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Extended gauge theory and gauged free differential algebras

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Abstract

Recently, Antoniadis, Konitopoulos and Savvidy introduced, in the context of the so-called extended gauge theory, a procedure to construct background-free gauge invariants, using non-abelian gauge potentials described by higher degree forms.

In this article it is shown that the extended invariants found by Antoniadis, Konitopoulos and Savvidy can be constructed from an algebraic structure known as free differential algebra. In other words, we show that the above mentioned non-abelian gauge theory, where the gauge fields are described by p-forms with $p \geq 2$, can be obtained by gauging free differential algebras.

1. Introduction

Higher gauge theory [1–8] is an extension of ordinary gauge theory, where the gauge potentials and their gauge curvatures are higher degree forms. It is believed that higher gauge theories describe the dynamics of higher dimensional extended objects thought to be the basic building blocks of fundamental interactions.

The basic field of the abelian higher gauge theory, originated in supergravity is a p-form gauge potential A, whose $(p+1)$-form curvature is given by $F = \mathrm{d}A$ from which the Lagrangian and

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the action of the theory can be constructed. This abelian theory is known in the specialized literature as p-form electrodynamics and it is endowed with a local gauge symmetry with the transformation law $A \to A' = A + \mathrm{d}\varphi$ for some $(p-1)$-form φ.

The natural question is: does there exist a non-abelian higher gauge theory? To answer this question it is interesting to remember that the points of a curve have a natural order and the definition of the parallel transport along a given curve indeed makes use of this order. However, for higher dimensional submanifolds such a canonical order is not available. This lack of natural order led to C. Teitelboim in Ref. [9] to the formulation of a no-go theorem, ruling out the existence of non-abelian gauge theories for extended objects.

Recent attempts to circumvent this theorem has been carried out in Refs. [1–8]. In particular, in Refs. [1–4], were found invariants similar to the Pontryagin–Chern forms $\mathcal{P}_{2n}$ in non-abelian tensor gauge field theory, denoted by Γ_{2n+p}, with $p = 3, 4, 6, 8$. Since $\mathrm{d}\Gamma_{2n+p} = 0$, we can write $\Gamma_{2n+p} = \mathrm{d}\mathfrak{C}^{(2n+p-1)}_{\mathrm{ChSAS}}$. In the same references were found explicit expressions for these invariants in terms of higher order polynomials of the curvature forms. As with standard Chern–Simons forms, the secondary forms $\mathfrak{C}^{(2n+p-1)}_{\mathrm{ChSAS}}$ are background-free, quasi-invariant and only locally defined (and therefore defined only up to boundary terms, $\mathfrak{C}^{(2n+p-1)}_{\mathrm{ChSAS}} \sim \mathfrak{C}^{(2n+p-1)}_{\mathrm{ChSAS}} + \mathrm{d}\sigma^{(2n+p-2)}$).

The purpose of this paper is to show that the invariants introduced in Refs. [1–4] can be constructed from a gauged free differential algebra.

This paper is organized as follows: In Section 2, we briefly review the extended gauge theory developed in Refs. [1–8]. In Section 3, we will make a short review about free differential algebras and their gauging. Section 4 contains the results of the main objective of this work, namely: to show that the algebraic structure known as free differential algebras (FDA), allows to formulate a theory of non-abelian gauge with gauge fields described by p-forms with $p \geq 2$ and to prove that the extended invariants found in Refs. [1–4] can be constructed by gauging free differential algebras. We finish in Section 5 with some final remarks and considerations on future possible developments.

2. Chern–Simons–Antoniadis–Savvidy (ChSAS) forms

In this section we briefly review the extended gauge theory developed in Refs. [1–4].

