African Journal Of Mathematical Physics Volume **10** *(2011)39-47*

Brans-Dicke scalar field and de Sitter Relativity

E. Benedetto

Mathematics and Computer Science Department
University of Salerno, Italy.

abstract

In this paper we investigate the problem to satisfy the Mach's principle in cosmology. Particularly we consider de Sitter-Fantappiè Relativity and Brans-Dicke theory. These two approaches, in fact, in natural way seem to incorporate this principle and the accelerating Universe.

I. INTRODUCTION

Einstein's General Theory of Relativity revolutionized our thinking about the nature of space and time and it explained gravity in a different way from Newton's law. Gravity is a manifestation of the geometry of spacetime and the gravitational force becomes a metric force, resulting from the local curvature of spacetime. One of the very first applications of General Relativity concerned the Universe itself and the first attempts towards applying General Relativity to cosmology were made by Einstein himself in 1917. The current models of cosmology are based on the following Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor, and Λ is the cosmological constant. By assuming a standard perfect fluid matter we may write

$$T_{\mu\nu} = (p+\rho)\, u_\mu u_\nu - p g_{\mu\nu} \tag{1.2}$$

where p is the pressure, ρ the energy density and u_μ the velocity. Therefore if we apply this equations to the whole Universe, we find the relativistic cosmology, in which the cosmological principle can be postulated and a model of constant spatial curvature obtained. In fact the Robertson-Walker metric describes a spacetime with homogeneous and isotropic spatial sections, so that the intrinsic spatial curvature is constant throughout the space and its general form is written as

$$ds^2 = c^2dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{1.3}$$

in which (r,θ,φ) are the comoving coordinates and $a(t)$ is the scale factor. The dynamical problem is completely set when matter evolution equations are given; they are the contracted Bianchi identities

$$T^{\mu}_{\nu;\mu} = 0 \tag{1.4}$$

A further equation has to be imposed in order to assign the thermodynamical state of matter. It, usually, is

$$p = \gamma c^2 \rho \tag{1.5}$$

where γ is a constant ($0 \leq \gamma \leq 1$ for standard perfect fluid matter). In such context we obtain the system

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p) + \frac{\Lambda c^2}{3} \tag{1.6}$$

$$(\frac{\dot{a}}{a})^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3} \tag{1.7}$$

$$\dot{\rho} + 3(\frac{\dot{a}}{a})(\rho + p) = 0 \tag{1.8}$$

Predictions of relativistic cosmology include the initial abundance of chemical elements formed in a period of primordial nucleosynthesis, the large-scale structure of the Universe and the existence and properties of a thermal echo from the early cosmos, the cosmic background radiation. Despite this, we have to pay close attention to General Relativity, where, inevitably, the application of Einstein's equations to cosmological problems requires an extreme extrapolation of their validity to very far regions of spacetime. The relativistic cosmology is unable to provide an explanation as to why the density of the Universe should be so close to the critical value. In fact we have

$$\Omega - 1 = \frac{k}{H^2 a^2} = \frac{k}{\dot{a}^2} \tag{1.9}$$

and as $\dot{a}^2$ decreases with time, $|\Omega - 1|$ must increase if k is non zero. This means that the Universe diverges from the flat case if $k \neq 0$ and the fact that it appears to be almost flat today means that Ω must have been very close to one in the early Universe. Besides at present, the cosmic microwave background is observed to be extremely homogeneous and isotropic on large scales, with temperature fluctuations of only $10^{-5}K$. This suggests that all regions of the sky were in causal contact at some time in the past, but is contradicted as follows. The horizon size is the distance light has travelled since the beginning of the Universe and is given by

$$d(t) = a(t) \int_{t_1}^{t_2} \frac{dt}{a(t)} \tag{1.10}$$

which remains finite as $a(t_1) \to 0$ if $\ddot{a} > 0$. When the microwave background was formed the region in causal contact would have been approximately $0.09Mpc$. With the subsequent expansion this corresponds to a patch of the present microwave background subtending an angle of only 2 degrees. Finally magnetic monopoles should have been created in large numbers during phase transitions in the early universe. They are very massive, stable and survive annihilation, to quickly come to dominate the Universe, resulting in matter domination before the epoch of nucleosynthesis and an oservable monopole density today. However, the light elements are observed in the abundances predicted by primordial nucleosynthesis in radiation dominated cosmology and no monopoles are observed, so a mechanism is nedeed to remove the monopoles prior to nucleosynthesis. The most principal problem is the singularity problem and, according to Hawking-Penrose theorems, the appearance of singularity in cosmological solutions of General

