SPECTRAL DEPENDENCE OF THE ANALYTICITY DOMAIN OF LOCAL VERTEX FUNCTIONS

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ABSTRACT

A method is outlined which allows to construct the envelope of holomorphy for general local vertex functions with arbitrary mass spectrum.

0.

Analyticity properties of the momentum space three point function V are considered in the frame-work of axiomatic field theory. More precisely, we are studying the Fourier transform of the vacuum expectation value of the retarded product of three in general different Wightman fields or currents.

The problem is to find the envelope of holomorphy $\mathcal{H} = \mathcal{H}(\mathcal{B}_v)$ of the primitive analyticity domain $\mathcal{B}_v$ of V, in its dependence on the spectral assumptions of the theory. We disregard non-linear properties of field theory, so the results are independent of an appearance of Martin catastrophies.

Since we can disregard one-particle poles in V, the boundary $\partial\mathcal{H}$ of $\mathcal{H}$ depends only on the lower limits of the continuous spectra of the mass operator in the different channels (i.e. the multi-particle thresholds M_k, $k = 1,2,3$): $\partial\mathcal{H}(\mathcal{B}_v(M_k)) \equiv \partial\mathcal{H}$. The primitive domain in the variables $z_k = p_k^2$, $k = 1,2,3$ with $\sum p_k = 0$ is

$$\mathcal{B}_v(M_k) = \bigcup_{k=1}^{3} \left\{ (z_k, z_l, z_m) \; ; \; 0 \leq z_k < M_k^2 \, , \, (z_l, z_m) \in \mathbb{C}^2, \, (z_k \pm i\varepsilon, z_l, z_m) \in \mathcal{B}_v(0) \; \forall \varepsilon > 0 \right\} \cup \mathcal{B}_v(0)$$

where (k,l,m) denotes a cyclic permutation of (1,2,3) and $\mathcal{B}_v(0)$ is the Källen - Wightman domain [1].

The procedure of constructing $\partial\mathcal{H}$ runs through following steps [2]:

1) choice of suitable variables,
2) determination of edges of $\partial\mathcal{H}(\mathcal{B}_v(M_k))$,
3) choice of a suitable parametrization of a family of smooth hypersurfaces $\mathcal{S}_\lambda$ which interpolate between these edges,
4) geometrical properties of $\mathcal{H}$,

5) characterization of the requested pseudoconvexity properties of $\partial\mathcal{H}$,
6) introduction of a constraint on $\mathcal{S}_\lambda$ through a differential equation (PSC-equation) on the defining functions of $\mathcal{S}_\lambda$ which is sufficient for $\mathcal{S}_\lambda$ to have the desired pseudoconvexity,
7) proof of the existence and uniqueness of a solution $\mathring{\lambda}$ of the resulting system of boundary value problems,
8) proof that the resulting unique hypersurface $\mathcal{S}_{\mathring{\lambda}}$ is the boundary of a (natural) domain of holomorphy,
9) construction of disks to establish $\partial\mathcal{H}(\mathcal{B}_v(M_k)) \equiv \mathcal{S}_{\mathring{\lambda}}$.

1.

For the choice of variables we first remark that in the special cases where $M_k \to 0$ $\forall\, k$ or $M_k \to \infty$ $\forall k$ *the holomorphy envelope of V is invariant under the two* transformation groups

$$v \to R_\varphi\, v \qquad \text{(simultaneous rotations: SR)}$$
$$v \to D_\chi\, v \qquad \text{(simultaneous dilatations: SD)}$$

where

$$v := \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \equiv \begin{pmatrix} j_1\,(p_{1o} - p_{11}) - j_2\,(p_{2o} - p_{21}) \\ j_1\,(p_{2o} - p_{21}) - j_2\,(p_{1o} - p_{11}) \\ j_1\,(p_{1o} + p_{11}) - j_2\,(p_{2o} + p_{21}) \\ j_1\,(p_{2o} + p_{21}) - j_2\,(p_{1o} + p_{11}) \end{pmatrix}$$

$$R_\varphi := \begin{pmatrix} e^{i\frac{2\pi}{3}\varphi} & & & 0 \\ & e^{-i\frac{2\pi}{3}\varphi} & & \\ & & e^{i\frac{2\pi}{3}\varphi} & \\ 0 & & & e^{-\frac{2\pi}{3}\varphi} \end{pmatrix}$$

