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# FORMALISM FOR PARTIAL WAVE ANALYSIS OF THE REACTION: MESON ($0^-$) + BARYON ($1/2^+$) $\rightarrow$ MESON ($0^-$) + BARYON ($3/2^+$)*

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ABSTRACT

We outline the formalism for making a partial wave analysis of the reactions $\pi + N \rightarrow \Delta(1236) + \pi$ and $\overline{K} + N \rightarrow \Sigma(1385) + \pi$. From such an analysis the decay rates of baryon resonances into these inelastic channels can be determined.

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# I. INTRODUCTION

We give the formulae for making a partial wave analysis of the reaction sequence:

$$\pi + N \rightarrow \Delta(1236) + \pi \qquad \text{(I. 1 a)}$$
$$\quad \hookrightarrow N + \pi \qquad \text{(b)}$$

and

$$\overline{K} + N \rightarrow \Sigma(1385) + \pi \qquad \text{(I. 2 a)}$$
$$\quad \hookrightarrow \Lambda + \pi \qquad \text{(b)}$$

The decay sequence (I. 1) is described by five independent variables, which may be chosen as k, $\Theta$, $\theta$, $\phi$, $\vec{P}$, where k is the momentum in the $\pi$N c.m. system, $\cos\Theta = \hat{\pi}_{in} \cdot \hat{\pi}_{out}$ in the c.m. system, $\theta$, $\phi$ are the polar and azimuthal angles of the decay nucleon in the $\Delta$ rest frame, and $\vec{P}$ is the nucleon polarization. By "partial wave" analysis we mean that the experimental distributions are used to infer the quantities $S^J_{\ell' \, 3/2, \, \ell \, 1/2}(k)$ which are the S matrix elements connecting the initial $\pi$N state with orbital angular momentum $\ell$ and spin J to the $\Delta\pi$ state with orbital angular momentum $\ell'$ and spin J. The $S^J_{\ell' \, 3/2, \, \ell \, 1/2}$ are complex scalar quantities which are functions only of the total c.m. energy.

Partial wave analyses of reactions (I. 1) and (I. 2) can be used to determine the decay rates of N* and Y* resonances into $\Delta(1238) + \pi$ and $\Sigma(1385) + \pi$ respectively, and to check the $(IJ^P)$ values inferred from analyses of elastic scattering data. Considerable data on these reactions is being amassed in current bubble chamber experiments at incident momentum below 3 GeV/c. Partial wave analyses of the reactions $\overline{K} + N \rightarrow \Sigma(1385) + \pi$, and $K + N \rightarrow \Lambda(1520) + \pi$ have already been made for the special case that the cross section is dominated by a single amplitude.[1,2,3] These analyses led to a determination of the $(IJ^P)$ quantum numbers for $\Sigma(1770)$ and $\Lambda(1815)$, and their decay rate into $\Lambda(1520) + \pi$ and $\Sigma(1385) + \pi$ respectively.

In this paper we relate the distributions in the variables k, $\Theta$, $\theta$, $\phi$, $\vec{P}$, which can be measured experimentally to the $S^J_{\ell' \, 3/2, \, \ell \, 1/2}$ amplitudes. Previous authors have studied certain aspects of this problem, (mainly the relation between the cos $\Theta$ distribution and the $S^J_{\ell' \, 3/2, \, \ell \, 1/2}$ amplitudes), but have not enumerated a complete set of equations which utilizes all the experimental information on the production and decay of $\Delta(\Sigma)$ to determine the $S^J_{\ell' \, 3/2, \, \ell \, 1/2}$.[3,4,5]

We have used the ($\ell$SJM) representation for the S matrix, rather than the helicity representation, since we are concerned with s-channel states of definite spin and parity. All formulas are relativistically exact.

In general, the reaction $\pi N \rightarrow \pi\pi N$ is a coherent sum of such amplitudes as $A(\pi N \rightarrow \pi N^*)$, $A(\pi N \rightarrow \rho N)$, $A(\pi N \rightarrow \pi\pi N)$, etc. The formalism outlined below is applicable only to a subset of events corresponding to the reaction $\pi N \rightarrow \pi\Delta$. In most experiments it is possible to isolate a sample of these events which are relatively free from interference of other amplitudes (in some cases the complete range of variables $\Theta$, $\theta$, $\phi$ will not be accessible due to experimental cuts).

Models which take into account all possible two particle interactions in the $\pi\pi N$ final state have been worked out.[5,6] However, the application of these models poses practical difficulties because of their mathematical complexity. We believe that the formalism outlined here is a useful tool for the preliminary analysis of the three body final states (I. 1) and (I. 2). This approach has already provided information on the coupling of baryon resonances to quasi two-body channels.[1,2]

## II. THE PRODUCTION ANGULAR DISTRIBUTION

In this section the differential cross section for the reaction

$$0^- + 1/2^+ \rightarrow 3/2 + 0^- \qquad \text{(II. 1)}$$

is expressed as a function of the matrix elements $S^J_{\ell' \, 3/2, \, \ell \, 1/2}$ (Eq. (II. 8)). A detailed derivation is given in order to show the origin of the various terms in Eq. (II. 8).

We follow the treatment of two-body scattering reactions by Goldberger and Watson,[7] with their notation:

| | Entrance Channel | Exit Channel |
|---|---|---|
| | c | c' |
| Momentum in c. m.[8] | k | $k_f$ |
| Total spin | S(1/2) | S'(3/2) |
| z component of spin | $\nu$ | $\nu'$ |
| Orbital angular momentum | $\ell$ | $\ell'$ |
| z component of orbital angular momentum | m | m' |
| Total angular momentum | J | J' |
| z component of total angular momentum | M | M' |

The differential cross section in the c.m. system for the reaction, (II.1) with initial state $|k, S, \nu\rangle$ and final state $|k_f, S', \nu'\rangle$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{k}\right)^2 |\langle c'; \hat{k}_f, S', \nu' | S(k) | c; \hat{k}, S, \nu\rangle|^2 \tag{II.2}$$

$$= |\langle S', \nu' | f(\hat{k}_f, c', \hat{k}, c) | S, \nu\rangle|^2 \tag{II.2a}$$

where $\langle c'; \hat{k}_f, S', \nu' | S(k) | c; \hat{k}, S, \nu\rangle$ is the unitary S matrix element in the barycentric subspace on the energy and momentum shell; Eq. (II.2a) defines the scattering amplitude f for the reaction. For an unpolarized target the average cross section for any final spin orientation is obtained by averaging over the initial spin orientations $\nu$ and summing over the final spin orientations $\nu'$:

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2S+1} \sum_{\nu',\nu} \frac{d\sigma}{d\Omega} (c'; \hat{k}_f, S', \nu'; c; \hat{k}, S, \nu)$$

$$= \frac{1}{2S+1} \mathrm{Tr}\,(f f^+) \tag{II.3}$$

The S matrix elements are used throughout rather than the T matrix elements, because there is no unique convention for the normalization of the T matrix. If the T matrix elements are defined by:

$$\langle c'; \hat{k}_f, S' \nu' | S(k) | c; \hat{k}, S, \nu\rangle = \delta_{c',c}\, \delta_{S',S}\, \delta_{\nu',\nu}\, \delta_{\hat{k}_f, k} + i \langle c'; \hat{k}_f, S', \nu' | T(k) | c; \hat{k}, S, \nu;\rangle$$

S may be replaced everywhere by iT, since we are dealing with an inelastic reaction. We will omit the channel suffix c from now on.

The S matrix element in (II.2) is transformed to the ($\ell$SJM) representation, using the transformation matrix:

$$\langle \hat{k}\ S,\nu | \ell m S\nu \rangle = Y_\ell^m(\hat{k}) \tag{II.4}$$

which resolves plane wave states into partial waves, and with the Clebsch-Gordon coefficients $\langle \ell, S, m, \nu | J, M \rangle$ which resolve states $|\ell m\, S\nu\rangle$ into states of total angular momentum $|J\, M\rangle$

$$\langle \hat{k}_f, S', \nu' | S(k) | \hat{k}, S, \nu \rangle = \sum_{\substack{\ell' m' J' M' \\ \ell m J M}} \langle \hat{k}_f, S'\ \nu' | \ell'\ m'\ S'\ \nu' \rangle \langle \ell', S'; m', \nu' | J', M' \rangle$$

$$\times \langle \ell'\ S'\ J'\ M' | S(k) | \ell\ S\ J\ M \rangle \langle \ell, S; m, \nu | J, M \rangle \langle \ell\ m\ S\ \nu\ | \hat{k}, S\ \nu \rangle \tag{II.5}$$

From rotational invariance

$$\langle \ell'\ S'\ J'\ M' | S(k) | \ell\ S\ J\ M \rangle = \delta_{J,J'}\, \delta_{M,M'}\, S^J_{\ell'\ S', \ell\ S}(k) \tag{II.6}$$

where $S^J_{\ell' S', \ell S}$ is the S matrix element in the ($\ell$SJM) representation.

