Pub 61

(To be submitted for publication in THE PHYSICAL REVIEW)

BOOTSTRAP CALCULATION OF THE ρ MESON REGGE POLE

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* Supported by the Atomic Energy Commission.

† Supported in part by the U.S. Air Force through Air Force Office of Scientific Research Grant AF-AFOSR-62-452.

ITP-110 2/64

ABSTRACT

The left hand discontinuities in the partial wave amplitudes for π-π scattering are assumed to be dominated by the exchange of the ρ meson in a form suggested by the Regge representation for a resonance. This Regge behavior provides the necessary high energy cutoff and allows the N/D equations to be solved. The partial wave $I = 1$ amplitudes are calculated for non-integer angular momenta $\ell < 1$ as well as $\ell = 1$. The trajectory $\alpha_\rho(s)$ as well as the residue $\beta_\rho(s)$ of the ρ meson Regge pole are evaluated. An attempt is made to obtain a self-consistent solution for the relevant parameters, namely the position and width of the ρ resonance and $\alpha_\rho(0)$. The results of this calculation give $\alpha_\rho(0) \gtrsim 0.9$. The $I = 0$ vacuum trajectory is also discussed.

I. INTRODUCTION

There have been a number of papers written on the problem of determining the position and width of the ρ meson self-consistently.[1,2] In essence, these bootstrap calculations of the ρ used the exchange of this $I = 1$, $\ell = 1$ resonance in the crossed channels to provide the force necessary to produce the ρ meson in the direct channel. The $\ell = 1$ part of the interaction is projected out and the partial wave dispersion relations are solved by the N/D method. The hope is that the solution yields a resonance having the same position and width as that of the exchanged one.

A major difficulty is due to the divergence arising from the exchange of a massive vector particle, with sufficiently large coupling, which necessitates the use of a cutoff. Instead of considering the ρ to be a vector particle even when the energy of the exchanged ρ is not close to the resonant energy, Wong[2] employed a form suggested by the Regge representation for a resonance. This then provides a cutoff at high energy, the relevant parameter being the angular momentum of the ρ trajectory at zero energy, $\alpha_\rho^{In}(0)$.

The purpose of this article is to carry Wong's ρ (bootstrap) calculation with a "Regge cutoff" a step further. For $\ell = 1$ we carry out a calculation similar to his but then continue the N/D equations for non-integer angular momenta and calculate $\alpha_\rho(s)$, comparing $\alpha_\rho(0)$ with the input parameter $\alpha_\rho^{In}(0)$. In other words, this is an attempt to bootstrap not only the position and width of the ρ resonance, but the slope of its Regge trajectory. The residue function $\beta_\rho(s)$ is also determined. The sensitivity of our results to some of the approximations made is examined. For example, the above calculation is compared to a similar one in which we take the exchanged ρ to have constant angular momentum and employ a straight cutoff. The I = 0 vacuum trajectory is also calculated.

Section II is devoted to a presentation of the relevant formalism. The results of the numerical calculations are given and discussed in Section III.

The results may be summarized as follows: In the same sense that the usual bootstrap calculations of the ρ are not self-consistent, i.e., the output width of the ρ (for reasonable values of the position of the ρ) is larger than the input width of the exchanged ρ,[1,2] so the calculated $\alpha_\rho(0)$ is larger than the input parameter $\alpha_\rho^{In}(0)$. For all cases, both $\alpha_\rho(0)$ are $\gtrsim 0.9$, in agreement with the results of Foley et al.[3] and the calculation of Chang and Sharp,[4] however in disagreement with other determinations of $\alpha_\rho(0) \sim 0.5$.[5] The residue of the ρ Regge trajectory, after removal of a threshold factor, turns out to be nearly constant in the scattering region ($s < 0$) and very close to the input β. The calculations of the I = 0 vacuum pole trajectory give a small slope: $\alpha_P'(0) \lesssim 1/500$.[6]

II. FORMULATION OF THE INTEGRAL EQUATIONS

We shall obtain amplitudes for pion-pion scattering by the familiar N/D solution[7] of the partial wave dispersion relations. The usual expressions for the scalar variables s,t, and u in terms of the momentum k and scattering angle θ in the center of mass system of the direct or s channel are[6] $s = 4(k^2 + 1)$, $t = -2k^2(1 - \cos\theta)$, and $u = 4 - s - t$. The invariant partial wave amplitude A_ℓ is defined in terms of the S matrix by

$$A_\ell(s) \equiv \frac{1}{2i\rho} \; (S_\ell - 1) \equiv B_\ell(s) + {}^R A_\ell(s) \tag{1}$$

where

$$\rho = \left(\frac{s-4}{s}\right)^{\frac{1}{2}} \tag{2}$$

and B_ℓ is regular for $s > 0$ and ${}^R A_\ell(s)$ has only a right hand cut. The right hand discontinuity in $A_\ell(s)$ is given by unitarity: We make the approximation that elastic unitarity holds for all physical k^2:

$$A_\ell(s) = B_\ell(s) + \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s} \; |A_\ell(s')|^2 \left(\frac{s'-4}{s'}\right)^{\frac{1}{2}} \quad . \tag{3}$$

The left hand discontinuity or generalized potential[8] is derived from application of an approximate form of crossing symmetry. We will first determine $B_\ell(s)$ and then discuss the N/D equations and their solution.

