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A THEORY OF DIRAC MONOPOLES WITH A NON-ABELIAN SYMMETRY*

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ABSTRACT

From the viewpoint of a global formulation of Yang-Mills fields, a Lagrangian theory of non-Abelian classical Dirac monopoles is proposed. Dirac strings are used instead of coordinate patches, they are defined as purely geometric constructs. The formalism is free from pathologies such as Dirac's Veto. While the analysis focuses on the gauge group $SU(N)/Z_N$ allowing for only (N-1) nontrivial and topologically distinct types of monopoles, it applies in general to any compact Lie group.

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I. INTRODUCTION

Aside from their intrinsic interest, Dirac monopoles in non-Abelian gauge theories,[1-5] may be of relevance to the issue of quark-confinement and provide a possible field theoretic foundation for dual string models.[3,6,7] In this context Mandelstam[3] pointed out that at most (N-1) topologically distinct types of Dirac monopoles exist in a local SU(N) gauge theory. These monopoles are in a one to one correspondence with the (N-1) nontrivial elements of the center Z_N of SU(N).[3] Later, Wu and Yang[4] emphasized the importance of the distinction between the local and global symmetry groups, taking the example of SU(2) and $O(3) = SU(2)/Z_2$. While locally isomorphic, they differ in their global structures, one is simply connected, the other two-fold connected. Accordingly, there are no monopole solutions for SU(2), but there is one for $SU(2)/Z_2$. In general, we can introduce only (n-1) varieties of Dirac monopoles into a gauge theory if its global group is n-fold connected.[8] In this paper, we present a Lagrangian theory of Dirac monopoles with a non-Abelian gauge symmetry. The gauge theory group is taken to be $SU(N)/Z_N$, but the generalization to any other compact groups is straightforward once its Lie algebra is known.

A Lagrangian formalism was once attempted[7] following the work of Mandelstam.[3] However the scheme is imperfect on two counts: (i) an unnecessary concept, the measuring operator, plays an essential role, and (ii) the problems inherent to Dirac strings, such as Dirac's veto[1,9] come about. The measuring operator was introduced to construct a gauge invariant magnetic charge. In the present paper, the path-dependent formulation[10] is used for this purpose. The problems associated with the Dirac strings are solved in the light of the global formulation of gauge theories.[4] Our paper is organized as follows.

In Section II we formulate a theory of Abelian Dirac monopoles with strings in the framework of the global formulation of gauge theories. The difference with the theory of Wu and Yang[5] is as follows. In their theory of Dirac monopoles without strings, two potentials A_μ and A'_μ are taken on two overlapping domains V(A) and V(A') about a monopole. The definition of these domains is quite flexible. We can immediately see that, without violating the global analysis, the domain V(A) is extensible to the (3+1) dimensional space-time <u>minus</u> a certain world sheet.[5] Such a maximum domain is used for a coordinate patch in the present formalism. The Dirac string is defined as the boundary of the coordinate patch at each time. It is purely a global geometrical concept. The potential $A_\mu(x)$ and the charged field $\phi(x)$ are not defined along Dirac strings. As the electromagnetic field $F_{\mu\nu}(x)$ is the gauge invariant representative $A_\mu(x)$, the path-dependent field $\phi(x, P)$ is the gauge invariant counterpart of $\phi(x)$. Only $F_{\mu\nu}(x)$ and $\phi(x, P)$ are defined all over space-time. It is pointed out that the present theory can be elegantly formulated in the theory of fibre bundles.[11]

In Section III, starting from elementary concepts in fibre bundles, we formulate a Lagrangian theory of Dirac monopoles with a non-Abelian compact gauge symmetry.

II. ABELIAN MONOPOLES

We first formulate a Lagrangian theory of Abelian monopoles. It will be seen to share all the essential features of the non-Abelian theory. Magnetic monopoles are assumed to be classical point particles tracing out world lines $z_\mu^{(i)}$.[5] The electromagnetic field $F_{\mu\nu}(x)$ is in interaction with the electrically charged field $\phi(x)$ and the monopoles. The Maxwell equations are

$$\partial_\nu F_{\mu\nu}(x) = j_\mu^e(x) \tag{2.1}$$

and

$$\partial_\nu F^*_{\mu\nu}(x) = j^m_\mu(x) \tag{2.2}$$

where the magnetic current $j^m_\mu(x)$ must have support only along the world lines $z^{(i)}_\mu$:

$$j^m_\mu(x) = \sum_i g^{(i)} \int \frac{dz^{(i)}_\mu}{ds} \delta^4(x-z^{(i)})\, ds \tag{2.3}$$

The magnetic charge $g^{(i)}$ satisfies the Dirac quantization condition

$$g^{(i)} = 2\pi n^{(i)}/e\ , \qquad n^{(i)} = \text{integer} \tag{2.4}$$

as we shall later show.

We define a manifold M to be the (3+1) dimensional space-time R^4 minus these world lines $z^{(i)}_\mu$. Because $\partial_\nu F^*_{\mu\nu}(x) = 0$ for $x \in M$, there must exist a gauge potential $A_\mu(x)$.