2.1. Chern–Simons forms

The Pontryagin–Chern forms $\mathcal{P}_{2n+2} = \langle F^{n+1} \rangle$ satisfy the condition $\mathrm{d}\mathcal{P}_{2n+2} = 0$, where $F = \mathrm{d}A + A^2$ is the 2-form field strength of the 1-form gauge field A. From the Poincaré lemma, we know that locally there exists a $(2n+1)$-form $\mathcal{C}_{2n+1}$ such that $\mathcal{P}_{2n+2} = \mathrm{d}\mathcal{C}_{2n+1}$. This $(2n+1)$-form $\mathcal{C}_{2n+1}$ is called a Chern–Simons form which is quasi-invariant under gauge transformations [10].

Using the Chern–Weil theorem we can find an explicit expression for the Chern–Simons forms. In fact: let $A^{(0)}$ and $A^{(1)}$ be two one-form gauge connections on a fiber bundle over a $(2n+1)$-dimensional base manifold M, and let $F^{(0)}$ and $F^{(1)}$ be the corresponding curvatures. Then, the difference of Pontryagin–Chern forms is exact,

$$\left\langle \left[F^{(1)}\right]^{n+1} \right\rangle - \left\langle \left[F^{(0)}\right]^{n+1} \right\rangle = \mathrm{d}\mathcal{T}^{(2n+1)}\left(A^{(1)}, A^{(0)}\right), \tag{1}$$

where

$$\mathcal{T}^{(2n+1)}\left(A^{(1)},A^{(0)}\right)=(n+1)\int_0^1 \mathrm{d}t\left\langle\Theta F_t^n\right\rangle, \tag{2}$$

is called a transgression $(2n+1)$-form, with $\Theta = A^{(1)} - A^{(0)}$ and $A_t = A^{(0)} + t\ \Theta$. The 2-form F_t stands for the field-strength of the 1-form connection A_t, $F_t = \mathrm{d}A_t + A_t A_t$. Setting $A^{(0)} = 0$ and $A^{(1)} = A$ in (2), we obtain the well known Chern–Simons $(2n+1)$-form

$$\mathcal{C}_{2n+1}(A)=\mathcal{T}^{(2n+1)}(A,0)=(n+1)\int_0^1 \mathrm{d}t\left\langle A\left(t\mathrm{d}A+t^2A^2\right)^n\right\rangle. \tag{3}$$

From the Chern–Weil theorem it is straightforward to show that under gauge transformations the Chern–Simons forms are quasi-invariant. However, it is important to stress that since a connection cannot be globally set to zero unless the bundle (topology) is trivial, Chern–Simons forms turn out to be only locally defined.

2.2. *Non-abelian tensor gauge fields*

The idea of extending the Yang–Mills fields to higher rank tensor gauge fields was used in Refs. [1–4] to construct gauge invariant and metric independent forms in higher dimensions. These forms are analogous to the Pontryagin–Chern forms in Yang–Mills gauge theory.

2.2.1. *ChSAS forms in* $(2n+2)$*-dimensions*

The first series of exact $(2n+3)$-forms is given by

$$\Gamma_{2n+3}=\left\langle F^n,F_3\right\rangle=\mathrm{d}\mathfrak{C}^{(2n+2)}_{\mathrm{ChSAS}}, \tag{4}$$

where $F_3 = \mathrm{d}A_2 + [A, A_2]$ is the 3-form field-strength tensor for the 2-rank gauge field $A_2 = \frac{1}{2}B_{\mu\nu} \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}B^a{}_{\mu\nu}T_a \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and satisfy the Bianchi identities, $\mathrm{D}F_3 + [A_2, F] = 0$. Under gauge transformations, the gauge potential A_2 and the corresponding curvature transform as [1]

$$\delta A_2=\mathrm{D}\xi_1+[A_2,\xi_0], \tag{5}$$

$$\delta F_3=\mathrm{D}(\delta A_2)+[\delta A,A_2], \tag{6}$$

where $\xi_0 = \xi^a T_a$ is a 0-form gauge parameter and $\xi_1 = \xi^a{}_\mu T_a \otimes \mathrm{d}x^\mu$ is a 1-form gauge parameter.