Using crossing symmetry, $B_\ell(s)$ is calculated from the scattering amplitude in the crossed t and u channels. We will consider only the exchange of the I = 1 ρ resonance in the t and u channels. Then in the s channel for I = 1 and ℓ equal to an integer we obtain

$$B_\ell^{I=1}(s) = \frac{1}{2}\int_{-1}^{1} P_\ell(\cos\theta)\; d\cos\theta \left[\tfrac{1}{2}A_R^{I=1}(t,s) - \tfrac{1}{2}A_R^{I=1}(u,s)\right] \tag{4}$$

which for ℓ odd becomes

$$B_\ell^1(s) = \frac{1}{(s-4)}\int_{-(s-4)}^{0} P_\ell\left(1 + \frac{2t}{s-4}\right)\; dt\; A_R^1\,(t,s) \tag{5}$$

where $A_R^1(t,s)$ is the part of the scattering amplitude in the t channel, $A^1(t,s)$, which has no singularities for $s > 0$, i.e., $t < 4$.

Taking a Breit-Wigner form for the ρ resonance, we have

$$A^1(t,s) \approx \frac{3\Gamma(t-4)}{m_\rho^{\,2} - t - i\Gamma(t-4)^{3/2}/t^{1/2}}\; P_1\left(1 + \frac{2s}{t-4}\right) \quad . \tag{6}$$

Further making the narrow width approximation, so that $A_R^1(t,s) = A^1(t,s)$, we have the simple form for ℓ equal to an odd integer:[9]

$$B_\ell^1(s) = \frac{6\Gamma}{s-4}\left(m_\rho^2 - 4 + 2s\right) Q_\ell\left(1 + \frac{2m_\rho^{\,2}}{s-4}\right) \quad . \tag{7}$$

Eq. (7) has an acceptable behavior in the ℓ plane as $|\ell| \to \infty$ and thus can be continued for non-integer ℓ even though both (4) and (5) cannot.[10] However $B_\ell(s)$ as given by (7) diverges like log (s) as $s \to \infty$ and the resulting N/D equations do not have a unique solution.

A mechanism that damps this singular high energy behavior is provided by the Regge motion of resonance poles. In the Regge description for the ρ resonance we take

$$A^1(t,s) = \frac{b_\rho(t)}{\sin\pi\,\alpha_\rho(t)}\ \frac{1}{2}\left[P_{\alpha_\rho}(t)\left(-1-\frac{2s}{t-4}\right) - P_{\alpha_\rho}(t)\left(1+\frac{2s}{t-4}\right)\right] \quad . \tag{8}$$

We are interested in B_ℓ for $s \geq 4$ and hence in the region $t \leq 0$ where $\alpha_\rho(t)$ is real and < 1. For large s, (8) is or order $s^{\alpha_\rho(t)}$ and hence an acceptable input to the N/D equations.

Since we do not know the behavior of $b_\rho(t)$ or $\alpha_\rho(t)$ except in the immediate vicinity of the ρ resonance, we will take a very simple form for (8) which reduces to the correct Breit-Wigner form (6) near $t = m_\rho^2$, yields the same $B^1_{\ell=1}(s=4)$ as Eq. (7), and gives the same high energy behavior in s (for small t) as the Regge pole:

$$A^1_R(t,s) \approx \frac{3\Gamma(t-4)}{(m_\rho^2 - t)}\left(1+\frac{2s}{t-4}\right)\left(\frac{s}{4}\right)^{\alpha'_\rho(0)(t-m_\rho^2)} \quad ; \tag{9}$$

With this approximation, $A^1_{\ell=1}(s)$ is readily calculated numerically.[11] However we are interested in continuing the partial wave amplitude for non-integer ℓ. Eq. (5) cannot be continued; there are alternate formulations for $B_\ell(s)$ which can be continued.[10] From the point of making the computations manageable, we again note that expression (7) can be continued in the ℓ plane. Thus we are led to make the further approximation that using (5) in making the partial wave projection $B^1_\ell(s)$ of (9) we evaluate the last factor $(s/4)^{\alpha_\rho'(0)(t-m_\rho^2)}$ at $t = 0$ (where it gives the maximum contribution). Hence our "Reggeized" $B^1_\ell(s)$ becomes[12]

$$B^1_\ell(s) = \frac{6\Gamma}{(s-4)} \left(m_\rho^2 - 4 + 2s\right) Q_\ell\left(1 + \frac{2m_\rho^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1} \quad . \tag{10}$$

This expression which is our approximate form for the left hand cut for the partial wave π-π amplitude in the $I = 1$ state and odd integer ℓ has acceptable behavior for large ℓ and can be continued in the ℓ plane.

Now in order to insure that $A^1_\ell(s)$ has the proper threshold behavior, i.e., $(s-4)^\ell$ and also remove this additional cut from $B^1_\ell(s)$ for non-integer ℓ, we define new amplitudes

$$\underset{\sim}{A}^1_\ell(s) \equiv \frac{1}{(s-4)^\ell}\; A^1_\ell(s) \equiv \frac{1}{2i\rho_\ell}\left(S_\ell - 1\right) \equiv \underset{\sim}{B}^1_\ell(s) + {}^R\underset{\sim}{A}^1_\ell(s) \tag{11}$$

where

$$\rho_\ell = \left(\frac{s-4}{s}\right)^{\frac{1}{2}} (s-4)^\ell \quad , \tag{12}$$

and

$$B_\ell^1(s) = \frac{6\Gamma}{(s-4)^{\ell+1}} \quad (m_\rho^2 - 4 + 2s)\ Q_\ell\left(1 + \frac{2m_\rho^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1} \quad . \tag{13}$$