From Eqs. (II.4), (II.5) and (II.6) we have

$$\langle \hat{k}_f, S', \nu' | S(k) | k, S, \nu \rangle = \sum_{\ell' m' \ell M J M} Y^{m'}_{\ell'}(\hat{k}_f) \langle \ell', S'; m', \nu' | J, M \rangle S^J_{\ell' S', \ell S}(k)$$

$$\times \langle \ell, S; m, \nu | J, M \rangle Y^{*m}_{\ell}(\hat{k})$$

$$= \sum_{\ell' \ell J} Y^{\nu-\nu'}_{\ell'}(\cos\Theta) \langle \ell', 3/2; \nu-\nu', \nu' | J, \nu \rangle S^J_{\ell' 3/2, \ell 1/2} \langle \ell, 1/2; 0, \nu | J, \nu \rangle \sqrt{\frac{2\ell+1}{4\pi}}$$

(II. 7)

specifying the incident beam direction $\hat{k}$ as axis of quantization,

i. e., $m = 0, M = \nu, Y^0_\ell = \sqrt{\frac{2\ell+1}{4\pi}}$ and $\cos\Theta = \hat{k} \cdot \hat{k}_f$.

Inserting (II. 7) into (II. 3) gives

$$\frac{d\bar{\sigma}}{d\Omega}=\frac{\pi}{k^2}\sum_{\nu'}\left|\sum_{\ell' J}(J+1/2)^{1/2}(-)^{\ell'-(J-1/2)}\langle\ell',3/2;1/2-\nu',\nu'|J,1/2\rangle S^J_{\ell' 3/2,\ell 1/2}(k)\,Y^{1/2-\nu'}_{\ell'}(\cos\Theta)\right|^2 \qquad \text{(II. 8)}$$

where we have dropped the summation over $\nu = 1/2, -1/2$ since, for the chosen axis of quantization, $d\sigma(\nu',\nu) = d\sigma(-\nu',-\nu)$. The summation over $\ell$ is superfluous because $\ell$ is uniquely determined by J and $\ell'$ and $\langle\ell, 1/2;0,1/2 \mid J,1/2\rangle =$

$$(-)^{\ell'-(J-1/2)}\left(\frac{J+1/2}{2\ell+1}\right)^{1/2}$$

If the relative intrinsic parities of the initial and final state particles are odd, then

$$\langle\ell,1/2;0,1/2 \mid J,1/2\rangle = (-)^{\ell'-(J+1/2)}\left(\frac{J+1/2}{2\ell+1}\right)^{1/2}$$

The total cross section is:

$$\bar{\sigma} = \sum_J \pi\lambda\!\!\!^{-2}(J+1/2)\sum_{\ell'}\left|S^J_{\ell' 3/2,\ell 1/2}(k)\right|^2 \qquad \text{(II. 9)}$$

The maximum value of $S^J_{\ell' 3/2,\ell 1/2}$ is unity, so that the maximum inelastic cross section for a single partial wave amplitude is $\pi\lambda\!\!\!^{-2}(J+1/2)$. More generally

$$\bar{\sigma} = \sum_J \pi\lambda\!\!\!^{-2}(J+1/2)\sum_{\ell'}\left|S^J_{\ell' 3/2,\ell 1/2}(k) - \delta_{c',c}\,\delta_{\ell',\ell}\,\delta_{S',S}\right|^2$$

and the maximum $\bar{\sigma}$ in the elastic channel for a single amplitude is therefore $4\pi\lambda\!\!\!^{-2}(J+1/2)$.

The S matrix element in the proceeding equations is in general a linear combination of two isotopic spin amplitudes. Denoting the isotopic spin of the initial state meson and baryon by I and T respectively we have for reactions (I. 1):

$$S = \langle I, T; I_3, T_3 | 3/2, I_3 + T_3 \rangle S_{3/2} + \langle I, T; I_3 T_3 | 1/2, I_3 + T_3 \rangle S_{1/2}$$

and for reactions (I. 2)

$$S = \langle I, T; I_3, T_3 | 1, I_3 + T_3 \rangle S_1 + \langle I, T; I_3 T_3 | 0, I_3 + T_3 \rangle S_0$$

The R. H. S. of Eq. (II. 8) can be written as an expansion in the Legendre polynomials $P_n(\cos\Theta)$

$$\frac{d\bar{\sigma}}{d\Omega} = \lambda^2 \sum_n A_n P_n (\cos\Theta) \qquad \text{(II. 10)}$$

where

$$A_n = \frac{(2n+1)}{4\pi\lambda^2} \int \frac{d\bar{\sigma}}{d\Omega} P_n (\cos\Theta)\, d\Omega$$

$$= \sum_{\ell', J} \sum_{\geq \lambda, J'} \mathrm{Re}(S^J_{\ell'\ell}\, S^{J'*}_{\lambda'\lambda})\, B^n_{\ell\ell' J;\, \lambda\lambda' J'} \qquad \text{(II. 11)}$$

The coefficients $B^n$, evaluated by inserting (II. 8) in (II. 11) are listed in Table I.

The scattering angle $\cos\Theta = \hat{k}\cdot\hat{k}_f$ is not uniquely defined. In elastic scattering the convention is that $\hat{k}$ and $\hat{k}_f$ refer to the same particle. Then the amplitude $S^J$ always lies in the upper half of the complex plane (Im $S > 0$). In inelastic scattering such as $\pi + N \to K + \Lambda$, where the outgoing baryon and boson belong to the same SU3 octets as the initial state particles, the same convention is maintained by invoking SU symmetry. However, the amplitude $S^J$ can now lie anywhere in the complex plane because of the sign of the SU3 Clebsch-Gordon coefficients.

In the reaction $\pi + N \rightarrow \pi + \Delta$, in which N and $\Delta$ belong to different SU3 multiplets, a higher symmetry is needed to make a correspondence between N and $\Delta$. Instead we make the simple convention that $\hat{k}$ and $\hat{k}_f$ are the directions of the initial and final state bosons. The formalism in Sect. II is indbpendent of the definition of scattering angle. If $\hat{k}_f$ is replaced by $(-\hat{k}_f)$, then $S^J_{\ell'\ell}$ goes to $(-)^{\ell'} S^J_{\ell'\ell}$.

## III. THE DECAY ANGULAR DISTRIBTUION

For clarity we specify reaction(I. 1) Notation

$\theta$, $\phi$ Polar and azimuthal decay angles in the $\Delta$ rest frame.

$\rho$ Spin density matrix of $\Delta$

$f_D$ Amplitude for the decay I. 1b

The differential decay distribution $W(\theta, \phi)$ can be expressed as a function of the partial wave amplitudes. This is most conveniently done through the density matrix formalism:

$$\rho = \frac{f f^+}{Tr(f f^+)} \tag{III. 1}$$

where f is the scattering amplitude for reaction(I. 1a), defined in Eq. (II. 2a). Here $\rho$ is the $\pi\Delta$ spin state density matrix in the overall center-of-mass. However since the pion has zero spin $\rho$ is simply the $\Delta$ density matrix in the same system. The spin state of a particle is invariant under a transformation from the rest frame of the particle to a moving frame,[9] hence $\rho$ is also the density matrix of $\Delta$ in its rest frame. Then

$$W(\theta, \phi) = T_r(f \rho f^+) = \frac{Tr\left(f_D f f^+ f_D^+\right)}{Tr\ (f f^+)} \tag{III. 2}$$

The decay amplitude for the p-wave decay $\Delta \rightarrow N + \pi$ is given by Eq. (II. 7):

$$\langle 1/2\nu' | f_D | 3/2\nu \rangle \propto Y_1^{\nu-\nu'}(\cos\Theta, \phi) \langle 1, 1/2; \nu - \nu', \nu' | 3/2\nu \rangle \tag{III. 3}$$

We ignore the energy dependent part of the amplitude, since it does not affect the angular distributions. With this definition of $f_D$ the decay distribution (III. 2) is normalized to unity.

