebra, which may be used to study the systems with $U_q(A_{n-1})$ symmetry. This interesting relation motivates us to study the Belavin Z_n symmetric model with nontrivial boundary condition.

The symmetry of the solution of QYBE plays a key role in procedure of studying the systems on finite interval with independent boundary condition on each end by means of the method proposed by Sklyanian [12] and developed by Mezincescu and Nepomechie [13]. The Z_n elliptic solution of QYBE associated with the Belavin $Z_n \times Z_n$ symmetric model, which is the generalization of the Baxter's eight-vertex model. Some symmetries of the model was revealed in ref.[14]. The structure constants, both for the classical and for the quantum algebras, and fusion representation in this model were discussed [15]. Hou et al [16] had shown that there exists the quantum symmetric algebra in the Z_n elliptic solution of QYBE under the trigonometric limit. The trigonometric solution of QYBE based on the algebra $A_n(n > 1)$ does not satisfy the restrictive conditions suggested in [12] and [13].

The purpose of this paper is to show some new symmetries and useful properties of the Z_n elliptic solution of QYBE, which can be used to investigate the Belavin $Z_n \times Z_n$ symmetric model with nontrivial boundary condition.

2. Belavin $Z_n \times Z_n$ symmetric model.

The Belavin $Z_n \times Z_n$ symmetric model is the elliptic function solution of QYBE. There exist two equivalent forms of its expression [10,14]

$$\begin{aligned} R(u) &= \exp\{-i\pi u\} \cdot \sum_{\alpha \in Z_n^2} W_\alpha(u) I_\alpha \otimes I_\alpha^h \\ &= \sum_{ij,i'j'} S(u)_{ij}^{i'j'} E_{ii'} \otimes E_{jj'}, \end{aligned} \tag{1}$$

where, superscript h stands for the Hermitian conjugation. E_{ij} and $I_\alpha = h^{\alpha_1} g^{\alpha_2}$ are $n \times n$ matrices with the matrix elements

$$(E_{ij})_{kl} = \delta_{ki}\delta_{jl}, \tag{2}$$

$$(h)_{ij} = \delta_{i(\mathrm{mod}.n)}^{j+1}, \qquad g_{ik} = \omega^k \delta_{ij}, \tag{3}$$

ω is equal to $\exp(\frac{i2\pi}{n})$ and $\alpha = (\alpha_1, \ \alpha_2), \alpha_i = 0, 1, \cdots, n-1$. The coefficient $S(u)_{ij}^{kl}$ in Eq.(1) is called the Boltzmann weight and can be parametrised in terms of

Jacobi theta function

$$S(u)_{ij}^{kl} = S(u)_{i+p,j+p}^{k+p,l+p}$$

$$= \delta_{i+j,k+l}^{(\mathrm{mod}.n)} \cdot n \cdot \exp\{-i\pi u\} \frac{\theta\begin{bmatrix} \frac{l-k}{k}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u+w,n\tau)}{\theta\begin{bmatrix} \frac{i-k}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(w,n\tau)\cdot\theta\begin{bmatrix} \frac{l-i}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,n\tau)}$$

$$\times \frac{\Pi_{\lambda=0}^{n-1}\theta\begin{bmatrix} \frac{\lambda}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,n\tau)}{\Pi_{\lambda=1}^{n-1}\theta\begin{bmatrix} \frac{\lambda}{n}+\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0,n\tau)}. \tag{4}$$

In another expression form, $W_\alpha(u)$ in (1) reads

$$W_\alpha(u) = \frac{\theta\begin{bmatrix} \frac{\alpha_1}{n}+\frac{1}{2} \\ \frac{\alpha_2}{n}+\frac{1}{2} \end{bmatrix}(u+\frac{w}{n},\tau)}{\theta\begin{bmatrix} \frac{\alpha_1}{n}+\frac{1}{2} \\ \frac{\alpha_2}{n}+\frac{1}{2} \end{bmatrix}(\frac{w}{n},\tau)}. \tag{5}$$

The definition of Jacobi theta function is given by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u,\tau) = \sum_{m\in\mathcal{Z}} \exp\{i\pi\tau(m+a)^2 + i2\pi(m+a)(u+b)\}. \tag{6}$$

R matrix (1) satisfies the QYBE

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{7}$$

Introducing an operator

$$L(u) = \sum_{\alpha\in\mathcal{Z}_n^2} W_\alpha(u)I_\alpha S_\alpha, \tag{8}$$

we can rewrite the QYBE as

$$R_{12}(u-v)L^1(u)L^2(v) = L^2(v)L^1(u)R_{12}(u-v). \tag{9}$$

The relation (9) is equivalent to

$$\sum_{\gamma\in Z_n^2} \omega^{(\beta_1-\gamma_1)(\gamma_2-\alpha_2)} W_{\alpha\beta\gamma}(u,v) S_{\alpha+\beta-\gamma} S_\gamma = 0, \tag{10}$$

here

$$W_{\alpha\beta\gamma}(u,v) = W_{\gamma-\alpha}(u-v) W_{\alpha+\beta-\gamma}(u) W_\gamma(v) - W_{\beta-\gamma}(u-v) W_\gamma(u) W_{\alpha+\beta-\gamma}(v). \tag{11}$$

In order to simplify the equation (10), one can introduce the formal solution of $W_{\alpha\beta\gamma}$

$$\begin{aligned} W_{\alpha\beta\gamma}(u,v) &= C_{\alpha\beta\gamma}(u,v)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(u,\tau)\theta\begin{bmatrix} \frac{\alpha_1}{n}+\frac{1}{2} \\ \frac{\alpha_2}{n}+\frac{1}{2} \end{bmatrix}(u+v+\tfrac{2w}{n},\tau) \\ &\cdot\theta\begin{bmatrix} \frac{\beta_1}{n}+\frac{1}{2} \\ \frac{\beta_2}{n}+\frac{1}{2} \end{bmatrix}(v,\tau). \end{aligned} \tag{12}$$

It can be shown that the constant $C_{\alpha\beta\gamma}(u,v)$ is independent of the spectral parameters u and v [15]. Hence, Eq.(10) can be rewritten as the following algebraic relation

$$\sum_\gamma C_{\alpha\beta\gamma} S_{\alpha+\beta-\gamma} S_\gamma = 0. \quad \text{for } \alpha,\beta\in Z_n^2 \tag{13}$$

Since Eq.(13) can be reduced to the Sklyanin algebra when $n=2$, it is regarded as the generalization of the Sklyanin algebra.

3. The symmetries and the properties of the Belavin $Z_n \times Z_n$ symmetric model.

Richey and Tracy [14] had discussed the symmetries based on the invariances of the model. These symmetries are not enough to construct the exactly solvable statistical model with nontrivial boundary conditions. We must investigate new symmetries and some useful properties of R matrix of the model.

By directly calculating, one can show the relations $R(0)A \otimes BR(0) = n^2(B \otimes A)$ and $R^2(0) = n^2$, in which A and B are two arbitrary $n \times n$ matrices. From these relstions, we have the following consequence:

Proposition 1. The value of R(u) at u=0 is in proportion to the permutation operator

$$R(0) = nP \tag{14}$$

Proposition 2.

$$P_{12}R_{12}(u)P_{12} = R_{12}^{h_1 h_2}(u) \tag{15}$$

where, h_i represents the hermitian conjugation of the i-th vector space and $P_{12}(a \otimes b) = b \otimes a$.

This property of the R matrix is obvious. It means that the Z_n elliptic solution of QYBE possess the PT invariance

Proposition 3. The matrix R(u) has the unitary property

$$R_{12}(u)R_{12}^{h_1 h_2}(-u) = N(u,\tau)\mathrm{id} \tag{16}$$

here, id stands for the identical operator and

$$N(u,\tau) \;=\; n^2 \frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u+w,\tau)\,\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u+w,\tau)}{\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)}. \tag{17}$$

Proof. According to the definition of R(u), the left hand side of Eq.(16) is read as

$$
\begin{aligned}
R_{12}(u)R_{12}^{h_1h_2}(-u) &= \sum_{\alpha,\beta} W_\alpha(u)W_\beta(-u)I_\alpha I_\beta^h \otimes I_\alpha^h I_\beta \\
&= \sum_{\alpha,\beta} W_\alpha(u)W_\beta(-u)I_{\alpha-\beta} \otimes I_{\alpha-\beta}^h \omega^{\alpha_1\beta_2-\alpha_2\beta_1} \\
&= \sum_{\alpha\in Z_n^2}\left[\sum_{\beta\in Z_n^2} W_{\alpha+\beta}(u)W_\beta(-u)\omega^{\alpha_1\beta_2-\alpha_2\beta_1}\right] I_\alpha \otimes I_\alpha^h.
\end{aligned} \tag{18}
$$

Because of Z_n symmetry it is sufficient to consider S_{0a+b}^{ab} defined by (4), for $a, b \in Z_n$, or $S^{a,b}$ as we shall henceforth abbreviate. From (1) and (2), we have

$$W_{(a,b)}(u,\tau) = \frac{1}{n}\sum_{c\in Z_n} S^{-a,c}\omega^{bc}. \tag{19}$$

Thus, (18) can be rewritten as

$$R_{12}(u)R_{12}^{h_1h_2}(-u) = \frac{1}{n}\sum_{\gamma_1,\gamma_2\in Z_n^2} \omega^{c\gamma_2}\{\sum_{\beta\in Z_n^2} S^{-(\gamma-c)-b,b}(u,w,\tau)\cdot S^{c-b,-b-\gamma_2}(-u,w,\tau)\}. \tag{20}$$

Let

$$\varphi_{\sigma\sigma'}(u) = \sum_{\beta\in Z_n^2} S^{-(\sigma'-\sigma)-b,b}(u,w,\tau)\cdot S^{\sigma-b,-b-\sigma'}(-u,w,\tau). \tag{21}$$

Now, we want to determine the explicit expression of $\varphi_{\sigma\sigma'}(u)$, which can be obtained by exploring the properties of its zeros. By means of the transformation properties of the theta function, one can show that

$$
\begin{aligned}
S^{a,b}(u+\xi_1\tau+\xi_2,w,\tau) &= exp\{-i2\pi\xi_1^2\tau - i2\pi\xi_1(u+\tfrac{w}{n}+\tfrac{\tau}{2}+\tfrac{1}{2})\} \\
&\cdot\omega^{-a\xi_2}S^{a,b+\xi_1}(u,w,\tau)
\end{aligned} \tag{22}
$$

$$S^{a,b}(u+\xi_1\tau+\xi_2,w,\tau) = exp\{-i2\pi\frac{\xi_1}{n}u\}\omega^{b\xi_2}S^{a-\xi_1,b}(u,w,\tau) \tag{23}$$

Combining (22) and (23) with (21), we get the transformation relations of $\varphi_{\sigma\sigma'}(u)$

$$\varphi_{\sigma\sigma'}(u+1) = \omega^{\sigma'}\varphi_{\sigma\sigma'}(u) \tag{24}$$

$$\varphi_{\sigma\sigma'}(u+n\tau) = exp\{-i2\pi(n^2\tau + 2nu\}\varphi_{\sigma\sigma'}(u) \tag{25}$$

It is an elementary complex analysis argument that if f is entire, not identically zero, and satisfies

$$f(z+\tau) = exp\{-2\pi i(A_1 + A_2 z)\}f(z),$$

$$f(z+1) = exp\{-2\pi iB\}f(z).$$

Then necessarily A_2 is a positive integer, and f has zeros in Λ_τ with

$$\sum \text{zero} = \frac{1}{2}A_2 + B\tau - A_1 \qquad (\text{mod } \Lambda_\tau) ,$$

where, $\Lambda_\tau = \{\xi_1\tau + \xi_2 | \xi_1, \xi_2 \in \mathcal{Z}\}$ the lattice generated by 1 and τ. We apply this to $\varphi_{\sigma\sigma'}(u)$ to conclude that there are $2n$ zeros in $\Lambda_{n\tau}$ with sum

$$\sum \text{zero} = nu - n^2\tau - \sigma'\tau \qquad (\text{mod } \Lambda_{n\tau}) . \tag{26}$$

In order to determine $\varphi_{\sigma\sigma'}(u)$, we must find the locations of the 2n zeros. It is from the $S^{a,b}(0,w,\tau) = 0$ for $b \neq 0$ and the relation (22) that

$$S^{a,b}(k\tau, w, \tau) \sim S^{a,b+k}(0) = 0 \qquad for\ k \neq -b(n) . \tag{27}$$

This means that $\varphi_{\sigma\sigma'}(k\tau)$ $(k = 0, \cdots, n-1)$ is identically equal to zero unless $\sigma' = 0$. By using of (25), one can write $\varphi_{\sigma\sigma'}(u)$ at $u = -w + k\tau$ as

$$\begin{aligned} \varphi_{\sigma\sigma'}(-w+k\tau) &= exp\{-i2\pi k^2\tau - i2\pi(k\tau - w)\} \\ &\cdot \sum_b S^{\sigma-\sigma'-b,b+k}(-w,w,\tau) S^{\sigma-b,-b-\sigma'-k}(w,w,\tau) \end{aligned} \tag{28}$$

Recall the $S^{a,b}(u)$ is zero at $u = (a-b)\tau - w$. Thus the first factor in each term in the sum of (28) is zero if $b + k = \sigma - \sigma' - b$. The b satisfies this condition, which is read as

$$b = \begin{cases} \frac{1}{2}(\sigma - k - \sigma')(mod\ n), & for\ \sigma - \sigma' - k = even,\ and\ n\ =\ odd\ or\ even \\ \frac{1}{2}[\sigma - \sigma' - k(mod\ n)], & for\ \sigma - \sigma' - k = even,\ and\ n = even \\ \frac{1}{2}(n + \sigma - k - \sigma'), & for\ \sigma - \sigma' - k = odd,\ and\ n\ =\ odd, \\ no\ solutions, & for\ \sigma - \sigma' - k = odd,\ and\ n\ =\ even. \end{cases} \tag{29}$$

The $b's$ values of the remaining nonzero terms in the sum are expressed by $\{b \in Z_n | 2b \neq \sigma - \sigma' - k\} = \bigcup_i^x (b_i, -\sigma' - b_i + \sigma)$, where $x = \left[\frac{n}{2}\right]$ if n is odd or $2b = \sigma - \sigma' + k$ has no solutions or $x = \left[\frac{n}{2}\right] - 1$ if $2b = \sigma - \sigma' - k$. Thus the sum in (28) can be rewritten

$$\begin{aligned} \sum_b S^{\sigma-\sigma'-b,b+k}(-w,w,\tau)S^{\sigma-b,-b-\sigma'-k}(w,w,\tau) \\ = \sum_i S^{\sigma-\sigma'-b_i,b_i+k}(-w,w,\tau)S^{\sigma-b_i,-b_i-\sigma'-k}(w,w,\tau) \\ + \sum_i S^{k+b_i,\sigma-\sigma'-b_i}(-w,w,\tau)S^{k+\sigma'+b_i,b_i-\sigma}(w,w,\tau). \end{aligned} \tag{30}$$

Noting the fact that

$$S^{a,b}(-w,w,\tau) = -S^{b,a}(-w,w,\tau)$$

and

$$S^{a,b}(w,w,\tau) = -S^{-b,-a}(w,w,\tau),$$

we obtain that

$$\varphi_{\sigma\sigma'}(-w + k\tau) = 0, \quad k = 0,\ 1, \cdots, n-1. \tag{31}$$

From the equations (27) and (31), we immediately see that $\varphi_{\sigma\sigma'}(u)$ has the 2n zeros with the sum $n(n-1)\tau - nu$ if $\sigma' \neq 0$. Comparing it with (26), we conclude that

$$\varphi_{\sigma\sigma'}(u) \equiv 0 \qquad k = 0, 1, \cdots, n-1. \tag{32}$$

that is

$$\varphi_{\sigma\sigma'}(u) \equiv \delta_{\sigma',0}\varphi_{\sigma 0}(u) = \delta_{\sigma',0} \sum_{b\in Z_n} S^{\sigma-b,b}(u,w,\tau) S^{\sigma-b,-b}(-u,w,\tau) \tag{33}$$

In order to calculate $\varphi_{\sigma 0}$, we introduce a function

$$\psi_{\sigma\sigma'}(u) = \varphi_{\sigma 0}(u) - \varphi_{\sigma' 0}(u)$$

Because $\varphi_{\sigma 0}(u)$ and $\varphi_{\sigma'}(u)$ have the same transformation properties, $\psi_{\sigma\sigma'}(u)$ has 2n zeros with $\sum$ zero $= 0 \bmod \Lambda_{n\tau}$. On other hand, at $u = k\tau$, $\psi_{\sigma\sigma'}(u)$ is equal to zero. By using of (31), one can see that $\psi_{\sigma\sigma'}(u)$ has $2n$ zeros with the sum $-nu \bmod \Lambda_{n\tau}$. This results in that $\psi_{\sigma\sigma'}(u)$ is independent of σ. From the definition of $\varphi_{\sigma 0}(u)$, it is obvious that $\varphi_{\sigma 0}$ has the following transformation properties

$$\begin{aligned} \varphi_{\sigma 0}(u+1) &= \varphi_{\sigma 0}(u), \\ \varphi_{\sigma 0}(u+2\tau) &= exp\{-i2\pi(4\tau+4u)\}\varphi_{\sigma 0}(u). \end{aligned} \tag{34}$$

Hence, there are four zeros of $\varphi_{\sigma 0}(u)$ with $\sum$ zero $= 0 \bmod \Lambda_{2\tau}$. Two zeros are determined by Eq.(31), which are $-w$ and $-w+\tau$. By using of the properties of $S^{a,b}(u,w,\tau)$, one can show that $u = w + \tau$ is the zero of $\varphi_{\sigma 0}(u)$. The condition $\sum$ zero $= 0$ results in that the rest zero is w. It is now straightforward to prove that $N(u,w,\tau)$ has the same transformation properties and zero set as $\varphi_{\sigma 0}(u)$. By applying the elementary arguments of the entire function, we know that the difference between $\varphi_{\sigma 0}(u)$ and $N(u,w,\tau)$ is a scalar factor, which does not depend on the spectral parameter u. The factor is obtained by comparing $\varphi_{\sigma 0}(0)$ with $N(0,w,\tau)$. The result is 1. Now, (16) follows immediately.

Proposition 4. The R matrix satisfies the crossing unitary symmetry

$$\left[R_{12}^{h_1}(u)R_{12}^{h_2}(-u-nw)\right]^{h_1} = M(u,w,\tau)\mathrm{id}, \tag{35}$$

where,

$$M(u,w,\tau) = n^2\exp\{i\pi nw\}\frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u,\tau)\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u-nw,\tau)}{\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)}. \tag{36}$$

The proof of proposition 4 is very similar to that of proposition 3. We shall omit some analogous to above but rather long calculation.

Proof. Recalling the definition and some algebra properties of the R matrix, we can write the left hand side of (35) as

$$r.h.s. \text{ of } (35) = \sum_\gamma\sum_\beta W_{\beta+\gamma}(u)W_\beta(-u-nw)I_\gamma \otimes I_\gamma^h. \tag{37}$$

Substituting (19) into (37), we find that

$$\begin{aligned} F_\gamma(u) &= \sum_\beta W_{\beta+\gamma}(u)W_\beta(-u-nw) \\ &= \frac{1}{n}\cdot\sum_{\beta_1,a} S^{-\beta_1-\gamma_1,a}(u)S^{-\beta_1,-a}(-u-nw)\omega^{-a\gamma_2}. \end{aligned} \tag{38}$$

The following conclusion can be obtained by applying the some formulas in Ref.[14] and performing the same procedure in the proof of proposition 3, which is

$$F_\gamma(u) = M(u,w,\tau)\delta_{\gamma_1,0}\delta_{\gamma_2,0}. \tag{39}$$

Inserting (39) into (37), we get (35).

A direct corollary of proposition 4 is read as

$$R_{12}^{h_1}(u)R_{12}^{h_2}(-u-nw) = M(u,w,\tau)\mathrm{id}. \tag{40}$$

4. Remarks.

Sklyanin's method [12] and its generalization can be used not only to solve the exactly solvable models on a finite interval with independent boundary conditions on each end, but also to construct the statistical systems with the quantum group symmetry. It plays an important role in Sklyanin's formalism that the solution R of QYBE satisfies the symmetric properties. Sklyanin assumes

$$\begin{aligned}
&P_{12}R_{12}(u)P_{12} = R_{12}(u),\\
&R_{12}^{t_1}(u) = R_{12}^{t_2}(u),\\
&R_{12}(u)R_{12}(-u) = \rho(u)\mathrm{id},\\
&R_{12}^{t_1}(u)R_{12}^{t_2}(u-2\eta) = \tilde{\rho}(u)\mathrm{id},
\end{aligned} \tag{41}$$

where t_i denotes transposition in the ith vector space. The $\rho(u)$ and $\tilde{\rho}(u)$ are some scalar functions. Mezincescu and Nepomechie [13] extend Sklyanin's formalism to the case of a 'non-symmetric' R matrix, which satisfies the less restrictive conditions

$$\begin{aligned}
&P_{12}R_{12}(u)P_{12} = R_{12}^{t_1 t_2}(u),\\
&R_{12}(u) = \overset{1}{V}\, R_{12}^{t_2}(-u-\eta)\, \overset{1}{V}^{-1},\\
&R_{12}(u)R_{12}^{t_1 t_2}(-u) = \rho(u)\mathrm{id},\\
&R_{12}^{t_1}(u)\, \overset{1}{M}\, R_{12}^{t_2}(-u-2\eta)\, \overset{1}{M}^{-1} = \tilde{\rho}(u+\eta)\mathrm{id},
\end{aligned} \tag{42}$$

here $\overset{1}{V} = V \otimes 1$ and the forth relation can be derived from the second and third relations. Comparing the propositions shown by us with (41) and (42), we can see that the Z_n symmetric elliptic solution of QYBE does not satisfy (41) and (42). Hence, the symmetries (15), (16) and (35) can be regarded as the starting point to construct the exactly solvable Belavin's model with nontrivial boundary terms to solve it by means of the quantum inverse scattering method. It is emphasized that for the case of the trigonometric limit the Hamiltonian of this system should have the quantum group symmetry of $U_q(A_{n-1})$. The detail is reported elsewhere [17].