Now define

$$\underset{\sim}{A}_\ell^1(s) = N_\ell(s)/D_\ell(s) \tag{14}$$

where N has only a left hand cut and D has only a right hand cut. Then in terms of the generalized potential $\underset{\sim}{B}_\ell^1(s)$ which is regular in the physical region, the N and D equations are[2,13]

$$D_\ell(s) = 1 - (s-s_0)\frac{P}{\pi}\int_4^\infty \rho_\ell(s')\ N_\ell(s')\ \frac{ds'}{(s'-s)(s'-s_0)}$$
$$- i\rho_\ell(s)\ N_\ell(s)\ \Theta(s-4), \tag{15}$$

$$N_\ell(s) = \underset{\sim}{B}_\ell^1(s) + \frac{1}{\pi}\int_4^\infty \left(\underset{\sim}{B}_\ell^1(s') - \frac{(s-s_0)}{(s'-s_0)}\ \underset{\sim}{B}_\ell^1(s)\right)\ \rho_\ell(s')\ N_\ell(s')\ \frac{ds'}{s'-s} \ . \tag{16}$$

Note that the solutions $\underset{\sim}{A}_\ell^1(s)$ are independent of the subtraction point s_0. As long as $0 < \ell < 2 - \alpha_\rho(0) < 2$, these equations have unique solutions. The Fredholm integral equation (16) for $N_\ell(s)$ was solved by matrix inversion on the Stanford 7090 computer.

For given input parameters $m_\rho{}^{In}$, Γ^{In} and $\alpha_\rho{}^{In}(0)$, which determine $B^1_\ell(s)$ ($\alpha_\rho{}^{In}(0)$ being fixed by the requirement that we get an $\ell=1$ resonance at $m_\rho{}^{In}$, i.e., Re $D_{\ell=1}(s = (m_\rho{}^{In})^2) = 0$) we calculate the width of the $\ell=1$ resonance. Then we solve (15) and (16) for non-integer $\ell < 1$ in order to determine the properties of the ρ trajectory. For a given ℓ, we look for the value of s ($\equiv s_\ell$) for which Re $D_\ell(s) = 0$:

$$\text{Re } D_\ell(s_\ell) = 0 \ . \tag{17}$$

For $s_\ell < 4$ this gives directly the Regge trajectory $\alpha_\rho(s)$, whereas for $s_\ell > 4$, in the limit of a narrow resonance, it gives approximately Re $\alpha_\rho(s)$. The residue $b_\rho(s)$ is determined as follows: Since Re $D_\ell(s_\ell) = 0$, in the vicinity of s_ℓ we have (for $s_\ell < 4$)

$$\underset{\sim}{A}{}^1_\ell(s) = \left(N_\ell(s) \Big/ \frac{\partial \text{ Re } D_\ell(s)}{\partial s} \right)_{s_\ell} \Big/ (s - s_\ell) \ . \tag{18}$$

The residue is real since $N_\ell(s_\ell)$ is simply given by

$$N_\ell(s_\ell) = \frac{P}{\pi} \int_4^\infty \underset{\sim}{B}{}^1_\ell(s') \ \rho(s') \ N_\ell(s') \frac{ds'}{s'-s_\ell} \ . \tag{19}$$

The partial wave projection of the ρ Regge pole of "odd j parity"[10] divided by the threshold factor $(s-4)^\ell$,

$$\frac{b_\rho(s)}{\sin\pi\,\alpha_\rho(s)\ (s-4)^\ell}\ P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right)\equiv\frac{\beta_\rho(s)\ \pi(2\alpha_\rho(s)+1)}{\sin\pi\,\alpha_\rho(s)}\ P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right), \quad (20)$$

then must be compared with (18).[14] Now

$$\frac{1}{2}\int_{-1}^{1} P_\ell(\cos\theta)\ P_{\alpha_\rho(s)}(-\cos\theta)\ d\cos\theta\ \beta_\rho(s)\ \frac{\pi(2\alpha_\rho(s)+1)}{\sin\pi\,\alpha_\rho(s)}$$

$$= \frac{\beta_\rho(s)\ (2\alpha_\rho(s)+1)}{(\alpha_\rho(s)-\ell)(\alpha_\rho(s)+\ell+1)} \underset{s\approx s_\ell}{\approx} \frac{\beta_\rho(s_\ell)}{\alpha'_\rho(s_\ell)(s-s_\ell)} \quad . \quad (21)$$

Thus for a given ℓ, we find $\alpha'(s_\ell)$ from $\alpha(s)$ (as found from (17)) and hence the residue is given by

$$\beta_\rho(s_\ell) = \left(N_\ell(s)\Big/\frac{\partial\ \mathrm{Re}\ D_\ell(s)}{\partial\ s}\right)_{s_\ell}\ \alpha'_\rho(s_\ell)\ . \quad (22)$$

III. RESULTS AND CONCLUSIONS

As discussed earlier, in addition to evaluating the I=1, ℓ=1 π-π scattering amplitude in an attempt to "bootstrap" the ρ meson, we calculate the ρ's Regge pole parameters for non-integer $\ell < 1$. We computed both the position, α_ρ, and residue, β_ρ, of the pole as functions of s.