Using the Chern–Weil theorem, we can find an explicit expression for the Chern–Simons form. In fact: Let $A^{(0)}$ and $A^{(1)}$ be two gauge connection 1-forms, and let $F^{(0)}$ and $F^{(1)}$ be their corresponding curvature 2-forms. Let $A_2^{(0)}$ and $A_2^{(1)}$ be two gauge connection 2-forms and let $F_3^{(0)}$ and $F_3^{(1)}$ be their corresponding curvature 3-forms. Then, the difference $\Gamma^{(1)}_{2n+3} - \Gamma^{(0)}_{2n+3}$ is an exact form

$$\Gamma^{(1)}_{2n+3}-\Gamma^{(0)}_{2n+3}=\left\langle\left[F^{(1)}\right]^n,F_3^{(1)}\right\rangle-\left\langle\left[F^{(0)}\right]^n,F_3^{(0)}\right\rangle=\mathrm{d}\mathfrak{T}^{(2n+2)}\left(A^{(0)},A_2^{(0)};A^{(1)},A_2^{(1)}\right), \tag{7}$$

where

$$\mathfrak{T}^{(2n+2)}\left(A^{(0)},A_2^{(0)};A^{(1)},A_2^{(1)}\right)=\int_0^1 \mathrm{d}t\left(n\left\langle F^{n-1},\Theta,F_{3t}\right\rangle+\left\langle F_t^n,\Phi\right\rangle\right), \tag{8}$$

with $\Phi = A_2^{(1)} - A_2^{(0)}$, is what we call Antoniadis–Savvidy (AS) transgression form.

Using the procedure followed in the case of Chern–Simons forms, we define the $(2n+2)$-ChSAS form as

$$\mathfrak{C}^{(2n+2)}_{\text{ChSAS}} = \mathfrak{T}^{(2n+2)}(A, A_2; 0, 0) = \int_0^1 \mathrm{d}t \langle n A F_t^{n-1} F_{3t} + A_2 F_t^n \rangle$$
$$= \langle F^n, A_2 \rangle + \mathrm{d}\varphi_{2n+1}. \tag{9}$$

This result is analogous to the usual Chern–Simons form (3), but in even dimensions [8]. It is interesting to notice that transgression forms (both, standard ones and the above generalization) are defined globally on the spacetime basis manifold of the principal bundle and are off-shell gauge invariant. Chern–Simons forms (both, standard ones and the AS generalization) are locally defined and are off-shell gauge invariant only up to boundary terms (i.e., quasi-invariants).

2.2.2. *ChSAS forms in $(2n+3)$-dimensions*

The second series of invariant forms is defined in $2n+4$ dimensions and is given by

$$\Gamma_{2n+4} = \langle F^n, F_4 \rangle = \mathrm{d}\mathfrak{C}^{(2n+3)}_{\text{ChSAS}}, \tag{10}$$

where the corresponding $(2n+3)$-form $\mathfrak{C}^{(2n+3)}_{\text{ChSAS}}$ is defined in terms of the 4-form $F_4 = \mathrm{d}A_3 + [A, A_3]$ field-strength tensor for the 3-rank gauge field A_3. In fact, following the procedure shown in the above subsection, we define the $(2n+3)$-ChSAS form as

$$\mathfrak{C}^{(2n+3)}_{\text{ChSAS}} = \int_0^1 \mathrm{d}t \langle n A F_t^{n-1} F_{4t} + A_3 F_t^n \rangle$$
$$= \langle F^n, A_3 \rangle + \mathrm{d}\varphi_{2n+2}. \tag{11}$$