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{2.5}$$

As originally discussed by Dirac,[1] at each time t, the potential $A_\mu(x)$ becomes singular along so-called Dirac strings L(A) each terminating at one monopole. The positions of strings can be changed arbitrarily, $L_i(A_1) \to L_i(A_2)$, by a gauge transformation

$$\begin{aligned} A_{2\mu}(x) &= A_{1\mu}(x) + \frac{i}{e}\,\partial_\mu S_{21}(x)\, S_{21}(x)^{-1} \\ \phi_2(x) &= S_{21}(x)\, \phi_1(x) \end{aligned} \tag{2.6}$$

with $S_{21}(x) = \exp\left\{-ie\Lambda_{21}(x)\right\}$.

Recently Wu and Yang[4] made a simple but important observation. According to them, the above encounter with singularities merely indicates that we cannot use a single potential $A_\mu(x)$ all over space-time. Obviously the domain on which

$A_\mu(x)$ is defined as a differentiable function is the manifold M minus the world sheets swept by Dirac strings. We denote this domain by V(A). Similarly the field $\phi(x)$ is defined as a differentiable function on the same open set V(A).

We call an open set V(A) a coordinate patch by regarding $V(A_1) \neq V(A_2)$ provided that $A_1(x) \neq A_2(x)$ even if $V(A_1)$ and $V(A_2)$ are the same as subsets of M. A coordinate patch $V(A_i)$ fixes uniquely a potential $A_i(x)$, an electric field $\phi_i(x)$ and Dirac strings $L(A_i)$. The quantities on different coordinate patches are related by the gauge transformation (2.6). The electromagnetic field $F_{\mu\nu}(x)$ is gauge invariant, so, it does not depend on the choice of coordinate patches to calculate it. $F_{\mu\nu}(x)$ is differentiable on M. In the terminology of fibre bundles,[11] the gauge potential A_μ and the field ϕ are respectively a connection and a cross section on the principal fibre bundle P(M, G), G = U(1), where the covering of M is provided by a set of coordinate patches $V(A_i)$. For the uninitiated, a fibre bundle dictionary is provided in Ref. 4 and the connection with gauge fields is amply discussed in Ref. 12. Here we shall assume familiarity with at least the basic concepts of fibre bundles.

Electrodynamics without monopoles are described by the principal fibre bundle $P(R^4, G)$, G = U(1), where a single coordinate patch covers R^4. The essence of the present theory is the singular behavior of $A_\mu(x)$ at the boundary of V(A), which prevents the extension of the covering V(A) over R^4. We shall now analyze this behavior.

We take a loop C around a Dirac string L(A) at fixed time. We choose another coordinate patch V(A') which covers the loop C as well as the string L(A). These two potentials are related by (2.6) or

$$A_\mu(x) = A'_\mu(x) + \partial_\mu \Lambda(x) \quad . \tag{2.7}$$

We integrate (2.7) along the loop C from a point x_0 to x_0:

$$\oint_C dy_\mu A_\mu = \oint_C dy_\mu A'_\mu + \bar{\Lambda}(x_0) - \Lambda(x_0) \quad . \tag{2.8}$$

The term

$$\bar{\Lambda}(x_0) - \Lambda(x_0) = \oint dy_\mu \partial_\mu \Lambda$$

does not vanish because $\Lambda(x)$ is in general multivalued. However, $S(x) = \exp\left\{ie\Lambda(x)\right\}$ must be single-valued for the gauge transformation (2.6) to be definable. We obtain

$$\oint_C dy_\mu A_\mu = \oint_C dy_\mu A'_\mu + 2\pi n/e \ , \qquad n = \text{integer} \quad . \tag{2.9}$$

Equation (2.9) is the essence of the Abelian monopole theory.[4]

Now let us continuously shrink the loop C to the string position L(a) in (2.10). Since A'_μ is finite along L(A), $\oint dy_\mu A'_\mu \to 0$, hence

$$\oint_{L(A)} dy_\mu A_\mu = 2\pi n/e \tag{2.10}$$

with the integration being performed along an infinitesimal loop C around L(A). This condition characterizes the singular behavior of $A_\mu(x)$ along the Dirac string L(A). Equations (2.9) and (2.10) are equivalent to one another. They define uniquely the fibre bundle P(M, G) with the connection A_μ.

The Maxwell equation (2.3) with (2.4) is also equivalent to (2.9). At each fixed time we take a closed surface $S = \partial V$ enclosing a single monopole with the loop C around it. We obtain

$$\oint_S d\sigma_{\mu\nu} F_{\mu\nu} = \oint_C dy_\mu (A_\mu - A'_\mu) = 2\pi n \tag{2.11}$$

by use of (2.9). $F_{\mu\nu}(x)$ is a differentiable function and $\partial_\nu F^*_{\mu\nu}(x) = 0$ holds for x in the volume V except for a point where the monopole exists. We define the distribution $\hat{F}_{\mu\nu}(x)$ for $x \in R^4$, such that

$$\hat{F}_{\mu\nu}(x) = F_{\mu\nu}(x) \ , \qquad x \in M \tag{2.12}$$

and

$$\int_V dV_\mu \, \partial_\nu F^*_{\mu\nu} = 2\pi m \tag{2.13}$$

Equations (2.12) and (2.13) uniquely define $\partial_\nu \hat{F}^*_{\mu\nu}$ in the form (2.3) with (2.4). Thus the Maxwell equation (2.3) is purely a kinematical equation.

