EINSTEIN FIELD EQUATIONS WITHIN LOCAL FRACTIONAL CALCULUS

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In this paper, we introduce the local fractional Christoffel index symbols of the first and second kind. The divergence of a local fractional contravariant vector and the curl of local fractional covariant vector are defined. The fractional intrinsic derivative is given. The local fractional Riemann-Christoffel and Ricci tensors are obtained. Finally, the Einstein tensor and Einstein field are generalized by involving the fractional derivatives. Illustrative examples are presented.

Key words: local fractional Christoffel index; local fractional Riemann-Christoffel tensor; local fractional Ricci tensor; local fractional Einstein field.

1. INTRODUCTION

Fractional calculus is an old subject and it recently found many applications in physics, mechanics, chaos, control, and so on [1–8]. The fractional derivative is non local which is not suitable in fractal medium. As it is well known the fractals have many application in science [9–14]. Therefore, the local fractional calculus has been defined [15–26]. The fractional calculus is used to generalized Newtonian mechanics, the Maxwell's equations and the Hamiltonian mechanics [27–30]. The one-dimensional heat equations with the local fractional derivative has been studied using Adomian decomposition method [31]. A new Neumann series method has been applied to find analytic solution for the family of local fractional Fredholm and Volterra integral equations [32]. Recently, the nonlocal fractional derivative was used to generalized general relativity [33, 34].

The plan of the paper is as follows. In section 2 we review the fractional calculus. In section 3 the local fractional Christoffel symbols are introduced. Divergence and curl of a local fractional contravariant vector are studied in section 4. We give the definition for the local fractional intrinsic derivative in section 5. In section 6 local fractional Riemann-Christoffel and Ricci tensors are explained. The local fractional Einstein field equation is suggested in section 7. Finally, the section 8 is devoted to our conclusions.

2. A REVIEW OF LOCAL FRACTIONAL DERIVATIVES

In this section we review the local fractional derivative, local fractional integral, and tensors in fractal orthogonal coordinates systems and their properties [18].

2.1. LOCAL FRACTIONAL DERIVATIVE

Suppose that $f(x) \in C_\alpha[a,b]$ and $x \in (x_0-\delta, x_0+\delta), \delta > 0$ then

$$D^\alpha_{x_0} f(x) = \frac{d}{dx^\alpha} f(x)|_{x=x_0} =: \lim_{x\to x_0} \frac{\Gamma(1+\alpha)[f(x)-f(x_0)]}{(x-x_0)^\alpha}, \tag{1}$$

if the limit exists.

2.2. LOCAL FRACTIONAL INTEGRAL

Let $f(x) \in C_\alpha[a,b]$, the local fractional integral of the function $f(x)$ is defined [18]

$$_aI^\alpha_b f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_i)(\Delta t_j)^\alpha, \tag{2}$$

where $0 < \alpha \leq 1$, $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ...\Delta t_j ...\}$.

2.3. TENSORS IN LOCAL FRACTIONAL ORTHOGONAL COORDINATES SYSTEMS

Tensors are the quantities obeying special transformations. Here we study the local fractional tensor notation. The local fractional covariant and contravariant linear vectors are as follows [18]

$$\begin{gathered}\overrightarrow{A}_\alpha(x) = x_{1\alpha}\hat{e}_{1\alpha} + y_{1\alpha}\hat{e}_{1\alpha} + z_{1\alpha}\hat{e}_{1\alpha}; \quad \overrightarrow{A}^\alpha(x) = x^{1\alpha}\hat{e}^{1\alpha} + y^{1\alpha}\hat{e}^{1\alpha} + z^{1\alpha}\hat{e}^{1\alpha}; \\ \hat{e}_{1\alpha} = (1^\alpha, 0^\alpha, 0^\alpha); \qquad \hat{e}_{2\alpha} = (0^\alpha, 1^\alpha, 0^\alpha)\ ; \qquad \hat{e}_{3\alpha} = (0^\alpha, 0^\alpha, 1^\alpha).\end{gathered} \tag{3}$$

The squared fractional distance between two points $y^{i\alpha} = y^{i\alpha}(x^{1\alpha}, x^{2\alpha}, x^{3\alpha}, ..., x^{N\alpha})$ and $y^{i\alpha} + d^\alpha y^i$ in local fractional Riemann space is given by

$$(d^\alpha s)^2 = g^\alpha_{rs} dx^{r\alpha} dx^{s\alpha} \quad g^\alpha_{rs} = \frac{1}{\Gamma^2(\alpha+1)} \frac{d^\alpha y^m}{dx^{r\alpha}} \frac{d^\alpha y^m}{dx^{s\alpha}} \quad r,s = 1,2,3,...,N. \tag{4}$$

where g^{α}_{rs} is the local fractional metric [18].