Acknowledgement

We would like to thank Profs. B.Y. Hou, K.J. Shi, K. Wu and P. Wang for valuable discussions. It is a pleasure to acknowledge the hospitality of CCAST(World Laboratory).

References

1. C.N. Yang, Phys. Rev. Lett. **19**(1967)1312.
2. R.J. Baxter, Ann. Phys. **70**(1972)193.
3. P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. **3** (1981) 393.
4. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl.Phys **B290** (1987)231.
5. B.Y. Hou, D.P. Lie and R.H. Yue, Phys. Lett. **B229**(1989)45.
6. G. Felder, J. Frohlich and G. Keller, Comm. Math. Phys. **124**(1989)717.
7. B.Y. Hou, K.J. Shi, P. Wang and R.H. Yue, Nucl. Phys. **B345**(1990)659.
8. G.E. Andrews, R.J. Baxter and P.J. Forester, J. Stat. Phys. **35**(1984)193.
9. E.K. Sklyanin, Funct. Anal. Appl. **16**(1982)263; **17**(1983)273.

10. A.A. Belavin, Nucl. Phys. **B180**(1980)109.

11. I.V. Cheredinik, Funct. Anal. Appl. **17**(1983)77; Theor. Math. Phys. **61**(1984)911.

12. E.K. Sklyanin, J. Phys. **A21**(1988)2375.

13. L. Mezincescu and R.I. Nepomechie, J. Phys. **A24**(1991)L17; Mod. Phys. Lett. **A6**(1991)2497.

14. M P Richey and C.A. Tracy, J. Stat. Phys. **42**(1986)311.

15. B.Y. Hou and H. Wei, J. Math. Phys. **30**(1989)2750.

16. B.Y. Hou, K.J. Shi and Z.X. Yang, preprint IMP-NWU-91.

17. R.H. Yue and Y.X. Chen, CCAST-92-12.

$\mathcal{ASITP}$-92-21
$\mathcal{CCAST}$-92-12
$\mathcal{ZIMP}$-92-06
March 1992

Integrable $Z_n \times Z_n$ Belavin model with nontrivial boundary terms[1]

Rui-Hong Yue†§ and Yi-Xin Chen †‡

†CCAST (World Laboratory)
P.O.Box 8730, Beijing 100080, China

§Institute of Theoretical Physics, Academia Sinica
P.O.Box 2735, Beijing 100080, China [2]

‡Institute of Modern Physics, Zhejiang University
Hangzhou 310027, China[3]

Abstract

The open chain corresponding to the Belavin model is constructed by generalizing Sklyanin's formalism to the case of the R matrix with Z_n symmetry.

[1]Work supported in part by the National Natural Science Foundation of China and CCAST (World Laboratory).

[2]Mailing address.

[3]Permanent address.

It is well known that the quantum Yang-Baxter equation(QYBE) plays a key role in the exactly solvable statistical models and the integrable field theory. Recently, the exact solutions of the QYBE have been studied fruitly [1-5]. One way to study exactly solvable statistical systems is the quantum inverse scattering method (QISM) which was initiated by Faddeev and Takhtajan [6]. Sklyanin [7] had solved the open spin-$\frac{1}{2}$ H_{xxz} model by generalizing QISM to the systems with independent boundary conditions on each end. This model with proper boundary conditions has the quantum group symmetry of $SU_q(2)$ [8]. Therefore, the sklyanin's method can be used to find the new exactly solvable statistical models with quantum group symmetries. In Sklyanin's paper [7], he assumes that R matrices possess the following properties

$$P_{12}R_{12}(u)P_{12} = R_{12}(u), \tag{1}$$

$$R_{12}^{t_1}(u) = R_{12}^{t_2}(u), \tag{2}$$

$$R_{12}(u)R_{12}(-u) = \rho(u)id, \tag{3}$$

$$R_{12}^{t_1}(u)R_{12}^{t_2}(u-2\eta) = \tilde{\rho}(u)id, \tag{4}$$

where t_i denotes transposition in the ith vector space and id an identical operator. The $\rho(u)$ and $\tilde{\rho}(u)$ are some scalar functions. Unfortunately, most of the solutions of QYBE do not satisfy the Sklyanin's assumption. Mezincescu and Nepomechie [9] extended Sklyanin's formalism to the systems with the PT symmetric R matrices. The restrictive conditions of this generalization are read as

$$P_{12}R_{12}(u)P_{12} = R_{12}^{t_1t_2}(u), \tag{5}$$

$$R_{12}(u) = \overset{1}{V} R_{12}^{t_2}(-u-\eta) \overset{1}{V}^{-1}, \tag{6}$$

$$R_{12}(u)R_{12}^{t_1t_2}(-u) = \rho(u)id, \tag{7}$$

$$R_{12}^{t_1}(u)\stackrel{1}{M}R_{12}^{t_2}(-u-2\eta)\stackrel{1}{M}^{-1}=\tilde{\rho}(u+\eta)id, \tag{8}$$

where $\stackrel{1}{V}$ stands for $V\otimes\mathbf{1}$, V is a matrix determined by the R matrix and $M=V^tV$. The condition (8) can be derived from (6) and (7). However, the R matrix based on A_n^1 for $n>1$ does not have the crossing symmetry (7). The spin open chains, which correspond to such R matrices not be treated directly by using of the Sklyanin's formalism and its generalization.

Because the Z_n symmetric solution of the QYBE is related with algebra A_{n-1}^1, to exploit the symmetric properties of the Belavin $Z_n\times Z_n$ symmetric model is helpful for solving the above open problem. We have recently shown [11] that the Belavin solution R of QYBE satisfies the following symmetries

$$P_{12}R_{12}(u)P_{12}=R_{12}^{h_1h_2}(u), \tag{9}$$

$$R_{12}(u)R_{12}^{h_1h_2}(-u)=\rho(u)id, \tag{10}$$

$$R_{12}^{h_1}(u)R_{12}^{h_2}(-u-nw)=\tilde{\rho}(u,w)id. \tag{11}$$

The superscript h_i denotes the hermitian conjugation in the i-th vector space and w is a new variable defined by $\eta=\frac{w}{n}+\frac{1}{2}+\frac{\tau}{2}$. It is obvious that the relations (9-11) are not equivalent with the Sklyanin's assumption and its generalization (5-8).

In this letter, we extend their formalisms to the case of the R matrix satisfying (9-11) to find the Hamiltonian of the Belavin model with independent boundary conditions. Recently, Hou et al had shown [12] that the quantum group $SL_q(n)$ can be considered as a limit of the quantum symmetric algebra in the $Z_n\times Z_n$ Belavin model, which is the generalized Sklyanin algebra [13]. Hence, the formalism developed in this paper can be used to construct the Hamilonian of spin chain with quantum group $SL_q(n)$ symmetry.

First of all, let us recall the fundmentals of the $Z_n \times Z_n$ Belavin model [2] and the major results in the our paper [11].

The Boltzmann weight of the $Z_n \times Z_n$ Belavin model can be written as

$$R_{jk}(u) = \sum_{\alpha \in Z_n^2} W_\alpha(u) I_\alpha^{(j)} I_\alpha^{(k)}, \tag{12}$$

where $I_\alpha^{(j)}$ acts on the subspace of the j-th site, $I_\alpha = h^{\alpha_1} g^{\alpha_2}$, h and g are the $n \times n$ matrices with elements

$$h_{ij} = \delta_{j \pmod n}^{k+1} \quad , \quad g_{jk} = \omega^k \delta_{jk}. \tag{13}$$

ω is equal to $exp\{\frac{i2\pi}{n}\}$. The Boltzmann coordinate $W_\alpha(u)$ can be expressed in terms of the Jacobi theta function of rational characteristics $(\frac{1}{2} + \frac{\alpha_1}{n}, \frac{1}{2} + \frac{\alpha_1}{n})$

$$\sigma_\alpha(u) \stackrel{\mathrm{def}}{=} \sum_{m=-\infty}^{\infty} exp\{i\pi\tau(m + \frac{1}{2} + \frac{\alpha_1}{n})^2 + i2\pi(m + \frac{1}{2} + \frac{\alpha_1}{n})(u + \frac{1}{2} + \frac{\alpha_2}{n})\}. \tag{14}$$

$W_\alpha(u)$ is read as

$$W_\alpha(u) = \frac{\sigma_\alpha(u+\eta)\sigma_0(\eta)}{\sigma_\alpha(\eta)\sigma_0(u+\eta)}. \tag{15}$$

The Boltzmann weights satisfy the QYBE

$$R_{12}(u-v)R_{13}(u)R_{23}(v) \quad = \quad R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{16}$$

The R matrix of the Belavin model satisfies the symmetries (9-11), in which the explicit expressions of the scalar functions are

$$\rho(u) \quad = \quad n^2 \frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u+w,\tau)\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u+w,\tau)}{\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)}, \tag{17}$$

and

$$\tilde{\rho}(u,w) = n^2 exp\{i\pi nw\} \frac{\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(u,\tau)\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-u-nw,\tau)}{\theta_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(w,\tau)}, \tag{18}$$

where $\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z,\tau) = \sigma_0(z)$.

In the operator representation, we can rewritten the QYBE (16) as

$$R_{12}(u-v)\overset{1}{L}(u)\overset{2}{L}(v) = \overset{2}{L}(v)\overset{1}{L}(u)R_{12}(u-v), \tag{19}$$

by introducing an operator

$$L(u) = \sum_{\alpha\in Z_n^2} W_\alpha(u)I_\alpha \otimes S_\alpha. \tag{20}$$

From Eq.(19), one can show that the quantum operator S_α is the operator of the generalized Sklyanin algebra [13].

In order to construct the Hamiltonian with the independent boundary conditions, we have to extend the Sklyanin formalism to the case of R matrix satisfying the restrictive conditions (9-11). We introduce two generalized algebras $\mathcal{T}_+$ and $\mathcal{T}_-$, which are defined by the following relations

$$\begin{aligned} R_{12}(u_-)\overset{1}{\mathcal{T}_-}(u_1)R_{12}^{h_1h_2}(u_+)\overset{2}{\mathcal{T}_-}(u_2) \\ = \overset{2}{\mathcal{T}_-}(u_2)R_{12}(u_+)\overset{1}{\mathcal{T}_-}(u_1)R_{12}^{h_1h_2}(u_-), \end{aligned} \tag{21}$$

and

$$\begin{aligned} R_{12}(-u_-)\overset{1}{\mathcal{T}_+^{h_1}}(u_1)R_{12}^{h_1h_2}(-u_+-nw)\overset{2}{\mathcal{T}_+^{h_2}}(u_2) \\ = \overset{2}{\mathcal{T}_+^{h_2}}(u_2)R_{12}(-u_+-nw)\overset{1}{\mathcal{T}_+^{h_1}}(u_1)R_{12}^{h_1h_2}(-u_-), \end{aligned} \tag{22}$$

where we have used the notation $u_\pm = u_1 \pm u_2$. These algebras, especially $\mathcal{T}_-$, are the fundamental of our construction. Our goal is to find the solution of Eqs.(21) and (22) for the R matrix given by (12) and (15). The Cherednik's work [14] gives an important hint to solve the problem. Define a matrix $\mathcal{K}(u)$ as

$$\mathcal{K}(u) = \frac{1}{n} \cdot \sum_{\alpha \in Z_n^2} W_{2\alpha}(u) \omega^{2\alpha_1\alpha_2} I_{2\alpha}. \tag{23}$$

The matrix $\mathcal{K}(u)$ satisfies the normalized condition $\mathcal{K}^2(0) = 1$. By using of the properties of Jacobi theta function, one can show that $\mathcal{K}_-(u) = \mathcal{K}(u)\mathcal{K}(0)$ is a representation of the algebra $\mathcal{T}_-$ and the mapping

$$\phi : \mathcal{K}_-(u) \longmapsto \mathcal{K}_+(u) = \mathcal{K}_-^h(-u - \frac{nw}{2}) \tag{24}$$

is isomorphic. The proof of the above conclusions is a direct but rather tedious calculation which we omit. It is pointed out that the existence of $\mathcal{K}(u)$ means that of the solution of Eqs.(21) and (22). If there exist the not equivalent solutions, they correspond to the spin chains with the different boundary terms.

As usual, the monodromy matrix $T(u)$ is given by

$$T(u) = L_n(u) \quad \cdots \quad L_1(u), \tag{25}$$

where

$$L_j(u) = \sum_{\alpha \in Z_n^2} W_\alpha(u) I_\alpha S_\alpha^{(j)} \tag{26}$$

and the superscript j denotes the quantum space acted by operator S_α. By directly calculating, we find that

$$\mathcal{T}_-(u) = T(u)\mathcal{K}_-(u)T^{-1}(u). \tag{27}$$

In the quantum inverse scattering method, the Hamiltonian of a system can be given by means of the transfer matrix. For the case of the open chain, we define the transfer matrix as

$$t(u) = Tr\mathcal{K}_+(u)\mathcal{T}_-(u). \tag{28}$$

By a suitable generalization of Sklyanin's arguments [7], it now follows that the $t(u)$ forms a commutative famely

$$[\ t(u)\ ,\ t(v)\ ] = 0. \tag{29}$$

The quantum space, acted by the operator $S^{(j)}$ is isomorphic to the auxiliary space and, furthermore, the operator $L_j(u)$ coincides with the matrix $R(u)$ on the direct product space of the quantum and auxiliary spaces, i.e.

$$L_j(u) = R_{0j}(u). \tag{30}$$

We know from the proposition 1 in [11] that if $R_{ij}(u)$ is normalized, the value of it at $u = 0$ is the permutation operator. Differentiating $t(u)$ with respect to u, one can find the Hamiltonian of the open chain

$$H = \sum_{j=1}^{n-1} H_{j,j+1} + \frac{1}{2}\overset{1}{\mathcal{K}'_-} + \frac{Tr_0 \overset{0}{\mathcal{K}_+}(0)H_{0,n}}{Tr\mathcal{K}_+(0)}, \tag{31}$$

where

$$H_{j,j+1} = P_{jj+1}R'_{jj+1}(u)|_{u=0}. \tag{32}$$

Substituting (12), (23) and $\mathcal{K}_\pm(u)$ into (31), we obtain the Hamiltonian of the Belavin model with independent boundary conditions

$$\begin{aligned} H \ = & \sum_{j=1}^{n-1} \sum_{\gamma,\beta\in Z_n^2} W'_\beta(0)\omega^{<\gamma,\beta>+\gamma_2\gamma_1} S_\gamma^{(j)} S_{-\gamma}^{(j+1)} \\ & +\frac{1}{2}\cdot\frac{1}{W_0(-\frac{nw}{2})}\cdot\sum_{\alpha,\beta,\gamma\in Z_n^2}(W_{2\alpha}(-\frac{nw}{2})W'_\beta(0) \\ & \cdot\delta_{2\alpha-\gamma,0\ (\mathrm{mod}\ n)}\omega^{2\alpha_1\alpha_2+\gamma_1\beta_2+\gamma_2\beta_1} S_\gamma^{(n)}). \end{aligned} \tag{33}$$

In conclusion, we have generalized Sklyanin's formalism for constructing integrable open chains to the case of R matrix satisfying (9-11). As a direct application of our extension, we have constructed the Hamiltonian of the open chain corresponding to the Belavin model.

Acknowledgement

We would like to thank Profs. B.Y. Hou, K.J. Shi, K. Wu and P. Wang for valuable discussions. It is a pleasure to acknowledge the hospitality of CCAST(World Laboratory).

References

1. R.J. Baxter,Exactly Solvable Models in Statistical Mechenics, (Academic London, 1982).
2. A.A. Belavin, Nucl. Phys. **B180**(1980)109.
3. B.Y. Hou, M.L. Yan and Y.K. Zhou, Nucl. Phys.**B324**(1989)715.
4. P.P. Kulish, N.Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. **3**(1981)393.
5. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. **B290**(1987) 231.
6. L.A. Takhtadzhan and L.D. Faddeev, Russ. Math. Surv.**34**(1979).
7. E.K. Sklyanin, J. Phys. **A21**(1988)2375.
8. N. Pasquier and H. Salcur, Nucl. Phys. **B330**(1990)523.
9. L. Mezincescu and R.I. Nepomechie, J. Phys. **A24**(1991)L17; Mod. Phys. Lett. **A6**(1991)2497.

10. V.V Bazhanov, Comm. Math. Phys. **113**(1987)471.

11. R.H. Yue and Y.X. Chen, AS-ITP-92-21.

12. B.Y. Hou, K.J. Shi and Z.X. Yang, preprint IMP-NWU-91.

13. B.Y. Hou and H. Wei, J. Math. Phys. **30**(1989)2750.

14. I.V. Cheredinik, **17**(1983)77; Theor. Math. Phys. **61**(1984)911.

QUANTUM DEFORMATION OF KDV HIERARCHIES AND THEIR INFINITELY MANY CONSERVATION LAWS

De-Hai ZHANG

Center of Theoretical Physics,CCAST (World Laboratory)
and
Physics Department
Graduate School, Academia Sinica
P.O.Box 3908, Beijing 100039, P.R.China

ABSTRACT

A reasonable q-deformed differential is defined. A set of operation rules are constructed for the q-deformed pseudo differential operators. The complete procedure of constructing the q-deformed KdV hierarchies is given. As an important example, we obtain the detailed structure and the infinite conservation laws of the simplest (3,2) system, i.e. the q-deformed KdV equations.

1.Introduction

Recently, 1+1 dimensional Korteweg-de Vries (KdV) hierarchy [1] arouses the strong interests of the theoretical physicists . A lot of researches demonstrate that the KdV hierarchy is closely related to the following popular topics:
1) The matrix models [2] and the non-perturbative treatment of 2 dimensional field theories [3], 2) The theories of 2 dimensional gravity coupled to matter systems [4], 3) The 2 dimensional topological field theories [5], 4) The conformal field theory [6] and W algebras [7]. The basic equations governing non-perturbative 2 dimensional gravity coupled to minimal models are the differential equations of KdV hierarchy. The partition function and the correlation functions of the 2 dimensional topological gravity coupled to minimal models are guessed to be described by the KdV hierarchy. The KdV hierarchy shows the miraculous power and mysterious relations in treatment of different mathematics and physics objects.

In the other hand, the interests of the quantum deformation (so-called q-deformation) of Lie algebra (quantum group) has been growing in the physical and mathematical regions [8]. The idea of quantum Lie algebras originated from the study of the solution of the quantum Yang-Baxter equation for the integrable lattice models [9]. The representation theory of the q-deformed simple Lie algebras has been investigating widely [10]. One of the methods well worth paying attention to in study of quantum group is the q-harmonic oscillator realization of quantum groups [11]. Several authors have extended the definition of q-differentiation [12][13].

The success of quantum groups stimulates people to look for new objects which can performed the analogous so-called q-deformation. For example, the q-deformed Virasoro algebra has been studied in refs. [14] and [15]. Chaichian, Popowicz and Presnajder even researched the q-deformed KdV system [16]. However, it is hard to say that all of these attempts has been accomplished perfectly.

For a long time past it is quite meaningful to discover a new kind of integrable systems. In this paper we shall do a new investigation about the q-deformed KdV hierarchies by defining the suitable q-differentiation. Through building a set of complete operations of the q-deformed pseudo differential operator and using the Lax pair, we present the constructing program of the q-deformed KdV hierarchies and obtain the q-deformed generalization of the ordinary KdV equation. It is believable that the q-deformed KdV hierarchies we gained is a new kind of the integrable system.