Our goal is to give the action principle for the Abelian monopole system. Dirac used a single coordinate patch for this purpose. This method as such results in some serious problems associated with the Dirac strings. The main difficulty is known as Dirac's Veto.[1] In order to derive the field equations and the proper Lorentz equations for both types of charges from his action principle, Dirac had to impose the extra condition that his strings must never cross an electrically charged particle. This veto is not derivable from his action principle.[13] From our global outlook of this problem, the source of this difficulty is apparent. The electrically charged field ϕ is being used in a region of space-time where it is not defined. This observation then implies the need to work with gauge invariant quantities in constructing a workable action principle. To write down $F_{\mu\nu}(x)$, Wu and Yang[4,5] use two potentials with a subsidiary condition (2.9). While this method solves the problem caused by the Dirac strings the actual formalism seems rather tedious especially when we generalize their scheme to non-Abelian monopoles. We choose to express $F_{\mu\nu}(x)$ in terms of a single potential and Dirac strings, as in Dirac's original paper,[1] but with a subsidiary condition (2.10). For the electrically charged field $\phi(x)$, we shall use a path-dependent formulation to construct its gauge invariant representative. Our essential observation is that the potential $A_\mu(x)$ is defined almost everywhere in R^4 and, that conditions (2.9) and (2.10) are equivalent. In fact, the measure of the domain R^4-V(A) is zero. First we notice that

$$F_{\mu\nu}(x, L(A)) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad , \tag{2.14}$$

with the global condition (2.10), is a distribution on R^4 such that

$$F_{\mu\nu}(x, L(A)) = F_{\mu\nu}(x) \ , \qquad x \in V(A) \ , \tag{2.15}$$

$$\lim_{S \to 0} \int_S d\sigma_{\mu\nu} F_{\mu\nu}(x, L(A)) = 2\pi n/e \ , \tag{2.16}$$

and

$$\int_V dV_\mu \partial_\nu F_{\mu\nu}(x, L(A)) = 0 \quad . \tag{2.17}$$

In (2.16), S is an infinitesimal area crossed by the string L(A). Equation (2.16) is a consequence of (2.10) while (2.15) and (2.10) follow from (2.14).

Now we introduce another distribution $G_{\mu\nu}(x, L(A))$:

$$G_{\mu\nu}(x, L(A)) = \sum g^{(i)} \int d\sigma\, d\tau \ \delta^4(x-y^{(i)}) \times \frac{\partial\left(y_\mu^{(i)}, y_\nu^{(i)}\right)}{\partial(\sigma, \tau)} \tag{2.18}$$

The $y_\mu^{(i)}(\sigma, \tau)$ labels the position of the points on the world sheet swept out by the Dirac strings $L_i(A)$'s. It is trivial to see that

$$G_{\mu\nu}(x, L(A)) = 0 \ , \qquad x \in V(A) \ , \tag{2.19}$$

$$\int_S d\sigma_{\mu\nu} G^*_{\mu\nu}(x, L(A)) = 2\pi n/e \quad , \tag{2.20}$$

and

$$\int_S dV_\mu \, \partial_\nu G^*_{\mu\nu}(x, L(A)) = 2\pi n/e \quad . \tag{2.21}$$

Combining (2.14) and (2.18), we deduce that

$$\hat{F}_{\mu\nu}(x) = F_{\mu\nu}(x, L(A)) - G^*_{\mu\nu}(x, L(A)) \tag{2.22}$$

with (2.10) is a distribution defined on R^4 such that (2.12) and (2.13) are satisfied. By a direct calculation,[1] (2.22) yields

$$\partial_\nu \hat{F}^*_{\mu\nu}(x) = \sum g^{(i)} \int \frac{dz_\mu^{(i)}}{ds} \delta^4(x-z^{(i)}) \quad . \tag{2.23}$$

This is nothing but (2.17) and (2.21).

The distribution $\hat{F}_{\mu\nu}(x)$ depends only on a potential A_μ and the monopole position $z_\mu^{(i)}$. The string L(A) is not a dynamical variable because of the topological constraint (2.10). Our formalism differs from Dirac's at this point. In fact, when we do a variation in L(A), the potential A_μ changes to guarantee (2.10) in addition to $G_{\mu\nu}(x, L(A))$. Then the net effect on $F_{\mu\nu}(x)$ in (2.22) is zero.

The electromagnetic field $F_{\mu\nu}(x)$ is expressible by a single potential, as we have shown, because the difference between two potentials $A_\mu(x)$ and $A'_\mu(x)$ is a pure gauge term. We cannot use a single field $\phi(x)$ all over space-time. We introduce the gauge invariant field by the path-dependent formalism.

Following Mandelstam,[10] we associate a path P to each space-time point x. P is a semi-infinite path leading from infinity to x. If the path P exists in a single coordinate patch, we define

$$\phi(x, P) = \exp\left\{ie \int^x dy_\mu A_\mu\right\} \phi(x) \tag{2.24}$$

where the integration is along the path P. For the path P to be covered by two coordinate patches $V(A_1)$ and $V(A_2)$, we take a point $x_0 \in V(A_1) \cap V(A_2)$ to define

$$\phi(x, P) = \exp\left\{ie \int^{x_0} dy_\mu A_{2\mu}\right\} S_{21}(x_0)$$
$$\times \exp\left\{ie \int_{x_0}^{x} dy_\mu A_{1\mu}\right\} \phi(x) \tag{2.25}$$

with $S_{21}(x)$ being the gauge transformation (2.6). Equation (2.25) does not depend on the choice of x_0. When we use more than two coordinate patches to cover the path P, we define $\phi(x, P)$ by an obvious generalization of (2.25). $\phi(x, P)$ is gauge invariant, or it does not depend on the choice of coordinate patches to define it.