Some formulas of local fractional calculus:

The Mittag-Leffler function on fractal set of dimension α is [18]

$$E_{\alpha}(\beta,x^{\alpha})=\sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(\beta+k\alpha)},\quad x\in R,\qquad 0<\alpha\leq 1.$$

The sine function on fractal set is defined as

$$\sin_{\alpha}x^{\alpha}=\sum_{k=0}^{\infty}(-1)^{k}\frac{x^{(2k+1)\alpha}}{\Gamma(1+\alpha(2k+1))},\qquad x\in R,\qquad 0<\alpha\leq 1$$

The tangent function on fractal set of dimension α is given by

$$\begin{gathered}\tan_{\alpha}x^{\alpha}=\frac{\sin_{\alpha}x^{\alpha}}{\cos_{\alpha}x^{\alpha}};\qquad D_{x}^{\alpha}c=0;\qquad D_{x}^{\alpha}\frac{x^{\alpha}}{\Gamma(1+\alpha)}=1;\\ D_{x}^{\alpha}\sin_{\alpha}x^{\alpha}=\cos_{\alpha}x^{\alpha};\qquad \int_{a}^{b}\cos_{\alpha}x^{\alpha}(dx)^{\alpha}=\Gamma(1+\alpha)(\sin_{\alpha}b^{\alpha}-\sin_{\alpha}a^{\alpha}).\end{gathered}\tag{5}$$

For more formulas see Ref. [18].

3. LOCAL FRACTIONAL CHRISTOFFEL INDEX SYMBOLS

The Christoffel symbols have an important role in the calculus on manifolds and general relativity in physics. It is used in the definition of the quantity of Riemann curvature tensor, divergence, curl, intrinsic derivative and Einstein tensor in N-dimensional manifold space. For the local coordinate system the Christoffel symbols have n^3 components. In the section, we generalize the Christoffel symbols by involving local fractional derivatives. Then we use them in the calculus of fractal manifolds.

3.1. LOCAL FRACTIONAL CHRISTOFFEL INDEX SYMBOL OF THE FIRST KIND

Suppose we have a fractional Riemannian manifold (M^{α},g^{α}) and a chart. So one can compute the fractional Christoffel index symbol of the first kind using the following definition

$$[ij,k]^{\alpha}=\frac{1}{2}(\frac{\partial g^{\alpha}_{ik}}{\partial x^{j\alpha}}+\frac{\partial g^{\alpha}_{jk}}{\partial x^{i\alpha}}-\frac{\partial g^{\alpha}_{ij}}{\partial x^{k\alpha}});\quad i,j,k=1,2,...,N\quad 0<\alpha\leq 1,\tag{6}$$

where $[ij,k]^{\alpha},g^{\alpha}_{ik}$ and α are called fractional Christoffel index symbol of the first kind, fractional fundamental metric tensor and fractal dimension, respectively.

3.2. LOCAL FRACTIONAL CHRISTOFFEL INDEX SYMBOL OF THE SECOND KIND

The local fractional Christoffel index symbol of the second kind is generalized using local fractional derivatives on local fractional Riemannian manifold (M^α, g^α) and a given chart

$$
{}^\alpha\Gamma^k_{ij} = {}^\alpha g^{km}[ij,m]^\alpha = \frac{1}{2}\,{}^\alpha g^{km}\left(\frac{\partial g^\alpha_{im}}{\partial x^{j\alpha}} + \frac{\partial g^\alpha_{jm}}{\partial x^{i\alpha}} - \frac{\partial g^\alpha_{ij}}{\partial x^{m\alpha}}\right), \qquad i,j,k = 1,2,...,N \qquad 0 < \alpha \leq 1, \quad (7)
$$

where ${}^\alpha g^{km}$ is the reciprocal tensor for the fundamental metric tensor g^α_{ij}.
Example 1. Let V^α_2 be a fractal space with fractal line element as