The decay angles $\theta$, $\phi$ refer to the coordinate frame in which the Z-axis is the axis of quantization. The amplitude f has been calculated in section II for $Z = \hat{k}$, the c.m. incident beam direction:

$$\langle 3/2\nu' | f | 1/2\nu \rangle = \frac{\sqrt{\pi}}{k} \sum_{\ell' J} (J+1/2)^{1/2} (-)^{\ell' - (J-1/2)} Y_{\ell'}^{\nu-\nu'}(\cos\Theta) \langle \ell' 3/2; \nu-\nu', \nu' | J\nu \rangle S^J_{\ell' 3/2, \ell 1/2}$$

(III. 4)

Defining a coordinate system $Z \equiv \hat{k}$, $y = \hat{k} \times \hat{k}_f$, Fig. 1a, the term $Y_{\ell'}^{\nu-\nu'}(\cos\Theta)$ above becomes $\sqrt{1/4\pi}\ \sqrt{(2\ell'+1)(\ell'-|\nu-\nu'|)!/(\ell'+|\nu-\nu'|)!}\ P_{\ell'}^{\nu-\nu'}(\cos\Theta)$ since the production azimuthal angle is always zero in this frame. This form of f, inserted in Eq. (III. 2), gives the decay angular distribution in the coordinate system of Fig. 1a.

The decay distribution in another frame of reference is obtained by rotation of the axis of quantization of the density matrix $\rho$. The rotation matrices which take the quantization axis from the beam direction to the helicity direction or to the production normal are given in the appendix.

The form of the decay distribution depends on the choice of quantization axis. For axis of quantization in the production plane (Fig. 1a or 1b) the general form for the density matrix is[10]

$$\begin{vmatrix} \rho_{33} & \rho_{31} & \rho_{3-1} & \rho_{3-3} \\ \rho^*_{31} & \rho_{11} & \rho_{1-1} & \rho^*_{3-1} \\ \rho^*_{3-1} & -\rho_{1-1} & \rho_{11} & -\rho^*_{31} \\ -\rho_{3-3} & \rho_{3-1} & -\rho_{31} & \rho_{33} \end{vmatrix} \qquad \text{(III. 5)}$$

Hermiticity requires that all diagonal elements are real and that $\rho_{3-3}$ and $\rho_{1-1}$ are purely imaginary. For a single amplitude all elements of the density matrix are real.

From Eqs. (III.2), (III.3), (III.5) the decay distribution for the z-axis in the production plane is

$$W(\theta\phi) = \frac{3}{4\pi}\Big[1/6 + 2/3\ \rho_{33} + (1/2 - 2\rho_{33})\cos^2\theta - \frac{2}{\sqrt{3}}\operatorname{Re}\rho_{3-1}\sin^2\theta\cos 2\phi - \frac{2}{\sqrt{3}}\operatorname{Re}\rho_{31}\sin 2\theta\cos\phi\Big] \tag{III.6}$$

For axis of quantization along the production normal $\hat{k} \times \hat{k}_f$ the general form for the density matrix is

$$\begin{vmatrix} \rho_{33} & 0 & \rho_{3-1} & 0 \\ 0 & \rho_{11} & 0 & \rho_{1-3} \\ \rho^{*}_{3-1} & 0 & \rho_{-1-1} & 0 \\ 0 & \rho^{*}_{1-3} & 0 & \rho_{-3-3} \end{vmatrix}$$

and the corresponding decay distribution is

$$W^N(\theta,\phi) = \frac{1}{8\pi}\Big[3\sin^2\theta + 2\left(\rho^N_{11} + \rho^N_{-1-1}\right)\left(2 - 3\sin^2\theta\right) - 2\sqrt{3}\sin^2\theta\left(\operatorname{Re}\left(\rho^N_{3-1} + \rho^N_{1-3}\right)\cos 2\phi - \operatorname{Im}\left(\rho^N_{3-1} + \rho^N_{1-3}\right)\sin 2\phi\right)\Big] \tag{III.7}$$

## IV. POLARIZATION OF THE SPIN 1/2 BARYON

Whereas Δ can be polarized only along the production normal the spin 1/2 baryon can be polarized in any direction. The density matrix of a spin 1/2 particle in its rest system can be written

$$\rho_{1/2} = 1/2 \begin{vmatrix} 1 + P_z , & P_x - i P_y \\ P_x + i P_y & 1 - P_z \end{vmatrix} \tag{IV.1}$$

where $P_x$, $P_y$ and $P_z$ are the three components of the polarization vector. The polarization can be expressed as a function of the partial wave amplitudes through the relation:

$$\rho_{1/2} = \left| \frac{f_D \, \rho \, f_D^{+}}{\mathrm{Tr}\left(f_D \, \rho \, f_D^{+}\right)} \right| = \frac{f_D \, \rho \, f_D^{+}}{W(\theta, \phi)} \tag{IV.2}$$

where $\rho$ is the density matrix of the Δ. Equation (IV.2) defines $\rho_{1/2}$ in the Δ rest frame with the same axis of quantization as $\rho$. The density matrix $\rho_{1/2}$ is unchanged by a transformation from the Δ system to the nucleon rest system. [When the reaction goes by a single partial wave amplitude both the Δ and the nucleon are unpolarized.]

## V. ANALYSIS OF EXPERIMENTAL DATA

The experimental data at a given momentum consists of a joint distruibtion in three independent variables, which may be $\Theta, \theta$ and $\phi$ as defined in Sect. I. (We neglect polarization which is usually not measured in reaction I.1). The distribuiton $I(\Theta, \theta, \phi)$ is related to the partial wave amplitudes $S^J_{\ell' 3/2, \ell 1/2}$ by:

$$I(\Theta, \theta, \phi) = \frac{d\bar{\sigma}(\Theta)}{d\Omega} \cdot W(\Theta, \theta, \phi) \tag{V.1}$$

There are two possible forms for $W(\Theta, \theta, \phi)$—Eqs. (III.6) and (III.7) depending on the coordinate frame in which $\theta, \phi$ are measured. In the coordinate frame $Z \equiv \hat{k}$, $y = \hat{k} \times \hat{k}_f$ (Fig. 1a), $\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2}$ Tr $(f f^+)$ and $W(\Theta, \theta, \phi) = \mathrm{Tr}\,(f_D f f^+ f_D{}^+)/(2\, d\bar{\sigma}/d\Omega)$ where f and $f_D$ are given by Eqs. (III.4) and (III.3) respectively. Note that in this case the Z axis is the incident beam direction in c.m. system (not in the $\Delta$ rest system).

For a given set of partial wave amplitudes, Eq. (V.1) predicts the distribution $I(\Theta, \theta, \phi)$. In general, the set of amplitudes which best fit the data is found by computer search, using the method of $\chi^2$ minimization or maximum likelihood.

The decay distribution of $\Delta$ or $\Sigma$ for a given production angle $\Theta$ is completely described by three parameters which are functions of the density matrix elements - $\rho_{33}(\Theta)$, Re $\rho_{3-1}(\Theta)$ and Re $\rho_{31}(\Theta)$ for the axis of quantization in the production plane, or $\left(\rho^N_{11}(\Theta) + \rho^N_{-1-1}(\Theta)\right)$, Re$\left(\rho^N_{31}(\Theta) + \rho^N_{1-3}(\Theta)\right)$ and Im$\left(\rho^N_{3-1}(\Theta) + \rho^N_{1-3}(\Theta)\right)$ for the quantization axis along the production normal. The experimental data at a single momentum can therefore be summarized in the form of four distributions in cos $(\Theta)$, —the production angle. These distributions are shown in Figs. 2,3 for the partial wave amplitudes DD5, FF7, and GD7 for the coordinate systems defined in Fig. 1a ($\rho^B$) and Fig. 1c ($\rho^N$). The correlations between the production and decay angles of the $\Delta$ are clearly sensitive to the spin and parity of the partial wave amplitudes.

The experimental density matrix elements are statistically correlated. This correlation must be taken into account if the comaprison between the experimental and calculated distributions is made in terms of density matrix elements.