We investigated the problem for several values of the input coupling constant Γ^{In} (or input width of the ρ) and for several input masses$(m_\rho^{In})^2$ ranging from 10 to the experimental value of 29. No self-consistent solution was obtained. The procedure was to evaluate the I=1, ℓ=1 amplitude for many values of $\alpha_\rho^{In}(0)$ until the mass of the input ρ was reproduced by a zero of Re $D_{\ell=1}(s)$ at $s = (m_\rho^{In})^2$, i.e., we always forced the mass of the produced ρ to be the same as that of the exchanged ρ. The output width could be determined either by evaluating the quantity $(N_{\ell=1}(s)/ \partial D_{\ell=1}(s)/\partial s)$ at the position of the resonance (which is a correct procedure for a narrow resonance), or by actually looking at the ℓ=1 phase shift as a function of s. In either the former case or the latter looking below the resonant energy the output width was larger than the input one by a factor of 3-6. Looking at the phase shift itself on the high energy side of the resonance the situation is even worse. The function $((s-4)^3/s)^{\frac{1}{2}}$ cot $\delta_1(s)$ is plotted in Fig. 1 together with the input value for this function. For energies larger than the position of the ρ resonance the function decreases too slowly for a resonant behavior. The input values for the exchanged ρ were $(m_\rho^{In})^2 = 29$ and $\Gamma^{In} = .145$ (which corresponds to a full width at half maximum of 110 MeV).

Hence for given m_ρ^{In} and Γ^{In}, $\alpha_\rho^{In}(0)$ is determined from the self-consistency requirement on m_ρ in the ℓ=1 calculation. Thus the generalized potential $\underset{\sim}{B}_\ell^1(s)$ is determined and we solve the full N_ℓ/D_ℓ equations (15) and (16) to determine the Regge trajectory and residue for the ρ. In Fig. 2 to

4 we present some of the results for $(m_\rho^{In})^2 = 29$. As the width of the produced ρ meson is rather large, the imaginary parts of the ρ trajectory will be large above $s = 4$. Since we have only looked for the zero of the real part of D_ℓ, we have obtained the actual trajectory only for $s < 4$. We emphasize this by plotting dashed curves for $s > 4$, e.g., the dashed $\alpha_\rho(s)$ curves correspond to an approximation to the real part of $\alpha_\rho(s > 4)$.

For $\Gamma^{In} = .145$ we show in Fig. 3 a comparison of α_ρ for a calculation as mentioned above to one in which a pure $\ell = 1$ ρ exchange (as given by Eq. (7)) was considered as a straight cutoff used in solving equations (15) and (16) (again the self-consistency requirement of the output ρ position equaling m_ρ^{In} determined the value of the cutoff). We see that although there is some quantitative difference, both trajectories have $\alpha_\rho(0)$ larger than 0.9. These calculations with the straight cutoff and other calculations specifically for $A^1_{\ell=1}(s)$, e.g., using (9) to calculate $\underset{\sim}{B}^1_{\ell=1}(s)$,[11] all gave very similar results for the $\ell = 1$ partial wave. We felt this was a fairly good test of a number of the approximations made in obtaining Eq. (10).

In addition to obtaining the output width larger than the input one, the output $\alpha_\rho(0)$ was larger than $\alpha_\rho^{In}(0)$.[12] The two discrepancies are correlated. Near the resonance, we have from (21), $(d\alpha_\rho/ds) = (\beta_\rho/\Gamma)$ so that a large Γ corresponds to a small slope for α and thus $\alpha_\rho(0)$ is larger at $s = 0$ than $\alpha_\rho^{In}(0)$. It is interesting to note that the output β_ρ, as shown in Fig. 4, is almost constant in the relevant scattering region $(s < 0)$ and is very close in magnitude to $\beta_\rho^{In} = (d\alpha_\rho^{In}/ds)\Gamma^{In}$.

We have also calculated the scattering amplitude in I=0 channel again using only ρ exchange in the crossed channels. If we use the same parameters as for the I=1 calculation[9] we find that there is a vacuum trajectory but that for s=0 it has an $\ell > 1$; specifically for $\ell=1$ the pole occurs for a very large negative s. Therefore we adjusted the cutoff parameters to force the I=0 trajectory to cross 1 at s=0 [15] and calculated the vacuum trajectory $\alpha_P(s)$. A typical curve is shown in Fig. 5. Note that the slope is quite small; $(d\alpha_P(s)/ds)_{s=0} \approx 10^{-3}$ and hence our results would not be consistent with the f^o [16] being on the vacuum trajectory. We also calculated the residue of the vacuum pole at s=0. The residue corresponding to the trajectory shown in Fig. 5 gave an asymptotic total π-π cross section of 3mb as compared to a value of the 15mb obtained using the factorization theorem[17] and the asymptotic πN and NN cross sections.

We feel that both the problem a) that the output ρ width is larger than the input ρ width and the problem b) that using the input ρ parameters which yield a ρ resonance to calculate the (I=0) vacuum trajectory give $\alpha_P(0) > 1$ are largely due to the one channel approximation. The effect of an inelastic channel below its threshold is to, i) always act as an attraction, and ii) tend to narrow a resonance. Hence if we include the inelastic effects in the I=1 channel, which we expect to be due largely to the $\pi\omega$ channel,[1] this would narrow the output ρ width, and increase the attraction so that a somewhat smaller $\alpha_\rho^{In}(0)$ would be required.[18] On the other hand, the $\pi\omega$ channel does not couple to the I=0 channel so that this additional attraction would not be present and hence we would have a smaller $\alpha_P(0)$.

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9. If we look at $I = 0$ and even angular momenta, the relevant Born term is of the same form as (7) with Γ replaced by 2Γ.

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11. Eq. (9) and other more complicated approximations to (8) were considered and used to calculate $A^1_{\ell=1}(s)$ even though these could not be continued to non-integer ℓ simply.

12. Thus the input cutoff parameter $\alpha_\rho^{In}(0)$ should be considered as some average value. Using form given by (8) would have necessitated a somewhat smaller $\alpha_\rho^{In}(0)$.