2. The q-deformed formal pseudo differential operator

At the first we introduce two operators, Q and $\tilde{Q}$, which are defined as

$$Qf(z) = f(zq), \tag{2.1}$$

$$\tilde{Q}f(z) = q^{z\partial} f(z), \tag{2.2}$$

where $\partial = \frac{\partial}{\partial z}$ and q is called a deformed parameter. For avoiding complexity q is limited to be a real parameter which is not −1. One can prove that $\tilde{Q}$ is equal to Q. In order to get this conclusion one rewrite the operator Q as a formal differential operator with infinite order of the ordinary differential operator ∂,

$$Qf(z) = f(z - \epsilon z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-\epsilon z)^n \partial^n f(z), \tag{2.3}$$

therefore

$$Q = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \epsilon^n z^n \partial^n, \tag{2.4}$$

where

$$\epsilon = 1 - q. \tag{2.5}$$

One can also rewrite the operator $\tilde{Q}$ as an infinite order differential operator

$$\tilde{Q} = \exp[z\partial \ln(1-\epsilon)] = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \left(\sum_{m=1}^{\infty} \frac{1}{m}\epsilon^m \right)^n (z\partial)^n. \tag{2.6}$$

Considering the commutator

$$[\partial, z] = 1, \tag{2.7}$$

one gets the relations between $z^n\partial^n$ and $(z\partial)^n$,

$$\begin{aligned} z^2\partial^2 &= (z\partial)^2 - z\partial, \\ z^3\partial^3 &= (z\partial)^3 - 3(z\partial)^2 + 2z\partial, \\ z^4\partial^4 &= (z\partial)^4 - 6(z\partial)^3 + 11(z\partial)^2 - 6z\partial, \end{aligned} \tag{2.8}$$

and so on. Expanding $\tilde{Q}$ of the formula (2.6) by ϵ powers and using the relations (2.8), one observes that the operator $\tilde{Q}$ is indeed same with the operator Q,

$$\tilde{Q} = Q. \tag{2.9}$$

After owning the infinite order differential operator Q (or $\tilde{Q}$ without distinction), we define the q-deformed differential operator [17]

$$\tilde{D} = \frac{1}{(1-q^{-2})z}(1-Q^{-2}), \tag{2.10}$$

which is also a formal differential operator with infinite order. It is easy to see that $\tilde{D}$ tends to ∂ when q tends to 1,

$$\lim_{q\to 1} \tilde{D} = \partial. \tag{2.11}$$

The commutative relation between Q and $\tilde{D}$ is described by the q-deformed commutator

$$[\tilde{D}, Q]_q = 0, \tag{2.12}$$

where

$$[A, B]_r = AB - rBA. \tag{2.13}$$

When r=1 one omits this 1 and denotes it simply as

$$[A, B] = AB - BA, \tag{2.14a}$$

and when r=–1 it becomes the ordinary anticommutator

$$\{A, B\} = AB + BA. \tag{2.14b}$$

According to the definition (2.10) of $\tilde{D}$, one can prove the q-deformed Leibniz rule

$$\tilde{D}(f(z)g(z)) = (\tilde{D}f(z))g(z) + (Q^{-2}f(z))(\tilde{D}g(z)), \tag{2.15}$$

which can be expressed in an operator form

$$\tilde{D} \circ f = f^{(0,-2)}\tilde{D} + f^{(1,0)}. \tag{2.16}$$

Here o represents a fact that the $\tilde{D}$ before o must acts on the other functions behind $f(z)$. In the above formulas one introduces the following symbol

$$f^{(n,m)}(z) = (\tilde{D}^n Q^m f(z)), \tag{2.17}$$

where $\tilde{D}$ and Q in the bracket do not act on the functions behind $f(z)$. For example, one has $z^{(1,0)} = 1$. Using the formula (2.16) one gets the q-deformed commutator between the q-deformed differential operator $\tilde{D}$ and the coordinate variable z,

$$[\tilde{D}, z]_{q^{-2}} = 1. \tag{2.18}$$

It is critical important to note that any q-deformed algebra must satisfy the q-deformed Jacobi identity[18],

$$[A, [B, C]_{q_1}]_{q_2} + q_2[B, [C, A]_{q_1}]_{q_2^{-1}} + [C, [A, B]_{q_1 q_2}] = 0. \tag{2.19}$$

By inserting $q_1 = q^{-2}$, $q_2 = 1$ and taking A, B, C as $\tilde{D}$, $\tilde{D}$, z obeyed the relation (2.18), one can check that the q-deformed Jacobi identity (2.19) is indeed satisfied. This fact demonstrates the consistency of the definition of the q-deformed differential operator $\tilde{D}$. Applying the q-deformed Leibniz rule (2.16) to high order case of the q-deformed differential operator, one obtains

$$\tilde{D}^n \circ f(z) = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right] q^{2m(n-m)} f^{(m,2m-2n)}(z) \tilde{D}^{n-m}, \tag{2.20}$$

where

$$\left[\begin{array}{c} n \\ m \end{array} \right] = \frac{[n]!}{[m]![n-m]!}, \tag{2.21}$$

$$[m]! = [m][m-1]\cdots[2][1], \tag{2.22}$$

$$[m] = \frac{1 - q^{-2m}}{1 - q^{-2}} = q^{1-m} \lfloor m \rfloor, \tag{2.23}$$

and

$$[0]! = 0! = 1. \tag{2.24}$$

In (2.23),the symbol

$$\lfloor m \rfloor = (q^m - q^{-m})(q - q^{-1})^{-1} \tag{2.25}$$

is just the ordinary q-deformed one used often in references. One can generalize (2.20) to the case of negative n,

$$\tilde{D}^{-n} \circ f(z) = \sum_{m=0}^{\infty} (-1)^m \left[\begin{array}{c} n+m-1 \\ m \end{array} \right] q^{-m(m+1)} f^{(m,2m+2n)}(z) \tilde{D}^{-n-m}. \tag{2.26}$$

Specially for the case of n=−1,one has

$$\tilde{D}^{-1} \circ f = f^{(0,2)} \tilde{D}^{-1} - q^{-2} f^{(1,4)} \tilde{D}^{-2} + q^{-6} f^{(2,6)} \tilde{D}^{-3} - \cdots. \tag{2.27}$$

It is necessary to note that the q-deformed differential operators differ from the ordinary difference operators. Although the q-deformed differential operator is likely a difference operator at the level of the first order form, the second order q-deformed differential operator

$$\tilde{D}^2 f(z) = \frac{f(z) - (1+q^2)f(zq^{-2}) + q^2 f(zq^{-4})}{(1-q^{-2})^2 z^2}$$

differs from a two order difference operator, due to its equal-ratio distances between points and its non-standard coefficients. Up to now we have described all of operations needed for construction of the q-deformed KdV hierarchies.

The q-deformed KdV hierarchy is described in terms of the q-deformed pseudo differential operator which is a formal expression

$$K = \sum_{n=-\infty}^{M} k_n \tilde{D}^n, \tag{2.28}$$

where the coefficients are functions $k_n(z)$ in a variable z and $\tilde{D}$ is defined as (2.10). The multiplicative rule of two q-deformed pseudo differential operators has been given by formulas (2.20) and (2.26) . One further introduces the decomposition

$$K_+ = \sum_{n=0}^{M} k_n \tilde{D}^n, \tag{2.29}$$

$$K_- = \sum_{n=1}^{\infty} k_{-n} \tilde{D}^{-n}, \tag{2.30}$$

$$r\tilde{e}sK = k_{-1}, \tag{2.31}$$

where "$r\tilde{e}s$" stands for the q-deformed residue.

3.The q-deformed KdV hierarchy and the q-deformed KdV equation

The N^{th} q-deformed KdV hierarchy consists of an infinite set of commuting q-deformed differential equations for the coefficients $V_n(z, t_p)$ $(n = 0, 1, \cdots, N-1)$ of a q-deformed differential operator L of order N that has been put in the canonical form

$$L = \tilde{D}^N + \sum_{n=0}^{N-1} V_n \tilde{D}^n. \tag{3.1}$$

In the algebra of q-deformed pseudo differential operators L has an unique N^{th} root $L^{1/N}$, and in the La·· representation [19] the p^{th} flow of the N^{th} q-deformed KdV hierarchy (called the (p, N) system) is given by

$$\frac{\partial}{\partial t_p} L = [L_+^{p/N}, L], \tag{3.2}$$

where t_p are called time parameters. Since L commutes with $L^{p/N}$, one has

$$[L_+^{p/N}, L] = [L, L_-^{p/N}]. \tag{3.3}$$

But since from the l.h.s. above the commutator can have only positive powers of $\tilde{D}$, and since from the r.h.s. above the highest order term is only up to one of $\tilde{D}^{N-1}$, the expression(3.3) is only the $N-1$ order q-deformed differential operator without negative powers of $\tilde{D}$. When expanded in powers of $\tilde{D}$ this operator equation (3.2) gives rise to a single q-deformed differential equation for each of the coefficients V_n. It should be expected that the q-deformed KdV hierarchies are the completely integrable systems which have the infinitely many conservation laws.

The simplest system of the (p, N) q-deformed KdV hierarchies must be the (3,2) system which is called q-deformed KdV equations. Let us give this system in order to illustrate the above procedure. This model is obtained by taking L to be the two order q-deformed differential operator

$$K = \tilde{D}^2 + V_1(z,t)\tilde{D} + V_0(z,t). \tag{3.4}$$

The formal expansion of $L^{1/2}$ in powers of D is given by

$$K^{1/2} = \tilde{D} + \sum_{n=0}^{\infty} W_{-n}\tilde{D}^{-n}. \tag{3.5}$$

Since one needs only the first five coefficients of W_{-n} in the later q-deformed KdV equations, one gives them in terms of V_1 and V_0 order by order,

$$W_0 = (1+Q^{-2})^{-1}V_1 = \sum_{n=0}^{\infty}(-1)^n V_1^{(0,-2n)}, \tag{3.6}$$

$$W_{-1} = -(1+Q^{-2})^{-1}(-V_0 + W_0^{(1,0)} + W_0^2), \tag{3.7}$$

$$W_{-2} = -(1+Q^{-2})^{-1}(W_{-1}W_0^{(0,2)} + W_{-1}^{(1,0)} + W_0W_{-1}), \tag{3.8}$$

$$\begin{aligned} W_{-3} = -(1+Q^{-2})^{-1}(&-q^{-2}W_{-1}W_0^{(1,4)} + W_{-2}W_0^{(0,4)} + W_{-1}W_{-1}^{(0,2)} \\ &+W_{-2}^{(1,0)} + W_0W_{-2}), \end{aligned} \tag{3.9}$$

$$W_{-4} = -(1+Q^{-2})^{-1}(q^{-6}W_{-1}W_0^{(2,6)} - q^{-2}[2]W_{-2}W_0^{(1,6)} + W_{-3}W_0^{(0,6)}$$

$$-q^{-2}W_{-1}W_{-1}^{(1,4)} + W_{-2}W_{-1}^{(0,4)} + W_{-1}W_{-2}^{(0,2)} + W_{-3}^{(1,0)} + W_0W_{-3}). \tag{3.10}$$

The first formula of (3.6) is regarded as one of the methods to calculate operator $(1+Q^{-2})^{-1}$. The q-deformed differential operator needed in the (3,2) system is $L_+^{3/2}$. Due to the identity (3.3), one needs only the first two terms of the q-deformed pseudo differential operator $L_-^{3/2}$,

$$K_-^{3/2} = U_{-1}\tilde{D}^{-1} + U_{-2}\tilde{D}^{-2} + \cdots, \tag{3.11}$$

where the coefficients U_{-1} and U_{-2} are given by following expressions,

$$U_{-1} = W_{-3} - q^{-2}W_{-1}V_1^{(1,4)} + W_{-2}V_1^{(0,4)} + W_{-1}V_0^{(0,2)}, \tag{3.12}$$

$$U_{-2} = W_{-4} + q^{-6}W_{-1}V_1^{(2,6)} - q^{-2}[2]W_{-2}V_1^{(1,6)} + W_{-3}V_1^{(0,6)} \\ -q^{-2}W_{-1}V_0^{(1,4)} + W_{-2}V_0^{(0,4)}. \tag{3.13}$$

Now we can obtain the q-deformed KdV equations by using equations (3.2), (3.3) and (3.11),

$$\frac{\partial V_1}{\partial t} = U_{-1}^{(0,-4)} - U_{-1}, \tag{3.14}$$

$$\frac{\partial V_0}{\partial t} = (U_{-2}^{(0,-4)} - U_{-2}) + V_1(U_{-1}^{(0,-2)} - U_{-1}) \\ -U_{-1}(V_1^{(0,-2)} - V_1) + q^2[2]U_{-1}^{(1,-2)}. \tag{3.15}$$

Through the equations (3.12-13) and (3.6-10) the right sides of the equations (3.14-15) can be expressed in terms of pure V_1 and V_0 finally. Because the q-deformed differential operators are not the ordinary difference operators, the q-deformed KdV equations are not the ordinary differencing of the ordinary KdV equation. It is a new kind of integrable systems.

4.The expanding expression of the q-deformed differential operators

Usually we know well the ordinary differential operators and are not familiar with the q-deformed differential operators . In this section our main task is to express the various kinds of q-deformed operators in terms of the ordinary differential operators. From the definition (2.1) of Q, we have

$$Q^m f(z) = f(z + (q^m - 1)z) = \sum_{n=0}^{\infty} \frac{1}{n!}(q^m - 1)^n z^n \partial^n f(z), \tag{4.1}$$

therefore we obtain

$$Q^m = \sum_{n=0}^{\infty} \frac{1}{n!}(q^m - 1)^n z^n \partial^n. \tag{4.2}$$

As for the operator $\tilde{D}$, according to (2.10), one gets

$$\tilde{D}^m = \left(\frac{1}{(1-q^{-2})z}(1 - Q^{-2})\right)^m, \tag{4.3}$$

which can be deformed in the following way,

$$\tilde{D}^m = \frac{1}{(1-q^{-2})^m z^m} \prod_{i=0}^{m-1} (1 - q^{2i} Q^{-2}) \tag{4.4}$$

$$= \frac{1}{(1-q^{-2})^m z^m} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{j(2m-j-1)} Q^{-2j} \tag{4.5}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} \frac{q^{j(2m-j-1)}}{(1-q^{-2})^m} (q^{-2j} - 1)^n \right) \frac{1}{n!} z^{n-m} \partial^n \tag{4.6}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^{k+n} [k+m]!}{(k+m)!(n-k)![k]!} \right) z^n \partial^{n+m}, (m \geq 0) \tag{4.7}$$

and finally one obtains

$$\tilde{D}^m = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^{k+n} \tilde{\Gamma}(k+m+1)}{\Gamma(k+m+1)(n-k)![k]!} \right) z^n \partial^{n+m}. \tag{4.8}$$

Here we introduce the ordinary Gamma function

$$\Gamma(n+1) = n\Gamma(n), \tag{4.9}$$

$$\Gamma(n+1) = n!, \quad (n \geq 0) \tag{4.10}$$

$$\Gamma(-n) = \frac{(-1)^n}{n!} \Gamma(0), \quad (n \geq 0) \tag{4.11}$$

and the q-deformed Gamma function

$$\tilde{\Gamma}(n+1) = [n]\tilde{\Gamma}(n), \tag{4.12}$$

$$\tilde{\Gamma}(n+1) = [n]!, \quad (n \geq 0) \tag{4.13}$$

$$\tilde{\Gamma}(-n) = \frac{(-1)^n}{[n]!} q^{-n(n+1)} \tilde{\Gamma}(0), \quad (n \geq 0) \tag{4.14}$$

where the $\Gamma(0)$ is infinity which is a formal symbol and the $\tilde{\Gamma}(0)$ is to be determined. Defining

$$\Gamma_q = \frac{\tilde{\Gamma}(0)}{\Gamma(0)}, \tag{4.15}$$

one has

$$\frac{\tilde{\Gamma}(-n+1)}{\Gamma(-n+1)} = \frac{(n-1)!}{[n-1]!} q^{-n(n-1)} \Gamma_q. \quad (n \geq 1) \tag{4.16}$$

The above formulae get ready for extending the expansion (4.8) to the case of negative powers of the q-deformed differential operator. We find

$$\tilde{D}^{-m} = \left(\sum_{n=0}^{m-1} \sum_{k=0}^{n} + \sum_{n=m}^{\infty} \sum_{k=0}^{m-1} + \sum_{n=m}^{\infty} \sum_{k=m}^{n} \right)$$
$$\frac{(-1)^{k+n} \tilde{\Gamma}(k-m+1)}{\Gamma(k-m+1)(n-k)![k]!} z^n \partial^{n-m} \tag{4.17}$$

$$= \left(\sum_{n=0}^{m-1} \sum_{k=0}^{n} + \sum_{n=m}^{\infty} \sum_{k=0}^{m-1} \right) \frac{(-1)^{k+n}(m-k-1)! q^{-(m-k)(m-k-1)} \Gamma_q}{[m-k-1]!(n-k)![k]!} z^n \partial^{n-m}$$
$$+ \sum_{n=m}^{\infty} \sum_{k=m}^{n} \frac{(-1)^{k+n}[k-m]!}{(k-m)!(n-k)![k]!} z^n \partial^{n-m}. \tag{4.18}$$

A task is to determine the quantity Γ_q. Let us to inspect a simple case

$$\tilde{D} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \omega^n z^n \partial^{n+1}, \tag{4.19}$$

where

$$\omega = q^{-2} - 1. \tag{4.20}$$

After moving a differential operator from right end to left end, we obtain

$$\tilde{D} = \partial \left(\sum_{n=1}^{\infty} a_n \frac{1}{n!} z^n \partial^n + a_0 \right), \tag{4.21}$$

where

$$a_n = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^{n+k}}{n+k+1}, \tag{4.22}$$

and

$$a_0 = \sum_{k=0}^{\infty} \frac{(-1)^k \omega^k}{k+1} = \frac{\ln(1+\omega)}{\omega}. \tag{4.23}$$

It is easy to get the inverse of $\tilde{D}$ from (4.21)

$$\tilde{D}^{-1} = \left(1 + \sum_{n=1}^{\infty} \frac{a_n}{n! a_0} z^n \partial^n\right)^{-1} a_0^{-1} \partial^{-1}. \tag{4.24}$$

Comparing the above result with the general expansion (4.18) one finds

$$\Gamma_q = a_0^{-1} = \frac{1 - q^{-2}}{2 \ln q}. \tag{4.25}$$

After getting an expansion expression in terms of the ordinary differential operators, one can take the ordinary residue

$$resB = b_{-1} \tag{4.26}$$

for an ordinary formal pseudo differential operator

$$B = \sum_{n=-\infty}^{\infty} b_n \partial^n. \tag{4.27}$$

According this definition of the ordinary residue and the results of (4.18), one can obtain

$$res(\tilde{D}^{-m}) = \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} q^{-(m-k)(m-k-1)} \Gamma_q}{[k]![m-k-1]!} z^{m-1}. \tag{4.28}$$

5.The infinite conservation laws

The one of the most important properties of some integrable systems is that they possess the infinite conservation laws. We shall prove that the q-deformed KdV hierarchies have the infinite conservation laws in this section. Our method comes from one of Drinfeld and Sokolov. Since the cases are very similar each other, reader can refer [20] for detail. We can prove that the flows determined by the Lax equations commute with one another. If

$$\frac{dL}{dt} = [A, L], \tag{5.1}$$

where L is a q-deformed differential operator (3.1), then one has

$$\frac{d}{dt} L^{r/k} = [A, L^{r/k}]. \tag{5.2}$$

Let us to consider the equations

$$\frac{\partial L}{\partial t} = [M_+, L],$$

$$M = \sum c_i L^{i/k}, \tag{5.3}$$

and

$$\frac{\partial L}{\partial \tau} = [\tilde{M}_+, L],$$

$$\tilde{M} = \sum \tilde{c}_i L^{i/k}. \tag{5.4}$$

It is can be verified that

$$\frac{\partial^2 L}{\partial t \partial \tau} = \frac{\partial^2 L}{\partial \tau \partial t} \tag{5.5}$$

which demonstrates the consistency of the q-deformed KdV hierarchies.

Drinfeld and Sokolov have pointed out that for P, Q being formal ordinary pseudo differential operators the $res[P,Q]$ is a total derivative of some differential polynomial in the coefficients of P and Q. In our case the polynomial will be infinite order. In order to understand this conclusion it is suffices to consider the case where

$$P = a\partial^m, \qquad Q = b\partial^l. \tag{5.6}$$

One then gets

$$res[P,Q] = g', \tag{5.7}$$

where

$$g = \frac{m(m-1)\cdots(1-l)(-l)}{(m+l+1)!} \sum_{n=0}^{m+l} (-1)^i a^{(i)} b^{(m+l-i)}, \tag{5.8}$$

$$a^{(i)} = (\partial^i a). \tag{5.9}$$

Taking the ordinary residue on both sides of equation (5.2) one obtains

$$\frac{d}{dt} res L^{r/k} = res[A, L^{r/k}] = \frac{\partial f}{\partial z}. \tag{5.10}$$

Integrating it and choosing suitable boundary condition one has

$$\frac{d}{dt} \int dz res L^{r/k} = \int \frac{\partial f}{\partial z} dz = 0. \tag{5.11}$$

Therefore we see that for any integer r , the residue

$$H_{r/k} = resL^{r/k} \tag{5.12}$$

is a density of conservation law for the Lax equation. Of course, nontrivial conservation laws

$$C_{r/k} = \int dzresL^{r/k} \tag{5.13}$$

correspond only to numbers r not a multiple of k. If one knows the expansion expression of the flows in terms of the q-deformed differential operators

$$L^{r/k} = \sum_{n=-\infty}^{r} a_n \tilde{D}^n, \tag{5.14}$$

one can obtain the density of the conservation law in terms of the coefficients of their expansion expression

$$H_{r/k} = resL^{r/k} = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} \frac{(-1)^{n-i-1} q^{-(n-i)(n-i-1)} \Gamma_q}{[i]![n-i-1]!} \right) a_{r+n} z^{n-1}. \tag{5.15}$$

An important example is for the q-deformed differential operator of two order

$$L = K = \tilde{D}^2 + V_1 \tilde{D} + V_0. \tag{3.4}$$

The nontrivial densities of conservation laws are supposed to be

$$K_-^{l-\frac{1}{2}} = R_l \tilde{D}^{-1} + S_l \tilde{D}^{-2} + \cdots. \tag{5.17}$$

We shall determine the recursion formulas of the q-deformed Gelfand-Dikii Potential R_l and S_l. From the obvious identity

$$[K^{l-\frac{1}{2}}, K] = 0, \tag{5.18}$$

one gets

$$[K_+^{l-\frac{1}{2}}, K] = [K, K_-^{l-\frac{1}{2}}]_+ \tag{5.19}$$

$$= (R_l^{(0,-4)} - R_l)\tilde{D} + (q^2[2]R_l^{(1,-2)} + S_l^{(0,-4)} - S_l + V_1 R_{l'}^{(0,2)} - R_l V_1^{(0,2)}). \tag{5.20}$$

On the other hand one has

$$K_+^{l+\frac{1}{2}} = \frac{1}{2}\{K_+^{l-\frac{1}{2}}, K\} + \frac{1}{2}\{K_-^{l-\frac{1}{2}}, K\}_+. \tag{5.21}$$

Doing

$$[K_+^{l+\frac{1}{2}}, K] = \frac{1}{2}\{[K_+^{l-\frac{1}{2}}, K], K\} + \frac{1}{2}[\{K_-^{l-\frac{1}{2}}, K\}_+, K], \tag{5.22}$$

we obtain

$$\begin{aligned} R_{l+1} &= (Q^{-4}-1)^{-1}\{(1-q^2[2])R_l^{(0,-4)}V_1^{(1,0)} \\ &+R_l^{(0,-4)}V_0^{(0,-2)} - R_lV_0 - R_l^{(2,0)} - R_l^{(1,0)}V_1 \\ &+S_l^{(0,-4)}V_1 - S_l^{(0,-2)}V_1 - q^2[2]S_l^{(1,-2)}\}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} S_{l+1} &= (Q^{-4}-1)^{-1}\{-q^2[2]R_{l+1}^{(1,-2)} + R_{l+1}V_1^{(0,2)} - R_{l+1}^{(0,-2)}V_1 \\ &+R_l^{(0,-4)}V_0^{(1,0)} - R_l^{(0,-4)}V_1^{(2,2)} + q^2[2]R_l^{(1,-2)}(V_0 - V_1^{(1,-2)}) \\ &-R_l^{(2,0)}V_1^{(0,2)} + R_l^{(0,-2)}V_1V_0 - R_lV_1^{(0,2)}V_0 - R_l^{(1,0)}V_1^{(0,2)}V_1 \\ &-R_l^{(0,-2)}V_1^{(1,2)}V_1 - S_l^{(2,0)} + S_l^{(0,-4)}V_0 - S_lV_0 - S_l^{(1,0)}V_1\}, \end{aligned} \tag{5.24}$$

which are just the recursion formulas of R_l and S_l.