The experimental data may be insufficient to determine all the correlations between $\Theta, \theta$ and $\phi$. In that case the question of how best to bin the data arises. Also the choice of coordinate frame in which the decay angles are measured may be important. There is no simple prescription - but a study of the density matrix elements for the hypothesis being tested will usually indicate the best procedure. For example, if one is trying to distinguish between the amplitudes FF7 and GD7 the correlations between $\Theta$ and $\theta^N$ or between $\Theta$ and $\phi^B$ are clearly very sensitive, as indicated by the plots of $\rho_{11}^N$ and Re $\rho_{3-1}^B$ in Fig. 2. These correlations are shown graphically in Figs. 4, 5, 6, and 7.

APPENDIX

## Rotation of the Axis of Quantization

If $\rho$ denotes the density matrix for axis of quantization z and $\rho'$ the same state for axis of quantization z', $\rho$ and $\rho'$ are related by the unitary transformation:

$$\rho' = R\,\rho\,R^{-1}$$

The change of axis of quantization is simply a change of basis states from $|JM\rangle$ to $|JM'\rangle$ where M' is an eigenvalue of $J_{z'}$.

Below we give the transformation matrixes $R_H$ and $R_N$ corresponding to a rotation of the axis of quantization from the incident beam direction (Fig. 1a) to (1) spin 3/2 particle direction (helicity direction) (Fig. 1b) and to (2) the production normal (Fig. 1c) respectively.

(1) Rotation to the helicity direction.

In the right-handed (x,y,z) coordinate frame $z = \hat{k}$ and $y = \hat{k} \times \hat{k}_f$. (Fig. 1a.) The Euler angles for the rotation which takes z into the helicity direction are $\alpha = 0$, $\beta = \Theta_H$ and $\gamma = 0$.[11] ($\Theta_H = 180^\circ - \Theta$.) The corresponding rotation matrix $R_H$ is:

$$\left|\begin{matrix}
\cos\frac{3\Theta_H}{2} + 3\cos\frac{\Theta_H}{2} & \sqrt{3}(\sin\frac{3\Theta_H}{2} + \sin\frac{\Theta_H}{2}) & \sqrt{3}(-\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2}) & -\sin\frac{3\Theta_H}{2} + 3\sin\frac{\Theta_H}{2} \\
-\sqrt{3}(\sin\frac{3\Theta_H}{2} + \sin\frac{\Theta_H}{2}) & 3\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2} & 3\sin\frac{3\Theta_H}{2} - \sin\frac{\Theta_H}{2} & \sqrt{3}(-\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2}) \\
\sqrt{3}(-\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2}) & -3\sin\frac{3\Theta_H}{2} + \sin\frac{\Theta_H}{2} & 3\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2} & \sqrt{3}(\sin\frac{3\Theta_H}{2} + \sin\frac{\Theta_H}{2}) \\
\sin\frac{3\Theta_H}{2} - 3\sin\frac{\Theta_H}{2} & \sqrt{3}(-\cos\frac{3\Theta_H}{2} + \cos\frac{\Theta_H}{2}) & -\sqrt{3}(\sin\frac{3\Theta_H}{2} + \sin\frac{\Theta_H}{2}) & \cos\frac{3\Theta_H}{2} + 3\cos\frac{\Theta_H}{2}
\end{matrix}\right|$$

(2) Rotation to production normal (see Fig. 1c).

The Euler angles which take (x,y,z) into (x',y',z') are $\alpha = 90^o$, $\beta = 90^o$ and $\gamma = 0$. The density matrix with axis of quantization z' is $R_N \rho R_N^{-1}$, where

$$R_N = \frac{1}{\sqrt{8}} \begin{vmatrix} e^{i3\pi/4} & \sqrt{3}\, e^{i\pi/4} & \sqrt{3}\, e^{-i\pi/4} & e^{-i3\pi/4} \\ -\sqrt{3}\, e^{i3\pi/4} & -\, e^{i\pi/4} & e^{-i\pi/4} & \sqrt{3}\, e^{-i3\pi/4} \\ \sqrt{3}\, e^{i3\pi/4} & -\, e^{i\pi/4} & -\, e^{-i\pi/4} & \sqrt{3}\, e^{-i3\pi/4} \\ -\, e^{i3\pi/4} & \sqrt{3}\, e^{-i\pi/4} & -\sqrt{3}\, e^{-i\pi/4} & e^{-i3\pi/4} \end{vmatrix}$$

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8. We use a system of units in which $\hbar = 1$, $K = 1/\lambda$.
9. G. C. Wick, "Group Theory, Invariance Principles, Symmetries", in High Engegy Physics, C. M. Dewitt and M. Jacob, eds., (Gordon and Breach N. Y. 1965).
10. See for example, N. Schmitz, Proceedings of the 1965 CERN Easter School Bad Kreuznach, April 1965, CERN-65-24.
11. The Euler angles are as defined in Rose's "Elementary Theory of Angular Momentum":[12] A rotation $\alpha$ about the original z-axis, followed by a rotation $\beta$ about the new y-axis, followed by a rotation $\gamma$ about the new z-axis. The rotation is performed in the positive sense in a right-handed coordinate system.
12. M. E. Rose, Elementary Theory of Angular Momentum, (John Wiley and Sons Inc., N. Y. 1957).

## LISTING OF TABLES

I, II and III:

The coefficients $B^{n}_{\ell\ell' J;\lambda\lambda' J'}$ defined by Eqs. (II.10) and (II.11) are given for all possible combinations of $\ell\ell'$, J, $\lambda$ $\lambda'$ and J' up to spin 7/2.

## FIGURE CAPTIONS

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1. Coordinate frames for decay angle of $\Delta$. The vectors $\hat{k}$ and $\hat{k}_f$ are the c.m. incident and final firm directions in the reaction $\pi + N \rightarrow \Delta + \pi$.

   a. incident beam direction in c.m. is Z axis, production normal is y axis.

   b. axis Z' is $-\hat{k}_f$ (see appendix).

   c. axis Z'' is production normal $\hat{k} \times \hat{k}_f$ (see appendix).

2. Production angular distributions for FF7, GD7 and DD5 amplitudes.

3. The decay parameters defined in Sect. V are shown as a function of the $\Delta$ production angle $\Theta$ for quantization axis of the density matrix as (a) beam direction and (b) normal to the production plane. For a single amplitude, as shown here, $\rho^N_{3-1} = \rho^N_{1-3}$.

4. Correlation between the FF7 production angular distribution and the normalized decay distribution $W(\theta^N, \phi^N)$ integrated over $\phi^N$. (Coordinate system as in Fig. 1c.)

5. Correlation between the FF7 production angular distribution and the normalized $W(\theta^B, \phi^B)$ integrated over $\cos\theta^B$. (Coordinate system as in Fig. 1a.)

6. Same as Fig. 4 but for the GD7 amplitudes.

7. Same as Fig. 5 but for the GD7 amplitudes.

TABLE I

| $\ell\ell'2J^{*}\lambda\lambda'2J'$ | $B^0$ | $B^1$ | $B^2$ | $B^3$ | $B^4$ | $B^5$ | $B^6$ | $B^7$ |
|---|---|---|---|---|---|---|---|---|
| SD1*SD1 | 0.25 | | | | | | | |
| PP1 | | 0.5 | | | | | | |
| PP3 | | -0.316 | | | | | | |
| PF3 | | 0.948 | | | | | | |
| DS3 | | | -0.707 | | | | | |
| DD3 | | | 0.707 | | | | | |
| DD5 | | | -0.567 | | | | | |
| DG5 | | | 1.388 | | | | | |
| FP5 | | | | -1.161 | | | | |
| FF5 | | | | 0.949 | | | | |
| FF7 | | | | -0.842 | | | | |
| FH7 | | | | 1.825 | | | | |
| GD7 | | | | | -1.604 | | | |
| GG7 | | | | | 1.195 | | | |
| PP1*PP1 | 0.25 | | | | | | | |
| PP3 | | | -0.316 | | | | | |
| PF3 | | | 0.949 | | | | | |
| DS3 | | -0.707 | | | | | | |
| DD3 | | 0.707 | | | | | | |
| DD5 | | | | -0.567 | | | | |
| DG5 | | | | 1.388 | | | | |
| FP5 | | | -1.161 | | | | | |
| FF5 | | | 0.949 | | | | | |
| FF7 | | | | | -0.842 | | | |
| FH7 | | | | | 1.825 | | | |
| GD7 | | | | -1.604 | | | | |
| GG7 | | | | 1.195 | | | | |
| PP3*PP3 | 0.5 | | -0.400 | | | | | |
| PF3 | | | -0.600 | | | | | |
| DS3 | | 0.447 | | | | | | |
| DD3 | | 0.358 | | -0.805 | | | | |
| DD5 | | 1.506 | | -1.147 | | | | |
| DG5 | | | | -0.878 | | | | |
| FP5 | | | 0.735 | | | | | |
| FF5 | | | 0.686 | | -1.286 | | | |
| FF7 | | | 1.992 | | -1.476 | | | |
| FH7 | | | | | -1.155 | | | |
| GD7 | | | | 1.014 | | | | |
| GG7 | | | | 1.007 | | -1.764 | | |