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14. For the evaluation of the residue function $b_\rho(s)$ we have factored out the threshold factor $(s-4)^\ell$. The partial wave projection of a single Regge pole does not have this (correct) threshold behavior but goes as $(s-4)^{\alpha_\rho(s)}$. As $\alpha_\rho(s)$ varies little over a large range of s, the above definition of β is adequate. A representation in which each pole has the correct partial wave threshold behavior has been given by N. Khuri, Phys. Rev. **130**, 429 (1963).

15. If we then recalculate the $I = 1$, $\ell = 1$ amplitude, no ρ resonance occurs.

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18. A relatively small change in $\alpha_\rho^{In}(0)$ produces a large shift in the output resonance position.

FIGURE CAPTIONS

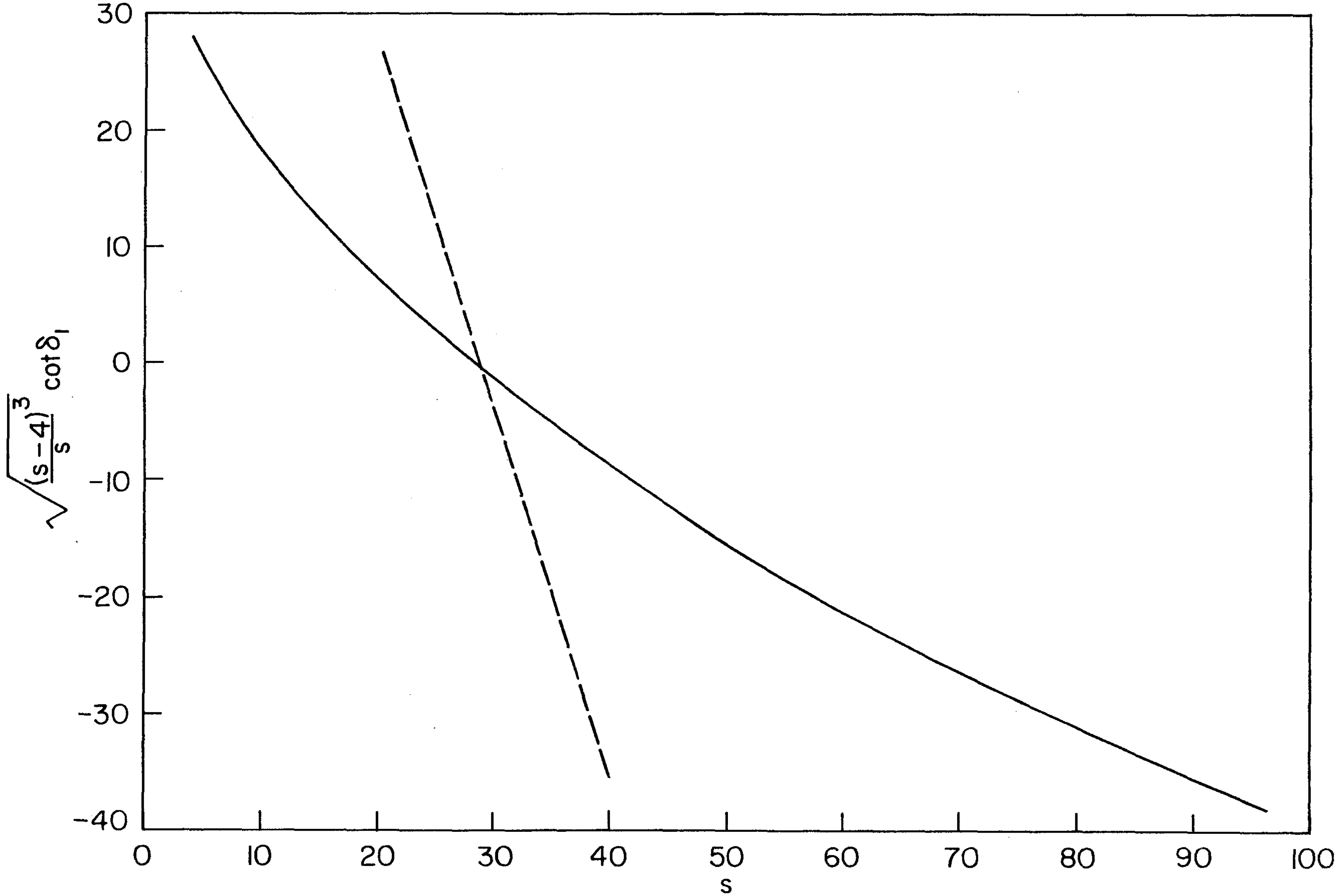
Fig. 1 Phase shift for $I=1$, $\ell=1$ amplitude versus s. The solid curve corresponds to the output, whereas the dashed curve comes from our input Breit-Wigner form. $\Gamma^{In} = .145$ and $\alpha_\rho^{In}(0) = .949$. For Fig. 1) - 4), $(m_\rho^{In})^2 = 29$ and the "cutoff parameter," i.e., $\alpha_\rho^{In}(0)$ is adjusted to force and $\ell=1$ resonance at m_ρ^{In}.

Fig. 2 $\alpha_\rho(s)$ for various input parameters. The dashed lines for $s > 4$ in Fig. 2-4 emphasize that we only investigated the vanishing of the real part of $D_\ell(s)$.

Fig. 3 Comparison of $\alpha_\rho(s)$ for a "straight cutoff" and a "Regge cutoff."

Fig. 4 The residue $\beta_\rho(s)$ for various input parameters. The arrows indicate the input $\beta_\rho^{In} = (d\alpha_\rho^{In}/ds)\Gamma^{In}$.

Fig. 5 The $I=0$ vacuum trajectory $\alpha_P(s)$ which has been adjusted to cross $s=0$ at $\ell=1$.