$$
(d^\alpha s)^2 = \frac{1^\alpha}{\Gamma^2(\alpha+1)}(dx^{1\alpha})^2 + \frac{x^{1\alpha}}{\Gamma^2(\alpha+1)}(dx^{2\alpha})^2 + \frac{x^{1\alpha}\sin^2_\alpha x^{2\alpha}}{\Gamma^2(\alpha+1)}(dx^{2\alpha})^2 \qquad (8)
$$

where a is a constant. Then, using the Eq. (4) we have the element of the local fractional metric tensor as

$$
\begin{gathered}
g^\alpha_{11} = \frac{1^\alpha}{\Gamma^2(\alpha+1)}; \qquad g^\alpha_{22} = \frac{(x^{1\alpha})^2}{\Gamma^2(\alpha+1)}; \\
g^\alpha_{33} = \frac{(x^{1\alpha})^2}{\Gamma^2(\alpha+1)}\sin^2_\alpha x^{2\alpha}; \qquad g^\alpha_{ij} = 0 \qquad i \neq j;
\end{gathered} \qquad (9)
$$

and the determinant of the metric tensor g^α_{ij} is

$$
g^\alpha = \begin{vmatrix} \frac{1^\alpha}{\Gamma^2(\alpha+1)} & 0^\alpha & 0^\alpha \\ 0^\alpha & \frac{(x^{1\alpha})^2}{\Gamma^2(\alpha+1)} & 0^\alpha \\ 0^\alpha & 0^\alpha & \frac{(x^{1\alpha})^2}{\Gamma^2(\alpha+1)}\sin^2_\alpha x^{2\alpha} \end{vmatrix} = \frac{(x^{1\alpha})^4 \sin^2_\alpha\, x^{2\alpha}}{\Gamma^6(\alpha+1)}. \qquad (10)
$$

The reciprocal of the local fractional metric tensor is obtained using ${}^\alpha g^{ij} = \frac{1}{g^\alpha_{ij}}$

$$
\begin{gathered}
{}^\alpha g^{11} = \frac{1}{g^\alpha_{11}} = \Gamma^2(\alpha+1); \\
{}^\alpha g^{22} = \frac{1}{g^\alpha_{22}} = \frac{\Gamma^2(\alpha+1)}{(x^{1\alpha})^2}; \qquad {}^\alpha g^{33} = \frac{1}{g^\alpha_{33}} = \frac{\Gamma^2(\alpha+1)}{(x^{1\alpha})^2\sin^2_\alpha\, x^{2\alpha}}.
\end{gathered} \qquad (11)
$$

Then, we calculate the local fractional Christoffel index symbol of the first kind as written below

$$\begin{aligned}
[12,2]^{\alpha} = [21,2]^{\alpha} = \frac{x^{1\alpha}}{\Gamma^2(\alpha+1)} \quad [13,3]^{\alpha} = [31,3]^{\alpha} = \frac{x^{1\alpha}}{\Gamma^2(\alpha+1)} \sin^2_{\alpha} x^{2\alpha} \\
[22,2]^{\alpha} = \frac{-x^{1\alpha}}{\Gamma^2(\alpha+1)} \qquad [23,3]^{\alpha} = \frac{(x^{1\alpha})^2}{\Gamma^2(\alpha+1)} \sin_{\alpha} x^{2\alpha} \cos_{\alpha} x^{2\alpha} \\
[33,1]^{\alpha} = \frac{-x^{1\alpha}}{\Gamma^2(\alpha+1)} \sin^2_{\alpha} x^{2\alpha} \quad [33,2]^{\alpha} = \frac{-(x^{1\alpha})^2}{\Gamma^2(\alpha+1)} \sin_{\alpha} x^{2\alpha} \cos_{\alpha} x^{2\alpha}.
\end{aligned} \tag{12}$$