Clearly the path P does not correspond to any new dynamical variable. It is not difficult to prove that the path-dependence is given by

$$\phi(x, P') = \exp\left\{ie \int_S d\sigma_{\mu\nu} \hat{F}_{\mu\nu}\right\} \phi(x, P) \tag{2.26}$$

where S is a surface bounded by P and P'. The consistency of the path-dependent formalism is assured by (2.9). Instead of using $\phi(x)$'s on each coördinate patches we can use a single $\phi(x, P)$, subject to the kinematical constraint (2.26).

Having thus made all the above technical preparations, we can give the action as

$$\mathscr{A} = \int d^4x \ \mathscr{L}(x) - m \sum \int dS^{(i)} \tag{2.27}$$

where the Lagrangian is taken, for instance, to be

$$\begin{aligned} \mathscr{L}(x) = -\frac{1}{4} \hat{F}_{\mu\nu}(x) \hat{F}_{\mu\nu}(x) + |\partial_\mu \phi(x, P)|^2 \\ + c_2 |\phi(x, P)|^2 - c_4 |\phi(x, P)|^4 \end{aligned} \tag{2.28}$$

with (2.22), (2.10) and (2.26). The dynamical variables are $A_\mu(x)$, $\phi(x, P)$ and the monopole position $z_\mu^{(i)}$. The variation $\delta\phi$ is taken independently to give

$$\partial_\mu^2 \phi(x, P) = -2\, c_2 \phi(x, P) + 4\, c_4 |\phi(x, P)|^2 \phi(x, P) \tag{2.29}$$

The variation δA_μ must be supplemented by $\delta\phi$ due to the kinematical constraint

$$\delta\phi(x, P) = -ie\, \phi(x, P) \int^x dy_\mu \delta A_\mu \quad .$$

We obtain:

$$\partial_\nu \hat{F}_{\mu\nu}(x) = ie\,\phi^*(x,P)\overleftrightarrow{\partial}_\mu \phi(x,P) \tag{2.30}$$

The variation $\delta z_\mu^{(i)}$ yields

$$m\,\frac{d^2 z_\mu^{(i)}}{dS^2} = g^{(i)} \hat{F}^*_{\mu\nu}(z)\,\frac{dz_\nu^{(i)}}{dS} \tag{2.31}$$

with $g^{(i)} = 2\pi m^{(i)}/e$. The global condition (2.10) with (2.22) is replacable by (2.23). The equations of motions are (2.29), (2.30), (2.31), (2.23) and (2.26). Naturally they are gauge invariant.

When we take $c_2 = c_4 = 0$ in (2.28), the system is made up of a massless electrically charged field $\phi(x,P)$, a massless gauge field $F_{\mu\nu}(x)$, and the classical monopoles with mass m. When we take $c_2 > 0$ and $c_4 > 0$ in (2.28), the system would undergo spontaneous symmetry breakdown. This could give rise to Nielsen-Olesen vortices[14] bridging monopoles; the vortices exemplifying possible coherent vacuum excitations.[7,8]

III. NON-ABELIAN MONOPOLES

We proceed to a theory with non-Abelian gauge symmetry. To emphasize the topological aspects of monopoles, we make use of the compact terminology of fibre bundles.[11, 12] However as the formalism is completely analogous to the Abelian case, readers will follow our arguments without much knowledge of the theory of fibre bundles.

We consider a principal fibre bundle P(M, G*) and a connection A_μ on it. Being simplest and physically most interesting, the structure group G* is taken to be the matrix Lie group SU(N). The base space M is defined by subtracting a certain number of world lines from the (3+1) dimensional space-time R^4. We call these world lines the trajectories of classical Dirac monopoles. At each time t, we attach a string L to a monopole. The position of the string is arbitrary. These strings trace out world sheets in M whose boundaries are the trajectories of monopoles. By subtracting these world sheets from M, we get an open set V_i. All possible such open sets $\{V_i\}$ make up a system of coordinate patches that covers M. For historical reasons[1] we call the string L the Dirac string.

A connection A_μ on P(M, G*) is the assignment of a differentiable function $A_{i\mu}(x)$ for each coordinate patch V_i. The function $A_{i\mu}(x)$ takes value in the Lie algebra of G*. It obeys the relation

$$A_{2\mu}(x) = S_{21}(x)\, A_{1\mu} S_{21}(x)^{-1} - \frac{i}{e}\, \partial_\mu S_{21}(x)\, S_{21}(x)^{-1} \tag{3.1}$$

for $x \in V_1 \cap V_2 \neq 0$ with $S_{21}(x) \in G$.

We define the curvature matrix

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ie\left[A_\mu(x),\ A_\nu(x)\right] \tag{3.2}$$

on each coordinate patch. The relation

$$F_{2\mu\nu}(x) = S_{21}(x)\ F_{\mu\nu}(x)\ S_{21}(x)^{-1} \tag{3.3}$$

follows. The covariant derivative is defined as

$$\nabla_\nu F_{\mu\nu}(x) = \partial_\nu F_{\mu\nu}(x) + ie\left[A_\nu(x),\ F_{\mu\nu}(x)\right]\ . \tag{3.4}$$

And the Bianchi identity

$$\nabla_\nu F^*_{\mu\nu}(x) = 0 \tag{3.5}$$

is the integrability condition for the existence of $A_\mu(x)$ on each coordinate patch.