TABLE II

| $\ell\ell'2J*\lambda\lambda'2J'$ | $B^0$ | $B^1$ | $B^2$ | $B^3$ | $B^4$ | $B^5$ | $B^6$ | $B^7$ |
|---|---|---|---|---|---|---|---|---|
| PF3*PF3 | 0.5 | | 0.4 | | | | | |
| DS3 | | | | -1.342 | | | | |
| DD3 | | 0.268 | | 1.073 | | | | |
| DD5 | | -0.215 | | -0.861 | | | | |
| DG5 | | 1.757 | | 0.878 | | | | |
| FP5 | | | -0.105 | | -2.10 | | | |
| FF5 | | | 0.515 | | 1.286 | | | |
| FF7 | | | -0.443 | | -1.106 | | | |
| FH7 | | | 2.474 | | 0.990 | | | |
| GD7 | | | | -0.226 | | -2.817 | | |
| GG7 | | | | 0.756 | | 1.512 | | |
| DS3*DS3 | 0.5 | | | | | | | |
| DD3 | | | -1.0 | | | | | |
| DD5 | | | 0.802 | | | | | |
| DG5 | | | | | -1.964 | | | |
| FP5 | | 1.643 | | | | | | |
| FF5 | | | | -1.342 | | | | |
| FF7 | | | | 1.155 | | | | |
| FH7 | | | | | | -2.582 | | |
| GD7 | | | 2.268 | | | | | |
| GG7 | | | | | 1.690 | | | |
| DD3*DD3 | 0.5 | | | | | | | |
| DD5 | | | 0.572 | | -1.375 | | | |
| DG5 | | | 0.561 | | 1.403 | | | |
| FP5 | | -0.329 | | -1.315 | | | | |
| FF5 | | 1.610 | | -0.268 | | | | |
| FF7 | | | | 0.770 | | -1.925 | | |
| FH7 | | | | 0.861 | | 1.721 | | |
| GD7 | | | -0.648 | | -1.620 | | | |
| GG7 | | | 2.173 | | -0.483 | | | |
| DD5*DD5 | 0.75 | | 0.306 | | -0.735 | | | |
| DG5 | | | -0.450 | | -1.125 | | | |
| FP5 | | 0.263 | | 1.054 | | | | |
| FF5 | | 0.246 | | 0.813 | | -2.134 | | |
| FF7 | | 2.381 | | 0.308 | | -1.763 | | |
| FH7 | | | | -0.690 | | -1.380 | | |
| GD7 | | | 0.519 | | 1.299 | | | |
| GG7 | | | 0.516 | | 1.003 | | 2.875 | |

TABLE III

| $\ell\ell'2J^*\lambda\ \lambda'2J'$ | $B^0$ | $B^1$ | $B^2$ | $B^3$ | $B^4$ | $B^5$ | $B^6$ | $B^7$ |
|---|---|---|---|---|---|---|---|---|
| DG5*DG5 | 0.750 | | 0.765 | | 0.413 | | | |
| FP5 | | | | -0.239 | | -2.988 | | |
| FF5 | | 0.188 | | 0.878 | | 1.568 | | |
| FF7 | | -0.162 | | -0.756 | | -1.350 | | |
| FH7 | | 2.535 | | 1.6905 | | 0.846 | | |
| GD7 | | | -0.035 | | -0.482 | | -3.936 | |
| GG7 | | | 0.395 | | 1.164 | | 1.7605 | |
| FP5*FP5 | 0.750 | | 0.600 | | | | | |
| FF5 | | | -0.630 | | -1.575 | | | |
| FF7 | | | 0.542 | | 1.355 | | | |
| FH7 | | | | | -0.385 | | -3.857 | |
| GD7 | | 2.485 | | 1.242 | | | | |
| GG7 | | | | -0.926 | | -1.852 | | |
| FF5*FF5 | 0.750 | | 0.472 | | -0.322 | | | |
| FF7 | | | 0.480 | | 0.905 | | -2.934 | |
| FH7 | | | 0.412 | | 1.215 | | 1.837 | |
| GD7 | | -0.217 | | -1.014 | | -1.811 | | |
| GG7 | | 2.430 | | 0.756 | | -0.907 | | |
| FF7*FF7 | 1.000 | | 0.794 | | -0.117 | | -1.010 | |
| FH7 | | | -0.355 | | -1.045 | | -1.581 | |
| GD7 | | 0.187 | | 0.873 | | 1.559 | | |
| GG7 | | 0.186 | | 0.798 | | 0.965 | | -3.901 |
| FH7*FH7 | 1.000 | | 1.111 | | 0.818 | | 0.404 | |
| GD7 | | | | -0.089 | | -0.751 | | -5.016 |
| GG7 | | | | 0.714 | | 1.511 | | 1.994 |
| GD7*GD7 | 1.000 | | 1.021 | | 0.551 | | | |
| GG7 | | | -0.456 | | -1.344 | | -2.033 | |
| GG7*GG7 | 1.000 | | 0.884 | | 0.150 | | -0.606 | |