30
20
10
0
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-40
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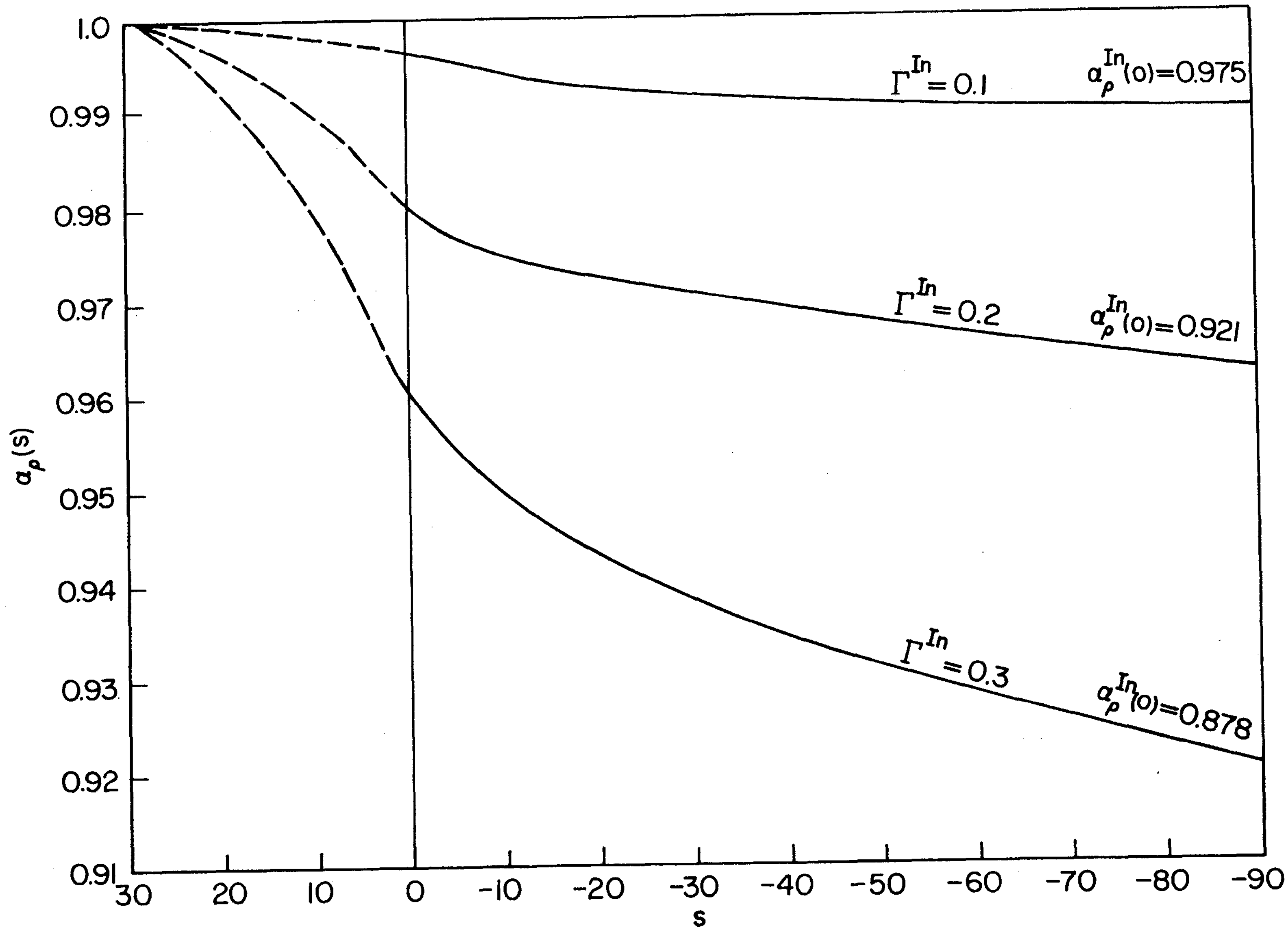
50
60
70
80
90
100
s
$\sqrt{\frac{(s-4)^3}{s}}\cot\delta_1$