It is important to remember that the group G*=SU(N) acts on the connection[11] A_μ through the adjoint representation. The matrix $S_{21}(x)$ satisfying (3.1) is not unique; there are N such matrices, $\omega^m S_{21}(x)$ where $\omega = \exp 2\pi_i/N$ and $m=0, 1, 2, \ldots, N-1$. In other words our structure group is actually $G = SU(N)/Z_N$,

Z_N being the center of SU(N). The discrete Abelian subgroup Z_N has the form

$$Z_N = \left(e,\ \omega e,\ \omega^2 e, \ldots, \omega^{N-1} e\right) \tag{3.6}$$

where e is the unit matrix.

For physicists, A_μ and $F_{\mu\nu}$ are known as the gauge potentials and the gauge field respectively. Equations (3.1) and (3.3) are the gauge transformations. Only the difference from the usual gauge theories is that a single coordinate patch cannot cover the manifold M. Indeed the existence of monopoles is correlated to the global structure of the gauge group.

Let us denote by $V(A_i)$ a coordinate patch to which a gauge potential $A_{i\mu}$ has been assigned. The coordinate patch $V(A_i)$ uniquely defines a Dirac string $L(A_i)$ at each fixed time. It is essential that $A_{i\mu}(x)$ is singular as $x \to L(A_i)$. Otherwise, the coordinate patch $V(A_i)$ would be extendable to cover M. This would result in a trivial topology of the field manifold and there will be no monopoles. In what follows we analyze mainly this singular behavior of $A_{i\mu}(x)$ as $x \to L(A_i)$.

We first define the parallel transport $U(x, x_0; P)$ which maps a path P in the base space M to a path P* in the group space G* as x moves from x_0 along the path P. In the case when P is covered by a single coordinate patch $V(A_i)$, we define

$$U_i(x, x_0; P) = T \exp\left\{ ie \int_{x_0}^{x} dy_\mu A_{i\mu} \right\} \quad . \tag{3.7}$$

Where T indicates the ordering of $A_{i\mu}$ along P. Taking three points x, x_1 and x_0 on P, we find

$$U_i(x, x_0; P) = U_i(x, x_1; P)\ U_i(x_1, x_0; P) \quad . \tag{3.8}$$

If two coordinate patches $V(A_1)$ and $V(A_2)$ cover P separately, the relation

$$U_2(x, x_0; P) = S_{21}(x)\, U_1(x, x_0; P)\, S_{21}(x_0)^{-1} \tag{3.9}$$

is proved. In general, when P is covered by n coordinate patches $V(A_i)$, $i=1,2,\ldots,n$, with $x \in V(A_1)$ and $x_0 \in V(A_n)$, define

$$U_{1n}(x,x_0;P) = U_1(x,x_1;P)\, S_{12}(x_1)\, U_2(x_1,x_2;P) \times S_{23}(x_2)\ldots S_{n-1,n}(x_{n-1})\, U_n(x_{n-1},x_0;P) \qquad (3.10)$$

by choosing $x_i \in P$ arbitrarily from $V(A_i) \cap V(A_i+1)$. The definition (3.10) does not depend on x_i nor on $V(A_i)$ except for x, x_0, $V(A_i)$ and $V(A_n)$. The proof is easy with the aid of (3.8) and (3.9).

We take a loop C around the Dirac string $L(A_i)$ at fixed time. We choose another coordinate patch $V(A_2)$ which covers the loop C as well as the string $L(A_1)$. $A_{1\mu}$ and $A_{2\mu}$ are related by (3.1), or

$$U_1(x,x_0;C) = S(x)\, U_2(x,x_0;C)\, S(x_0)^{-1} \qquad (3.11)$$

when x makes a complete turn along the loop G, we obtain

$$U_1(x_0,x_0;C) = \hat{S}(x_0)\, U_2(x_0,x_0;C)\, S(x_0)^{-1} \quad . \qquad (3.12)$$

Here, we have distinguished $\hat{S}(x_0)$ from $S(x_0)$ because S(x) is not single-valued in general. This is so since S(x) is an element of $G^*=SU(N)$ but our structure group is $G=SU(N)/Z_N$. S(x) and $\hat{S}(x)$ can be different by an element belonging to the center Z_N:

$$\hat{S}(x) = \omega^m S(x) \qquad (3.13)$$

with $\omega = \exp[2\pi_i/N]$. In our fibre bundle, S(x) and $\hat{S}(x)$ cannot be distinguished. Later, we shall give an example in which these two are distinguishable.