2.2.3. *ChSAS forms in $(2n+5)$-dimensions*

The third series of exact $(2n+6)$-forms is given by [4]

$$\Xi_{2n+6} = \langle F^n, F_6 \rangle + n \langle F^{n-1}, F_4^2 \rangle = \mathrm{d}\mathfrak{C}^{(2n+5)}_{\text{ChSAS}}, \tag{12}$$

where the corresponding $(2n+5)$-form $\mathfrak{C}^{(2n+5)}_{\text{ChSAS}}$ is defined in terms of the 6-form $F_6 = \mathrm{D}A_5 + [A_3, A_3]$ field-strength for the rank-5 gauge field A_5. As in subsection 2.2.1 we can now also define the $(2n+5)$-ChSAS form as

$$\mathfrak{C}^{(2n+5)}_{\text{ChSAS}} = \langle F^n, A_5 \rangle + n \langle F^{n-1}, F_4, A_3 \rangle. \tag{13}$$

2.2.4. *ChSAS forms in $(2n+7)$-dimensions*

The fourth series of invariant closed forms Γ_{2n+8} in $(2n+8)$ dimensions is given by [3]

$$\Upsilon_{2n+8} = \langle F^n, F_8 \rangle + 3n \langle F^{n-1}, F_4, F_6 \rangle + n(n-1) \langle F^{n-2}, F_4{}^3 \rangle = \mathrm{d}\mathfrak{C}^{(2n+7)}_{\text{ChSAS}}, \tag{14}$$

where the corresponding $(2n+7)$-form $\mathfrak{C}^{(2n+5)}_{\text{ChSAS}}$ is defined in terms of the 8-form $F_8 = \mathrm{D}A_7 + 3[A_3, A_5]$ field-strength for the rank 7 gauge field A_7. From (14) it is possible to find the called $(2n+7)$-ChSAS form

$$\mathfrak{C}^{(2n+7)}_{\text{ChSAS}} = \langle F^n, A_7 \rangle + n(n-1) \langle F_4, F_4, A_3, F^{n-2} \rangle + n \langle F_6, A_3, F^{n-1} \rangle + 2n \langle F_4, A_5, F^{n-1} \rangle. \tag{15}$$

3. Free differential algebras

In this section, we will make a short review on free differential algebras and their gauging [11–14].

The dual formulation of Lie algebras provided by the Maurer–Cartan equations [13] can be naturally extended to p-forms ($p > 1$). Let's consider an arbitrary manifold M and a basis of exterior forms $\{\Theta^{A_1(p_1)}, \Theta^{A_2(p_2)}, \ldots, \Theta^{A_n(p_n)}\}$ defined on M, labeled by the index A and by the degree p of the form, which may be different for different values of A. This means that each p_i takes values $0, 1, 2, \ldots, N$, while i takes the values $1, 2, ..., n$.

The external derivative $\mathrm{d}\Theta^{A(p)}$ can be expressed as a combination of the elements of the base, which leads to write a generalized Maurer–Cartan equation of the following type [11–14]

$$\mathrm{d}\Theta^{A(p)} + \sum_{n=1}^{N} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} \Theta^{B_1(p_1)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0, \tag{16}$$

where the coefficients $C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}$ are called generalized structure constants. The symmetry of these constants in the lower index is induced by the permutation of the forms $\Theta^{A(p)}$ in the product wedge and are different from zero only if

$$p_1 + p_2 + \cdots + p_n = p + 1. \tag{17}$$

Here, the number N is equal to $p_{\max} + 1$, where $p_{\max}$ is the highest degree in the set $\{\Theta^{A(p)}\}$. One can say that Eq. (16) is a generalized Maurer–Cartan equation and that it describes a FDA if and only if the integrability condition $\mathrm{d}^2\Theta^{A(p)} = 0$ follows automatically from (16). Explicitly, the condition for (16) to be a FDA is given by

$$\begin{aligned} \mathrm{d}^2\Theta^{A(p)} = &- \sum_{n,m=1}^{N} \frac{1}{m} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} C^{B_1(p_1)}_{D_1(q_1)\cdots D_m(q_m)} \\ &\Theta^{D_1(q_1)} \wedge \cdots \wedge \Theta^{D_m(q_m)} \wedge \Theta^{B_2(p_2)} \wedge \cdots \wedge \Theta^{B_n(p_n)} \\ = &\; 0. \end{aligned} \tag{18}$$