6.The first order formalism of q-deformed KdV equation

The form (3.14-15) of the q-deformed KdV equations is difficult to be understood. In order to compare it with the ordinary KdV equation, we must inspect the difference between the ordinary KdV equation and its q-deformed version when the deformed parameter q tends to 1. Letting $q = 1-\epsilon$, up to the second order of infinitesimal parameter ϵ, one has from (4.2)

$$Q^n = 1 - \epsilon nz\partial + \frac{\epsilon^2}{2}(n(n-1)z\partial + n^2z^2\partial^2) + O(\epsilon^3), \tag{6.1}$$

and from (4.8) up to the first order

$$\tilde{D}^m = \partial^m + \epsilon(mz\partial^{m+1} + \frac{1}{2}m(m-1)\partial^m) + O(\epsilon^2). \tag{6.2}$$

Then one obtains

$$\tilde{D}^mQ^n = \partial^m + \epsilon\left(\frac{1}{2}m(m-2n-1)\partial^m + (m-n)z\partial^{m+1}\right) + O(\epsilon^2). \tag{6.3}$$

The first equation (3.14) of the q-deformed KdV equations becomes

$$\frac{\partial V_1}{\partial t} = (Q^{-4} - 1)U_{-1} = 4\epsilon z U'_{-1}, \tag{6.4}$$

therefore one learns that V_1 is the quantity of one order of ϵ. Using (6.3) one can simplify the relations (3.6-10) and (3.12-13) up to the first order of ϵ,

$$W_0 = \frac{1}{2}V_1 + O(\epsilon^2), \tag{6.5}$$

$$W_{-1} = \frac{1}{2}V_0 - \frac{1}{4}V_1' - \frac{1}{2}\epsilon z V_0' + O(\epsilon^2), \tag{6.6}$$

$$W_{-2} = -\frac{1}{4}V_0' - \frac{1}{4}V_0V_1 + \frac{1}{8}V_1'' + \frac{1}{4}\epsilon V_0' + \frac{1}{4}\epsilon z V_0'' + O(\epsilon^2), \tag{6.7}$$

$$W_{-3} = -\frac{1}{8}V_0^2 + \frac{1}{8}V_0'' + \frac{3}{8}V_0V_1' + \frac{1}{4}V_1V_0' - \frac{1}{16}V_1'''$$
$$+\frac{3}{4}\epsilon z V_0V_0' - \frac{1}{4}\epsilon V_0'' - \frac{1}{8}\epsilon z V_0''' + O(\epsilon^2), \tag{6.8}$$

$$W_{-4} = \frac{3}{8}V_0V_0' - \frac{1}{16}V_0''' + O(\epsilon), \tag{6.9}$$

and

$$U_{-1} = \frac{3}{8}V_0^2 + \frac{1}{8}V_0'' - \frac{3}{8}V_0V_1' - \frac{3}{4}\epsilon z V_0V_0' - \frac{1}{16}V_1'''$$
$$-\frac{1}{4}\epsilon V_0'' - \frac{1}{8}\epsilon z V_0''' + O(\epsilon^2), \tag{6.10}$$

$$U_{-2} = -\frac{3}{8}V_0V_0' - \frac{1}{16}V_0''' + O(\epsilon), \tag{6.11}$$

Substituting these quantities into the q-deformed KdV equations, we obtain their first order form

$$\dot{V}_1 = \frac{1}{2}\epsilon z(6V_0V_0' + V_0'''), \tag{6.13}$$

$$\dot{V}_0 = \frac{1}{4}(6V_0V_0' + V_0''') - \frac{3}{4}V_0'V_1' - \frac{3}{4}V_0V_1'' - \frac{1}{8}V_1''''$$
$$+\frac{3}{2}\epsilon z V_0V_0'' + \frac{1}{4}\epsilon z V_0'''' - \frac{1}{2}\epsilon V_0''' + \frac{3}{2}\epsilon z V_0'^2, \tag{6.14}$$

where $\dot{V} = \frac{\partial V}{\partial t}$.

Now we expand V_0 and V_1 in ϵ

$$V_0 = X_0 + \epsilon X_2 \ , \qquad V_1 = \epsilon X_1.$$

The equation (6.13-14) becomes

$$\dot{X}_0 = \frac{1}{4}(X_0''' + 6X_0X_0'), \tag{6.15}$$

$$\dot{X}_1 = \frac{1}{2}z(X_0''' + 6X_0X_0'), \tag{6.16}$$

and

$$\dot{X}_2 = \frac{1}{4}(X_2''' + 6X_0X_2' + 6X_0'X_2) - \frac{3}{4}X_0'X_1' - \frac{3}{4}X_0X_1'' - \frac{1}{8}X_1''''$$
$$+\frac{1}{4}zX_0'''' - \frac{1}{2}X_0''' + \frac{3}{2}zX_0'^2 - \frac{3}{2}zX_0X_0''. \tag{6.17}$$

The above second equation is total differential $\dot{X}_1 = 2z\dot{X}_0$, its solution is $X_1 = 2zX_0 + f(z)$. For convenience we only consider the case of $f(z) = 0$, we have finally

$$\dot{X}_2 = \frac{1}{4}(X_2''' + 6X_0X_2' + 6X_0'X_2) - \frac{3}{2}(X_0''' + 3X_0X_0'). \tag{6.18}$$

The equation (6.15) is just the ordinary KdV equation. The first order q-deformed modification X_2 can be solved from (6.18) and X_1 is given by $2zX_0$.

7.The conservation quantities of the first order q-KdV equation

Up to the first order

$$\Gamma_q = 1 + \epsilon, \tag{7.1}$$

and

$$resL^{r/k} = (1+\epsilon)a_{r+1} - 2\epsilon z a_{r+2} + O(\epsilon^2), \tag{7.2}$$

therefore the densities of conservation laws is

$$H_{l-1/2} = resL^{l-1/2} = (1+\epsilon)R_l - 2\epsilon z S_l + O(\epsilon^2). \tag{7.3}$$

Let us to expand them in the q-deformed infinitesimal parameter ϵ. Let

$$R_l = r_l + \epsilon p_l + O(\epsilon^2), \tag{7.4}$$

$$S_l = h_l + \epsilon g_l + O(\epsilon^2). \tag{7.5}$$

From the recursion relation (5.23) of R_l one has for the zero order of ϵ

$$h_l = -\frac{1}{2}r_l', \tag{7.6}$$

and for the first order

$$g_l = -\frac{1}{2}p_l' - zX_0r_l. \tag{7.7}$$

From the recursion relation (5.24) of S_l one has for the zero order

$$r_{l+1}' = \frac{1}{4}r_l''' + r_l'X_0 + \frac{1}{2}r_lX_0', \tag{7.8}$$

and for the first order

$$p_{l+1}' = (-2zr_lX_0' + p_lX_0 + r_lX_2 + \frac{1}{4}p_l'' - r_lX_0)'$$
$$+r_l(zX_0'' + \frac{3}{2}X_0' - \frac{1}{2}X_2') - \frac{1}{2}p_lX_0'. \tag{7.9}$$

Taking $r_0 = 1$, $p_0 = 0$ one gives the first three rank results

$$r_1 = \frac{1}{2}X_0, \tag{7.10}$$

$$p_1 = -\frac{1}{2}X_0 - zX_0' + \frac{1}{2}X_2, \tag{7.11}$$

$$r_2 = \frac{3}{8}X_0^2 + \frac{1}{8}X_0'', \tag{7.11}$$

$$p_2 = -\frac{3}{2}zX_0X_0' + \frac{3}{4}X_0X_2 - \frac{3}{4}X_0^2 + \frac{1}{8}X_2'' - \frac{5}{8}X_0'' - \frac{1}{4}zX_0''', \tag{7.12}$$

$$r_3 = \frac{5}{32}X_0'^2 + \frac{5}{16}X_0X_0'' + \frac{1}{32}X_0'''' + \frac{5}{16}X_0^3, \tag{7.13}$$

$$p_3 = -\frac{15}{8}zX_0^2X_0' - \frac{5}{4}zX_0'X_0'' + \frac{15}{16}X_0^2X_2 - \frac{15}{16}X_0^3 + \frac{5}{16}X_0X_2''$$
$$-\frac{15}{8}X_0X_0'' - \frac{5}{8}zX_0X_0''' + \frac{5}{16}X_0''X_2 - \frac{15}{16}X_0'^2 + \frac{5}{16}X_0'X_2'$$
$$+\frac{1}{32}X_2'''' - \frac{9}{32}X_0'''' - \frac{1}{16}zX_0^{(5)}. \tag{7.14}$$

We see the r_l is just the ordinary Gelfand-Dikii potentials. At the first order approximation,

$$-2\epsilon zS_l = \epsilon zR_l', \tag{7.15}$$

therefore

$$H_{l-1/2} = (1+\epsilon)R_l + \epsilon z R_l' = R_l + \epsilon(zR_l)'. \tag{7.16}$$

We learn the $H_{l-1/2}$ differs from R_l only by a total differential. From (7.3) we have

$$H_{l-1/2} = r_l + \epsilon(r_l + p_l + zr_l'). \tag{7.17}$$

For the first three rank we obtain

$$H_{1/2} = \frac{1}{2}V_0 - \frac{1}{2}\epsilon z V_0' + O(\epsilon^2), \tag{7.18}$$

$$H_{3/2} = \frac{3}{8}V_0^2 + \frac{1}{8}V_0'' - \frac{3}{8}\epsilon V_0^2 - \frac{1}{2}\epsilon V_0'' - \frac{3}{4}\epsilon z V_0 V_0' - \frac{1}{8}\epsilon z V_0''' + O(\epsilon^2), \tag{7.19}$$

$$\begin{aligned} H_{5/2} = & \frac{5}{32}V_0'^2 + \frac{5}{16}V_0V_0'' + \frac{1}{32}V_0'''' + \frac{5}{16}V_0^3 \\ & -\epsilon(\frac{15}{16}zV_0^2V_0' + \frac{5}{8}zV_0'V_0'' + \frac{5}{8}V_0^3 + \frac{25}{16}V_0V_0'' + \frac{5}{16}zV_0V_0''' \\ & +\frac{25}{32}V_0'^2 + \frac{1}{4}V_0'''' + \frac{1}{32}zV_0^{(5)}) + O(\epsilon^2). \end{aligned} \tag{7.20}$$

Using the definition of variation

$$\frac{\delta H}{\delta V_0} = \sum_{n=0}^{N}(-1)^n\left(\frac{\partial H}{\partial V_0^{(n)}}\right)^{(n)}, \tag{7.21}$$

one can directly verify that the first three rank results satisfied the above relation

$$\frac{\delta H_{l+1/2}}{\delta V_0} = (l+\frac{1}{2})(H_{l-1/2} + \epsilon z R_l'). \tag{7.22}$$

Due to

$$H_{l-1/2} = R_l + \epsilon(zR_l)'$$

then

$$\frac{\delta H_{l-1/2}}{\delta V_0} = \frac{\delta R_l}{\delta V_0}. \tag{7.23}$$

From (3.15),(3.11) and (5.17) one has

$$\begin{aligned} \frac{\partial V_0}{\partial t} &= 4\epsilon z S_2' + (2-2\epsilon)(R_2' + \epsilon(2R_2' + 3zR_2'')) \\ &= 2(1-2\epsilon)((1+\epsilon)R_2 + 2\epsilon z R_2'), \end{aligned} \tag{7.24}$$

one can rewrite the motion equation as

$$\frac{\partial V_0}{\partial t} = \left(\frac{\delta}{\delta V_0}[\frac{4}{5}(1-2\epsilon)H_{5/2}]\right)' + O(\epsilon^2). \tag{7.25}$$

The conservation quantities are

$$C_{l-1/2} = \int dx H_{l-1/2} = \int dx R_l. \tag{7.26}$$

The first three ones are

$$C_{1/2} = \int dx \frac{1}{2}(1+\epsilon)V_0, \tag{7.27}$$

$$C_{3/2} = \int dx \frac{3}{8}V_0^2, \tag{7.28}$$

$$C_{5/2} = \int dx \frac{5}{32}(-V_0'^2 + 2V_0^3(1-\epsilon)). \tag{7.29}$$

8.Remarks

The q-deformed KdV equations (3.14-15) are in fact the non-linear integrable evolution equations with infinite order of ordinary differential. By comparing the ordinary KdV equation (6.15) one finds the complexity of the q-deformed KdV equations increases largely. To look for their solutions may need to develop a set of new methods. Our investigation is only preliminary and a lot of new interesting problems are awaiting to be studied. Since the wide researching subjects connect with the ordinary KdV hierarchy or with the quantum deformation , we expect that the q-deformed KdV hierarchy should possess wide applications.

This work was supported in part by the Foundation of National Education Committee of China and in part by the National Natural Science Foundation of China.

REFERENCES

[1] As a review, see, for example, P.G.Drazin and R.S.Johnson, Soliton: an Introduction, Cambridge University Press 1989.

[2] D.J.Gross and A.A.Migdal, Nucl.Phys.B340(1990)333.

[3] E.Brezin and V.Kazakov, Phys.lett.236B(1990)144.

[4] V.Knizhnik, A.Polyakov and A.Zamolodchikov, Mod.Phys.Lett.A3(1988)819.

[5] R.Dijikgraaf and E.Witten, IASSNS-HEP-90/18 (1990).

[6] A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl.Phys.B241(1984)333.

[7] A.B.Zamolodchikov, Theor.Math.Phys.65(1985)1205. D.H.Zhang, Phys.Lett.232B(1989)323.

[8] V.G.Drinfeld, Proc.Intern.Congress of Mathematicians (Berkeley), Vol.1 (Academic Press, New York. 1986) p.793.

[9] R.Baxter,"Exactly Solved Models in Statistical Mechanics" Academic Press, London, 1982.

[10] M.Jimbo, Lett.Math.Phys.11([illegible]

[11] M.Chaichian and [illegible]Kulish, Phys.Lett.234B(1990)72.

[12] D.Berna[illegible] A.LeClair, Phys.Lett.227B(1989)417. [illegible]Kulish and E.V.Damaskinsky, J.Phys.A:Math.Gen.23(1990)L415.

[13] R.W.Gray and C.A.Nelson, J.Phys.A:Math.Gen.23(1990)L945.

[14] N.Aizawa and H.Sato, HUPD-9012.

[15] T.Curtright and C.Zachos, Phys.Lett.243B(1990)237.

[16] M.Chaichian, Z.Popowicz and P.Presnajder Phys.Lett.249B(1990)63.

[17] F.H.Jackson, Quart.J.Math.41(1910)193.

[18] M.Chaichian, P.Kulish and J.Lukierski, Phys.Lett.237B(1990)401.

[19] P.P.Lax, Comm.Pure Appl.Math.21(1969)467.

[20] V.G.Drinfeld and V.V.Sokolov,Journ.Sov.Math.30(1985)1975.

AS-ITP-92-23
BIHEP-TH-92-22
CCAST-92-21
April 18, 1992

On the Equivalence of Non-critical Strings and G_k/G_k Topological Field Theories

HONG-LIANG HU[a,b*,c] and MING YU[a,c*]

[a] *CCAST (World Laboratory)*
P.O.Box 8730, Beijing 100080, P.R.China

[b] *Institute of High Energy Physics, Academia Sinica*
P.O.Box 918(4), Beijing 100039, P.R.China

[c] *Institute of Theoretical Physics, Academia Sinica*
P.O.Box 2735, Beijing 100080, P.R.China

Abstract

The $(p,\ q)$ noncritical strings are shown to be equivalent to the twisted $SL(2,R)_{\frac{p}{q}-2}/SL(2,R)_{\frac{p}{q}-2}$ gauged WZNW model. The underlying $N=2$ superconformal symmetries are shown explicitly.

* Permanent Address.

The recent years have seen much progress in understanding coupling 2D gravity to minimal matter, the so called non-critical strings[1,2]. In parallel, the matrix models[3,4] have been found to be useful tools in discretizing the 2D random surfaces and making some calculations in the continuum limit accessible. About the same time, topological field theories have been introduced in describing the 2D gravity (the so called topological gravity) [5]. It is understood[6], in fact, that the $(p+1)$-matrix model is equivalent to the topological gravity coupled to the p -minimal topological matter, a theory obtained by twisting[7] the $N=2$ minimal superconformal field theory of central extension $c=\frac{3p}{p+2}$ [8,9]. The non-critical strings are obtained by some suitable perturbations of the topological conformal field theories[10]. The perturbed minimal topological matter theories coupled to the 2D topological gravity at the multicritical points are believed to be equivalent to the $(p,\ q)$ minimal conformal field theory coupled to the conformal factor of the 2D metric, which can be rephrased as a Liouville field theory[11].

In this letter, we shall give an alternative way of describing the non-critical strings. Indeed, we shall show that the $(p,\ q)$ minimal model coupled to the Liouville field theory is equivalent to the twisted G/G topological WZNW model with $G=SL(2,R)$ and level $k=\frac{p}{q}-2$. The latter can also be described as a kind of twisted $N=2$ superconformal field theory. It is reasonable to generalize our results to the case of W_N minimal matter coupled to the W_N gravity with $G=SL(N)$.

There are at least two advantages in our formalism. First, the non-critical strings are represented in a fully covariant form in terms of tensored WZNW models of different levels. The corresponding Kac-Moody algebra and BRST symmetry may be used to determine the physical space and to calculate the physical correlation functions. Second, our formalism can also be rephrased as the topological minimal matter coupled to the topological gravity. Therefore there is a hidden $N = 2$ superconformal symmetry which can be used to determine the selection rules, etc.. More detailed calculations to substantiate our arguments will be presented elsewhere[12].

To couple the minimal conformal field theory to 2D gravity, we choose the conformal gauge and consider the following path integral

$$W = \int [d\phi_M d\phi_L dbdc] \exp\{-S_M(\phi_M) - S_L(\phi_L) - S_{gh}(b,c)\}, \qquad (1)$$

where b, c are the usual reparametrization ghosts of spin 2 and -1, ϕ_L the Liouville field, ϕ_M the matter part of the theory with $c_M = 1 - \frac{(p-q)^2}{6pq}$, $(p, q) = 1$, $p, q \in Z^+$. Here we shall assume a translationally invariant measure $[d\phi_L]$, see ref.[2].

For convenience, we could enlarge our field space and consider both ϕ_L and ϕ_M as the residual components of the hamiltonian reduction of the $SL(2,R)$ WZNW model [13]

$$\begin{aligned} \int [d\phi_M] \exp\{-S_M(\phi_M)\} =& \int [dgdAd\bar{A}] \exp\{-S_k(g) + \frac{1}{4\pi} \int \mathrm{Tr}\, (\bar{A}(J-\mu) \\ &+ A(\bar{J} - \nu) + k\bar{A}gAg^{-1})\}, \\ \int [d\phi_L] \exp\{-S_L(\phi_L)\} =& \int [dhdBd\bar{B}] \exp\, \{-S_{\tilde{k}}(h) + \frac{1}{4\pi} \int \mathrm{Tr}\, (\bar{B}(\tilde{J} - \tilde{\mu}) \\ &+ B(\tilde{\bar{J}} - \tilde{\nu}) + \tilde{k}\bar{B}hBh^{-1})\}, \end{aligned} \qquad (2)$$

where $g,\ h \in SL(2,R)$, A (B) and $\bar{A}$ $(\bar{B})$ are valued on the Borel subalgebras generated by $\tau^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\tau^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, resp., $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ the constant matrices, and

$$\begin{aligned} J &= -k\partial g \cdot g^{-1}, \qquad \tilde{J} = -\tilde{k}\partial h \cdot h^{-1}, \\ \bar{J} &= kg^{-1} \cdot \bar{\partial} g, \qquad \bar{\tilde{J}} = \tilde{k} h^{-1} \cdot \bar{\partial} h. \end{aligned} \tag{3}$$

$S_k(g)$ is the WZNW action of level k

$$S_k(g) = -\frac{k}{8\pi} \mathrm{Tr} \int_\Sigma g^{-1}\partial g \cdot g^{-1}\bar{\partial} g + \frac{k}{12\pi} \int_{M,\partial M=\Sigma} \mathrm{Tr}\ (g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg). \tag{4}$$

The ϕ_L part can be defined similarly but with level $\tilde{k}$.