Let us continuously shrink the loop C to the string position $L(A_1)$ in (3.12). Because $A_{2\mu}$ is finite along $L(A_1)$, $U_2(x_0,x_0;C) \to 1$, hence

$$U_1(x_0,x_0;C \to 0) = \omega^m \quad . \qquad (3.14)$$

This result can be used to fix the singularity of $A_\mu(x)$ as $x \to L(A)$. Equation (3.14) reads

$$T \exp\left\{ie \oint dy_\mu A_\mu\right\} = \omega^m \tag{3.15}$$

where the integration is along an infinitesimal circle around L(A) at fixed time. This relation is satisfied if and only if[6]

$$\begin{gathered}\oint dy_\mu A_\mu = \frac{2\pi n}{e}\lambda(x) \\ n = m + N\ell\ , \qquad \ell = \text{integer}\ .\end{gathered} \tag{3.16}$$

where $\lambda(x) = S(x)\,\lambda\,S(x)^{-1}$, S(x) being an arbitrary element of SU(N) and λ being the last generator of the Lie algebra, viz.,

$$\lambda = \frac{1}{N}\,\mathrm{diag}\ (1, 1, \ldots, 1\text{–}N)\ . \tag{3.17}$$

Thus there is a gauge transformation which set the singularity in the last component of $A_\mu(x)$:

$$\begin{aligned}&\oint dy_\mu A^i_\mu = 0\ , \qquad i \neq N^2-1 \\ &\oint dy_\mu A^i_\mu = \frac{2\pi n}{e}\ , \quad i = N^2-1\end{aligned} \tag{3.18}$$

First it appears that there exists an infinite varieties of string singularities, corresponding to all integers n as in the Abelian theory. This turns out to be a gauge illusion.[8] For completeness we prove this fact.

We consider

$$U(x, x_0; C) = T\ \exp\left\{ie \int_{x_0}^{x} dy_\mu A_\mu\right\} \tag{3.19}$$

where C is an infinitesimal loop around the Dirac string L(A). There is a one-to-one correspondence between $A_\mu(x)$ and $U(x, x_0;C)$:

$$ie\,A_\mu(x) = \partial_\mu\, U(x, x_0;C)\cdot U(x, x_0;C)^{-1} \tag{3.20}$$

$U(x, x_0; C)$ maps the loop C to a curve $C(A)^*$ in SU(N), whose end-points are the unit element e and ω^m, an element of Z_N. Suppose there are two potentials $A_\mu(x)$ and $A'_\mu(x)$, yielding (3.16) with m and n, $n = m + \ell N$ ($\ell \neq 0$), respectively. The corresponding curves $C(A)^*$ and $C(A')^*$ have the joint end-points e and ω^m. Because SU(N) is a simply connected space, these curves are homotopic to one another. This implies the existence of a continuous gauge transformation which connects A_μ and A'_μ. Therefore, the two string singularities are not distinguishable. In all there are only N-1 varieties of topologically distinct string singularities.[8]

Having completed the analysis of the singular behavior of $A_\mu(x)$, we proceed to formulate a Lagrangian theory of classical monopoles.

We take a semifinite path P leading from infinity to a point x. Two arbitrary points on P are space-like separated. Following Mandelstam[10] and Bialynicki-Birula[15] we define

$$F_{\mu\nu}(x, P) = U(x, P)^{-1} F_{\mu\nu}(x) \, U(x, P) \tag{3.21}$$

where $U(x, P) = U(x, x_0 \to \infty; P)$ with (3.10). It is proved that $F_{\mu\nu}(x, P)$ is gauge invariant, or that it does not depend on the choice of coordinate patches to define it.

The path P does not correspond to a new dynamical variable. For another path P' leading to x, we find

$$\begin{aligned} F_{\mu\nu}(x, P') &= W(P, P') \, F_{\mu\nu}(x, P) \, W(P, P') \\ W(P, P') &= U(x, P)^{-1} \, U(x, P') \end{aligned} \tag{3.22}$$

When P' differs from P by an infinitesimal area $\sigma_{\alpha\beta}$ at the point y, (3.22) reads

$$\delta_z F_{\mu\nu}(x, P) = ie \left[F_{\mu\nu}(x, P), F_{\alpha\beta}(x, \hat{P}) \right] \sigma_{\alpha\beta} \tag{3.23}$$

The path $\hat{P}$ is the portion of P leading to y. In the gauge-invariant formulation the path-dependence is determined by the kinematical equation (3.23); it has nothing to do with the action principle.

The Bianchi identity (2.5) is rewritten as

$$\partial_\nu F^*_{\mu\nu}(x, P) = 0 \tag{3.24}$$

This implies the existence of a path-dependent gauge potential:

$$F_{\mu\nu}(x, P) = \partial_\mu A_\nu(x, P) - \partial_\nu A_\mu(x, P) \tag{3.25}$$

such that

$$[\partial_\alpha, \partial_\beta] A_\mu(x, P) = 0 \quad . \tag{3.25'}$$

$A_\mu(x, P)$ is written in terms of $A_\mu(x)$:

$$A_\mu(x, P) = U(x, P)^{-1} A_\mu(x)\, U(x, P) + \Gamma_\mu(x, P) \quad , \tag{3.26}$$

$$\Gamma_\mu(x, P) = \frac{i}{2e} \int_{-P}^{x} dy_\mu \left\{ \partial_\mu U^{-1} \partial_\nu U - \partial_\nu U^{-1} \partial_\mu U \right\} \; . \tag{3.26'}$$

The coordinate patch to which $A_\mu(x, P)$ is assigned is the same as V(A). The gauge transformation is Abelian:

$$A_{2\mu}(x, P) = A_{1\mu}(x, P) + \partial_\mu \Lambda(x, P) \tag{3.27}$$

such that

$$[\partial_\mu, \partial_\nu] \Lambda(x, P) = 0 \quad . \tag{3.27'}$$

Thus all the equations are linearized in the path-dependent formalism once the path P is fixed.