Relativity is inevitable[1],[2]. Many physicists and cosmologists are inclined to believe that classical General Relativity must be revised in the case of extremely high energy densities, pressures and temperatures. The singularity must mean for cosmology that the classical Einsteinian theory is inapplicable in the beginning of cosmological expansion of the Universe. Due to these facts alternative theories have been considered as, for example, Extended Theories of Gravity which have become a sort of paradigm in the study of gravitational interaction based on the enlargement and the correction of the traditional Einstein scheme[3]. The paradigm consists in adding higher-order curvature invariants and scalar fields into dynamics which come out from quantum terms in the effective action of gravity. The minimal extension, discussed by Einstein himself to obtain a static Universe model, is the general relativity with a cosmological constant and in this case the action is the following

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R + 2\Lambda)d^4x \tag{1.11}$$

Gravity lagrangians with terms of quadratic or higher order in the Ricci scalar have also been studied in cosmology. In a Riemannian spacetime, let be given the action of the fourth-order as[4]

$$S = \frac{1}{16\pi G}\int \sqrt{-g}(R + \alpha R^2 + 16\pi G L_{matter})d^4x \tag{1.12}$$

In a Riemannian spacetime this action is minimized with respect any variation of the metric tensor if and only if we have that

$$(1 + 2\alpha R)(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 8\pi G T_{\mu\nu} - \frac{\alpha}{2}R^2 g_{\mu\nu} + 2\alpha(g^{\lambda}_{\mu}g^{k}_{\nu} - g_{\mu\nu}g^{\lambda k})R_{;\lambda k} \tag{1.13}$$

The previous relation is the fundamental field equation of the fourth-order gravity in a Riemannian spacetime. We can take into account the most general class of higher–order-theories in four dimensions derived from lagrangians that are functions not only of R but also $\Box R$ or $\Box^n R$, where $\Box$ is the d'Alembertian. They can be generated by the action

$$S = \frac{1}{16\pi G}\int \sqrt{-g}[F(R, \Box R, ..., \Box^n R) + 16\pi G L_{matter}]d^4x \tag{1.14}$$

The field equations are obtained by varying action with respect to the metric getting

$$\Theta(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 8\pi G T^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(F - \Theta)R + (g^{\mu\lambda}g^{\nu k} - g^{\mu\nu}g^{\lambda k})\Theta_{;\lambda k}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}(g^{\mu\nu}g^{\lambda k} + g^{\mu\lambda}g^{\nu k})(\Box^{j-1}R)_{;k}(\Box^{i-j}\frac{\partial F}{\partial \Box^i R})_{;\lambda}$$

$$-g^{\mu\nu}g^{\lambda k}[(\Box^{j-1}R)_{;k}\Box^{i-j}\frac{\partial F}{\partial \Box^i R}]_{;\lambda} \tag{1.15}$$

with

$$\Theta \equiv \sum_{j=0}^{n}\Box^j(\frac{\partial F}{\partial \Box^j R}) \tag{1.16}$$

Other motivations come from the Mach's Principle, that played an important role in the development of General Relativity, as, for example, the Brans-Dicke theory in which the gravitational interaction is mediated by a scalar field as well as the tensor field of General Relativity and the gravitational constant G is not presumed to be constant but instead $1/G$ is replaced by a scalar field which can vary from place to place and with time[5]. Another theory in accord with Mach's Principle is de Sitter-Fantappiè Relativity initially proposed by Fantappiè and subsequently developed by Arcidiacono and other authors [6-27]. At present, both Brans-Dicke theory and de Sitter Relativity are generally held to be in agreement with observation.