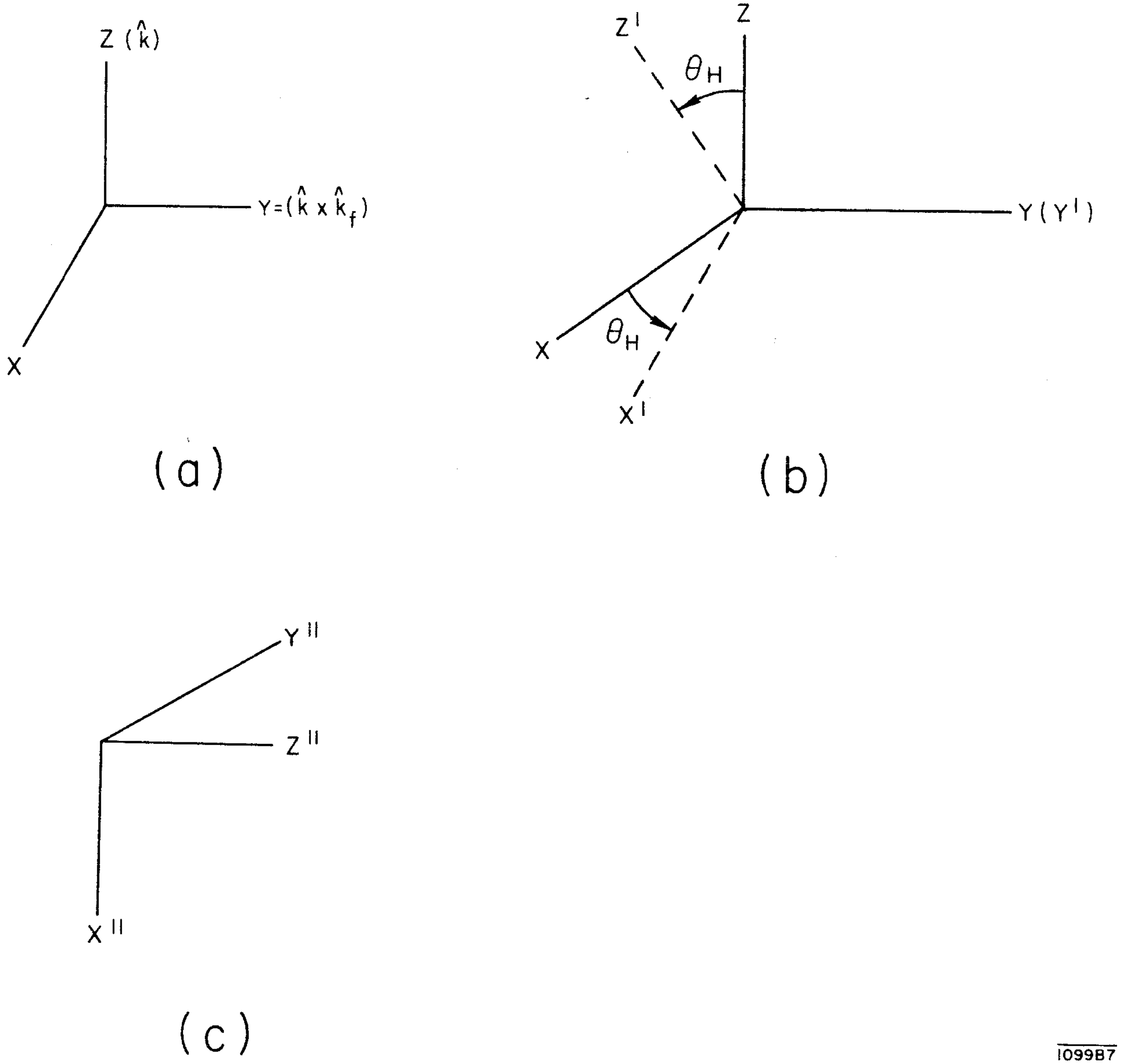



Fig. 1

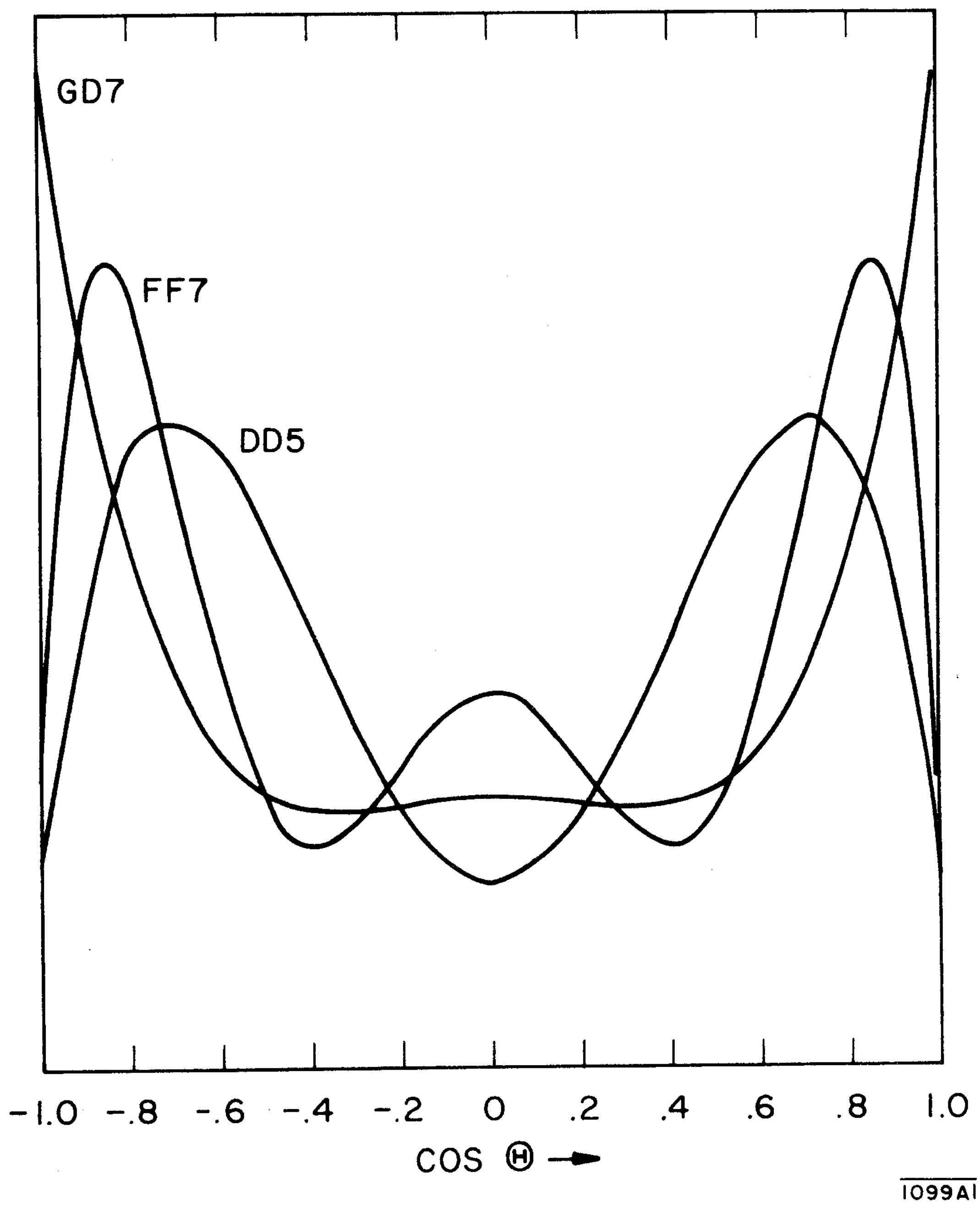
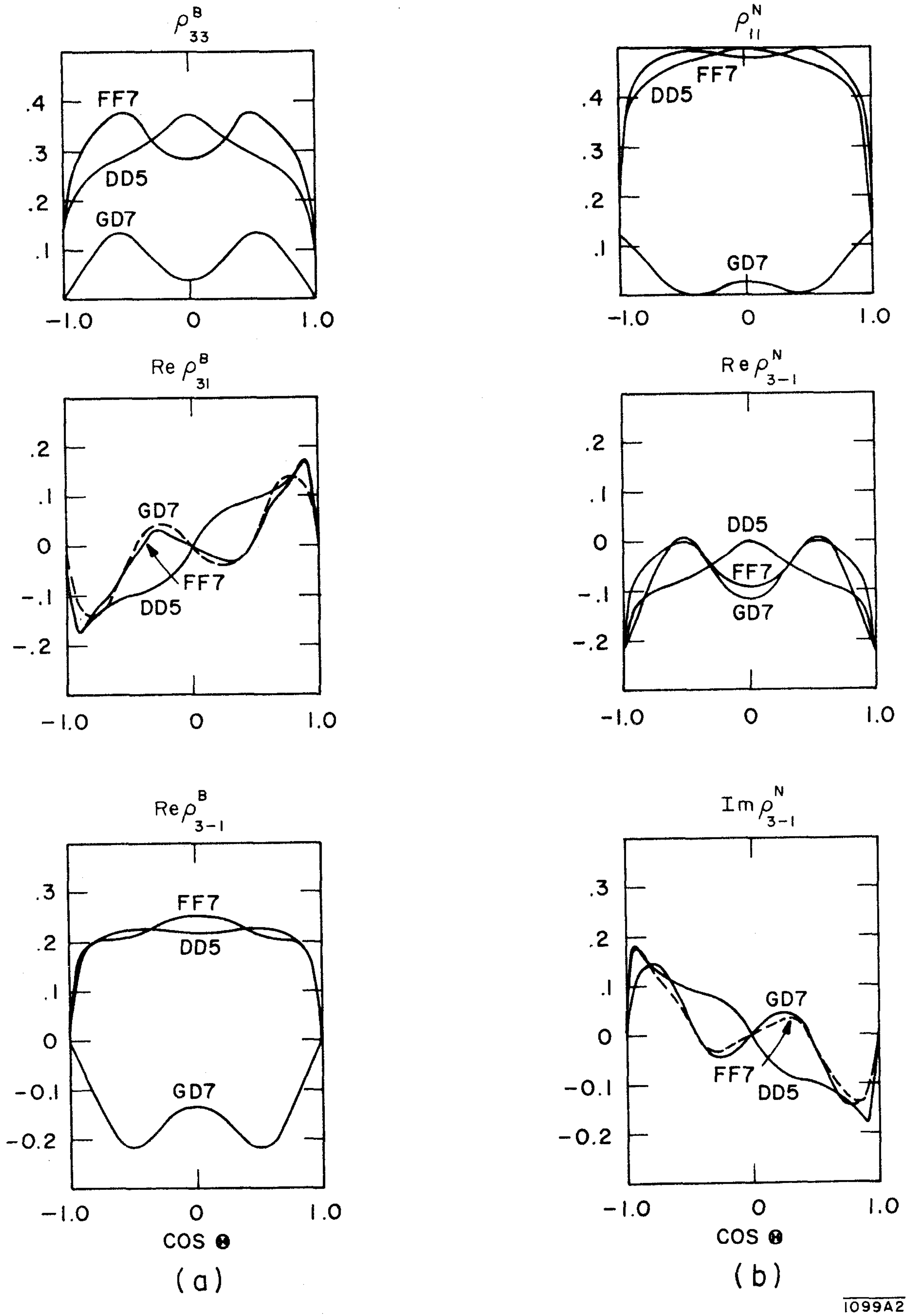