In view of Eq. (7), we arrive at the local fractional Christoffel index symbol of the second kind as

$$\begin{aligned}
{}^{\alpha}\Gamma^1_{22} = -x^{1\alpha}; \quad {}^{\alpha}\Gamma^1_{33} = -x^{1\alpha} \sin^2_{\alpha} x^{2\alpha}; \quad {}^{\alpha}\Gamma^2_{12} =^{\alpha} \Gamma^2_{21} = \frac{1^{\alpha}}{x^{1\alpha};} \\
{}^{\alpha}\Gamma^2_{33} = -\sin_{\alpha} x^{2\alpha} \cos_{\alpha} x^{2\alpha}; \quad {}^{\alpha}\Gamma^3_{23} =^{\alpha} \Gamma^3_{32} = \cot_{\alpha} x^{2\alpha}; \\
{}^{\alpha}\Gamma^3_{13} =^{\alpha} \Gamma^3_{31} = \frac{1^{\alpha}}{x^{1\alpha}}.
\end{aligned} \tag{13}$$

All the results for this example in a fractal space will lead to standard results by choosing $\alpha = 1$.

4. DIVERGENCE AND CURL OF A FRACTIONAL CONTRAVARIANT VECTOR

Sink and source of vector field can be measured by divergence of vector field which is a vector operator. The curl is the infinitesimal rotation in a vector space. Here we expand the standard curl and divergence to the fractal space that involve the local fractional derivatives. Now let us define local fractional contravariant and covariant Riemann-Christoffel tensors which are denoted by $A^{i\alpha}_{,j}$ and $A_{j\alpha,k}$, respectively, as follows

$$A^{i\alpha}_{,j} = \frac{\partial A^{i\alpha}}{\partial x^{j\alpha}} +^{\alpha} \Gamma^i_{kj} A^{k\alpha}; \quad A_{j\alpha,k} = \frac{\partial A_{j\alpha}}{\partial x^{k\alpha}} -^{\alpha} \Gamma^r_{jk} A_{r\alpha} \tag{14}$$

where $A^{i\alpha}$, $A_{j\alpha}$ are components of arbitrary local fractional contravariant and covariant tensors. Using the Eq. (14) the local fractional divergence is given by

$$\operatorname{div}^{\alpha} A^{i\alpha} = \nabla^{\alpha}_i A^{i\alpha} = A^{i\alpha}_{,j}; \qquad \operatorname{div}^{\alpha} A_{i\alpha} = {}^{\alpha}g^{jk} A_{j\alpha,k}. \tag{15}$$

We now introduce curl of a local fractional covariant tensor in the fractal space V^{α}_N as

$$\operatorname{curl}^{\alpha} A_{i\alpha} = \nabla^{\alpha} \times A_{i\alpha} = A_{i\alpha,j} - A_{j\alpha,i} = \frac{\partial A_{i\alpha}}{\partial x^{j\alpha}} - \frac{\partial A_{j\alpha}}{\partial x^{i\alpha}}, \tag{16}$$

where $\operatorname{curl}^{\alpha} A_{i\alpha}$ is called local fractional curl operator. We can get the standard results in Eqs. (15) and (16) by putting $\alpha = 1$.

5. FRACTIONAL INTRINSIC DERIVATIVE

It is known that the covariant differentiation in a Riemannian space is regarded as a generalization of partial differentiation. Intrinsic or absolute differentiation is considered as the generalization of ordinary differentiation. Let C^α be a certain fractal space curve that is described by the parametric equations in V_N^α such as

$$C^\alpha : x^{i\alpha} = x^{i\alpha}(t^\alpha); \qquad i = 1, 2, ..., N. \tag{17}$$

For any local fractional contravariant vector along the local fractional C^α we can define intrinsic or absolute local fractional derivative as

$$\frac{\delta A^{i\alpha}}{\delta t^\alpha} = A^{i\alpha}_{,k} \frac{dx^{k\alpha}}{dt^\alpha} = \left[\frac{\partial A^{i\alpha}}{\partial x^{k\alpha}} + A^{m\alpha}\,{}^\alpha\Gamma^i_{mk}\right]\frac{dx^{k\alpha}}{dt^\alpha} = \frac{dA^{i\alpha}}{dt^\alpha} + A^{m\alpha}\,{}^\alpha\Gamma^i_{mk}\frac{dx^{k\alpha}}{dt^\alpha}, \tag{18}$$

and for local fractional covariant vector will be

$$\frac{\delta A_{i\alpha}}{\delta t^\alpha} = A_{i\alpha,k}\frac{dx^{k\alpha}}{dt^\alpha} = \frac{dA_{i\alpha}}{dt^\alpha} - A_{m\alpha}\,{}^\alpha\Gamma^{ik}_m\frac{dx^{k\alpha}}{dt^\alpha}. \tag{19}$$