The paper is organized as follows: In Sect.2 we introduce the Mach's Principle, while in Sect.3 we analyze the Brans-Dicke Cosmology; Sect.4 is devoted to the Brans-Dicke theory in the context of de Sitter Relativity; in Sect.5 we give the conclusions.

II. MACH'S PRINCIPLE

Acceleration appears absolute and in his *Philosophiae Naturalis Principia Mathematica*, Newton tried to demonstrate that one can always decide if one is rotating with respect to the absolute space, measuring the apparent forces that arise only when an absolute rotation is performed. In his famous example of the rotating bucket filled with water, Newton deduced the existence of an absolute, nonrotating space from the observation of the curved surface the water forms. If a bucket is filled with water, and made to rotate, initially the water remains still, but then, gradually, the walls of the vessel communicate their motion to the water, making it curve and climb up the borders of the bucket, because of the centrifugal forces produced by the rotation. Newton says that this experiment demonstrates that the centrifugal forces arise only when the water is in rotation with respect to the absolute space. It is know that Mach proposed a radical criticism of Newton's absolute space, more than thirty years before Einstein's first paper on relativity and he concluded that the inertia would be an interaction that requires other bodies to manifest itself, so that it would have no sense in a Universe consisting of only one mass. This same thought had been expressed by the philosopher George Berkeley in his *De Motu* published in 1721 and he can be considered the precursor of Mach and Einstein. In Berkeley and Mach idea the concept of absolute motion should be substituted with a total relativism in which every motion, uniform or accelerated, has sense only in reference to other bodies. However, from Mach's principle it has been further deduced that the numerical value of the gravitational constant must be determined by the mass distribution in the Universe, while in Newton's theory it is just an arbitrary constant. According to this principle, the inertia of a body is not an intrinsic property of its own, it rather depends on the mass distribution of the rest of the universe.It is well known that Mach's ideas about the relativity of inertia played an important role in the development of general relativity. Einstein aimed at an explanation of inertia which would eliminate the privileged role of the class of inertial frames in classical mechanics, and which was based on the premise that the results of measurements should not depend on the choice of coordinates assigned to events. With his principle of equivalence, Einstein recognized that gravity was simply acceleration in disguise. Moreover, Einstein's equations indicate that matter is the source for gravity. But if acceleration and gravity are linked, and if gravity depends on matter, then can acceleration be attributed to matter? The origin of inertia and Mach's principle provided the motivation for Thirring to investigate the gravitational field inside a rotating hollow shell[28]. If the rotation of astronomical bodies is relative to the distant masses in the Universe, then one might expect to recover inertial forces inside a rotating hollow shell. Thirring showed the existence of a Coriolis-type force that has been qualitatively interpreted as a Machian dragging effect. Moreover, Lense and Thirring gave a general treatment of orbital precession due to the proper rotation of a central source[29]. Ironically, even if general relativity contains Machian elements, contrary to its name it still contains absolute elements and does not resolve the problem of the origin of inertia.

III. BRANS-DICKE COSMOLOGY

The Brans-Dicke theory of gravity is a natural extension of Einstein's gravity and it takes into account a variable Newton gravitational coupling whose dynamics is governed by a scalar field $\phi = A/G$. In such a way Mach's principle is better implemented and the action for this theory is the following

$$S=\int\sqrt{-g}Ld^4x=\int\sqrt{-g}[\frac{1}{16\pi}(\phi R-\frac{\omega\nabla_\mu\phi\nabla^\mu\phi}{\phi})+L_M]d^4x \tag{3.1}$$

where ω is a generic dimensionless parameter of the theory and the Lagrangian L_M represents the perfect fluid matter. Note that the Einstein' General Relativity will be recovered in the $\omega\to\infty$ limit of the Brans-Dicke theory. The equations of motion for the metric and the Brans-Dicke scalar field are

$$\phi G_{\alpha\beta}+g_{\alpha\beta}\Box\phi-\phi_{,\alpha;\beta}=\chi T_{\alpha\beta}+\frac{\omega}{\phi}(\phi_{,\alpha}\phi_{,\beta}-\frac{1}{2}g_{\alpha\beta}\phi_{,\sigma}\phi^{,\sigma}) \tag{3.2}$$