$$D_\chi := \chi \cdot \mathbb{1}$$

with momenta (p_{ko}, p_{k1}), $k = 1,2,3$, $\sum_k p_{ko} = \sum_k p_{k1} = 0$ in a two-dimensional Minkowski space,with $j_k = e^{i\frac{2\pi}{3}k}$, $k = 1,2,3$ and with $\varphi \in \mathbb{Z}$, $\chi \in \mathbb{R}_+$. For $M_k \to \infty$ the holomorphy envelope is even invariant under a larger group, namely $\varphi \in \mathbb{R}$. In general both symmetries are broken if $0 < M_k < \infty$ for at least one k, i.e. if at least one of the multiparticle thresholds is finite. However, if the multiparticle thresholds in the three channels are equal, one has still invariance under SR, but not under SD.

Because we are interested in the dependence of the analyticity domain on M_k it is desirable to choose instead of z_k three other Lorentz invariants x, y, α in such a way that only one of them is not SD-invariant (X) respectively not SR-invariant (α). Such a choice is possible :

$$x := (v_1 v_2 v_3 v_4)^{-1/2},\ y := v_1 v_4 x,\ \alpha := v_1 v_3 x.$$

Then the Källén-Wightman manifolds C_k and $F'_{k\ell}$ have similar representations: With

$$\Psi_k(\tau, t) := \frac{\tau\alpha}{j_k} + \frac{j_k}{\tau\alpha} - \frac{y}{\tau} - \frac{\tau}{y} + t\,x$$

both C_k and $F'_{k\ell}$ can be represented by $\Psi_k(\tau, t) = 0$ with $t \in \mathbb{R}$ and different choices of τ, $\tau \in S_2$.

Furthermore for fixed $y \in \mathbb{C}$ the singularity region of V is restricted to the union of the 3 cut manifolds C_k and a region $\Delta(y) \subset \mathbb{C}^2$, $\Delta(y) := \{(x, \alpha);\ x \in \Omega(y), \alpha \in d(y,x)\}$, which is finite of $M_k \neq o\ \forall k$. Here $\Omega(y)$ depends analytically on y in the sense that it can be represented by

$$f(x,y;\sigma,\sigma') = 0 \qquad (\sigma, \sigma') \in \Omega_1$$

where f is holomorphic in x and y and $C^{(\infty)}$ in σ, σ' in Ω_1 with

$$\Omega_\varrho := \left\{(\sigma, \sigma') \in \mathbb{R}^2;\ (\sigma \cdot \min_k M_k^2\, \varrho) \in [0,1],\ \sigma'^2 \in [0,1],\ \varrho \text{ fixed} \in \mathbb{R}\right\}.$$

<u>2.</u>

By construction of perturbation theoretical examples it can be shown that certain portions of C_k, $F'_{k\ell}$ are edges E_ϱ of $\partial\mathcal{H}$, that for arbitrary $y_0 \in \mathbb{C}$ and arbitrary $x_0 \in \partial\Omega(y_0)$ the union of these edges has a non-vanishing intersection with $\{x, y, \alpha;\ x = x_0,\ y = y_0,\ \alpha \in d(y_0, x_0)\}$ and that the edges are analytic in y.

<u>3.</u>

One can provide that a family $\{\mathcal{S}_\lambda\}$ of hypersurfaces exists which for $(\sigma, \sigma') \in \partial\Omega_\varrho$ $\varrho \in \mathbb{R}$ pass through the edges E_ϱ of $\partial\mathcal{H}$, are quasi-analytic of rank 1 and can be represented by

$$\mathcal{S}_\lambda : \begin{cases} f(x,y;\sigma,\sigma') = 0 \\ \lambda(x,y,\alpha;\varrho,\sigma,\sigma') + \Lambda(x,y,\alpha;\varrho,\sigma,\sigma') = 0 \end{cases}$$

where f is the special function characterizing the domain $\Omega(y)$ and λ, Λ are holomorphic in (x,y,α) for $y \in \mathbb{C}$, $(x, \alpha) \in \Delta(y)$ and twice continuously differentiable in $(\varrho, \sigma, \sigma')$ for $\varrho \in \mathbb{R}$, $(\sigma, \sigma') \in \overline{\Omega}_\varrho$. The function Λ can be constructed in such a way that the hypersurface $f = \Lambda = 0$ interpolates between the edges E_ϱ of $\partial\mathcal{H}$ and that E_ϱ is approached on $f = \Lambda = 0$ in the limit dist $[(\sigma, \sigma'), \partial\Omega_\varrho] \to 0$.