1.0
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
$\alpha_\rho(s)$
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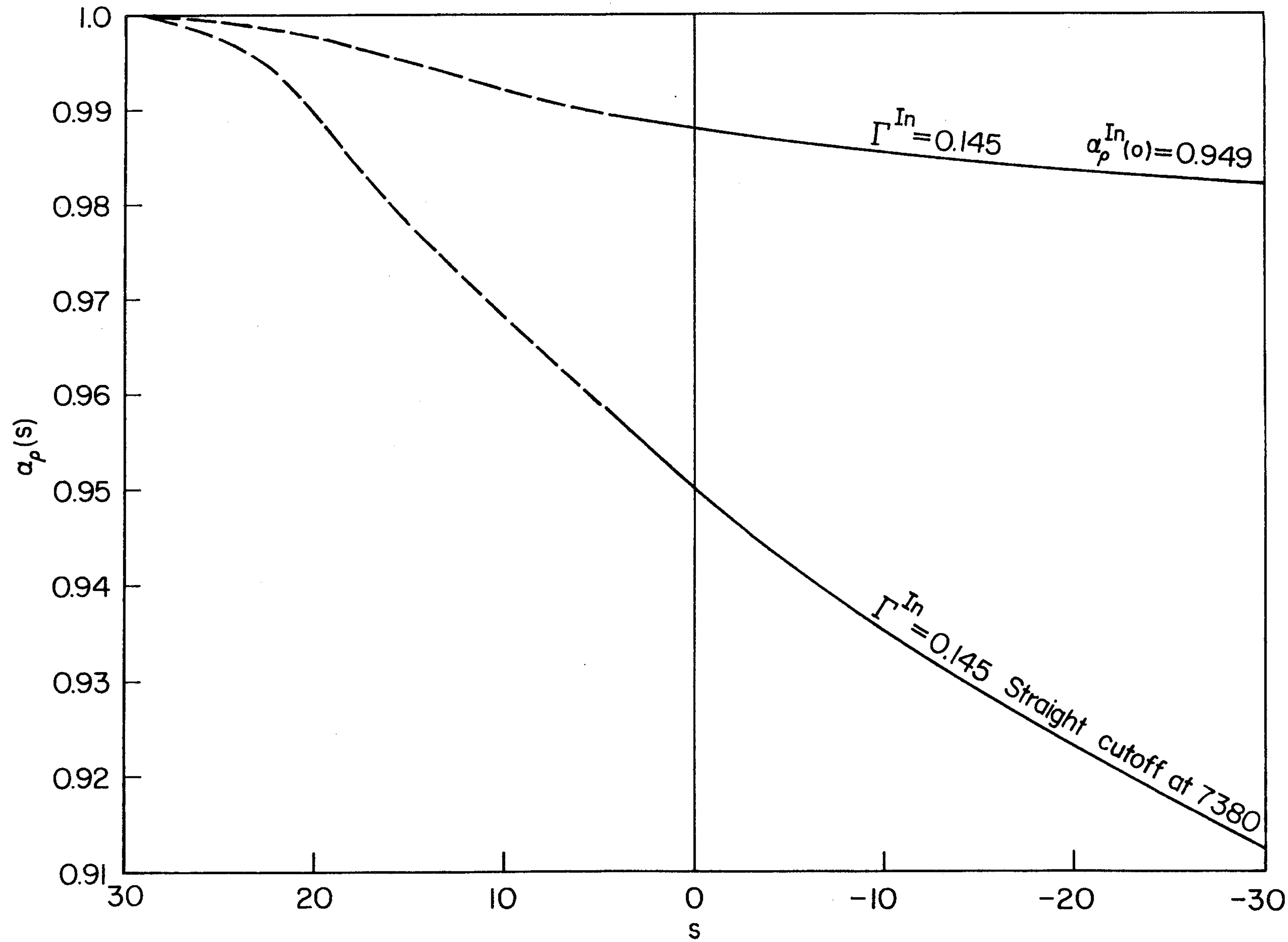
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
s
$\Gamma^{In}=0.1$
$\alpha_\rho^{In}(0)=0.975$
$\Gamma^{In}=0.2$
$\alpha_\rho^{In}(0)=0.921$
$\Gamma^{In}=0.3$
$\alpha_\rho^{In}(0)=0.878$