The Eqs. (18) and (19) are written in the local fractional differential form

$$\delta A^{i\alpha} = dA^{i\alpha} + {}^\alpha\Gamma^{kj}_i A^{j\alpha}dx^{k\alpha}; \quad \delta A_{i\alpha} = dA_{i\alpha} - {}^\alpha\Gamma^{ki}_j A_{j\alpha}dx^{k\alpha} \tag{20}$$

where $\delta A^{i\alpha}$ and $\delta A_{i\alpha}$ are intrinsic or absolute local fractional derivatives of contravariant and covariant local fractional tensors, respectively.

Example 2. Let a particle moves along a fractal curve $x^{k\alpha} = x^{k\alpha}(t^\alpha)$ where t^α is the parameter in the fractal time space. Then, the generalized local fractional velocity of a particle on fractal manifold is

$$v^{k\alpha} = \frac{dx^{k\alpha}}{dt^\alpha}, \quad k = 1, 2, 3, ..., N, \tag{21}$$

and the fractal acceleration will be

$$a^{k\alpha} = \frac{\delta v^{k\alpha}}{\delta t^\alpha} = \frac{dv^{k\alpha}}{dt^\alpha} + {}^\alpha\Gamma^k_{qp}\,v^{p\alpha}\,\frac{dx^{q\alpha}}{dt^\alpha} = \frac{d^2x^{k\alpha}}{(dt^\alpha)^2} + {}^\alpha\Gamma^k_{qp}\frac{dx^{p\alpha}}{dt^\alpha}\frac{dx^{q\alpha}}{dt^\alpha}, \quad k = 1, 2, 3, ..., N. \tag{22}$$

These definitions are the standard ones if one choose $\alpha = 1$.

6. LOCAL FRACTIONAL RIEMANN-CHRISTOFFEL AND RICCI TENSORS

In this section we extend the Riemann-Christoffel, Ricci tensors, and scalar curvature involving local fractional derivative. Let $B_{i\alpha}$ be an arbitrary covariant vector then its local fractional covariant derivative with respect to $x^{j\alpha}$ is given by

$$B_{i\alpha,j} = \frac{\partial B_{i\alpha}}{\partial x^{j\alpha}} - {}^\alpha\Gamma^m_{ij}B_{m\alpha}, \tag{23}$$

which is a local fractional covariant tensor of rank 2. Riemann-Chrisstoffel tensor is given by

$$ {}^{\alpha}R^{m}_{ijk} = \begin{vmatrix} \frac{\partial}{\partial x^{j\alpha}} & \frac{\partial}{\partial x^{k\alpha}} \\ {}^{\alpha}\Gamma^{m}_{ij} & {}^{\alpha}\Gamma^{m}_{ij} \end{vmatrix} + \begin{vmatrix} {}^{\alpha}\Gamma^{m}_{\beta j} & {}^{\alpha}\Gamma^{m}_{\beta k} \\ {}^{\alpha}\Gamma^{\beta}_{ij} & {}^{\alpha}\Gamma^{\beta}_{ik} \end{vmatrix}. \tag{24} $$

The Riemann-Chrisstoffel tensor ${}^{\alpha}R^{m}_{ijk}$ can be contracted in three ways with respect to m and any one of its lower indices, i.e., ${}^{\alpha}R^{m}_{mjk}$, ${}^{\alpha}R^{m}_{imk}$, ${}^{\alpha}R^{m}_{ijm}$. The contracted tensor ${}^{\alpha}R^{m}_{ijm}$, which is not identically zero is called the Ricci tensor of the first kind and its components are denoted by ${}^{\alpha}Rij = {}^{\alpha}R^{m}_{ijm}$

$$ {}^{\alpha}Rij = \frac{\partial^2}{\partial x^{j\alpha}\partial x^{i\alpha}}(\log_{\alpha}\sqrt{g^{\alpha}}) - \frac{\partial}{\partial x^{m\alpha}}\Gamma^{m}_{ji} + \Gamma^{m}_{\beta j}\Gamma^{\beta}_{mi} - \Gamma^{m}_{\beta m}\Gamma^{\beta}_{ji}. \tag{25} $$