To obtain the explicit form for the scalar field, we calculate the trace of all members of the Brans-Dicke previous equation and we get

$$\Box\phi=\frac{\chi}{2\omega+3}T \tag{3.3}$$

This equation shows that the scalar field ϕ is produced by the matter T in accord with the Mach's Principle. Assuming our Universe is homogeneous and isotropic on large scale, we work with the Robertson-Walker spacetime

$$ds^2=c^2dt^2-a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\right) \tag{3.4}$$

We suppose the scalar field to be homogeneous in expanding homogeneous isotropic Universe so its energy density and pressure depends only on time $\phi=\phi(t)$. Then the field equations take the forms

$$H^2+H\frac{\dot{\phi}}{\phi}-\frac{\omega}{6}(\frac{\dot{\phi}}{\phi})^2=\frac{8\pi}{3\phi}\rho-\frac{k}{a^2} \tag{3.5}$$

$$\ddot{\phi}+3H\dot{\phi}=\frac{8\pi}{2\omega+3}(\rho-3p) \tag{3.6}$$

$$\dot{\rho}+3H(\rho+p)=0 \tag{3.7}$$

where H is the Hubble parameter and the overdot stands for the derivative with respect to the cosmic time. The first equation corresponds to the Friedmann relation, the second equation is the equation of motion of the Brans-Dicke scalar field. The last is the conservation law for the matter fluid. In[30], the authors have shown some interesting solutions. For example, by following Dirac and his relationship between fundamental constants[31],we have $G\propto H$ where the symbol $\propto$ is taken to mean 'proportional to'. In fact we have the mysterious empirical Weinberg formula

$$m=(\frac{H\hbar^2}{Gc})^{1/3} \tag{3.8}$$

where m is the pion mass. Therefore we can write

$$\begin{cases} G=\gamma H \\ \phi=\frac{A}{G}=\frac{A}{\gamma H}\end{cases} \tag{3.9}$$

Let us remember that $H=\frac{\dot{a}}{a}\Rightarrow\dot{H}=\frac{\ddot{a}a-\dot{a}^2}{a^2}$ and therefore we have

$$\dot{\phi} = \frac{d}{dt}(\frac{A}{\gamma H}) = -\frac{A}{\gamma}\frac{\dot{H}}{H^2} = -\frac{A}{\gamma}\frac{\ddot{a}a - \dot{a}^2}{a^2}\frac{a^2}{\dot{a}^2} = \frac{A}{\gamma}\frac{\dot{a}^2 - \ddot{a}a}{\dot{a}^2} = \frac{A}{\gamma}(1+q) \tag{3.10}$$

where $q = -\frac{\ddot{a}a}{\dot{a}^2}$ is the deceleration parameter. From the previous relation we get $\frac{\dot{\phi}}{\phi} = (1+q)H$ and if we assume $\rho \cong \frac{3H^2}{8\pi G}$ the Friedmann relation becomes

$$H^2 + (1+q)H^2 - \frac{\omega}{6}(1+q)^2 H^2 = H^2 - \frac{k}{a^2} \tag{3.11}$$

Therefore we obtain the following equation

$$1 + q - \frac{\omega}{6} - \frac{\omega}{3}q - \frac{\omega}{6}q^2 + \frac{6k}{\dot{a}^2} = \omega q^2 + (2\omega - 6)q + \omega - 6 - \frac{36k}{\dot{a}^2} = 0 \tag{3.12}$$

In the flat Universe we have the following solutions

$$q = -1 \text{ and } q = \frac{6}{\omega} - 1. \tag{3.13}$$

We can see that $-1 \leq q \leq 0$ and this implicates that the Universe is accelerated.