Fig. 2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DD5
GD7
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.3
.2
.1
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GD7
Re ρ^B_31
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FF7
DD5
Re ρ^N_3−1
DD5
FF7
GD7
Re ρ^B_3−1
FF7
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Im ρ^N_3−1
GD7
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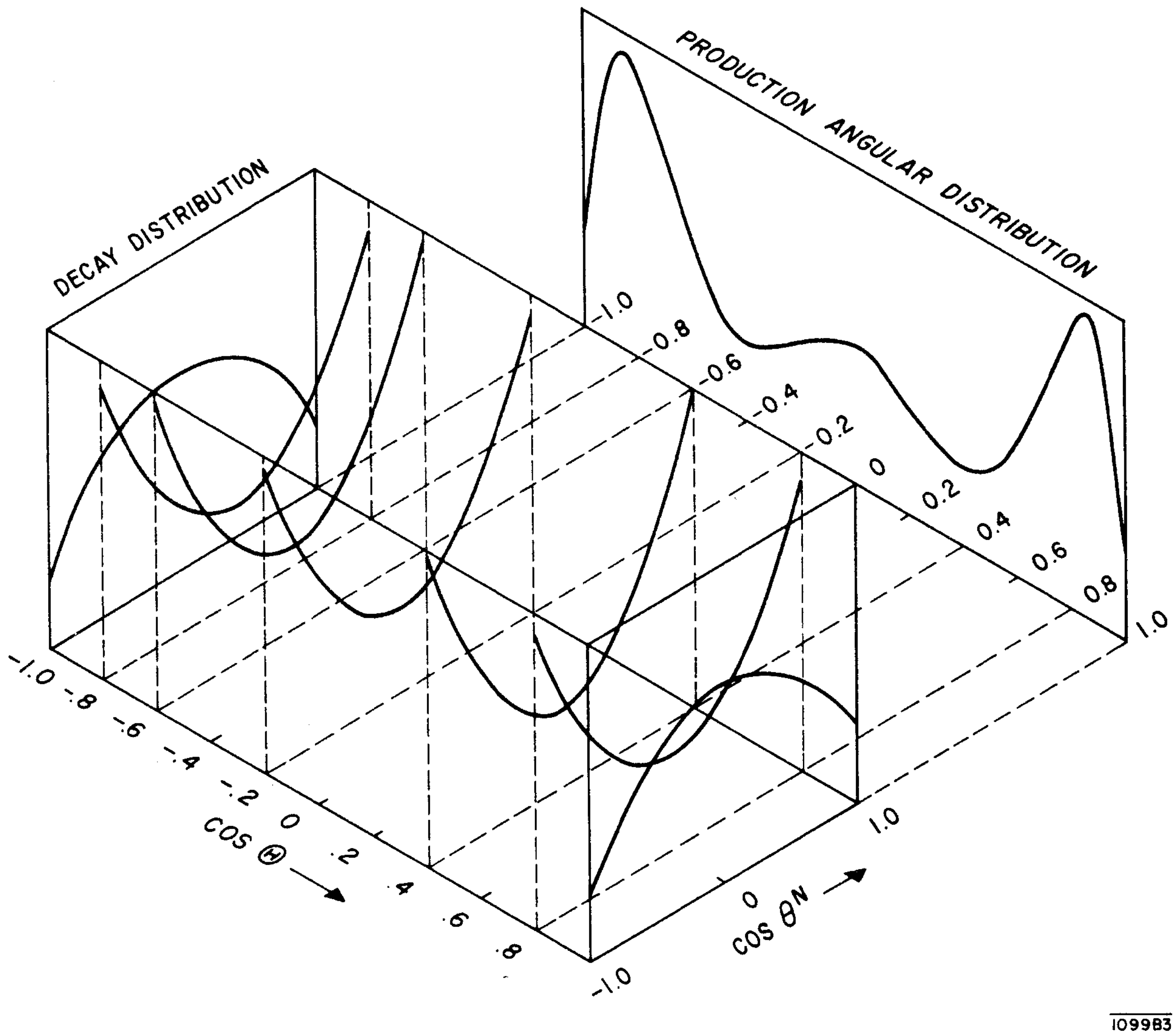
COS Θ
(a)
(b)
1099A2

Fig. 3

FF7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-1.0 0 1.0
COS θN
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Fig. 4

FF7

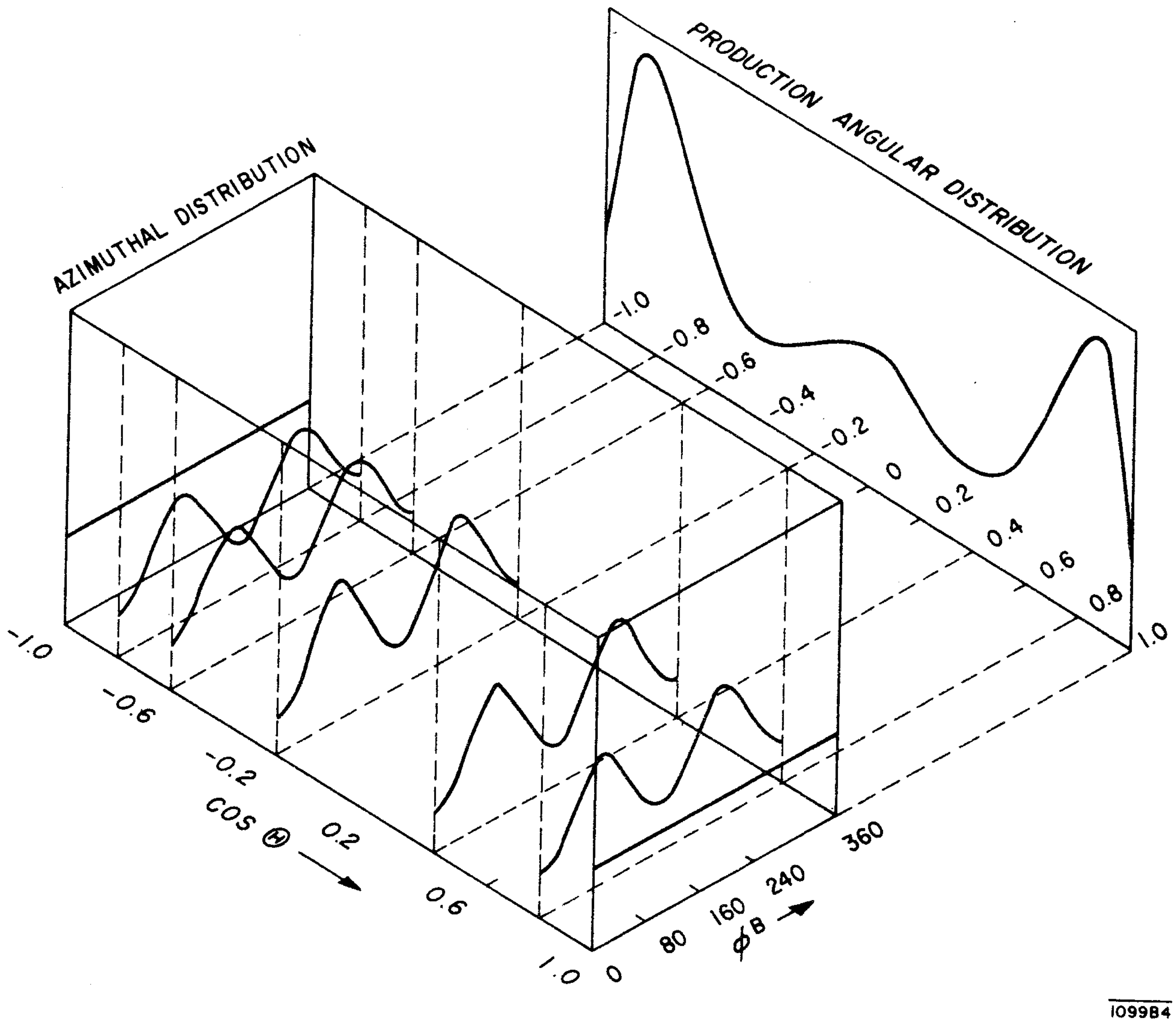
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Fig. 5