The local fractional scalar curvature in fractal space R^{α} is defined as

$$ R^{\alpha} = {}^{\alpha}g^{im}R^{\alpha}_{mi}. \tag{26} $$

Example 3. Let us consider the local fractional metric tensor in $E^{3\alpha}$ space such as

$$ (d^{\alpha}s)^2 = \frac{a^{2\alpha}}{\Gamma^2(\alpha+1)}(dx^{1\alpha})^2 + \frac{a^{2\alpha}\sin^2_{\alpha}x^{1\alpha}}{\Gamma^2(\alpha+1)}(dx^{2\alpha})^2. \tag{27} $$

Using fractional linear vector Eq. (27) we obtain fundamental local fractional metric tensor as

$$ \begin{aligned} g^{\alpha}_{11} &= \frac{a^{2\alpha}}{\Gamma^2(\alpha+1)}; \quad g^{\alpha}_{12} = g^{\alpha}_{21} = 0^{\alpha}; \\ g^{\alpha}_{22} &= \frac{a^{2\alpha}\sin^2_{\alpha}x^{1\alpha}}{\Gamma^2(\alpha+1)}; \quad g^{\alpha} = \frac{a^{4\alpha}\sin^2_{\alpha}x^{1\alpha}}{\Gamma^6(\alpha+1)}, \end{aligned} \tag{28} $$

and the reciprocal local fractional metric tensor

$$ {}^{\alpha}g^{11} = \frac{\Gamma^2(\alpha+1)}{a^{2\alpha}}; \qquad {}^{\alpha}g^{12} = {}^{\alpha}g^{21} = 0^{\alpha}; \qquad {}^{\alpha}g^{22} = \frac{\Gamma^2(\alpha+1)}{a^{2\alpha}\sin^2_{\alpha}x^{1\alpha}}. \tag{29} $$

The local fractional Christoffel index symbol of the second kind in this case will be

$$ {}^{\alpha}\Gamma^{1}_{22} = -\sin_{\alpha}x^{1\alpha}\cos_{\alpha}x^{1\alpha}; \quad {}^{\alpha}\Gamma^{2}_{21} = {}^{\alpha}\Gamma^{2}_{12} = \cot_{\alpha}x^{1\alpha}. \tag{30} $$

For the local fractional metric tensor Eq. (27) local fractional Ricci tensor will be

$$ \begin{aligned} R^{\alpha}_{11} &= \frac{\partial^2}{(\partial x^{1\alpha})^2}\log_{\alpha}\sqrt{g^{\alpha}} - \frac{\partial}{\partial x^{m\alpha}}\,{}^{\alpha}\Gamma^{m}_{11} + {}^{\alpha}\Gamma^{m}_{\beta 1}\,{}^{\alpha}\Gamma^{\beta}_{m1} - {}^{\alpha}\Gamma^{m}_{\beta m}\,{}^{\alpha}\Gamma^{\beta}_{11}; \\ &= -\operatorname{cosec}^2_{\alpha}x^{1\alpha} + {}^{\alpha}\Gamma^{2}_{21}\,{}^{\alpha}\Gamma^{2}_{21} = -\operatorname{cosec}^2_{\alpha}x^{1\alpha} + \cot^2_{\alpha}x^{1\alpha} = -1^{\alpha}, \end{aligned} \tag{31} $$

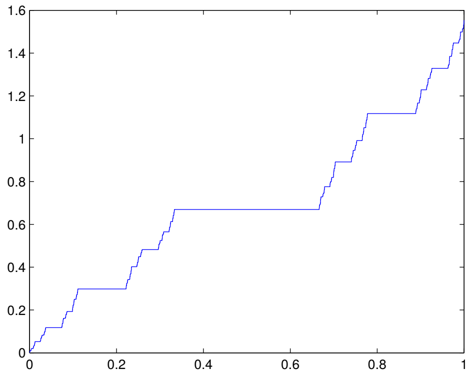

Fig. 1 – The graph of function $f(a,\alpha)=\frac{1}{k^{\alpha}}$ for the parameters $\alpha=\frac{\ln 2}{\ln 3}$, and $x\in[0,1]$.