IV. BRANS-DICKE THEORY IN DE SITTER SPACETIME

De Sitter-Fantappiè Relativity was initially proposed by Fantappiè obtaining as spacetime symmetry group $S(O_5)$ and that is the group of rotation of the Euclidean 5-dimensional space. He wrote a new group of transformations which had as limit Poincaré's group and he was able also to demonstrate that his group was not able to be the limit of any continuous group of 10 parameters. Fantappié's group is characterized by two constants: speed of light c and a radius of space-time $r = ct_b$ where t_b is the temporal distance from the Big Bang. This group determines an Universe endowed with a perfect symmetry: de Sitter's Universe. The Fantappiè group generalizes Poincaré for long distance kinematics, meaning that when magnitudes of all translations are small compared to the de Sitter radius, the Fantappiè group becomes the Poincarè group. By considering the homogeneous coordinates so defined

$$x_k = r\overline{x}_k/\overline{x}_5 \ , \tag{4.1}$$

we get the 5-dimensional pitagorics metric and the following Beltrami metric in projective 4-dimensional spacetime

$$L^2 ds^2 = L(\sum_{i=1}^{4} dx_i dx_i) - (\sum_{i=1}^{4} \frac{x_i}{r} dx_i)^2 \tag{4.2}$$

with

$$L = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - \frac{c^2}{r^2}t^2 + 1 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - (\frac{t}{t_b})^2 + 1 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} - \eta^2 + 1. \tag{4.3}$$

At the relativistic limit, that is for $r \to \infty$, this metric is reduced to Minkowski's metrics in fact

$$\begin{cases} L \to 1 \\ \sum \frac{x_i}{r} dx_i \to 0 \end{cases} \tag{4.4}$$

Instead, de Sitter-Fantappiè General Relativity is based on the following Arcidiacono 5-dimensional equations that generalize the 4-dimensional Einstein equations

$$G_{AB} = \chi T_{AB} \tag{4.5}$$

We shall use the indices A, B for the values $0, 1, 2, 3, 4$ and the indices i, k for the values $0, 1, 2, 3$.Let us remember that, following the definition of Cartan, any Riemann manifold is associated with an infinite family of Euclidean spaces tangent to it in each of its P points. These infinity spaces are joined by a connection law and are individuated by a holonomy group. By introducing a local coordinates system y^i and a linear forms, ω^i, of differential dy^i we can write $ds^2 = \omega_s \omega^s$. If we consider, on the tangent space to a point P, four ortogonal vectors e_i we have[10]

$$\begin{cases} dP = \omega^i e_i \\ de_i = \omega_i^k e_k \\ e_i e_k = \delta_{ik} \end{cases} \tag{4.6}$$

where $\omega_k^i = \gamma_{ks}^i \omega^s$ and γ_{ks}^i are the Ricci rotation coefficients. If the point P and the associated reference frame describe a closed infinitesimal cycle on the tangent space, in general the vector e_i' doesn't coincide with e_i and the cycle is open. It can be closed through a translation Ω^i and a rotation Ω_k^i on the tangent space and we have

$$\begin{cases} \Omega^i = d\omega^i + \omega_s^i \wedge \omega^s \\ \Omega_k^i = d\omega_k^i + \omega_s^i \wedge \omega_k^s \end{cases} \tag{4.7}$$

where Ω^i is the torsion and Ω_k^i is the curvature. To develop de Sitter General Relativity we have to introduce a $5-$dimensional Riemann manifold which allows as holonomy group the Fantappié one, isomorphic to the $5-$dimensional rotations group, and the gravitation equations are the previous Arcidiacono equations. Given a Riemannian manifold M and u and v, two linearly independent tangent vectors at the same point x_0, we can define

$$K(u,v) = [\frac{\sum R_{\alpha\beta\gamma\delta} u^\alpha v^\beta u^\gamma v^\delta}{\sum (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) u^\alpha v^\beta u^\gamma v^\delta}](x_0). \tag{4.8}$$