The conformal invariance requires a modified energy momentum tensor

$$\begin{aligned} T_M &= \frac{J^a J^a}{k+2} - \partial J^3, \\ T_L &= \frac{\tilde{J}^a \tilde{J}^a}{\tilde{k}+2} - \partial \tilde{J}^3. \end{aligned} \tag{5}$$

Now, consider only the matter part

$$W_M = \int [dgdAd\bar{A}] \exp\{-S_k(g,\ A,\ \bar{A})\}. \tag{6}$$

Here

$$S_k(g,\ A,\ \bar{A}) = S_k(g) - \frac{1}{4\pi} \int \mathrm{Tr}\ (\bar{A}(J-\mu) + A(\bar{J}-\nu) + k\bar{A}gAg^{-1})$$

is invariant under the following gauge transformation

$$\begin{aligned} g &\rightarrow \Gamma g \bar{\Gamma}, \\ A &\rightarrow A + \partial\lambda\tau^+, \\ \bar{A} &\rightarrow \bar{A} + \bar{\partial}\bar{\lambda}\tau^-, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Gamma &= e^{\lambda\tau^+}, \\ \bar{\Gamma} &= e^{\bar{\lambda}\tau^-}, \end{aligned} \tag{8}$$

To eliminate the gauge freedom, we choose the gauge

$$A = \bar{A} = 0, \tag{9}$$

and introduce a pair of ghosts ξ, η of spin 1 and 0 resp. to compensate the change of measure $[dAd\bar{A}]$. We arrive at the following path integral

$$W_M \propto \int [dgd\xi d\eta] \exp\{-S_k(g) + \int \hat{R}\phi - \int (\xi\bar{\partial}\eta + \bar{\xi}\partial\bar{\eta})\}. \tag{10}$$

where $\partial\phi = J^3(z)$, $\hat{R}$ the 2D background scalar curvature and the term $\int \hat{R}\phi$ arises because of the shift in the energy momentum tensor.

Similar procedure can be applied to the Liouville part, and we denote the corresponding ghost as $(\tilde{\xi},\ \tilde{\eta})$, again with spin 1 and 0 resp.,

$$W_L = \int [dhd\tilde{\xi}d\tilde{\eta}] \exp\{-S_{\tilde{k}}(h) + \int \hat{R}\tilde{\phi} - \int (\tilde{\xi}\bar{\partial}\tilde{\eta} + c.c.)\}. \tag{11}$$

where $\partial\tilde{\phi} = \tilde{J}^3$.

Now combining the matter, the Liouville and the reparametrization ghost parts all together, we have the following path integral

$$\begin{aligned} W &= \int [dgdhd\xi d\eta d\tilde{\xi}d\tilde{\eta}dbdc] \exp\{-S_k(g) - S_{\tilde{k}}(h) \\ &\quad + \int \hat{R}(\phi + \tilde{\phi}) - \int (\xi\bar{\partial}\eta + \tilde{\xi}\bar{\partial}\tilde{\eta} + b\bar{\partial}c + c.c.)\}. \end{aligned} \tag{12}$$

The total energy momentum tensor including the ghost part is

$$T^{tot} = T_M + T_L + T_{gh}, \tag{13}$$

with central extension

$$c^{tot} = \left(\frac{3k}{k+2} - 6k - 2\right) + \left(\frac{3\tilde{k}}{\tilde{k}+2} - 6\tilde{k} - 2\right) - 26. \tag{14}$$

The condition that the total central extension vanish gives the following constraint[1]

$$\tilde{k} = -k - 4. \tag{15}$$

Eq.(12) looks very similar to the G/G gauged WZNW models with $G = SL(2, R)$ [14]. In fact, the main point of our paper is that eq.(12) is a twisted version of the G_k/G_k theory[15]. Recall that the G/G gauged WZNW model is defined as

$$\int [dgdAd\bar{A}] \exp\{-S_k(g) + \frac{1}{4\pi}\int \mathrm{Tr}\,(J\bar{A} - \bar{J}A + kgAg^{-1}\bar{A} - kA\bar{A})\}, \tag{16}$$

which is invariant under the following gauge transformation

$$\begin{aligned} g &\rightarrow \rho g \tilde{\rho}^{-1}, \\ A &\rightarrow \rho A \rho^{-1} + \partial\rho \cdot \rho^{-1}, \\ \bar{A} &\rightarrow \tilde{\rho}\bar{A}\tilde{\rho}^{-1} + \tilde{\rho}\bar{\partial}\tilde{\rho}^{-1}. \end{aligned} \tag{17}$$

where $\rho,\ \tilde{\rho} \in SL(2, R),\ A,\ \bar{A} \in sl_2$.

We could parametrize A and $\bar{A}$ as

$$A = h^{-1} \cdot \partial h, \qquad \bar{A} = \tilde{h}^{-1} \cdot \partial \tilde{h}, \tag{18}$$

[1]Another solution $\tilde{k} = -\frac{1}{k+2} - 2$ is discarded as it is inconsistent with the classical limit $k \rightarrow \infty, \tilde{k} \rightarrow -\infty$.

and the change of measure from $[dAd\bar{A}]$ to $[dhd\tilde{h}]$ introduces a fermionic determinant involving the ghosts c^a, and the antighosts b^a, where $a = 1, \cdots, \mathrm{dim}G$. It is possible to rewrite eq.(16) as following (see ref.[14]),

$$W_G = \int [dgdhd\tilde{h}db^a dc^a] \exp\{-S_k(g) - S_{-k-4}(h\tilde{h}^{-1}) - \int (b^a \bar{\partial} c^a + c.c)\}. \quad (19)$$

Finally we fix the gauge at $\tilde{h} = 1$ and the vacuum to vacuum amplitude now looks like

$$W_G \propto \int [dgdhdb^a dc^a] \exp\{-S_k(g) - S_{-k-4}(h) - \int (b^a \bar{\partial} c^a + c.c.)\}, \quad (20)$$

where b^a and c^a have spins 1 and 0 respectively.

To go from eq.(20) to eq.(12) requires some kind of twisting which we now proceed. In G/G gauged WZNW model, there are three sectors of the Kac-Moody algebras, J^a, $\tilde{J}^a$ and $J^{gh,a}$ which are defined as

$$\begin{aligned} J^a &= k\partial g \cdot g^{-1}, \\ \tilde{J}^a &= \tilde{k}\partial h \cdot h^{-1}, \\ J^{gh,a} &= i f^{abc} b^b c^c. \end{aligned} \quad (21)$$

The levels of these Kac-Moody algebras are k, $-k-4$ and 4 resp.. The requirement that our formalism is independent of the gauge choice means that

$$J^{tot,a} = J^a + \tilde{J}^a + J^{gh,a} = 0. \quad (22)$$

Similarly, the total energy momentum tensor is the sum

$$T^{tot}(z) = T(z) + \tilde{T}(z) + T^{gh}(z), \quad (23)$$

where

$$T(z) = \frac{J^a J^a}{k+2}, \qquad \tilde{T}(z) = \frac{\tilde{J}^a \tilde{J}^a}{-k-2}, \qquad T^{gh}(z) = 2b\partial c + \partial bc. \tag{24}$$

It is easy to verify that the total central extension of the energy momentum tensor is zero.

Now consider an improved energy momentum tensor T^{impr} twisted by $J^{tot,3}(z)$ component

$$T^{impr}(z) = T^{tot}(z) - \partial J^{tot,3}(z). \tag{25}$$

Correspondingly, the path integral eq.(20) requires a dilaton background

$$W_G^{twisted} = \left\langle e^{\int \phi^{tot} \hat{R}} \right\rangle, \tag{26}$$

where $\phi^{tot} = \phi + \tilde{\phi} + \phi^{gh}$, $\partial\phi^{tot} = J^{tot,3}$. Under such a twisting, the total central extension is still zero but the central extensions of the different sectors do change. Another point is that any Virasoro primary field of non-zero $J_0^{tot,3}$ charge requires a modification in the conformal dimension. Specifically, $\Delta \to \Delta + j^3$, j^3 being the the eigenvalue of $J_0^{tot,3}$. In particular, the ghosts and antighosts c^a, b^a have different spins before and after twisting (see the table below).

ghost field	c^+	c^3	c^-	b^+	b^3	b^-
spin Δ	0	0	0	1	1	1
isospin j^3	1	0	-1	1	0	-1
$\Delta + j^3$	1	0	-1	2	1	0

Table. The conformal dimensions of (anti-)ghost fields before and after twisting.

Now we recognize that b^+, c^- have the same spins as the reparametrization ghosts b, c in eq.(12), while b^3, c^3 and c^+, b^- have the same spins as the ghosts ξ, η, $\tilde{\xi}$, $\tilde{\eta}$ in eq.(12). Trading the ghost dilaton expectation value for the change of the ghost spins, we have established the equivalence of the ghost parts in eq.(20) and eq.(12). For the non-ghost sectors, the equivalence is manifest.

Thus we arrive at the main result of our paper, namely, the $(p,\ q)$ minimal model coupled to the 2D gravity is equivalent to the twisted $SL(2,R)_k/SL(2,R)_k$ gauged WZNW theory with $k = \frac{p}{q} - 2$.

Notice that the constraint, eq.(22), is of first class. Hence we can define the BRST operator

$$\begin{aligned} G &= (J^a + \tilde{J}^a + \frac{1}{2}J^{gh,a})c^a, \\ Q &= \oint G(z), \qquad Q^2 = 0. \end{aligned} \tag{27}$$

Both the total energy momentum tensor $T(z)$ and the BRST current G are Q commutators

$$\begin{aligned} T^{tot}(z) &= \{Q,\ \bar{G}\}, \qquad \bar{G} = \frac{1}{k+2}(J^a - \tilde{J}^a)b^a, \\ G(z) &= \{Q,\ J_{N=2}\}, \qquad J_{N=2} = c^a b^a, \end{aligned} \tag{28}$$

Now we have established that G/G model is a topological conformal field theory. Since $J^{tot,3}$ is a Q commutator, the twisted G/G models are still topological conformal field theories. This can be seen as we choose the following substitutions

$$\begin{aligned}
T^{tot}(z) &\to T^{impr}(z) = T^{tot}(z) - \partial J^{tot,3}(z) = \{Q,\ \bar{G} - \partial b^3\}(z), \\
\bar{G}(z) &\to \bar{G}^{impr}(z) = \bar{G}(z) - \partial b^3(z), \\
G(z) &\to G^{impr}(z) = G(z), \\
J_{N=2}(z) &\to J^{impr}(z)_{N=2} = J_{N=2}(z) - J^{tot,3}(z).
\end{aligned} \tag{29}$$

As usual, the above $N = 2$ algebra has the structure of the tensor product of the "parafermion" algebra and the $U(1)$ current algebra[9]. The $U(1)$ part is just the ghost current

$$J_{N=2}(z) = J_{gh}(z) = c^a b^a, \tag{30}$$

with the anomalous background charge $d_G = 3$.

The "parafermion" part $\psi(z)$ is the tensor product of WZNW models of different sectors

$$\psi(z) \in SL(2,R)_k \otimes SL(2,R)_{-k-4} \otimes SL(2,R)_4. \tag{31}$$

Under the $N = 2$ twisting procedure, the $U(1)$ current becomes anomalous

$$\begin{aligned}
T_{U(1)}(z) &= \frac{1}{6} J_{gh}(z) J_{gh}(z) + \frac{1}{2}\partial J_{gh}(z), \\
c_{U(1)} &= -8,
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
T_{\psi}(z) &= \frac{J^a J^a}{k+2} - \frac{\tilde{J}^a \tilde{J}^a}{k+2} + \frac{J^{gh,a} J^{gh,a}}{6} \\
c_{\psi} &= \frac{3k}{k+2} + \frac{3(k+4)}{k+2} + 2 = 8.
\end{aligned} \tag{33}$$

Since $T^{tot} = T_{U(1)} + T_\psi$, we have $c^{tot} = 0$ as desired.

The physical states in the twisted G/G models are identified as the Q -cohomology representatives. As our first remark, notice that since $J^{tot,3}$ is a Q commutator, the Q -cohomology representatives in the twisted G/G theories are the same as in the untwisted ones. Thus it suffices to consider the standard G/G theory as long as we are interested in the physical subspace of the theory. Recently, the G/G models have also been studied by some other groups[15], albeit from the point of view different from ours. Our second remark is that in our theories, $k = \frac{p}{q} - 2$ could be fractional where there exists the so-called admissible representations[16] of the Kac-Moody current algebra[2]. However, the role of the admissible level k Kac-Moody algebra in the topological field theory has not been paid much attention to in the literature. According to the authors of ref.[10], there are "critical" points in the space of topological field theories where there exist $N = 2$ superconformal symmetry. At these critical points, the corresponding theories are called topological conformal field theories, which are obtained by twisting the $N = 2$ minimal models of central extension $c = \frac{3k}{k+2}$, $k \in Z^+$. In this paper we have shown that it is possible to extend their results to the case of fractional k. The inclusion of the fractional levels k in the G_k/G_k theory is crucial in our understanding of the space of the topological field theories. It means that there are new "critical" points in such a space, besides those found in ref.[10]. At these critical points, the topological field theories become twisted $N = 2$ SCFT's and the corresponding theories are precisely non-critical strings. It is desirable to find the Landau-Ginsburg

[2]This is very similar to the case of (p, q) minimal conformal field theory with $p - q = \pm 1$ as a unitary series. Here, however, in topological field theories, a fractional level k does not necessarily mean the non-unitarity.

description for such models. We leave such analyses for our future work[12]. Here, however, we shall point out another underlying $N = 2$ structure in our theory, which is different from the one given in eqs.(27-29).

To see this new $N = 2$ structure, let us first find out the physical states in the ghost number zero sector of the G_k/G_k theory. In this sector, we have the constraint

$$\begin{aligned} J_n^a + \tilde{J}_n^a &= 0, \qquad n \geq 0, \\ \Delta + \tilde{\Delta} &= 0, \end{aligned} \tag{34}$$

where Δ ($\tilde{\Delta}$) are the conformal dimensions of the primary fields in the J^a ($\tilde{J}^a$) sector. Writing those primary fields as ϕ^l and $\tilde{\phi}^{\tilde{l}}$ resp., the constraint eq.(34) means that $l = \tilde{l}$.

ϕ^l can be expressed as

$$\phi^l = \phi^l_{para} e^{\sqrt{\frac{2}{k}}\phi}, \tag{35}$$

where ϕ^l_{para} is the primary field of the parafermionic current ψ. For the $\tilde{J}^a$ algebra we choose the free field representation[13],

$$\begin{aligned} \tilde{J}^- &= \beta, \\ \tilde{J}^3 &= \beta\gamma + \sqrt{\frac{-k-2}{2}}\,\partial\varphi, \\ \tilde{J}^+ &= -\beta\gamma^2 - (k+4)\partial\gamma - \sqrt{2(-k-2)}\,\gamma\partial\varphi, \\ \tilde{\phi}^l &= e^{-l\sqrt{\frac{2}{-k-2}}\varphi} \end{aligned} \tag{36}$$

Combining the two sectors together, we have the following physical state

$$\chi^l \equiv \phi^l\tilde{\phi}^l = \phi^l_{para} e^{2l\sqrt{\frac{1}{2(k+2)}}\varphi_1}, \tag{37}$$

where

$$
\begin{aligned}
\varphi_1 &= \sqrt{\frac{k+2}{2}}\phi + i\sqrt{\frac{k}{2}}\varphi, \\
\varphi_2 &= -i\sqrt{\frac{k}{2}}\phi + \sqrt{\frac{k+2}{2}}\varphi,
\end{aligned}
\tag{38}
$$

φ_1 and φ_2 are orthonormal.

Consider now the energy momentum tensor for the ϕ and φ part

$$
\begin{aligned}
T_{\phi,\varphi} &= \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}\partial\varphi\partial\varphi - i\sqrt{\frac{1}{2(k+2)}}\partial^2\varphi \\
&= \frac{1}{2}\partial\varphi_1\partial\varphi_1 + \sqrt{\frac{k}{4(k+2)}}\partial^2\varphi_1 + \frac{1}{2}\partial\varphi_2\partial\varphi_2 - \frac{i}{2}\partial^2\varphi_2.
\end{aligned}
\tag{39}
$$

If we define

$$
\begin{aligned}
T_1 &= \frac{1}{2}\partial\varphi_1\partial\varphi_1 + \sqrt{\frac{k}{4(k+2)}}\partial^2\varphi_1, \\
T_2 &= \frac{1}{2}\partial\varphi_2\partial\varphi_2 - \frac{i}{2}\partial^2\varphi_2,
\end{aligned}
\tag{40}
$$

we find $c_1 = 1 - \frac{3k}{k+2}$ and $c_2 = 4$.

Now we see that the parafermion part of $SL(2,R)_k$ and the scalar field φ_1 form the topological matter theory

$$
T_M = T_{para} + T_1, \tag{41}
$$

while β, γ, φ_2 and the ghosts b^a, c^a form the topological gravity theory after twisting. Thus, as a final remark we shall emphasize that the non-critical strings can also be regarded as the topological matters coupled to the topological gravity. The topological matter is described by twisting the second $N = 2$ structure with fractional $k = \frac{p}{q} - 2$ [17]. We notice that in the special cases of $q = 1, k = p - 2$, this equivalence has also been studied by the authors of ref.[18] via a different approach.

Notice that

i). In the topological matter sector, there are primary fields which correspond to the admissible representations of the $SL(2,R)$ current algebra at the level $k = \frac{p}{q} - 2$, while the topological gravity sector contains only one primary field.

ii). Under $J^{tot,3}$ twisting, $T_M(z)$ is invariant, and

$$\hat{T} \equiv T^{tot} - T_M = T_{\beta,\gamma} + T_2 + T_{gh} \rightarrow \tilde{T}_{\beta,\gamma} + \tilde{T}_2 + \tilde{T}_{gh}, \tag{42}$$

we have

$$\begin{aligned} \tilde{T}_{\beta,\gamma} &= T_{\beta,\gamma} + \partial(\beta\gamma) \\ \tilde{c}_{\beta,\gamma} &= c_{\beta,\gamma} = 2, \\ \tilde{T}_2 &= \frac{1}{2}\partial\varphi_2\partial\varphi_2 - i\frac{3}{2}\partial^2\varphi_2, \\ \tilde{c}_2 &= 28. \end{aligned} \tag{43}$$

We find that the spins of the fields and the central charges in the $\hat{T}$ sector are in agreement with Verlindes' description of the topological gravity [10]. In fact, as we can see now from eqs.(42-43), the topological gravity can be described as the twisted G_0/G_0 theory, which consists of the $\tilde{k} = -4$ current algebra and the ghost fields. The $\tilde{k} = -4$ current algebra is given by eq.(36) with $k = 0$ and $\varphi \rightarrow \varphi_2$.

Finally, we would like to briefly summarize the main points contained in our paper. By considering the path integral formalism of the G_k/G_k gauged WZNW theory and the representations of the corresponding chiral algebra, we found the following relations concerning the physical subspaces of the underlying theories

1. (p,q) non-critical string $\simeq$ twisted G_k/G_k.

2. Twisted $G_k/G_k \simeq G_k/G_k$.

3. $G_k/G_k \simeq$ Twisted $N = 2$ SCFT $\simeq \otimes$ WZNW's $\otimes$ twisted $U(1)$.

4. Twisted $G_k/G_k \simeq$ topological matter $\otimes$ topological gravity.

5. Topological matter $\otimes$ topological gravity $\simeq$ twisted $N = 2$ superconformal field theory of $c = \frac{3k}{k+2}|_{k=\frac{p}{q}-2} \otimes G_0/G_0$.

Acknowledgement: We are indebted to Y.X.Chen, H.Y.Guo, M.Li, K.Wu, Y.Yu and Z.Y.Zhu for valuable discussions during the course of our present work. We thank CCAST (World Lab.) for its hospitality during the final stage of writing the paper. We are grateful to M.Li for sending us the last two refs. of ref.[15]. This work is supported in part by the National Science Foundation of China, the Grant LWTZ-1398 from Academia Sinica, a fellowship (to M.Yu) from the Science Foundation for Young Researchers of Academia Sinica, and a fellowship (to H.L.Hu) from the National Postdoctoral Science Foundation.

References

[1] A.Polyakov, Mod. Phys. Lett.A2 (1987) 893; V.Knizhnik, A.Polyakov and A.B.Zamolodchikov, Mod. Phys. Lett.A3 (1988) 819.

[2] J.Distler and H.Kawai, Nucl. Phys. B321 (1989) 509; F.David, Mod. Phys. Lett.A3 (1988) 1651; E. D'Hoker, Mod. Phys. Lett. A6(1991) 745.

[3] D.Weingarten, Phys. Lett. B90 (1980) 285; V.Kazakov, Phys. Lett. B150 (1985) 282; F.David, Nucl. Phys. B257 (1985) 45; J.Ambjorn, B.Durhuus and J.Fröhlich, Nucl. Phys. B257 (1985) 433; J. Ambjorn, B. Durhuus, J. Frohlich and P.Orland, Nucl.Phys. B257 (1986) 457; B275 (1986) 543; V.A.Kazakov, I.K.Kostov and A.A.Migdal, Phys. Lett. B157 (1985) 295.

[4] E.Brezin and V.A.Kazakov, Phys. Lett. 236B (1990) 144; M.Douglas and S.Shenker, Nucl. Phys. B335 (1990) 635; D.J.Gross and A.Migdal, Phys. Rev. Lett. 64 (1990) 717; Nucl. Phys. B340 (1990) 333.

[5] J.Labastida, M.Pernici and E.Witten, Nucl. Phys. B310 (1988) 611; E.Witten, Nucl. Phys. B340 (1990) 281; R.Dijkgraaf and E.Witten, Nucl. Phys. B342 (1990) 486; J.Distler, Nucl. Phys. B342 (1990) 523; E.Verlinde and H.Verlinde, Nucl. Phys. B348 (1991) 457; E. Witten, Surveys in Diff. Geom. 1 (1991) 243; Nucl. Phys. B371(1992)191;

[6] K.Li, Nucl. Phys. B354 (1991) 711; Nucl. Phys. B354 (1991) 725.

[7] E. Witten, Commun. Math. Phys. 118 (1988) 411; T.Eguchi, S.Hosono and S.-K.Yang, Mod. Phys. Lett. A5 (1990) 1653.

[8] W.Boucher, D.Friedan and A.Kent, Phys. Lett. B172 (1986) 316; P.Di Vecchia, J.L.Petersen, M.Yu and H.B.Zheng, Phys. Lett. B174 (1986) 280.

[9] V.A. Fatteev and A.B.Zamolodchikov, Zh. Eksp. Teor. Fiz. 90(1986)1553; Z.Qiu, Phys. Lett. B188 (1987) 207.

[10] R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl. Phys. B352 (1991)59; *Notes on topological string theory and 2d quantum gravity*, Proceedings of the Trieste spring school 1990, M. Green et al eds. (World Scientific, 1991); R.