1.0
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
$a_\rho(s)$
30
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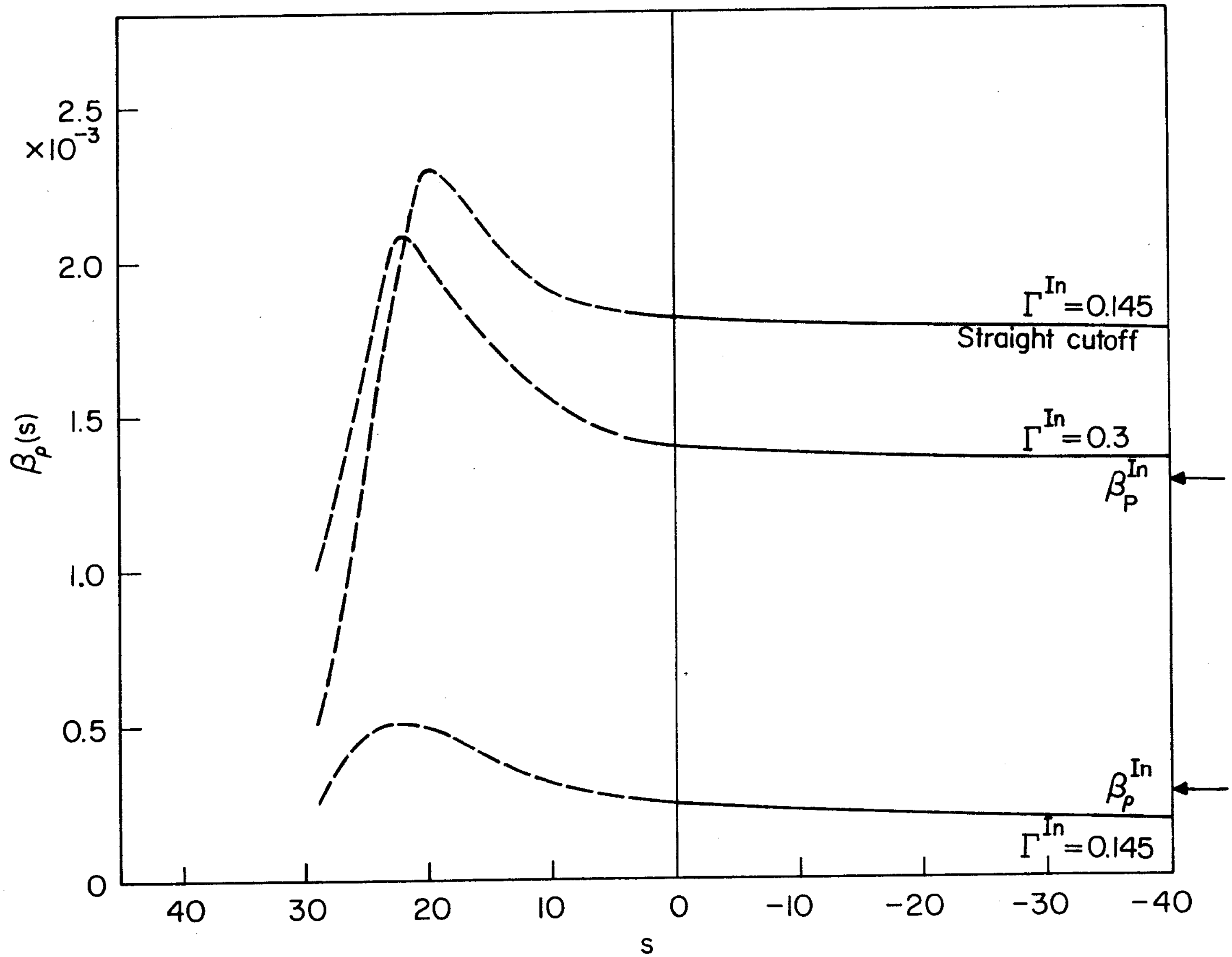
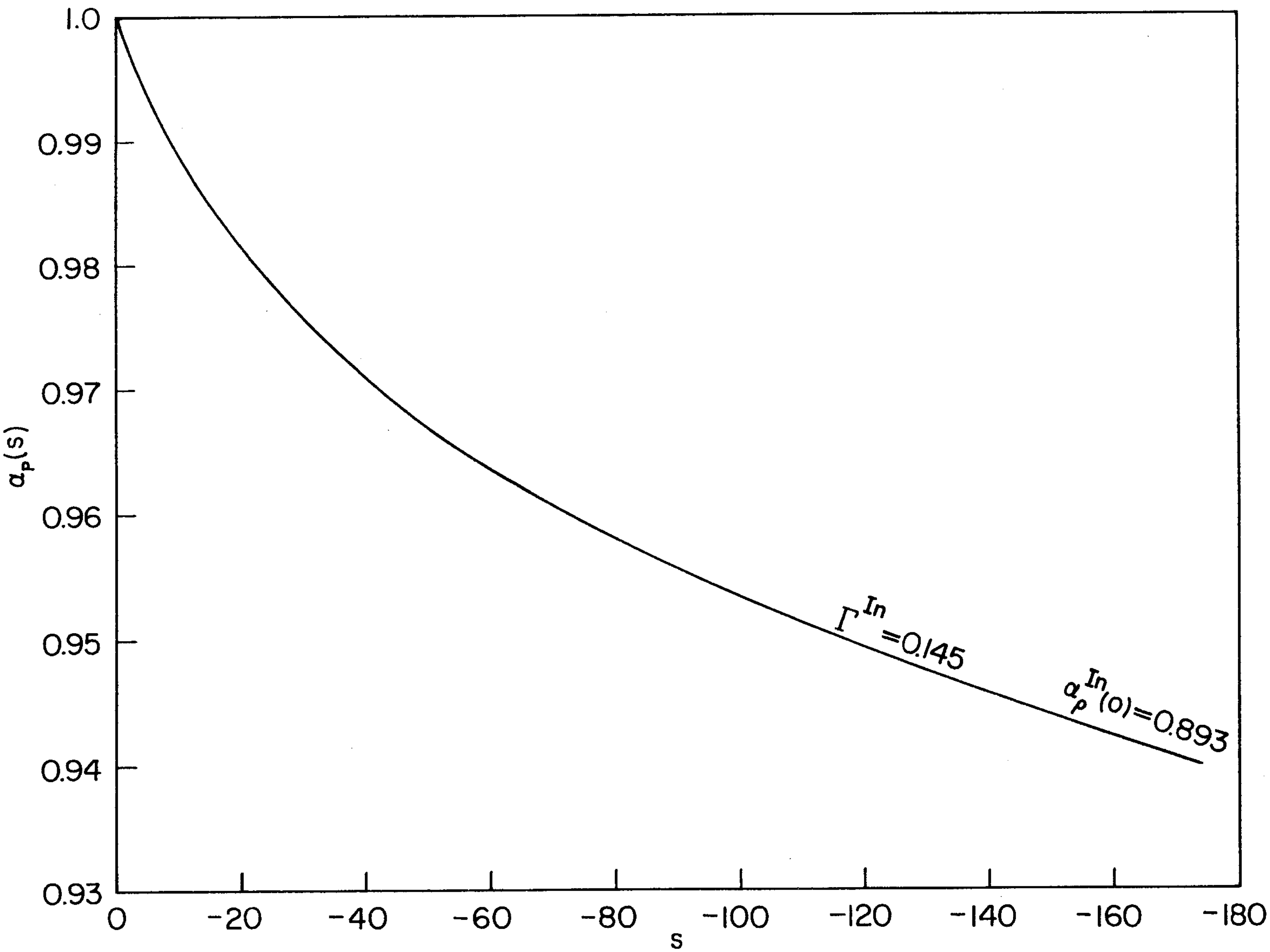
10
0
−10
−20
−30
s
$\Gamma^{In}=0.145$
$a_\rho^{In}(o)=0.949$
$\Gamma^{In}=0.145$ Straight cutoff at 7380



2.5
×10^-3
2.0
1.5
1.0
0.5
0
$\beta_p(s)$
40
30
20
10
0
−10
−20
−30
−40
s
$\Gamma^{In}=0.145$
Straight cutoff
$\Gamma^{In}=0.3$
β_P^{In}
β_p^{In}
$\Gamma^{In}=0.145$


1.0
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0.96
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0.93
$\alpha_p(s)$
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
s
$\Gamma^{In}=0.145$
$\alpha_\rho^{In}(0)=0.893$