This equation is just the analogue of the Jacobi identities of an ordinary Lie algebra. It is very instructive to have a look at the most general form of a FDA as it emerges from theorems of Sullivan. From Ref. [13] we know that: (i) a FDA is called "minimal algebra" when it is true that $C^{A(p)}_{B(p+1)} = 0$. This means that all forms appearing in the expansion of $\mathrm{d}\Theta^{A(p)}$ have at most degree p, being the degree $(p + 1)$ ruled out; (ii) a FDA is called a contractible algebra when the only form appearing in the expansion of $\mathrm{d}\Theta^{A(p)}$ has degree $(p + 1)$, namely

$$\mathrm{d}\Theta^{A(p)} = \Theta^{A(p+1)}, \text{ i.e., } \mathrm{d}\Theta^{A(p+1)} = 0. \tag{19}$$

Sullivan's fundamental theorem. *The most general free differential algebra is the direct sum of a contractible algebra with a minimal algebra.*

3.1. Gauging free differential algebras

Physical applications of FDA require a generalization of the concepts of soft 1-forms and curvatures introduced gauging of Maurer–Cartan equations [12–14].

Let $\{A^{B_1(p_1)}, A^{B_2(p_2)}, \dots, A^{B_n(p_n)}\}$ be a set of p-forms gauge potential, labeled by the index B and by the degree p of the form, which may be different for different values of B. If we consider the p-forms $A^{B_i(p_i)}$ as the gauge potentials of a FDA, in the same way as the components A^a are the gauge potentials of an ordinary Lie algebra described by the ordinary Maurer–Cartan equations, then the curvatures associated with the $A^{B_i(p_i)}$ potentials are given by

$$F^{A(p+1)} = \mathrm{d}A^{A(p)} + \sum_{n=1}^{N} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} A^{B_1(p_1)} \wedge \cdots \wedge A^{B_n(p_n)}. \tag{20}$$

If we apply the exterior derivative to both sides of Eq. (20), we obtain a generalization of the Bianchi identity [13]

$$\nabla F^{A(p+1)} = \mathrm{d}F^{A(p+1)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} F^{B_1(p_1+1)} \wedge A^{B_2(p_2)} \wedge \cdots \wedge A^{B_n(p_n)} = 0. \tag{21}$$

In complete analogy to what one does in ordinary group theory, we say that the left side of (21) defines the covariant derivative ∇ of an adjoint set of $(p+1)$-forms. With this definition, the Bianchi identity (21) just states that the covariant derivative of the curvature set $F^{A(p+1)}$ is zero as it happens for ordinary groups.

4. Extended gauge theory and gauged FDA

Let us now consider the explicit form of the equations (20), (21). In the case of a minimal FDA, the explicit form of equations (20), (21) for $p = 1, 2, 3, 5, 7, 9$, is given in Appendices A and B respectively. Here we will list, using the nomenclature of Refs. [1–4], only the equations we will use later. In fact, from (79) we can see that, if we restrict ourselves to the case of an FDA whose structure constants satisfy the condition $C^{A(q+r-1)}_{B(q)C(r)} = C^{A}_{BC}$ for any $r < q$, where C^{A}_{BC} correspond to the structure constants of a Lie algebra,[1] then the equations (78), (79) can be written in the form (see Appendix A)

$$\begin{aligned}
F &= \mathrm{d}A + A^2, \\
F_3 &= \mathrm{d}A_2 + [A, A_2], \\
F_4 &= \mathrm{d}A_3 + [A, A_3], \\
F_6 &= \mathrm{d}A_5 + [A, A_5] + \frac{1}{2}[A_3, A_3], \\
F_8 &= \mathrm{d}A_7 + [A, A_7] + [A_3, A_5], \\
F_{10} &= \mathrm{d}A_9 + [A, A_9] + [A_3, A_7] + \frac{1}{2}[A_5, A_5].
\end{aligned} \tag{22}$$