Dijkgraaf, *Intersection theory, integrable hierarhics and topological field theory*, Lecture at the Cargèse summer school on new symmetry principles in quantum field theory, July 16-27, 1991.

[11] For reviews, see, for example:
N. Seiberg, Suppl. Prog. Theor. Phys. No.102 (1990) 319; E.D'Hoker, Lectures at the VI-th Swieca Summer School, Brasil, January 17–28, 1991, UCLA/91/TEP?35; L.Alvarez-Gaumé and C.Gomez, Lectures at the Trieste Spring School, April, 1991, CERN-TH 6175/91.

[12] H.L.Hu, and M.Yu, in preparation.

[13] M.Bershadsky, and H.Ooguri, Phys. Lett. 229B (1989) 374; Commun. Math. Phys. 126 (1989)49; A.Alekseev, and S.Shatashvili, Nucl. Phys. B323 (1989) 719; J.Balog, L.Feher, P.Forgacs, L.O'Raifeartaigh and A.Wipf, Phys. Lett. B227 (1990) 214; B244 (1990)435; Ann. Phys.(N.Y.) 203(1990)76.

[14] K.Gawedzki and A.Kupiainen, Phys. Lett. 215B (1988) 119; Nucl. Phys. B320 (1989) 649; K.Bardacki, E.Rabinovici and B.Saring, Nucl. Phys. B299(1988) 157; A.Altschuler, K.Bardacki, and E.Rabinovici, Commun. Math. Phys. 118 (1988) 241; D.Karabali, Q.-H.Park, H.J.Schnitzer and Z.Yang, Phys. Lett. 216B (1989) 307; D.Karabali and H.J.Schnitzer, Nucl. Phys. B329 (1990) 649.

[15] E.Witten, Commun. Math. Phys. 144 (1992) 189; M. Spiegelglas, Phys. Lett. B274 (1992) 21; M.Spiegelglas and S.Yankielowicz, Technion PH-34-90; D.Nemeschansky and N.P.Warner, USC-91/031.

[16] V.G. Kac and M. Wakimoto, Proc. Nat'l Acad. Sci. USA 85 (1988) 4956;

Adv. Ser. Math. Phys. 7 World Sci. (1989) 138; D. Bernard and G. Felder, Commun. Math. Phys. 127 (1990) 145; S. Mukhi and S. Panda, Nucl. Phys. B338 (1990) 263; M. Kato and Y. Yamada, KEK-TH-293;

[17] J.D. Lykken, Nucl. Phys. B313 (1989) 473.

[18] N. Ishibashi and M. Li, Phys. Lett. 262 (1991) 398.

ASITP-92-26
CCAST-92-24

A Note on the Equivalence between Canonical and BRST Quantization for Chern-Simons field Theory

W.F. Chen, Y.Yu and Z.Y. Zhu

CCAST(World Laboratory)
P.O. Box 8730, Beijing 100080, P.R. China
and
Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, P.R. China

ABSTRACT

The BRST quantization of Chern-Simons field theory is performed. With aid of the nilpotency property of BRST charge, the physical state condition $\hat{Q}_B|phys>=0$ *is reduced to the Gauss law constraints associated with the case that Wilson loops are present or not. Further the physical states are shown to satisfy Knizhnik-Zamolodchikov equation and hence the equivalence between canonical and BRST quantization approaches are exhibited.*

During the past several years, the quantization of Chern-Simons field theory has been greatly studied in different ways[1–5]. On one hand, Witten performed canonical quantization of Chern-Simons field theory[1], he found that the Hilbert space of the theory can be identical to the space of conformal block of WZW model. On this basis, he evaluated the expectation value of Wilson loop operators and reproduced the celebrated Jones polynomial. On the other hand, Guadagnini *etal* took the quantization of Chern-Simons theory in a alternative way[2]. They performed BRST gauge fixing of the theory and computed expectation value of Wilson loops to second order with aid of perturbation theory, and obtained the same result as Witten's. This fact suggest the problem that whether these two different quantization approaches are equivalent, for just as what Witten did, first impose the constraints, the topology of classical physical phase space may become very complicated, in general geometric quantization must be resorted to deal with this circumstance. Whilist that first quantize the system and then consider the constraints is another completely different story. Hence for a general constrained system, it is very difficult to prove the equivalence between these two quantization schemes. Fortunately for Chern-Simons field theory, Witten has already pointed out that the Hilbert space obtained by first imposing constraints and then quantizing the system can be identical to the space of conformal blocks, this statement was verified by many authors[6][7][8] and make it possible for us to prove the equivalence between these two quantization schemes. The aim of this paper is just give a explicit exhibition of the affirmative answer.

The action for Chern-Simons field theory takes the following form

$$S_{cs} = \frac{k}{4\pi}\int_M (A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{1}$$

where $A{=}A^a_\mu T^a dx^\mu$ with T^a the generators in some representation of gauge group $\mathcal{G}$, k is chosen to be an interger in order to make the system possess gauge invariance under the action of an arbitrary gauge group. Without loss of generality, we choose $\mathcal{G}=SU(N)$ and there exsits normalization $Tr(T^aT^b){=}\frac{1}{2}\delta^{ab}$. Choosing Lorentz gauge condition $\partial_\mu A^\mu{=}0$ and performing the BRST gauge fixing, we obtain the following effective action

$$\begin{aligned} S_{eff} &= S_{cs} + \int \tfrac{1}{\epsilon} Tr\delta_B[\bar{C}(\partial_\mu A^\mu + B)] \\ &= \int d^3x \tfrac{k}{16\pi}[\epsilon^{\mu\nu\rho}A^a_\mu(\partial_\nu A^a_\rho - \partial_\rho A^a_\nu) + if^{abc}\tfrac{2}{3}A^a_\mu A^b_\nu A^c_\rho \\ &\quad -\tfrac{ik}{8\pi}A^{\mu a}\partial_\mu B^a + \tfrac{ik}{8\pi}B^{a2} - \tfrac{1}{2}\partial_\mu\bar{C}^a D^\mu C^a]. \end{aligned} \tag{2}$$

The BRST transformations for every fields are read as following

$$\begin{aligned} \delta_B A^a_\mu &= \epsilon D_\mu C^a \quad , \quad \delta_B B^a = 0, \\ \delta_B C^a &= -\epsilon \frac{1}{2} f^{abc} C^b C^c \quad , \quad \delta_B \bar{C}^a = \epsilon \tfrac{ik}{4\pi} B^a. \end{aligned} \tag{3}$$

Evidently, they are nilpotent: ${\delta_B}^2=0$, Now the classical configuration space is enlarged by the introduction of new fields—ghost field C, antighost field $\bar{C}$ and multiplier field B, their canonical conjugate momenta defined by $\Pi_\Phi=\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}$, $\Phi = (A_1, B, C, \bar{C})$ can be obtained:

$$\begin{aligned} \Pi^a_{A_1} &= \tfrac{k}{8\pi} A^a_2, \Pi^a_B = -\tfrac{ik}{8\pi} A^{0a}, \\ \Pi^a_{\bar{C}} &= -\tfrac{1}{2} D^0 C^a, \Pi^a_C = \tfrac{1}{2}\dot{\bar{C}}. \end{aligned} \tag{4}$$

These fields and their canonical conjugate momenta satisfy Poisson bracket (for bosonic fields) or antibracket (for fermionic ones) relation

$$\begin{aligned} [\Pi^I_\Phi(\mathbf{x},t), \Phi_J(\mathbf{y},t)]_{\pm P.B} &= -i\delta^I_J \delta^{(3)}(\mathbf{x}-\mathbf{y}), \\ [\Pi^I_\Phi(\mathbf{x},t), \Pi^J_\Phi(\mathbf{y},t)]_{\pm P.B} &= [\Phi_I(\mathbf{x},t), \Phi_J(\mathbf{y},t)]_{\pm P.B} = 0. \end{aligned} \tag{5}$$

The BRST charge corresponding to the invariance under BRST transformation eq.(3) can be computed

$$\begin{aligned} Q_B &= \textstyle\int d^2x [\tfrac{k}{8\pi}\epsilon^{ij} D_i C^a A^a_j - \tfrac{1}{4} f^{abc} \dot{\bar{C}}^a C^b C^c - \tfrac{ik}{8\pi} B^a D^0 C^a] \\ &= \textstyle\int d^2x [-\tfrac{k}{8\pi} C^a F^a_{12} - \tfrac{1}{4} f^{abc} \Pi^a_C C^b C^c + \tfrac{ik}{4\pi} B^a \Pi^a_{\bar{C}}]. \end{aligned} \tag{6}$$

By a direct calculation, it is easy to prove that

$$\begin{aligned} [Q_B, \Phi]_{\pm P.B} &= \tfrac{1}{\epsilon}\delta_B \Phi, \Phi = (A_1, B, C, \bar{C}), \\ \tfrac{1}{2}[Q_B, Q_B]_{+,P.B} &= Q^2_B = 0. \end{aligned} \tag{7}$$

The following is performing quantization. According to the general procedure for quantization, the classical observables are replaced by operators and Poisson (anti-)brackets by (anti-)commutation Lie brackets. Especially the polarization must be specified, here we choose $\hat{\Phi}$=$(\hat{A}_1, \hat{B}, \hat{C}, \hat{\bar{C}})$ to play the role of canonical coordinates and the Hilbert space is composed of square integrable functional in term of $\hat{\Phi}$. Since there exists no anomaly in Chern-Simons field theory, the BRST charge $\hat{Q_B}$ are well defined and the BRST algebra relations

$$\frac{1}{2}[\hat{Q}_B, \hat{Q}_B] = \hat{Q}^2_B = 0 \tag{8}$$

are still satisfied (the problem of operator ordering is ignored here and in what follows). It is well known that the state space at present possesses indefinite metric due to the introduction of non-physical fields. According to the general principle of BRST quantization, the physical states satisfy that

$$\hat{Q}_B|phys>=0. \tag{9}$$

Notice that the above condition determines physical states up to a zero-norm state due to the nilpotency property of BRST charge operator $\hat{Q}_B$, that is, $|phys> \longrightarrow |phys> + |X>, |X>= \hat{Q}_B|phys>$. Obviously the zero-norm states have the property that they are normal to all physical states including themselves, $<X|phys>=<X_1|X_2>=0$. Hence they make no contribution to the observables and they are in essential unphysical. The genuine physical state space should contains no such states. Kugo and Ojima, making use of the quartet representation of BRST algebra provided by the Hilbert space, found that non-physical states always appear as the zero-norm state combination[9], hence non-physical states are confined and the genuine physical state sector can be reached. Here we adopt the viewpoint of physical operator proposed by Marnelius[10]. Physical operators are defined to be those transform a physical state into another one, according to their action on the physical states, physical operators can be divided into two types: A-type and B-type ones. A A-type physical operator is a genuine physical operator and it transforms a genuine physical operator into the other one, i.e. $\hat{A}|phys>=|phys>'$. A B-type operator take the following form

$$\hat{B}=[\hat{C},\hat{Q}_B]_\pm \tag{10}$$

where $\hat{C}$ a nonphysical operator. Evidently A B-type operator transforms a physical state into a zero-norm state and it can be regarded as the generator of a new type of gauge transformation for its action on a physical operator $\hat{\Phi}$

$$[\hat{B}_i,\hat{\Phi}]_\pm = f_{ij}(\hat{\Phi})\hat{B}_j. \tag{11}$$

B-type operators form a ideal in the operator algebra[10], the product of an arbitrary operator $\hat{K}$ (physical or non-physical) with a B-type operator is also the generator of gauge transformation for the fact

$$[(\hat{K}\hat{B})_i,\hat{\Phi}]_\pm = f'_{ij}(\hat{\Phi})(\hat{K}\hat{B})_j \tag{12}$$

Followed BRST charge eq.(6), it can be proved that

$$\begin{aligned} \hat{B}^a_1 &= [\hat{Q}_B,\hat{\Pi}^a_C] = -\tfrac{k}{8\pi}\hat{F}^a_{12} - \tfrac{1}{2}f^{abc}\hat{\Pi}^b_C\hat{C}^c, \\ \hat{B}^a_2 &= [\hat{Q}_B,\hat{\Pi}^a_B] = \tfrac{ik}{4\pi}\hat{\Pi}^a_C, \\ \hat{B}^a_3 &= [\hat{Q}_B,\partial_\mu\hat{A}^{\mu a}] = M_{ab}\hat{C}^b, \end{aligned} \tag{13}$$

where M_{ab}=-$\frac{k}{8\pi}[\hat{F}^a_{12}, \partial_\mu \hat{A}^{\mu b}]$. Note that the matrix M=(M_{ab}) is nonsingular for the reason that -$\frac{k}{8\pi}F_{12}$ and $\partial_\mu A^\mu$ constitute a pair of second-class constraint, in classical sense the Poisson bracket between them doesn't vanish. It is worthy to stress the point that during above process we have used on-shell condition. The nonsingularity of M states that $\hat{C}^a$=$(M^{-1})^{ab}\hat{B}_{3b}$ are also generators of gauge transformation.

From eqs.(6) and (13), one can see that the three parts consisting of BRST charge $\hat{Q}_B$ are all gauge transformation generators, but the second and third terms suit for non-physical fields. As for the first part, one can see that there exists no coupling between the unphysical gauge transformation generators $\hat{C}^a$ and the physical ones $\frac{k}{8\pi}\hat{F}^a_{12}$, so if there exist no Wilson loop operators in the universe, the physical state condition $\hat{Q}_B|Phys>$=0 reduces to that

$$\frac{k}{8\pi}\hat{F}^a_{12}|phys>= 0. \tag{14}$$

In the case that Wilson loop operators are present, since they are not local operators, they locate in a finite region of space time, they only produce the excitation from the vacuum to vacuum. If one want consider their effect, one must choose a proper time at which Wlison loop operators excite the vacuum, and the generated state can be illustrately represented by a punctured surface, which is obtained geometrically by the intersection of links, on which Wilson loop operators are defined, with surface Σ determined by some time t. Under above polarization choice, the gauge invariant state functional at some time t takes the following form

$$\begin{aligned}\Psi_{phys} &= \int \mathcal{D}X\ exp(i\int^t_{t_0=-\infty} S_{eff}dt + \int_\Sigma d^2x \textstyle\sum^2_{i=1} A^a_i A^{ia})\Pi^N_{n=1} exp\int^{Q_n}_{P_n} A^{(n)}\Psi_0[A],\\ &\quad X = (A, C, \bar{C}, B),\end{aligned} \tag{15}$$

where n denotes the different components of links and P_n,Q_n the punctured points created by the intersection of nth components of links with Σ. $\Psi_0[A]$ is the vacuum wave functional at time t=-∞, which is determined by eq.(14) and will be explicitly shown later. In the language of a operator form, eq.(15) can be rewritten as following

$$|phys>= (\Pi_n Pexp\int^{Q_n}_{P_n\Gamma} \textstyle\sum_{i=1,2} \hat{A}_i(x_1, x_2)dx^i)|0> \tag{16}$$

with Γ the projection on transverse surface Σ of the links locating in the three-dimensional space-time region of less time t. Hence we have

$$\begin{aligned}\hat{Q}_B|phys> &= (\int d^2x[-\tfrac{k}{8\pi}\hat{C}^a\hat{F}^a_{12}+\tfrac{k}{4\pi}\hat{B}^a\hat{\Pi}^a_{\bar{C}}+\tfrac{1}{2}f^{abc}\hat{\Pi}^a_{\bar{C}}\hat{C}^b\hat{C}^c] \\ &\quad \times\Pi_{n=1}^N Pexp\textstyle\int_{P_n}^{Q_n}\sum_{i=1,2}\hat{A}_i dx^i)|0> \\ &= (\int d^2x[-\hat{C}^a[\tfrac{k}{8\pi}\hat{F}^a_{12}-(\textstyle\sum_{n=1}^N\delta^{(2)}(\mathbf{x}-\mathbf{x}_{P_n})T^a_{(n)}-\delta^{(2)}(\mathbf{x}-\mathbf{x}_{Q_n})] \\ &\quad +\tfrac{k}{4\pi}\hat{B}^a\hat{\Pi}^a_{\bar{C}}-\tfrac{1}{2}f^{abc}\hat{\Pi}^a_{\bar{C}}\hat{C}^b\hat{C}^c])|phys>,\end{aligned} \tag{17}$$

where we have used that

$$\begin{aligned}\partial_{\mathbf{x}_{P_n}}\frac{\delta}{\delta A^a_1(\mathbf{x}_{P_n})}|phys> &= -\tfrac{8\pi}{k}T^a_{(n)}|phys>, \\ \partial_{\mathbf{x}_{Q_n}}\frac{\delta}{\delta A^a_1(\mathbf{x}_{Q_n})}|phys> &= \tfrac{8\pi}{k}T^a_{(n)}\delta(\mathbf{x}-\mathbf{x}_{Q_n})|phys>.\end{aligned} \tag{18}$$

Note that in the locations of punctures, $\hat{Q}_B$ becomes a matrix-valued operator, so do every fields containing in $\hat{Q}_B$, but out the punctures Q_B and the fields containing in it restore to the ordinary ones. Repeating above process, one can reduce the physical state condition $\hat{Q}_B|phys>=0$ to that

$$[\frac{k}{8\pi}F^a_{12}-\sum_{n=1}^N(\delta^{(2)}(\mathbf{x}-\mathbf{x}_{P_n})T^a_n-\delta^{(2)}(\mathbf{x}-\mathbf{x}_{Q_n}])T^a_n]|phys>=0. \tag{19}$$

From Gauss law constraints eqs.(14) and (19) satisfied by physical states, we can show that the physical states indeed can be explained as conformal blocks. Let's first see the case that there exist no Wilson loop operators. As stated above, we choose the polarization that $\hat{A}_1$ plays the role of canonical coordinate and the physical states are represented by the functional with respect to $A_1(x)$, the eigenvalue of operator $\hat{A}_1$. Introducing the new fields $U(\mathbf{x})$ defined by[9]

$$T^aA^a_1(\mathbf{x})=-iU^{-1}(\mathbf{x})\partial_1U(\mathbf{x}) \tag{20}$$

or in other words,

$$U(\mathbf{x})=U(x_1,x_2)=Pexpi\int_{-\infty}^{x_1}dtT^aA^a_1(t,x_2), \tag{21}$$

one can find the solution to the constraint equation eq.(14) takes the following form

$$\begin{aligned}\Psi_0[A] &= exp[-i\tfrac{k}{12\pi}\textstyle\int_M d^3y\epsilon^{\alpha\beta\gamma}TrU^{-1}\partial_\alpha U^{-1}\partial_\beta U^{-1}\partial_\gamma-i\tfrac{k}{8\pi}\int_\Sigma d^2xtr\partial_iU^{-1}\partial^iU] \\ &= exp-S_{WZW}\end{aligned} \tag{22}$$

Hence one see that physical vacuum state corresponds to a special conformal block of WZW model — the scattering amplitude from vacuum to vacuum .

$$< 0\ out|0\ in >= \int \mathcal{D}U\ exp - iS_{WZW} \tag{23}$$

If there exist Wilson loops in the universe, in order to exhibit the explicit identification of conformal block with physical state, we adopt complex coordinate description with the convention $z=\frac{1}{\sqrt{2}}(x_1+ix_2)$, $\bar{z}=\frac{1}{\sqrt{2}}(x_1-ix_2)$; $\hat{A}_z=\frac{1}{\sqrt{2}}(\hat{A}_1-i\hat{A}_2)$, $\hat{A}_{\bar{z}}=\frac{1}{\sqrt{2}}(\hat{A}_1+i\hat{A}_2)$. It can be easily proved that the $\hat{A}_{\bar{z}}$ polarization in term of complex language is consistent with the real $\hat{A}_1$ polarization

$$\begin{aligned} [\hat{A}_z, \hat{A}_w] &= [\hat{A}_{\bar{z}}, \hat{A}_{\bar{w}}] = 0 \\ [\hat{A}^a_z, \hat{A}^b_{\bar{w}}] &= -i\tfrac{8\pi}{k}\delta^{(2)}(z-w)\delta^{ab} \end{aligned} \tag{24}$$

Correspondingly, the constraint equation eq.(20) takes the following form

$$\frac{k}{8\pi}F^a_{z\bar{z}}|phys>=\sum_{n=1}^{N}[\delta^{(2)}(z-z_{P_n})T^a_{(n)}-\delta^{(2)}(z-z_{Q_n})T^a_{(n)}]|phys> \tag{25}$$

Starting from this, one can verify that

$$\begin{aligned} \frac{\partial}{\partial z_{P_n}}\Psi_{phys}[A_{\bar{z}}] &= \tfrac{2}{k+C_V}\Big(\sum_{n\neq m}^{N}\tfrac{T^a_{(n)}T^a_{(m)}}{z_{P_n}-z_{P_m}}-\sum_{m=1}^{N}\tfrac{T^a_{(n)}T^a_{(m)}}{z_{P_m}-z_{Q_n}}\Big)\Psi_{phys}[A_{\bar{z}}], \\ \frac{\partial}{\partial z_{Q_n}}\Psi_{phys}[A_{\bar{z}}] &= \tfrac{2}{k+C_V}\Big(\sum_{m\neq n}^{N}\tfrac{T^a_m T^a_n}{z_{Q_n}-z_{Q_m}}-\sum_{m=1}^{N}\tfrac{T^a_{(m)}T^a_{(n)}}{z_{Q_n}-z_{P_m}}\Big)\Psi_{phys}[A_{\bar{z}}], \end{aligned} \tag{26}$$

that is,

$$\frac{\partial}{\partial z_k}\Psi_{phys}[A_{\bar{z}}]=\frac{2}{k+C_V}\sum_{k\neq l}^{2N}\frac{T^a_{(k)}T^a_{(l)}}{z_k-z_l}\Psi_{phys}[A_{\bar{z}}], k=1,2,...2N. \tag{27}$$

They are exactly the Knizhnik-Zamolodchikov equations satisfied by the conformal block for $2N$ primary fields

$$\frac{\partial}{\partial z_k}<\Phi_1(z_1)...\Phi_{2N}(z_{2N})>=\frac{2}{k+C_V}\sum_{k\neq l}^{2N}\frac{T^a_{(k)}T^a_{(l)}}{z_k-z_l}<\Phi_1(z_1)...\Phi_{2N}(z_{2N})> \tag{28}$$

Hence from eqs.(23) and (27), we come to the conclusion that physical states can be explained as the conformal blocks of WZW model.