For this aim the similarity of the representations of C_k and $F'_{k\ell}$ in (x, y, α) -space can be exploited: The desired interpolation between the edges E_ϱ is established by an ansatz

$$\Lambda := s^{-1} \prod_{k=1}^{3} \Psi_k \ (\tau, t_k)$$

where $S = S\,(x, y, \alpha\ ;\ \varrho,\ \sigma,\ \sigma')$ never vanishes in $\triangle(y)$ and where $\tau = \tau(\varrho, \sigma, \sigma')$, with $|\tau| = 1$, and $t_k = t_k\,(\varrho, \sigma, \sigma')$, $t_k \geqslant 0$ are suitably chosen, twice continuously differentiable ($\varrho \in \mathbb{R}$, $(\sigma, \sigma') \in \overline{\Omega}_\varrho$) and depend on the lower limits M_k, $k = 1,2,3$, of the mass spectrum in the three channels.(For details see ref. 2). In the case when we still have invariance under SR (i.e. in the equal mass case),

$$t_k \equiv t \quad \forall\, k\ .$$

The hypersurfaces $\mathcal{S}_\lambda$ yield interpolations of the desired type for arbitrary λ with $\lambda = 0$ on $\partial\Omega_\varrho$.

4.

Using the fact that a common scaling of the three mass spectra results in a scaling of x with y, α unchanged (more precisely: of σ with y, α, σ' unchanged) the boundary $\partial\mathcal{H}$ of $\mathcal{H}$ must be such that on $\partial\mathcal{H}$ the coordinate σ is uniquely determined, $\sigma = \sigma_0\,(y, \alpha, \sigma')$, by y, α, σ'. Thus we may use the notion: below (above) $\partial\mathcal{H}$, for characterizing points with $\sigma < \sigma_0$ ($\sigma > \sigma_0$).

5.

From the geometrical property of $\mathcal{H}$ just stated a necessary condition for a hypersurface to be in $\partial\mathcal{H}$ is that it be Cartan pseudoconvex from below.

6.

If a hypersurface $\phi = 0$, $\phi : \mathbb{C}^3 \to \mathbb{R}$, $\phi \in C^{(2)}$ possesses in a neighbourhood of some point P supporting analytic manifolds of lower dimension (and if the rank of the Hesse form restricted to the analytic tangent space fulfils a further condition, both conditions being satisfied for quasi-analytic hypersurfaces of rank 1) then pseudoconvexity in the sense of Cartan and pseudoconvexity in the sense of Hesse forms are equivalent. Thus for pseudoconvexity from below of the hypersurface $\mathcal{S}_\lambda$ at some point P semi-definiteness of the corresponding Hesse form at P (instead of definiteness in the general case) and $\lambda + \Lambda \in C^{(2)}$ with respect to ϱ, σ, σ' (instead of $C^{(4)}$ in the general case) are already sufficient. In particular a system of second order elliptic differential equations (pseudoconvexity equation) for λ in the variables σ, σ' can be stated:

$$\text{(PSC)} \qquad \frac{\partial^2}{\partial\sigma^2}\lambda + \frac{\partial^2}{\partial\sigma'^2}\lambda + a\left(y, \alpha, \varrho \,;\, \sigma, \sigma' \,;\, \frac{\partial}{\partial\sigma}\lambda, \frac{\partial}{\partial\sigma'}\lambda\right) = 0$$

which is sufficient for $\mathcal{S}_\lambda$ to be pseudoconvex from below (for any smooth solution λ with $\frac{\partial}{\partial\sigma}\lambda + \frac{\partial}{\partial\sigma}\Lambda \neq 0$ on $\mathcal{S}_\lambda$).