In the same way, for the equation (80) we find (see Appendix B),

$$\begin{aligned}
\mathrm{D}F &= 0, \\
\mathrm{D}F_3 + [A_2, F] &= 0, \\
\mathrm{D}F_4 + [A_3, F] &= 0,
\end{aligned}$$

[1] We will consider this condition in the rest of this paper.

$$\begin{aligned} \mathrm{D}F_6 + [A_3, F_4] + [A_5, F] &= 0, \\ \mathrm{D}F_8 + [A_3, F_6] + [A_5, F_4] + [A_7, F] &= 0, \\ \mathrm{D}F_{10} + [A_3, F_8] + [A_5, F_6] + [A_7, F_4] + [A_9, F] &= 0. \end{aligned} \tag{23}$$

It should be noted that equations (22) and (23) match those found in Refs. [1–4], except for numerical coefficients. However, they coincide exactly after an appropriate transformation of the gauge fields (see Appendix F).

4.1. Gauge transformations

Let $\{\lambda^{B_1(p_1)}, \dots, \lambda^{B_{n-1}(p_{n-1})}\}$ be a set $(p-1)$-forms gauge parameters and let $\{A^{B_1(p_1)}, \dots, A^{B_n(p_n)}\}$ be a set of p-forms gauge potentials labeled by an index B and by the degree p. Under a gauge transformation, the gauge potential transforms as

$$\delta A^{A(p+1)} = \mathrm{d}\lambda^{A(p)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)B_2(p_2)\cdots B_n(p_n)} A^{B_1(p_1)} \wedge \lambda^{B_2(p_2)} \wedge \cdots \wedge \lambda^{B_n(p_n)}. \tag{24}$$

In the case of a minimal FDA, the explicit form of equation (24) for $n = 2$ and $p = 1, 2, 3, 5, 7, 9$, is given in Appendix C. From (83) we can see that

$$\begin{aligned} \delta A &= \mathrm{D}\lambda, \\ \delta A_2 &= \mathrm{D}\lambda_1 + [A_2, \lambda], \\ \delta A_3 &= \mathrm{D}\lambda_2 + [A_3, \lambda], \\ \delta A_5 &= \mathrm{D}\lambda_4 + [A_3, \lambda_2] + [A_5, \lambda], \\ \delta A_7 &= \mathrm{D}\lambda_6 + [A_3, \lambda_4] + [A_5, \lambda_2] + [A_7, \lambda], \\ \delta A_9 &= \mathrm{D}\lambda_8 + [A_3, \lambda_6] + [A_5, \lambda_4] + [A_7, \lambda_2] + [A_9, \lambda]. \end{aligned} \tag{25}$$

4.2. Gauge transformations for curvatures

Following the definition of the usual gauge theory, we have

$$\delta F^{A(p+1)} = \nabla\left(\delta A^{A(p)}\right), \tag{26}$$

so that

$$\begin{aligned} \delta F^{A(p+1)} &= \nabla\left(\delta A^{A(p)}\right) = \mathrm{d}\left(\delta A^{A(p)}\right) \\ &\quad + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)B_2(p_2)\cdots B_n(p_n)} \delta A^{B_1(p_1)} \wedge A^{B_2(p_2)} \wedge \cdots \wedge A^{B_n(p_n)}. \end{aligned} \tag{27}$$

In the case of a minimal FDA, the explicit form of equation (27) for $p = 1, \dots, 9$ is given Appendix D. When a FDA has structure constants that satisfy the condition $C^{A(q+r-1)}_{B(q)C(r)} = C^{A}_{BC}$, we find that the equations (85) can be written in the form (see Appendix D),

$$\begin{aligned} \delta F &= [F, \lambda], \\ \delta F_4 &= [F_4, \lambda] + [F, \lambda_2], \\ \delta F_6 &= [F_6, \lambda] + [F_4, \lambda_2] + [F, \lambda_4], \end{aligned}$$

$$\delta F_8 = [F_8, \lambda] + [F_6, \lambda_2] + [F_4, \lambda_4] + [F, \lambda_6],$$
$$\delta F_{10} = [F_{10}, \lambda] + [F_8, \lambda_2] + [F_6, \lambda_4] + [F_4, \lambda_6] + [F, \lambda_8]. \tag{28}$$

The equations (25), (28) match those found in Refs. [1–4], after an appropriate redefinition of the gauge fields (see Appendix F).