It can be shown that $K(u, v)$ depends only on the plane spanned by u and v and it is called sectional curvature. We have that a Riemannian manifold is locally projectively flat if and only if the sectional curvature is constant. Therefore while in classical General Relativity the curvature tensor equal to zero means Minkowski spacetime, in projective general relativity curvature tensor equal to zero means de Sitter spacetime. The metric of spacetime in de Sitter General Relativity is the following generalized Beltrami metric which was found by Arcidiacono

$$L'^2 ds^2 = [L' g_{ik} + (Y_i - X_i)(Y_k + X_k)] dx^i dx^k \tag{4.9}$$

where $X_i = g_{i0} + g_{ik} x^k$, $Y_i = \frac{1}{2}(\partial_i g_{00} + x^s \partial_i g_{s0} + x^r x^s \partial_i g_{rs})$ and $L' = g_{00} + 2g_{i0} x^i + g_{ik} x^i x^k$.

Therefore the metric tensor is

$$\tilde{g}_{ik} = \frac{L' g_{ik} + (Y_i - X_i)(Y_k + X_k)}{L'^2} \tag{4.10}$$

and it is not symmetric. The symmetric component is

$$\tilde{g}_{(ik)} = \frac{L' g_{ik} - X_i X_k + Y_i Y_k}{L'^2} \tag{4.11}$$

instead the antisymmetric part is

$$\widetilde{g}_{[ik]} = \frac{X_i Y_k - X_k Y_i}{L'^2} \tag{4.12}$$

In the case of de Sitter special relativity we have $g_{ik} = \delta_{ik}, g_{i0} = 0, g_{00} = 1, L' = L, X_i = x_i$ and $Y_i = 0$ and we get again the classical Beltrami metric. If we consider negligible x/r and t/t_b,by setting $g_{ik} = a_{ik}$, $g_{i0} = \phi_i$ and $g_{00} = \phi^2$ we obtain $L' = \phi$, $X_i = \phi_i$, $Y_i = \phi\psi_i$ with $\psi_i = \partial_i\phi$ and we get

$$\widetilde{g}_{(ik)} = \frac{a_{ik} - \phi_i\phi_k + \psi_i\psi_k}{\phi^2} \tag{4.13}$$

$$\widetilde{g}_{[ik]} = \frac{\phi_i\psi_k - \phi_k\psi_i}{\phi^3} \tag{4.14}$$

If we have $\phi_i = 0$, we get

$$\widetilde{g}_{(ik)} = \frac{a_{ik} + \psi_i\psi_k}{\phi^2} \tag{4.15}$$

$$\widetilde{g}_{[ik]} = 0 \tag{4.16}$$

obtaining a scalar-tensor theory similar to classical Brans-Dicke theory and we can write

$$\begin{cases} R_{ik} - \frac{1}{2}Rg_{ik} = \chi T_{ik} \\ R_{i0} = \chi T_{i0} \\ R_{00} - \frac{1}{2}\phi^2 R = \chi T_{00} \end{cases} \tag{4.17}$$

V. CONCLUSION

The Brans-Dicke theory of gravity and de Sitter-Fantappiè Relativity are the best theories that incorporate the Mach's Principle. The Mach idea played an important role in the development of the Einstein theory but, General Relativity, doesn't verify such principle. Like General Relativity, Brans-Dicke theory predicts light deflection and the precession of perihelia of planets orbiting the Sun. However, the precise formulas which govern these effects, according to Brans-Dicke theory, depend upon the value of the coupling constant ω. The value of ω consistent with experiment has risen with time. Initially $\omega > 5$ was consistent with known data, while, currently, the experimental measures show that the value of ω must exceed 40. In the Brans-Dicke theory, in addition to the metric, which is a rank two tensor field, there is a scalar field, ϕ, which has the physical effect of changing the gravitational constant. Arcidiacono has drawn a scalar-tensor theory in the context of Projective Relativity obtaining a geometric scalar field.

A. Acknowledgements

The author wishes immensely to thank Prof. Ignazio Licata and Prof. Leonardo Chiatti for the stimulating discussions.