In summary, we perform BRST quantization for Chern-Simons field theory and reduce the physical state condition $\hat{Q}_B$=0 to the Gauss law constraints associated with the case that Wilson loops are present or not, then we exhibit the identification of constraint equation for physical states with Knizhnik-Zamolodchikov equation satisfied by conformal blocks, this is consistent with the facts coming from Witten's canonical quantization approach. In this sense, we state that for Chern-Simons field theory, its BRST and canonical quantization are equivalent.

Acknowledgement: This work is supported partially by Chinese National Natural Science Foundation and the Grant LWTZ–1298 of Chinese Academy of Science and this work was completed during the workshop of Quantum Group and Low-dimensional Field theory of CCAST, 1992. The authors would like to thank all the staff of CCAST for their hospitality

References

[**1**] E.Witten, Commu. Math. Phys. **121** (1989), 351.

[**2**] E.Guadagnini, M.Martellini and M.Mintchev, Phys.Lett. **B 227**(1989), 111.Nucl. Phys.B 330(1990), 575.

[**3**] L.Alvarez-Gaume, J.M.F.Labastida and A.V.Ramallo, Nucl. phys. **B 334**(1990) 103.

[**4**] A.Blasi and R.Collina. Nucl. Phys. B 345(1990) 472.

[**5**] F.Delduc, C.Lucchesi, O.Piguet and S.P.Sorella, Nucl. Phys. **B 346** 2,3(1990) 313.

[**6**] M.Bos ansd V.P. Nair, Int. J. Mod. Phys.A 5(1990) 959.

[**7**] S.Elitzur, G.Moore, A. Schwimmer and N.Siberg, Nucl. phys. **B 326**(1989) 108.

[**8**] J.M.F.Labastida and A.V.Ramallo, Phys.Lett.B 228(1989) 214.

[**9**] E.Guadagnini, M.Martellini and M.Mintchev, Nucl.Phys. **B336**(1990) 581.

[**10**] T.Kugo and I.Ojima, Supp.Prog. Theor. 66(1979) 1.

[**11**] R.Marnelius, Phys.Lett.B99(1981) 467.

CCAST-92-9

π_1-Effects in Quantum Theory via Geometric Quantization and Quantum Statistics

Dehai Bao and Zhong-Yuan Zhu

CCAST (World Laboratory)

P.O.Box 8730, Beijing 100080, P.R.China

Institute of Theoretical Physics,Academia Sinica

P.O.Box 2735,Beijing 100080, P.R.China

Abstract

General classical systems with phase spaces being cotangent bundles are quantized in geometric quantization formalism. Under Schrodinger coordinate polarization, it is shown that quantum Hilbert spaces are composed of appropriate sections of some vector bundles with flat connections over classical configuration spaces. Concequently, we have shown that different adoptions of irreducible unitary representations of the fundamental group of the classical configuration space will lead to different quantum Hilbert spaces, and hence, different quantum theories.

Applying the result to identical partical systems, we have shown that various quantum statistics can be naturally derived. Physical interpretations of various quantum statistics are also discussed.

PACS: 03.65. -w

1

[1]Works supported by NSF of China and the Grant LWTZ-1298 of Academia Sinica.

1. Introduction

The π_1-effect in quantum theory is referred to such phenomenons that, when the fundamental group of a classical configuration space is nontrivial, various new effects which are unobservable at classical level may appear at quantum level. An example of π_1-effect in quantum theory is the famous Aharanove-Bohm effect[1].

Many people have investigated the problem and its concequances in the framework of Feynmann's path integral quantization[2]. However, with an observation that π_1-effect is a kind of pure quantum effect related to global properties and should emerge naturally in quantization procedure, it seems reasonable and helpful to treat the problem in view of geometric quantization which is essentially the "globalization" of canonical quantization. Some papers where discussions are along this line but limited at scalar quantum mechanics has been published previously[3].

In this paper, we will discuss the problem in a more general contents and show that general π_1-effect can be naturally derived via geometric quantization. We will also discuss the application of our result to identical partical systems, where various quantum statistics will appear as natural concequances of the π_1-effect. Physical interpretations of various quantum statistics are to be discussed, either.

2. The π_1-Effect via Geometric Quantization

In Hamiltonian formalism of classical mechanics, a classical system is described by a phase space which is a symplectic manifold (M, ω) plus a set of classical observables $f \in C^\infty(M, R)$. The purpose of geometric quantization is to construct

CCAST-92-9

π_1-Effects in Quantum Theory via Geometric Quantization and Quantum Statistics

Dehai Bao and Zhong-Yuan Zhu

CCAST (World Laboratory)

P.O.Box 8730, Beijing 100080, P.R.China

Institute of Theoretical Physics, Academia Sinica

P.O.Box 2735, Beijing 100080, P.R.China

Abstract

General classical systems with phase spaces being cotangent bundles are quantized in geometric quantization formalism. Under Schrodinger coordinate polarization, it is shown that quantum Hilbert spaces are composed of appropriate sections of some vector bundles with flat connections over classical configuration spaces. Concequently, we have shown that different adoptions of irreducible unitary representations of the fundamental group of the classical configuration space will lead to different quantum Hilbert spaces, and hence, different quantum theories.

Applying the result to identical partical systems, we have shown that various quantum statistics can be naturally derived. Physical interpretations of various quantum statistics are also discussed.

PACS: 03.65. -w

1

[1]Works supported by NSF of China and the Grant LWTZ-1298 of Academia Sinica.

1. Introduction

The π_1-effect in quantum theory is referred to such phenomenons that, when the fundamental group of a classical configuration space is nontrivial, various new effects which are unobservable at classical level may appear at quantum level. An example of π_1-effect in quantum theory is the famous Aharanove-Bohm effect[1].

Many people have investigated the problem and its concequances in the framework of Feynmann's path integral quantization[2]. However, with an observation that π_1-effect is a kind of pure quantum effect related to global properties and should emerge naturally in quantization procedure, it seems reasonable and helpful to treat the problem in view of geometric quantization which is essentially the "globalization" of canonical quantization. Some papers where discussions are along this line but limited at scalar quantum mechanics has been published previously[3].

In this paper, we will discuss the problem in a more general contents and show that general π_1-effect can be naturally derived via geometric quantization. We will also discuss the application of our result to identical partical systems, where various quantum statistics will appear as natural concequances of the π_1-effect. Physical interpretations of various quantum statistics are to be discussed, either.

2. The π_1-Effect via Geometric Quantization

In Hamiltonian formalism of classical mechanics, a classical system is described by a phase space which is a symplectic manifold (M, ω) plus a set of classical observables $f \in C^\infty(M, R)$. The purpose of geometric quantization is to construct a Hilbert space $\mathcal{H}$ from this symplectic manifold and establish a map $f \to \hat{f}$ so that $\hat{f}$ is a linear operator acting on $\mathcal{H}$ and satisfies following Dirac's quantum conditions:

1). $f \to \hat{f}$ is a monomonphism.

2). If $\{f, g\} = k$, then $[\hat{f}, \hat{g}] = -i\hbar\hat{k}$.

where $\{.,.\}$ is Poisson bracket and $[.,]$ is commutator.

Considering the cross-sections of a vector bundle possece a natural structure of vector space, quantization for a classical system stated above could be achieved by introducing a Hermit vector bundle B over M with connection. The set of square-integrable smooth sections will compose a Hilbert space $\mathcal{H}$ and the operator $\hat{f}$ corresponding to $f \in C^{\infty}(M, R)$ can be constructed as follows[4]. The one-parameter group ϕ_f^t of canonical transformations generated by f has a unique lift to a one-parameter group of connection preserving transformations of B which defines the action of ϕ_f^t on the space of sections of B. The operator $\hat{f}$ is then defined by

$$\hat{f}[s] = i\hbar \frac{d}{dt}(\phi_f^t s)|_{t=0} \tag{1}$$

where s are sections of vector bundle B.

Analogue to the derivations for line bundle[3,4], above expression can be envaluated more explicitively as:

$$\hat{f} = -i\hbar \nabla_{X_f} + f \cdot I \tag{2}$$

where I is the identity operator, ∇ is the covariant differential operator defined on the bundle and X_f is the Hamiltonian vector field associated with f, defined as $i_{X_f}\omega = -df$.

It is obvious that above construction of $\hat{f}$ satisfies Dirac's conditions 1). In order that it obeys condition 2), substituting equation (2) into the condition 2), a straightforward calculation shows that,

$$i\hbar([\nabla_{X_f}, \nabla_{X_g}] - \nabla_{[X_f, X_g]}) = X_f(g) \tag{3}$$

Namely, the curvature of the vector bundle B (lhs. of (3)) should be proportion to the symplectic structure ω of the base manifold (M, ω).(Note that,$X_f(g) = 2\omega(X_f, X_g)$.)

However, above constructed Hilbert space $\mathcal{H}$ can not be directly used as a correct physical quantum Hilbert space, since "wave functions" (identified with smooth sections of the vector bundle over M) generally depend on both coordinates and conjagated momenta in phase space, which would lead to violation of uncertainty principle.(It is for this reason that $\mathcal{H}$ is called prequantization Hilbert space[4].)

A typical way to circumvent this problem is to reduce the prequantization space by "polarization" mechanism[4], that is, only those "polarized" smooth sections are admitted to be physical "wave functions". When $M = T^*Q$ with Q is the configuration space, the simplest polarization is the so-called vertical polarization(or Schrodinger coordinate polarization) and the polarized sections are those depend only on coordinates of Q.

Under the vertical polarization, physical "wave-functions" can be equally regarded as smooth sections of a vector bundle E over Q, which is actually the project bundle of previous bundle B over T^*Q. Since the curvature of B is proposional to the symplectic structure ω of T^*Q and due to the fact that $\omega|_Q = 0$, we arrive at following lemma:

Lemma: When a classical system is quantized via geometric quantization under Schrodinger coordinate polarization, its quantum Hilbert space is composed of smooth sections of some vector bundle E over classical configuration space Q, whose curvature must be vanish. In other words, the bundle E admits only flat connections.

Now, according to Milnor's theorem[5], which said that a vector bundle over Q with standard fibre C^n admit a flat connection if and only if it is of the form $\tilde{Q} \times_R C^n$, where $\tilde{Q}$ is the universal covering space of Q, R is a n-dimensional complex representation of Q's fundamental group $\pi_1(Q)$ and $\tilde{Q} \times_R C^n$ is the associated vector bundle of $\tilde{Q}$ (which itself is a principle $\pi_1(Q)$-bundle over Q). As a result, physical "wave functions" can be identified with smooth sections of vector bundle $\tilde{Q} \times_R C^n$.

Furthermore, it can be proved that the smooth sections of vector bundle $\tilde{Q} \times_R C^n$ can be equivalently described by some appropriate functions $\tilde{\Psi} \in C^\infty(\tilde{Q}, C^n)$. To be concrete, let us denote $\Gamma(\tilde{Q} \times_R C^n)$ as the set of smooth sections of $\tilde{Q} \times_R C^n$.

Given a smooth section $S \in \Gamma(\tilde{Q} \times_R C^n)$, we may define a corresponding function $\tilde{\Psi}_s \in C^\infty(\tilde{Q}, C^n)$ by the relation,

$$S(pr(\tilde{q})) = [\tilde{q}, \tilde{\Psi}_s(\tilde{q})] \tag{4}$$

where $pr : T^*Q \to Q$.

Since $S(pr(\tilde{q})) = S(pr(\tilde{q}\gamma))$ for all $\gamma \in \pi_1(Q)$, from (4) and the equivalence relation $[\tilde{q}, y] = [\tilde{q}\gamma, R(\gamma^{-1})y]$ of associated bundle, we have,

$$[\tilde{q}, \tilde{\Psi}_s(\tilde{q})] = [\tilde{q}\gamma, \tilde{\Psi}_s(\tilde{q}\gamma)] = [\tilde{q}, R(\gamma)\tilde{\Psi}_s(\tilde{q}\gamma)] \tag{5}$$

Namely, the function $\tilde{\Psi}$ should obey following property,

$$\tilde{\Psi}_s(\tilde{q}) = R(\gamma)\tilde{\Psi}_s(\tilde{q}\gamma) \tag{6}$$

Conversely, given a function $\tilde{\Psi} \in C^\infty(\tilde{Q}, C^n)$ satisfing (6), we can define a corresponding smooth section of $\tilde{Q} \times_R C^n$ as follows,

$$S_\Psi(q) = [\tilde{q}, \tilde{\Psi}(\tilde{q})] \tag{7}$$

Since

$$[\tilde{q}, \tilde{\Psi}(\tilde{q})] = [\tilde{q}\gamma, R(\gamma^{-1})\tilde{\Psi}(\tilde{q})] = [\tilde{q}\gamma, \tilde{\Psi}(\tilde{q}\gamma)] \tag{8}$$

$S_\Psi(q)$ is independent of choices $\tilde{q} \in pr^{-1}(q)$ and thus, is well-defined.

Therefore, we arrive at an improtant result that,

$$\Gamma(\tilde{Q} \times_R C^n) \simeq \{\tilde{\Psi} \in C^\infty(\tilde{Q}, C^n) | \tilde{\Psi}(\tilde{q}\gamma) = R(\gamma^{-1})\tilde{\Psi}(\tilde{q}). \quad \forall \gamma \in \pi_1(Q)\} \tag{9}$$

As a result, when performing quantization in multi-connected configuration space Q, physical wave functions can be described by functions $\tilde{\Psi} \in C^\infty(\tilde{Q}, C^n)$ satisfying $\tilde{\Psi}(\tilde{q}\gamma) = R(\gamma^{-1})\tilde{\Psi}(\tilde{q})$, $\forall \gamma \in \pi_1(Q)$, where R is any unitary representation of the foundamental group $\pi_1(Q)$.

Note that the wavefunctions $\tilde{\Psi}(\tilde{q})$ are defined over the universal covering space $\tilde{Q}$ of classical configuration space Q and are singled-valued. The quantum system may also be described by wavefunctions $\Psi(q)$ over Q through a suitable projection[*]. In this case, the wavefunctions $\Psi(q)$ have to be multi-valued. When a point q is moved around a loop $\gamma \in \pi_1(Q)$, the wavefunction $\Psi(q)$ will gain a phase factor $R(\gamma^{-1})$.

Clearly, when $\pi_1(Q)$ has various nonequivalent irreducible unitary representations, each of them will bear a distinct quantum theory. Therefore, there is generally an ambiguity in quantizing a classical system with nontrivial configuration space. This kind of ambiguity is totally different from dynamical ones such as ordering ambiguity in conventional quantum mechanics[6] and is commonly referred as the π_1-effect in quantum theory. It is recognized that many quantum phenomenons are straight concequances of the effect. In next section, we will show how various quantum statistics can be naturally explained and derived from the result here.

3. The π_1-effect and Quantum Statistics

At its beginnings, the concept of quantum statistics was limited at Bose-Einstein statistics or Fermi-Dirac statistics of identical partical systems, which require the wavefunctions of the systems to be symmetry or antisymmetry under permutation of two particles' positions. Though the requirements upon wavefunctions are basically from physics considerations, it has been recognized that they can be understood topologically. The essential idea is to consider the identicalness of particles before quantization, which leads to following classical configuration space for N identical particles in R^3[2],

$$C_N(R^3) = (R^{3N} - \Delta)/S_N \tag{10}$$

where S_N is permutation group and Δ is the diagolized set which should be cut off in order to make $C_N(R^3)$ a smooth manifold.

The fundamental group of $C_N(R^3)$ is readily proved to be S_N. There are only two 1-dimensional irreducible unitary representations of S_N. The first of them is such that $R(\gamma) = 1$, $\forall \gamma \in S_N$; the second one is such that $R(\gamma) = -1$ if γ is a odd permutation and $R(\gamma) = 1$ if γ is an even permutation. Obviously, according to the result of last section, they lead to Fermionic and Bosonic statistics, respectively.

When Wilczeck proposed anyon systems which was shown to be of importance in understanding FQHE and perhaps high-T_c superconductivity, fractional statistics emerged as a commonly interested issue[7]. Though Wilczeck's anyon system was based on concrete model(composite of electric charge and magnetic flux), it is not hard to see that anyon can be understood more naturally in the framework of π_1-effect. Let us consider a system of N identical particles in R^2, whose classical configuration space is $C_N(R^2) = (R^{2N} - \Delta)/S_N$.

Its fundamental group is proved to be Artin's braid group B_N composed of generators $\{\sigma_i | i = 1, 2, ..., N\}$ with following algebraic relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \forall |i - j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{11}$$

where σ_i means the operation moving the ith particle around the (i+1)th particle once without enclosing any other particles. The 1-dimensional irreducible unitary representations of the B_N are easily shown to be labelled by a parameter $\theta \in [0, 2\pi)$ and the forms of σ_i's under the representation labelled by θ are as follows:

$$R^\theta(\sigma_i) = e^{i\theta}, \quad \forall \sigma_i \in B_N \tag{12}$$

Therefore, referring to the result of last section, general θ-statistics intermediated between Boson and Fermion are allowed in two dimensional plane. (For more detailed analysis please see references [3] or Y.S.Wu's paper in [2].)

With these succeses, it could be suggested that this kind of topological approach of understanding general quantum statistics might be a correct attemption. Instead of limitting on Euclidean space, we may consider identical particle systems on general manifolds. Furthermore, we may investigate the results of nonscalar quantization, which will involve high dimensional irreducible representations of the foundamental group[8].

The configuration space of N spinless identical particles in a general manifold M is $C_N(M) = (M^N - \Delta)/S_N$. Its fundamental group is to be denoted as $B_N(M)$, commonly named as N-string braid group on M.

According to the discussions last section, different adoptions of irreducible unitary representations of $B_N(M)$ will bear distinct quantum theories. When M is simply connected, these distinct quantum theories can be equivalently interpretated as quantum theories of N identical particles system with distinct quantum statistics, since the elements of $B_N(M)$ are nothing but the exchanging operations among particles. However, if M itself is multi-connected, there is a quantization ambiguity already present for $N = 1$ (and therefore has nothing to do with statistics) which will manifest itself again in $B_N(M)$ for any N. To get the set which labels different quantum statistics, we must "mod out" those resulted from $\pi_1(M)$ in an appropriate way.

To illustrate the ideas explicitively, let us give a concrete example—three identical particles on a torus T^2. According to the general discussions of braid groups by Ladegaillerie[9], the foundamental group of the system's configuration space, denoted as $B_3(T^2)$, is readily shown to be composed of generators $\{\sigma_1, \sigma_2; \alpha, \beta\}$, where σ_i means the operation moving the ith particle around the (i+1)th particle once without enclosing any other particles and α (β) mean the operations moving the 1th particle around the meridinal(longitadinal) loop of the torus once. The algebraic relations among the generators are as follows:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad \sigma_1\sigma_2^2\sigma_1\beta\alpha^{-1}\beta^{-1}\alpha = e,$$

$$\sigma_2\alpha\sigma_2^{-1}\alpha^{-1} = \sigma_2\beta\sigma_2^{-1}\beta^{-1} = e, \qquad (\sigma_1\alpha)^2 = (\alpha\sigma_1)^2 \tag{13}$$

$$(\sigma_1\beta)^2 = (\beta\sigma_1)^2, \qquad \sigma_1^{-1}\alpha\sigma_1\beta = \beta\sigma_1\alpha\sigma_1$$

. It should be noted that though $B_3(T^2)$ has four generators, only two of them, σ_1 and σ_2, are directely related to quantum statistics, since the other two, α and β, are actually the generators of $\pi_1(T^2)$ and have nothing to do with exchanging operations. We will call the subgroup composed of σ_1 and σ_2 as "statistics subgroup".

Now, by examining the irreducible unitary representations of $B_3(T^2)$, we will see that various quantum statistics are allowed for such three identical particles system:

i). Scalar θ-statistics,

For one dimensional unitary representations of $B_3(T^2)$, the algebraic relations (13) can be simplified as,

$$\sigma_1 = \sigma_2 = \sigma, \qquad \sigma^2 = e.$$

Therefore, σ have to be (+1) or (-1), which means that only allowed scalar statistics of the system are Bosonic or Fermionic statistics.

ii). Nonscalar abelian statistics,

Since $B_3(T^2)$ is a nonabelian group, its high dimensional irreducible representations must be nonabelian. However, generally speaking, the representations of the "statistics subgroup" composed of σ_1 and σ_2 under some high dimensional irreducible representations of $B_3(T^2)$ could be abelian. Therefore, due to the nontrivialness of $\pi_1(T^2)$, there appears the concept of nonscalar abelian statistics. As a simplest example, let us consider a certain high dimensional irreducible unitary representation of $B_3(T^2)$ where $\sigma_1 = \sigma_2 = e^{i\theta} I$ while α, β take appropriate forms to make the representation being irreducible. According to (13), one can find that θ must obey the condition $e^{6i\theta} = 1$. Namely, θ can take only one of values 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$. Adoption of $\theta = 0$ or π means Bosonic or Fermionic statistics, while other choices of θ's values correspond to various fractional statistics.

A fact to be emphasized is that, if limited at scalar quantum mechanics, the system allows only Bosonic or Fermionic statistics as mentioned in i); however, if we include nonscalar quantum mechanics into considerations, then new possibilities of fractional statistics are allowed.

iii). Nonabelian statistics,

Under most of the high dimensional irreducible representations of nonabelian group $B_3(T^2)$, the representations of "statistics subgroup" composed of σ_1 and σ_2 are nonabelian, either. In these cases, the corresponding quantum statistics are commonly named as nonabelian statistics, where the "phase factors" resulted from exchanging operations among identical particles are noncommutable matrix.