5. Extended invariants

In this section it is shown that the extended invariants found by Antoniadis and Savvidy in Refs. [1–4] can be constructed from a gauged free differential algebra.

5.1. Chern–Pontryagin invariants

Let $A = A^a T_a$ be a 1-form connection evaluated in the Lie algebra $\mathfrak{g}$ of the group G and let $F = F^a T_a = \mathrm{d}A + A^2$ be its corresponding 2-form curvature. The Chern–Pontryagin topological invariant in $2n + 2$ dimensions is given by [15]

$$\mathcal{P}_{2n+2} = \langle F \wedge \cdots \wedge F \rangle = g_{a_1 \cdots a_{n+1}} F^{a_1} \wedge \cdots \wedge F^{a_{n+1}}, \tag{29}$$

where the bracket $\langle \cdots \rangle$ is a symmetric multilinear form that represents an appropriately normalized trace over the algebra defined by

$$g_{a_1 \cdots a_{n+1}} = \langle T_{a_1}, \ldots, T_{a_{n+1}} \rangle. \tag{30}$$

5.2. Generalized Chern–Pontryagin invariants

Let's consider now the generalization of the Chern–Pontryagin topological invariant to the case where Lie algebra $\mathfrak{g}$ is replaced by a free differential algebra. Let $\{F^{B_1(p_1)}, \ldots, F^{B_{n+1}(p_{n+1})}\}$ be a set of p-forms field intensities. It is possible to construct topological invariants analogous to the Chern–Pontryagin invariant as follows

$$\begin{aligned}\tilde{\mathcal{P}} &= \sum_{\{p_i\}} \left\langle F^{(p_1)} \wedge \cdots \wedge F^{(p_{n+1})} \right\rangle \\ &= \sum_{\{p_i\}} g_{B_1(p_1) \cdots B_{n+1}(p_{n+1})} F^{B_1(p_1)} \wedge \cdots \wedge F^{B_{n+1}(p_{n+1})},\end{aligned} \tag{31}$$

where for each order of the form $\tilde{\mathcal{P}}$, the sum runs over all possible combinations.

5.2.1. Case $p_1 + \cdots + p_{n+1} = 2n + 2$

If $p_1 + \cdots + p_{n+1} = 2n + 2$, the only possible choice is $p_1 = \cdots = p_{n+1} = 2$. Then we find

$$\begin{aligned}\tilde{\mathcal{P}} &= g_{B_1(p_1) \cdots B_n(p_{n+1})} F^{B_1(p_1)} \wedge \cdots \wedge F^{B_{n+1}(p_{n+1})} \\ &= g_{B_1(2) \cdots B_{n+1}(2)} F^{B_1(2)} \wedge \cdots \wedge F^{B_{n+1}(2)} \\ &= \left\langle F^{(2)} \wedge \cdots \wedge F^{(2)} \right\rangle = \left\langle \left[F^{(2)} \right]^{n+1} \right\rangle.\end{aligned} \tag{32}$$

Using the nomenclature used in Refs. [1–4] we can write,

$$\mathcal{P} = \left\langle F^{n+1} \right\rangle, \tag{33}$$

which coincides with the usual Chern–Pontryagin invariant $\mathcal{P}_{2n+2}$.