7.

The system of boundary value problems consisting of (PSC) and the boundary condition $\lambda \equiv 0$ on $\partial\Omega_\varrho$ can be shown by use of the Leray-Schauder theorem (and of Ladyzhenskaya-Ural'tseva estimates) to have a solution $\mathring{\lambda}$ which is unique and which, moreover, is analytic in y, α, ϱ with $y \in \mathbb{C}$, $\alpha \in d\,(y, x(y, \sigma, \sigma'))$, $\varrho \in \mathcal{N}(\mathcal{R})$.

8.

The uniquely determined hypersurface $\mathcal{S}_{\mathring{\lambda}}$ is pseudoconvex from below at all points where $\frac{\partial}{\partial\sigma}\mathring{\lambda} + \frac{\partial}{\partial\sigma}\Lambda \neq 0$. From the boundary condition and the properties of the solution $\mathring{\lambda}$ the assumption that $\frac{\partial}{\partial\sigma}\mathring{\lambda} + \frac{\partial}{\partial\sigma}\Lambda = 0$ somewhere on $\mathcal{S}_{\mathring{\lambda}}$ implies the existence of points where $\mathcal{S}_{\mathring{\lambda}}$ is convex from above and $\frac{\partial}{\partial\sigma}\mathring{\lambda} + \frac{\partial}{\partial\sigma}\Lambda \neq 0$ what leads to a contradiction by the fact that (PSC) is a sufficient condition for pseudoconvexity from below at this point. It follows that $\mathcal{S}_{\mathring{\lambda}}$ is the boundary of <u>some</u> domain of holomorphy.

9.

The last step is to show that $\mathcal{S}_{\mathring{\lambda}}$ coincides with part of $\partial\mathcal{H}$ by use of the continuity theorem. In the proof the construction of disks is based on the fact that the rank 1 quasi-analytic hypersurface $\mathcal{S}_{\mathring{\lambda}}$ has some analyticity properties: through any of its points P passes a 1-dimensional analytic manifold completely belonging to $\mathcal{S}_{\mathring{\lambda}}$ in a full neighbourhood of P . The boundary condition imposed on $\mathring{\lambda}$ on $\partial\Omega_\varrho$ allows to construct a one-parameter sequence of disks $\Delta(\kappa)$, $\kappa \in [0,1]$ in such a way that for arbitrary $\kappa \in [0,1]$ the large α region on $\Delta(\kappa)$ and for $\kappa = 1$ the whole of Δ is in the analyticity domain.

lo.

To conclude, for $(x,\alpha) \in \triangle(y)$, $y \in \mathcal{C}$ the envelope of holomorphy of the general local vertex function for arbitrary mass spectrum is bounded by a quasi-analytic hypersurface $\mathcal{S}_{\mathring{\lambda}}$ of rank 1. Here $\mathring{\lambda}$ is the unique solution of the boundary value problem consisting of (PSC) and the boundary condition $\mathring{\lambda} = 0$ on $\partial\Omega_{\varrho}$ and can be determined numerically. The dependence of $\mathcal{S}_{\mathring{\lambda}}$ on the mass spectrum is displayed by the functions $\mathring{\lambda}$ and Λ.

REFERENCES

1) G. Källén , A. Wightman, Math. Fys. Skv. Dan. Vid. Selsk. 1, no.6 (1958). For further references see 2).

2) G. Sommer, On the envelope of holomorphy of the general local vertex function in momentum space (with arbitrary mass spectrum), University of Bielefeld preprint, Bi-73/16.

DISCUSSION

V.S. Vladimirov (question):

Is it possible to calculate an explicit form of the surfaces at the boundary of $\mathcal{H}$ in p-space, as in x-space (Källén-Wightman), at least for equal masses?

G. Sommer (answer):

Unfortunately not. The dependence of the pseudoconvexity equations on the three mass-parameters as well as on the variables σ, σ' and the other parameters is still rather involved. This is,because the hypersurface has to interpolate between the sets given by the boundary conditions, and these are complicated. All this gets only slightly simplified in the equal mass case. The structure of the pseudoconvexity equations remains unchanged, because essentially the only simplification in this case is that all three t_k in the parametrization of Λ are equal.