4. Conclusions and Discussions

In this paper, general classical systems with phase space being cotangent bundles have been quantized in geometric quantization formalism. The quantization procedure has been performed generally without limitting on scalar quantum mechanics. The resulted quantum wavefunctions are generally multi-valued, multi-component functions over configuration space. The multi-valueness and multi-componentness of quantum wavefunctions are determined by the irreducible unitary representations of the foundamental group of the classical configuration spaces, which leads to so-called π_1-effect in quantum theory.

Applying our result to identical particles systems, we have shown that various quantum statistics could be interpretated as natural concequances of the π_1-effect. Furthermore, we have discussed identical particles systems in general manifold where the topological properties of the base manifold are shown to have nontrivial influences upon system's possible quantum statistics. It's shown that a system which does not allow fractional statistics in the framework of scalar quantum mechanics may allow fractional statistics if we adopt multi-component quantum wavefunctions(The number of components is just the dimension of the adopted representation of system's fundamental group). In addition, nonabelian statistics which do not exist in scalar quantum mechanics generally emerge in nonscalar quantum mechanics.

Here, we have seen that the generation from conventional scalar quantum wavefunctions to multi-component quantum wavefunctions is not a trivial one. Though this generation is straightforward in mathematical constructions of geometric quantization as described in section 2, its physics interpretations deserve more discussions. As usual, the multi-componentness of a system's wavefunctions can be attributed to some "internal freedom" of the system. However, if we are interested in the origin of the "internal freedom", there exist two very different cases: 1). One case is that the number of components of system's wavefunctions d is equal to m^N (where N is the number of identical particles of the system). In this case, the "internal freedom" of the system is understood as causing by the internal freedom of individual particle whose internal space is of dimension m. The physical realization of this case could be to regard individual particle as a composite of some nonabelian charge with corresponding flux-tube. (Similar to Wilczeck's anyon model.) 2). The other case is that $d \neq m^N$. In this case, the system's d "internal" degrees of freedom can not be attributed to individual particles, and one has to say that they are associated only to the many-body system. The physical realization of this case is not a simple work and is under researching by many people[10].

Acknowledgement: The authors are grateful to Dr.Y.Yue and W.F.Chen for many helpful discussions.

References:

1. Y.Aharonov and D.Bohm, *Phys.Rev.115(1959)485.*

   Y.Aharonov and A.Casher, *Phys,Rev.Lett.53(1984)319.*

2. L.Schulman, *Phys.Rev.176(1968)1558.*

   M.Laidlaw and C.M.DeWitt, *Phys.Rev.D3(1971)1375.*

   J.S.Dowker, *J.Phys.A5(1972)936.*

   Y.S.Wu, *Phys.Rev.Lett.52(1984)2103.*

   G.Date, T.Govindarajan, P.Sankavan and R.Shankar, *Commun.Math.Phys.132(1990)293.*

3. P.A.Horvathy, *Phys.Lett.76A(1980)11.*

   Y.Yue and Z.Y.Zhu, *Commun.Theor.Phys.15(1991)339.*

4. N.Woodhouse, *Geometric Quantization(Oxford Press, N.Y.1980)*

J.Sniatycki, *Geometric Quantization and Quantum Mechanics (Springer-Verlag, N.Y.1980)*

5. J.Milnor, *Commun.Math.Helv.32(1957)215.*

6. F.Berezin, *Theor.Math.Phys.6(1971)194.*

D.Bao and Z.Y.Zhu, *Commun.Theor.Phys.16(1991)201.*

7. F.Wilczeck, *Phys.Rev.Lett.49(1982)957.*

X.G.Wen, F.Wilczeck and A.Zee, *Phys.Rev.B39(1989)11413.*

8. E.Sudarshan, T.Imbo and T.Govindarajan, *Phys.Lett.B213(1988)471.*

Y.S.Wu, *Int.J.Mod.Phys.B5(1991)1649.*

9. Y.Ladegaillerie, *Bull.Sci.Math.100(1976)255.*

10. For example, see X.G.Wen, *Int.J.Mod.Phys.B5(1991)1641.*

or C.L.Ho and Y.Hosotani, *IP-astp-10-91.*

CCAST-92-13
ASITP-92-17
March 1992

Quantum Hall Effect under Non-uniform External Fields

Yue Yu

CCAST(World Laboratory) P.O.Box 8730, Beijing 100080,
and
Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China [1]

ABSTRACT

In this Letter, we predict that the quantum Hall effects, including both integral and fractional cases, can be observed when the magnetic field and the applied electric field are not uniform. In the present model, the non-uniformity breaks the higher Landau levels into energy bands while it may not affect the degeneracy of the lowest Landau level.

[1] Mailing address

The integral and fractional quantum Hall effects (IQHE and FQHE) were discovered in 1980 [1] and 1982[2]. Many investigations in this field have been provided.[3] In most of the experimental and theoretical proposals, one considers the effect under the uniform external fields because the uniformity of the external fields makes the simplity both of experiment and theory. However, one can ask if the quantum Hall effect can be observed when the magnetic field and the applied electric field are not uniform. In this letter, we discuss this subject. We find that when the uniform magnetic field is perturbed the ground state wave function of the single-electron is exactly soluble if there is a suitable electric field. Thus, the variational ground state wave function of the N-electrons may be constructed by means of a similar way to the construction in the common FQHE. The filling factor of this state is not uniform. But, the non-uniformity of the filling factor is same as that of the magnetic field in our case such that the Hall coefficient is still fractional. Under such external fields, however, the excited states of the single electron are not exactly soluble. To the first order of the non-uniformity, we find that the degeneracy of the Landau levels is lifted completely. Therefore, if the non-uniformity is regarded as some kind of imperfections, the IQHE should be observed in such case.

Consider a free electron moving on a two-dimensional plane. Assume there are non-uniform magnetic and electric fields. For specification[4], we investigate following two cases:

$$\mathbf{B}_1 = -B_0(1+8g_1r^2)\hat{k}, \quad \mathbf{E}_1 = \frac{8g_1\hbar B_0}{m_e c} r\hat{r}; \tag{1a}$$

$$\mathbf{B}_2 = -B_0(1-\frac{g_2}{2r})\hat{k}, \quad \mathbf{E}_2 = \frac{g_2\hbar B_0}{2m_e cr^2}\hat{r}, \tag{1b}$$

where g_i, $i=1,2$, are the parameters with dimensions L and L^{-2} respectively; B_0 are constant; $\hat{k}, \hat{r}$ are the unit vectors in the normal to the plane and the radius direction. The fields (1a) is related to the normal matrix model with the potential $V(M) = (1/4)Tr(M^\dagger M + (g_1/2)Tr(M^\dagger M)^2$ (M is an $N\times N$ normal matrix).[5] The fields (1b) may be prefered by experimentists. For example, a ring sample called Corbino disk is put closely between two sets of solenoids. In each set, the solenoids are ring upon ring. Then, by adjusting the electric currents through these solenoids, the magnetic field will have a reduced distribution along the radius. The electric field $\mathbf{E}_2$ may be made by putting a point charge on the center of the ring, whose charge, of course, should coincide with eq.(1b). In symmetric gauge, the vector potentials and the scalar potentials of eq.(1) are $\mathbf{A}^{(1)} = \frac{cB_0}{2e}[y(1+4g_1r^2)\hat{i} - x(1+4g_1r^2)\hat{j}], \varphi^{(1)} = \frac{4g_1\hbar B_0}{m_e c}r^2$ and $\mathbf{A}^{(2)} = \frac{cB_0}{2e}[y(1-g_2/r)\hat{i} - x(1-g_2/r)\hat{j}], \varphi^{(2)} = -\frac{g_2\hbar B_0}{2m_e cr}$, the single-electron Hamiltonians are

$$H_{S_i} = \frac{1}{2m_e}[(\frac{\hbar}{i}\frac{\partial}{\partial x} + \frac{e}{c}A_x^{(i)})^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y} + \frac{e}{c}A_y^{(i)})^2] - e\varphi^{(i)}. \tag{2}$$

It is easy to see that the Schrodinger's equations $H_{S_i}\phi^{(i)} = \mathcal{E}\phi^{(i)}$ have the exact ground state wave functions $\phi^{(1)} = z^m exp\{-(1/4l^2)r^2 - (g_1/2l^2)r^4\}$ and $\phi^{(2)} = z^m exp\{-(1/4l^2)r^2 - (g_2/2l^2)r\}$ $(l = (\hbar c/eB_0)^{1/2})$ and the energy of the groud states is $\mathcal{E}_0 = (1/2)\hbar\omega_c^0$, where $z = x + iy$ and $\omega_c^0 = eB_0/m_e c$ is the cyclotron frequency associated with the uniform field B_0.

Now, we consider the N-electron system described by the Hamiltonian

$$H^{(i)} = \sum_{j=1}^{N}[\frac{1}{2m_e}(\frac{\hbar}{i}\nabla_j + \frac{e}{c}\mathbf{A}_j^{(i)})^2 - e\varphi^{(i)}(r_j)] + U, \tag{3}$$

where U includes the potential generated by a neutralizing background and the Coulomb interactions among the electrons. And the interactions are weak *i.e.* $|U| \ll \hbar\omega_c^0$. The variational ground state wave functions, thus, can be constructed by the Laughlin-Jastraw's form[6]

$$\begin{aligned}\psi_m^{(1)}(z_1, ..., z_N) &= \prod_{i<j}^{N}(z_i - z_j)^m exp\{-\frac{1}{4}\sum_i |z_i|^2 - \frac{g_1}{2}\sum_i |z_i|^4\},\\ \psi_m^{(2)}(z_1, ..., z_N) &= \prod_{i<j}^{N}(z_i - z_j)^m exp\{-\frac{1}{4}\sum_i |z_i|^2 - \frac{g_2}{2}\sum_i |z_i|\},\end{aligned} \tag{4}$$

where the magnetic length $l = (\hbar c/eB_0)^{1/2}$ is set equal to unit and m is an odd integer.

Along the Laughlin's formulation in the discussions of the common FQHE,[6] one can write the square of the modulus of $\psi_m^{(i)}$ as a classical Boltzmann distribution

$$|\psi_m^{(i)}(z_1, ..., z_N)|^2 = exp\{-\beta\Phi_{eff}^{(i)}(z_1, .., z_N)\}, \tag{5}$$

where $1/\beta = m$ is the fictitious temperature and the effective potentials

$$\begin{aligned}\Phi_{eff}^{(1)} &= -2m^2\sum_{i<j} ln|z_i - z_j| + m\sum_k[\frac{|z_k|^2}{2} + g_1|z_k|^4],\\ \Phi_{eff}^{(2)} &= -2m^2\sum_{i<j} ln|z_i - z_j| + m\sum_k[\frac{|z_k|^2}{2} + g_2|z_k|].\end{aligned} \tag{6}$$

These effective potentials describe such a system: the first term in the potential represents the repulsion between particles of charge m via two dimensional Coulomb interaction; the second term is the attraction of these particles to the origin due to a neutralizing background on the same ring of charge densities $\rho_0^{(1)} = (1+8g_1r^2)/(2\pi l^2)$ and $\rho_0^{(2)} = (1-g_2/2r)/(2\pi l^2)$, *i.e.*

$$\Phi_{eff}^{(i)} = -2m^2\sum_{i<j} ln|z_i - z_j| + m\sum_k \int dz^2 \rho_0^{(i)}(|z|)ln|z-z_k|. \tag{7}$$

The neutrality of the system tells us that the electron density in states $\psi_m^{(i)}$ is equal to

$$\rho_m^{(i)} = \frac{\rho_0^{(i)}}{m} = \frac{1}{m}\frac{eB_i(r)}{hc}. \tag{8}$$

This results in the filling factors of these states have a radius distribution, $\nu^{(1)} = (1/m)(1+8g_1r^2)$ and $\nu^{(2)} = (1/m)(1-g_2/2r)$. Rigorously, we should prove that the perturbation to the incompressible quantum fluid theory do not change too many properties of the unperturbed theory. But if the perturbation is sufficient small we believe that the present system remains the main properties of the two-dimensional one-component plasma. For instance, in the small $\Gamma = 2m$, the system is a liquid. We shall give a detailed discussion for this matter in a separate proposal.

To understand the physical implication of the above result, we consider the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_e}\cdot\frac{\partial f}{\partial \mathbf{v}} = (\frac{\partial f}{\partial t})_{coll.}, \tag{9}$$

where $\mathbf{F} = e\mathbf{E}^{tot} + (e/c)\mathbf{v}\times\mathbf{B}$ with $\mathbf{E}^{tot}$ being the sum of the $\mathbf{E}_i$ and a uniform applied electric field $\mathbf{E}_0$. By definition, $\rho_m^{(i)} = \int f(\mathbf{r},\mathbf{v})d\mathbf{v}$ so that $\partial f/\partial\mathbf{r}$ is of the g_i-order. The velocity $\mathbf{v}$ in the lowest order is proportional to the electric field. Thus, in the weak field approximation, the second term of the left-hand side of eq.(9) may be ignored because it is proportional to $g_i|\mathbf{E}^{tot}|$. By using the relaxation time approximation, therefore, the conductivity tensor still has the familiar form $\sigma_{xx} = \sigma_0/[1+(\omega_c\tau)2], \sigma_{xy} = \rho_m^{(i)}ce/B_i - \sigma_{xx}/\omega_c\tau$ where $\sigma_0 = \rho_m^{(i)}e^2\tau/m$ and ω_c is the cyclotron frequency associated with B_i. When the relaxation time $\tau\to\infty$, we find that the Hall conductivity $\sigma_{xy} = \rho_m^{(i)}ce/B_i = \frac{1}{m}e^2/h$ is fractional while the longitude conductivity $\sigma_{xx} = 0$. As a result, one expects that the FQHE can be observed when the external magnetic an electric fields are not uniform.

The quasiparticles of the present model can also be discussed by the similar way to describe the quasiparticles in the common FQHE.[6] From eq.(7), we see that the only difference between our model and the two-dimensional one-component plasma is the density of the state. The charge of the particles is not changed. This, and the fractional Hall coefficient, imply that the quasiparticles in our model should have the fractional charge. In concluding our description to the FQHE, we world like to mention that the general fractional quantum

Hall state in our model may be constructed by the Haldane's hierarchical scheme.[7] We do not give the details here.

We turn to understand the excited states of the single-electron Schrodinger's equation. It is found that the excited states of the Schrodinger's equation are not exactly soluble. Up to the first order of perturbation, the higher Landau levels are broken into the energy bands. The standard perturbative theory in the course of quantum mechanecis may be employed. However, it is too complicated to solve our problem because the Landau levels are high degeneracy. Fortunately, the ground state of the single-electron is exactly known, which may be taken as a starting point to do the perturbative theory. Take the Hamiltonian H_{S_1} as an example, we present the results of the first order perturbation. Also consider the symmetric gauge, the wave functions are denoted by $\phi_{n,m}$ where n is the index of energy band and m is corresponding to the wave vector. The ground states $\phi_{0,m} = z^m e^{-f}$ ($f = (1/4)|z|^2 + (g_1/2)|z|^4$) are still degenerate as we have mentioned. Then, it may be check that the lowest state in each energy band, up to the first order of g_1,

$$\phi_{n,0} = L^n e^{-f}, \quad L = (1 - 4g_1|z|^2)\bar{z}, \tag{10}$$

and the eigenvalue of the first perturbation is (in the unit of l)

$$\mathcal{E}_{n,0} = [(1 + 16g_1)n + 1/2]\hbar\omega_c^0. \tag{11}$$

This implies that the lowest energy in each band is higher the original Landau energy. The other states may be constructed in the following way. For example, assuming the sought wave function $\phi_{n,1}$ is

$$\phi_{n,1} = L^{n-1}[a_n(\partial + \bar{z}) + L]ze^{-f}, \tag{12}$$

then a consistent coefficient a_n and eigen-energy are

$$a_n = -\frac{2n}{3 + (n+1)32g_1}, \quad \mathcal{E}_{n,1} = [(1 + (1+1)16g_1)n + \frac{1}{2}]\hbar\omega_c^0. \tag{13}$$

The state

$$\phi_{1,m} = [a_{1,m}(\partial + \bar{z}) + L]z^m e^{-f} \tag{14}$$

is, for a suitable $a_{1,m}$, corresponding to

$$\mathcal{E}_{1,m} = [(1 + (m+1)16g_1) + \frac{1}{2}]\hbar\omega_c^0. \tag{15}$$

Repeat the previous process, one can obtain the state $\phi_{n,m}$ and the corresponding energy $\mathcal{E}_{n,m} = [(1 + (m+1)16g_1)n + \frac{1}{2}]\hbar\omega_c^0$. The conclusion is the higher Landau levels are completely separated into energy bands. The theory of the IQHE is essentially based on the non-interaction two-dimensional electron gas with weak impurity or imperfection. In our case, the non-uniform parts of the external fields can be regarded as a type of weak imperfection. This imperfection breaks completely the Landau levels into energy bands. Furthermore, the existence of the electric field leads to that the extended states could exist. Thus, one expects that the IQHE can be observed in the present case.

The author thanks Wen-Feng Chen, Shao-Min Fei, Han-Ying Guo, Zhong-Yi Lu ,Jian-Bin Wu and Zhong-Yuan Zhu for useful discussions. This work was supported in part by the Chinese National Nature Science Foundation and the Grant LWTZ-1298 of Chinese Academy of Science. The work was completed during the Workshop of Quantum Group and Low-dimensional Field Theory of CCAST, 1992.The author would like to thank the Staff of CCAST for the warm hospitality.

References

[1] K. von Klitzing, G. Dorda and M. Pepper, Phys. Rev. Lett. **45** (1980) 494.

[2] D. C. Tsui, H. L. Stomer and A. C. Gossard, Phys. Rev. Lett. 48(1982) 1559.

[3] For a review book, see *The Quantum Hall Effect,* ed. by R. E. Prange and S. M. Girvin (Springer, New York, 1987).

[4] In general, the magnetic field may be taken as $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{A} = eB_0/2c[y(1+f(r)) - x(1+f(r))]$ and the electric field $\mathbf{E} = \nabla\varphi$, $\varphi = \hbar B_0/m_e c(r^2\partial f/\partial r^2 + f)$.

[5] L. L. Chau and Y. Yu, to appear in Phys.Lett. **A**.

[6] R. B. Laughlin, Phys. Rev. Lett. **50**(1983) 1395.

[7] F. D. Haldane, Phys. Rev. Lett. **51** (1983)605.

Solving of the Yang-Baxter Equation (II) [1]

Shi He

Institute of Systems Science, Academia Sinica
Beijing 100080, China

Abstract

Let the special matrix R and the permutation matrix P be

$$R = \begin{pmatrix} x_{15} & x_1 & x_1 & x_{11} \\ x_5 & x_{13} & x_9 & x_2 \\ x_6 & x_9 & x_{13} & x_2 \\ x_{12} & x_6 & x_6 & x_{16} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where x_i are complex variables.

Let $\check{R} = RP$. Then the Yang-Baxter equation is

$$M = (m_{ij}) = \check{R}_{12}\check{R}_{23}\check{R}_{12} - \check{R}_{23}\check{R}_{12}\check{R}_{23} = 0, \qquad (YBE)$$

where $\check{R}_{12} = \check{R} \otimes I$ and $\check{R}_{23} = I \otimes \check{R}$.

The Yang-Baxter equation (YBE) is a system of algebraic equations. By using Wu Elimination, it is solved for this special case. A series solutions of YBE including multiparameters are obtained.

[1]Work supported in part by CCAST (World Laboratory), ITP and NSF of China.

$\mathcal{ASITP}$-92-27
$\mathcal{CCAST}$-92-22

Quantum Algebra as Deformed Symmetry

Shao-Ming Fei and Han-Ying Guo

CCAST (World Laboratory)
P. O. Box 8730, Beijing 100080, China

and

Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China[1]

Abstract

The general algebraic (including deformation) maps among algebras with three generators are sytematically investigated in terms of symplectic geometry and geometric quantization on 2-D mainfolds. From which the explicit Hamiltonian of Heisenberg model with $SU_q(2)$ symmetry and arbitrary spin values are given. The deformed symmetries in differential dynamical systems and the q-deformation of $SO(3)$ group transformations in usual $\mathbf{R}^3$ are also discussed.

[1]Mailing address

Quantum Doubles and Quantum Double Pairs

The Relationship Between Quantum Algebras and Quantum Groups [1]

Han-ying Guo, Ke Wu and Renjie Zhang

CCAST(World Laboratory)
P. O. Box 8730, Beijing 100080, China

and

Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China [2]

Abstract

The concept of quantum double pairs was introduced and by this a method of realizing the quantum groups from quantum algebras was proposed.

[1]This work was supported by the Chinese National Natural Science Foundation.
[2]Mailing address.

Spin Chains Associated with Multiparameter R-Matrices of Six-Vertex Type [1]

Shao-Ming Fei[⋆,†], Han-Ying Guo[⋆,†], Lu-Yu Wang[⋆,‡] and Hong Yan[⋆,†]

⋆ CCAST (World Laboratory)
P.O.Box 8730, Beijing 100080, China

and

†Institute of Theoretical Physics, Academia Sinica
P.O.Box 2735, Beijing 100080, China[2]

‡Physics Department, Xinjiang University
Urumqi 830046, Xinjiang, China[2]

Abstract

The general Baxterizations and spin chain Hamiltonians related to the multiparameter solutions of Yang-Baxter equation are given. The quantum integrable spin-chain models are investigated via quantum inverse scattering method and the Bethe ansatz equations are presented. The spin-chains of XY type, associated to the nonstandard R-matrices are emphasized and the corresponding Bethe ansatz equations are shown to be analytically solvable. In the algebraic Bethe ansatz approach, the quantum universal enveloping algebras and quantum pseudo groups are analysed in a systematic way.

[1] Work supported in part by the National Natural Science Foundation of China, CCAST and ICTP.
[2] Mailing address.