and

$$\begin{aligned}R_{12}^{\alpha}&=\frac{\partial^2}{(\partial x^{2\alpha})^2}\log_\alpha\sqrt{g^\alpha}-\frac{\partial}{\partial x^{m\alpha}}\,{}^\alpha\Gamma_{22}^{m}+{}^\alpha\Gamma_{\beta 2}^{m}\,{}^\alpha\Gamma_{m2}^{\beta}-{}^\alpha\Gamma_{\beta m}^{m}\,{}^\alpha\Gamma_{22}^{\beta}\\&=\cos_\alpha 2x^{1\alpha}+{}^\alpha\Gamma_{22}^{1}\,{}^\alpha\Gamma_{12}^{1}+{}^\alpha\Gamma_{22}^{1}\,{}^\alpha\Gamma_{12}^{2}+{}^\alpha\Gamma_{12}^{2}\,{}^\alpha\Gamma_{22}^{1}-{}^\alpha\Gamma_{12}^{2}\,{}^\alpha\Gamma_{22}^{1}\\&=-\sin_\alpha^2 x^{1\alpha}.\end{aligned}\tag{32}$$

The local fractional scalar curvature for Eq. (27) is

$$R^\alpha={}^\alpha g^{11}R_{11}^{\alpha}+{}^\alpha g^{22}R_{22}^{\alpha}=-\frac{2\Gamma^2(\alpha+1)}{a^{2\alpha}}.\tag{33}$$

The local fractional Riemannian curvature tensor can be written as

$$\begin{aligned}{}^\alpha R_{212}^{1}&=-\frac{\partial}{\partial x^{2\alpha}}\,{}^\alpha\Gamma_{21}^{1}+\frac{\partial}{\partial x^{1\alpha}}\Gamma_{22}^{1}+{}^\alpha\Gamma_{\beta 1}^{1}\,{}^\alpha\Gamma_{22}^{\beta}-{}^\alpha\Gamma_{\beta 2}^{1}\,{}^\alpha\Gamma_{21}^{\beta}\\&=\frac{\partial}{\partial x^{1\alpha}}\Gamma_{22}^{1}-{}^\alpha\Gamma_{22}^{1}\,{}^\alpha\Gamma_{21}^{2}=\sin_\alpha^2 x^{1\alpha}\qquad {}^\alpha R_{212}^{2}=0.\end{aligned}\tag{34}$$

Then local fractional non-vanishing covariant curvature tensor is

$${}^\alpha R_{1212}=g_{1m}^{\alpha}\,{}^\alpha R_{212}^{m}=g_{11}^{\alpha}\,{}^\alpha R_{212}^{1}+g_{12}^{\alpha}\,{}^\alpha R_{212}^{2}=\frac{a^{2\alpha}\sin_\alpha^2 x^{1\alpha}}{\Gamma^2(\alpha+1)}.\tag{35}$$

Finally, local fractional Riemannian curvature k^α will be

$$k^\alpha=\frac{{}^\alpha R_{1212}}{g^\alpha}=\frac{\Gamma^4(\alpha+1)}{a^{2\alpha}}.\tag{36}$$

We have sketched the function $f(a,\alpha)=\frac{1}{k^{\alpha}}$ for the parameters $\alpha=\frac{\ln 2}{\ln 3}$, and $x\in[0,1]$ in Fig. 1.

7. FRACTIONAL EINSTEIN FIELD

In this section we define the Einstein local fractional tensor therefore we have suggested the local fractional Einstein field equation. First, let us define the local fractional covariant tensor G^{α}_{ij}, which is called Einstein tensor as following

$$G^{\alpha}_{ij} = R^{\alpha}_{ij} - \frac{1}{2} R^{\alpha} g^{\alpha}_{ij}, \tag{37}$$

where ${}^{\alpha}g^{ij} R^{\alpha}_{ij} = R^{\alpha}$ is the scalar curvature. Then we define the local fractional Einstein field equation as

$$G^{\alpha}_{ij} + g^{\alpha}_{ij} L = P T^{\alpha}_{ij}, \tag{38}$$

where L and P are local fractional space constants.

8. CONCLUSIONS

In this work we have considered the geometry of the real world as fractal. Therefore, we generalized the tensor calculus by using the local fractional derivatives. The local fractional Christoffel index symbols are suggested. We defined the divergence and curl of tensors on N-dimensional fractal space. The local fractional Riemann-Christoffel and Ricci tensors in fractal space are obtained. The Einstein field equations within the local fractional derivatives are also obtained.

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