REFERENCES

[1] R. Penrose, Structure of Space-Time. (Mir, Moscow, 1972).
[2] S. Hawking, G. Ellis, The Large Structure of Space-Time. (Nauka, Moscow, 1979).
[3] S. Capozziello, M. Funaro, Introduzione alla Relatività Generale, Liguori Editore (2005).
[4] S.Capozziello, M. Francaviglia, Extended Theories of Gravity and their Cosmological and Astrophysical Applications, Gen.Rel.Grav.40:357-420, (2008).
[5] C. H. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).
[6] L.Fantappie', Su una nuova teoria di relatività finale, Rend. Lincei. Vol. 17, 5, (1954).
[7] G. Arcidiacono, Rend. Accad. Lincei XVIII, fasc. 4 (1955).
[8] G. Arcidiacono, Projective Relativity, Cosmology and Gravitation. Hadronic Press, Nonantum (1986).
[9] G. Arcidiacono, La teoria degli universi, vol. I. Di Renzo, Rome, (1997).
[10] G. Arcidiacono, La teoria degli universi, vol. II. Di Renzo, Rome, (2000).
[11] G.Arcidiacono, Hadronic Journal, 9, 197 (1986).
[12] G.Arcidiacono, Hadronic Journal, 11, 287 (1988).
[13] T.Singh, G.P.Singh and G.Arcidiacono, Hadronic Journal, 12, 129 (1989).
[14] G.Arcidiacono and T.Singh, Hadronic Journal, 13, 483 (1990).
[15] G.Arcidiacono, General Relativity and Gravitation, 7, 885 (1976).
[16] G.Arcidiacono, General Relativity and Gravitation, 8, 865 (1977).
[17] G.Arcidiacono, General Relativity and Gravitation, 9, 949 (1978).
[18] I. Licata, Universe Without Singularities A Group Approach to de Sitter Cosmology, EJTP 3, No. 10 211-224 (2006).
[19] I. Licata, Osservando la Sfinge, Di Renzo Rome (2003).
[20] L. Chiatti, Fantappié-Arcidiacono Theory of Relativity Versus Recent Cosmological Evidences : A Preliminary Comparison, EJTP 4, No. 15 17–36 (2007).
[21] I. Licata, L. Chiatti, The Archaic Universe: Big Bang, Cosmological Term and the Quantum Origin of Time in Projective Cosmology, Int. Journ. Theor. Phys., vol. 47, n.12, (2008).
[22] G. Iovane, E. Benedetto, El Naschie $\epsilon^{(\infty)}$ Cantorian spacetime, Fantappié's group and applications in cosmology, Int.Jou.Nonlinear Scienze and Numerical Simulations 6(4) 357-370 (2005).
[23] G. Iovane, E. Benedetto, A projective approach to dynamical systems, applications in cosmology and connections with El Naschie $\epsilon^{(\infty)}$ Cantorian spacetime. Chaos Solitons and Fractals, 30, 2, 269-277, (2006).
[24] G. Iovane, S. Bellucci, E. Benedetto, Projected spacetime and varying speed of light, Chaos Solitons and Fractals, 37, 1, 49-59, (2008).
[25] S. Bellucci, E. Benedetto, G. Iovane, El Naschie's E-Infinity Cantorian Spacetime and Group Theory for Projective Relativity, accepted International Journal of E-Infinity and Complexity Theory in High Energy Physics & Engineering (2009)
[26] E.Benedetto, " E. Benedetto, Fantappiè-Arcidiacono Spacetime and Its Consequences in Quantum Cosmology, Int. Journ. Theor. Phys. 48 (6), 1603-1621 (2009)
[27] S. Cacciatori, V. Gorini, A. Kamenshchik, "Special Relativity in the 21st century", Annalen der Physik, Volume 17 Issue 9-10, Pages 728 - 768
[28] H. Thirring, Phys. Z. 19, 33 (1918); 22, 29 (1921)
[29] J. Lense and H. Thirring, Phys. Z. 19, 156 (1918)o, Phys. Rev. D 65, 063504 (2002).
[30] S. Carneiro, A.E. Montenegro Jr, Exact Solutions of Brans-Dicke Cosmology and the Cosmic Coincidence Problem, Brazilian Journal of Physics, vol. 35, no. 4B, (2005)
[31] Dirac, P. A. M., Nature, 139, 323 (1937).