As $A_\mu(x)$ becomes singular along the Dirac string L(A), so does $A_\mu(x, P)$. The singularity is calculated from (3.26) and (3.16). We calculate it in a special gauge for $A_\mu(x)$, i.e., (3.18). It is not difficult to derive

$$\oint_{L(A)} dy_\mu A_\mu(y, P) = \frac{2\pi m}{e} \lambda \, , \qquad 0 \leq n \leq N-1 \tag{3.28}$$

where the integration is along an infinitesimal loop around the Dirac string L(A) at each fixed time.

The principal fibre bundle P(M, G) with (3.16) is equivalent to the path-dependent system with (3.28). We have derived the latter uniquely from the former. Conversely, we can reconstruct the former from the latter. Noticing the Abelian character, we first make P(M, U(1)). Using the open covering $\{V_i\}$ of this P(M, U(1)), we construct uniquely P(M, G). As in the Abelian case, we can define the distributions

$$F_{\mu\nu}(x, P; L(A)) = \partial_\mu A_\nu(x, P) - \partial_\nu A_\mu(x, P) \tag{3.29}$$

and

$$\hat{F}_{\mu\nu}(x, P) = F_{\mu\nu}(x, P, L(A)) - \lambda G^*_{\mu\nu}(x, L(A)) \tag{3.30}$$

for $x \in R^4$, $G^*_{\mu\nu}(x, L)$ being given by (2.18), with

$$F_{\mu\nu}(x, P; L(A)) = F_{\mu\nu}(x, P) \tag{3.31}$$

for $x \in V(A)$ and

$$\hat{F}_{\mu\nu}(x, P) = F_{\mu\nu}(x, P) \tag{3.32}$$

for $x \in M$. Here again the Dirac string L(A) is not a dynamical variable due to the global constraint (3.28). Equation (3.30) gives

$$\partial_\nu \hat{F}^*_{\mu\nu}(x, P) = \lambda \sum g^{(i)} \int \frac{dz^{(i)}_\mu}{dS} \delta^4(x-z)\, dS \tag{3.33}$$

which is a kinematical equation resulting from the topology of P(M, G).

Having set the stage we now give the action for the non-Abelian monopole system:

$$\mathscr{A} = \int d^4x\, \mathscr{L}(x) - m \sum \int dS^{(i)} \tag{3.34}$$

$$\mathscr{L}(x) = -\frac{1}{4} \operatorname{Tr}\left\{\hat{F}_{\mu\nu}(x, P)\, \hat{F}_{\mu\nu}(x, P)\right\} \tag{3.35}$$

with (3.23), (3.28) and (3.30). The dynamical variables are $A_\mu(x, P)$ and the monopole positions $z_\mu^{(i)}$. The variations δA_μ and $\delta z_\mu^{(i)}$ yield

$$\partial_\nu \hat{F}_{\mu\nu}(x, P) = 0 \tag{3.36}$$

$$m \frac{d^2 z_\mu^{(i)}}{dS^2} = g^{(i)} \hat{F}_{\mu\nu}^{(N^2-1)*}(z, P) \frac{dz_\nu^{(i)}}{dS} \tag{3.37}$$

with $g^{(i)} = 2\pi m^{(i)}/e$. Equations (3.33) and (3.36) correspond to the Maxwell equations in the Abelian theory.

Dirac monopoles are possible in pure gauge theories because the gauge field itself is charged. We can introduce Higgs fields into the scheme. They provide an expedient means of implementing spontaneous symmetry breakdown, and would give rise to non-Abelian vortices bridging the monopoles.[3,6,7]

We consider a fibre bundle B(M, Y, G) in addition to P(M, G*) Y is the fibre space on which the structure group G acts effectively. We give two typical examples. When G = SU(N), Y is taken to be a vector space made up of N-component vectors. When $G = SU(N)/Z_N$, Y is taken to be a vector space made up of N×N-matrices.

A cross section of B(M, Y, G) is a collection of differentiable mappings $\phi_i(x)$ from each coordinate patch V_i into Y. They obey the relation

$$\phi_2(x) = S_{21}(x)\, \phi_1(x) \tag{3.38}$$

for N-component vectors, i.e., G = SU(N), and

$$\phi_2(x) = S_{21}(x)\, \phi_1(x)\, S_{21}(x)^{-1} \tag{3.39}$$

for N×N-matrices, i.e., $G = SU(N)/Z_N$. Here we hasten to add that Dirac monopoles are incompatible with G = SU(N). This is so because S(x) should be single-valued. Only the case m=0 is realizable in (3.13) or (3.28), as results in the trivial topology.[8]

So we choose $\phi(x)$ to be a N×N matrix. $\phi(x)$ corresponds to the familiar Higgs field of unified gauge theories. The covariant derivative is defined as

$$\nabla_\mu \phi = \partial_\mu \phi + ie\left[A_\mu, \phi\right] \tag{3.40}$$

on each coordinate patch. All the analysis concerning the singularities of the gauge potential follows without any modification.