GD7

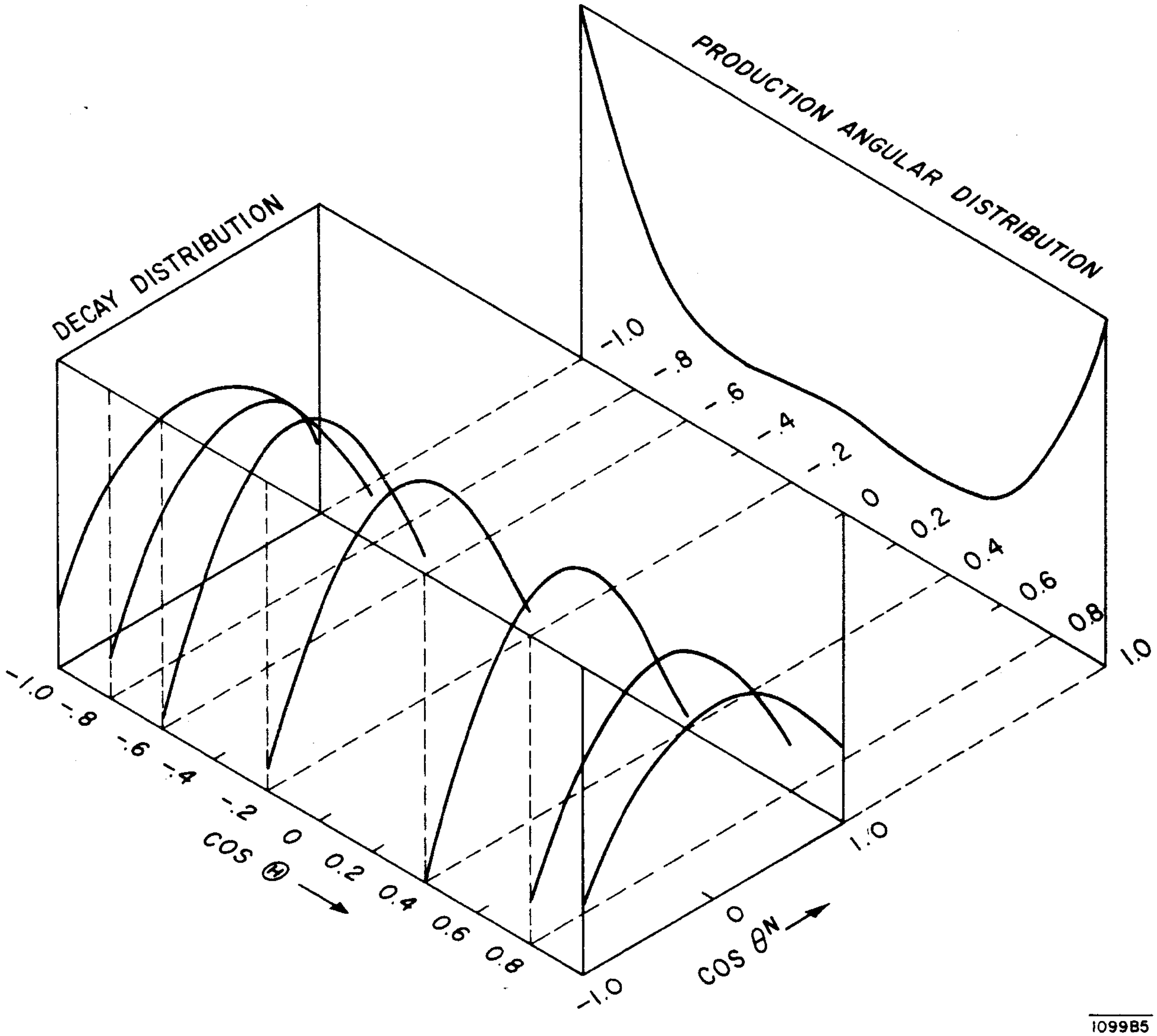

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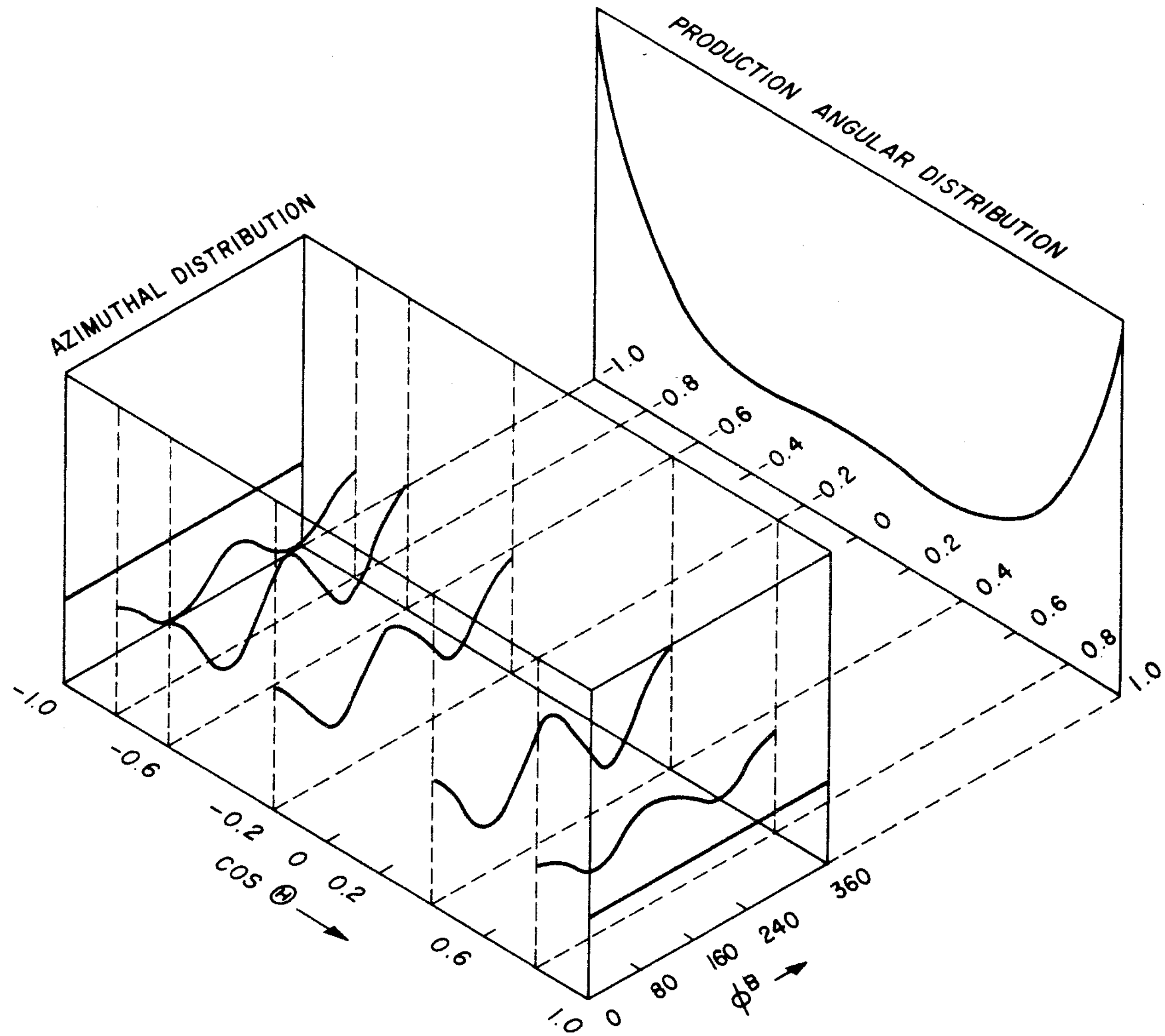
Fig. 6

GD7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-0.2
0
0.2
0.6
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COS Θ
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160
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φB
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Fig. 7