The path-dependent field is defined similarly to (3.21).

$$\phi(x,P) = U(x,P)^{-1}\phi(x)\, U(x,P) \tag{3.41}$$

$\phi(x,P)$ is gauge-invariant. Its path-dependence is a kinematical equation;

$$\delta_z \phi(x,P) = ie\left[\phi(x,P),\ F_{\alpha\beta}(x,\hat{P})\right]\sigma_{\alpha\beta} \tag{3.42}$$

as (3.23) for $F_{\mu\nu}(x,P)$.

The action is given by (3.34); the Lagrangian is taken, for instance, to be

$$\begin{aligned}\mathscr{L}(x) = &-\frac{1}{4}\,\mathrm{Tr}\left\{\hat{F}_{\mu\nu}(x,P)\,\hat{F}_{\mu\nu}(x,P)\right\}\\ &+\frac{1}{2}\,\mathrm{Tr}\left\{\partial_\mu\,\phi(x,P)\,\partial_\mu\,\phi(x,P)\right\}\\ &+\frac{1}{2}\,c_2\,\mathrm{Tr}\left[\phi(x,P)^2\right]-\frac{1}{2}\,c_4\,\mathrm{Tr}\left[\phi(x,P)^4\right]\end{aligned} \tag{3.43}$$

The dynamical variables are $A_\mu(x,P)$, $z_\mu^{(i)}$, and one set of Higgs fields $\phi(x,P)$. The variations δA_μ, $\delta\phi$, and $\delta z_\mu^{(i)}$ give

$$\partial_\mu \hat{F}_{\mu\nu}(x,P) = ie\left[\phi(x,P), \partial_\mu\phi(x,P)\right] \tag{3.44}$$

$$\partial_\mu^2\ \phi(x,P) = -2\,c_2\phi(x,P) + 4\,c_4\phi(x,P)^3 \tag{3.45}$$

and (3.37). Our system is composed of dynamical equations (3.44), (3.45), (3.37) and kinematical equations (3.33), (3.23), (3.42). The introduction into the Lagrangian (3.43) of several Higgs fields $\phi_k(x,P)$ necessary to have well-defined vortices is straightforward but is not relevant to the main theme of this work.

IV. CONCLUDING REMARKS

By way of elementary concepts in the topology of fibre bundles, we have formulated a Lagrangian theory of classical non-Abelian Dirac monopoles. As a modification of Dirac's original approach,[1] our string formulation is in harmony with the Wu-Yang global approach to gauge theories. In Wu and Yang's work,[5] Dirac strings have been discarded as being pathological means to describe Dirac monopoles. By properly defining the Dirac strings as global geometrical objects, we have thus restored their status as very useful non-pathological devices to accommodate monopoles.

In the foregoing analysis, we have introduced path-dependent fields in order to write the action in a compact form. However, to solve the equations of motion, we have to go back to the path-independent fields, for instance,

$$\phi(x) = \exp\left\{-ie \int^{x} dy_\mu A_\mu\right\} \phi(x, P) \tag{4.1}$$

as defined by (2.24) in the Abelian case. The function $\phi(x)$ is not defined along the Dirac string L(A). We must work in each coordinate patch V(A). Once a set of solutions $A_\mu(x)$ and $\phi(x)$ is obtained for a coordinate patch V(A), it is in principle easy to construct sets of solutions for any other coordinate patches. All these sets of solutions with coordinate patches make up the principal fibre bundle P(M, G) with connection A_μ.

In the present formalism the monopoles are point particles with definite world lines. Their replacement by a magnetically charged field $\psi(x)$ is yet to be done. This step is necessary if one wishes to construct a second quantized monopole theory. This program was once attempted by Cabibbo and Ferrari for the Abelian theory.[16] They define the field $\psi(x)$ via its path-dependent form

$$\psi(x, P') = \exp\left\{ig \int_S d\sigma_{\mu\nu} F^*_{\mu\nu}\right\} \psi(x, P) \tag{4.2}$$

After some arguments they postulate a set of equations

$$\not{\partial}\, \psi(x, P) = m\, \psi(x, P) \tag{4.3}$$

and

$$\partial_\nu F^*_{\mu\nu}(x) = ig\, \bar{\psi}(x, P)\, \gamma_\mu \psi(x, P) \tag{4.4}$$

which replace (2.31) and (2.23), respectively. We can imitate their approach for non-Abelian monopoles.[3] Remembering that the monopole interacts directly only with the last element $F^{(N^2-1)}_{\mu\nu}(x, P)$ of the gauge field $F_{\mu\nu}(x, P)$, we define the magnetic field $\psi(x, P)$ by its path-dependence:

$$\psi(x, P') = \exp\left\{ig \int_S d\sigma_{\mu\nu} F^{(N^2-1)*}_{\mu\nu} \quad \psi(x, P)\right\} \tag{4.5}$$

Following Cabibbo and Ferrari, we postulate

$$\not{\partial}\psi(x, P) = m\psi(x, P) \tag{4.6}$$

and

$$\partial_\nu F^*_{\mu\nu}(x, P) = ig\lambda\, \bar{\psi}(x, P)\, \gamma_\mu \psi(x, P) \tag{4.7}$$

to replace (3.37) and (3.33), respectively. However, the self-consistency of (4.6), (4.7) and the other equations is an <u>open</u> question. This is so because the Cabibbo-Ferrari scheme lacks an action principle in the presence of both the electrically and the magnetically charged fields. Our efforts in this direction are continuing.

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