5.2.2. Case $p_1 + \cdots + p_{n+1} = 2n + 3$

If $p_1 + \cdots + p_{n+1} = 2n + 3$ the only possible choice is $p_1 = \cdots = p_n = 2$; $p_{n+1} = 3$. According to the permutations law, there must exist $n + 1$ terms of the form

$$g_{B_1(2)\cdots B_n(2)B_{n+1}(3)} F^{B_1(2)} \wedge \cdots \wedge F^{B_n(2)} \wedge F^{B_{n+1}(3)} \\ = \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(3)} \rangle = \langle [F^{(2)}]^n, F^{(3)} \rangle, \tag{34}$$

so that, the corresponding extended Chern–Pontryagin invariant is given by

$$\tilde{\mathcal{P}} = (n+1) \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(3)} \rangle = (n+1) \langle F^{(2)n}, F^{(3)} \rangle. \tag{35}$$

Using the nomenclature used in Refs. [1–4] we find

$$\mathcal{P}_{2n+3} = \langle F^n, F_3 \rangle. \tag{36}$$

Since $\mathrm{d}\mathcal{P}_{2n+3} = 0$ we have $\mathcal{P}_{2n+3} = \mathrm{d}\mathfrak{C}^{(2n+2)}$. Following the usual procedure we have

$$\mathfrak{C}^{(2n+2)} = \langle F^n, A_2 \rangle + \mathrm{d}\varphi_{2n+1}. \tag{37}$$

These results coincide with the extended Chern–Pontryagin $(2n + 3)$-dimensional and with the $(2n + 2)$-Chern–Simons forms $\mathfrak{C}^{(2n+2)}_{\mathrm{ChSAS}}$ found by Antoniadis and Savvidy in Refs. [1–3].

5.2.3. Case $p_1 + \cdots + p_{n+1} = 2n + 4$

According to the permutations law, there must exist $n + 1$ terms of the form

$$g_{B_1(2)\cdots B_n(2)B_{n+1}(4)} F^{B_1(2)} \cdots \wedge F^{B_n(2)} \wedge F^{B_{n+1}(4)} \\ = \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(4)} \rangle = \langle [F^{(2)}]^n, F^{(4)} \rangle, \tag{38}$$

so that, the corresponding extended Chern–Pontryagin invariant is given by

$$\tilde{\mathcal{P}} = (n+1) \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(4)} \rangle = (n+1) \langle [F^{(2)}]^n, F^{(4)} \rangle. \tag{39}$$

Using the nomenclature used in Refs. [1–4] we can write as

$$\mathcal{P}_{2n+4} = \langle F^n, F_4 \rangle. \tag{40}$$

Since $\mathrm{d}\mathcal{P}_{2n+4} = 0$ we have $\mathcal{P}_{2n+4} = \mathrm{d}\mathfrak{C}^{(2n+3)}$, where

$$\mathfrak{C}^{(2n+3)} = \langle F^n, A_3 \rangle + \mathrm{d}\varphi_{2n+2}. \tag{41}$$

These results coincides with the extended topological invariant and with the $(2n + 3)$-Chern–Simons forms $\mathfrak{C}^{(2n+3)}_{\mathrm{ChSAS}}$ found in Refs. [1–3].

5.2.4. Case $p_1 + \cdots + p_{n+1} = 2n + 6$

In this case we will choice two combinations which will be analyze separately.

5.2.4.1. Term with $p_1 = \cdots = p_n = 2$ *and* $p_{n+1} = 6$ In this case we have that, according to the permutations law, there must exist $n + 1$ terms of the form

$$g_{B_1(2)\cdots B_n(2)B_n(6)} F^{B_1(2)} \wedge \cdots \wedge F^{B_n(2)} \wedge F^{B_{n+1}(6)} \\ = \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(6)} \rangle = \langle [F^{(2)}]^n, F^{(6)} \rangle, \tag{42}$$

so that

$$\tilde{\mathcal{P}}_1 = (n+1) \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(6)} \rangle = (n+1) \langle [F^{(2)}]^n, F^{(6)} \rangle. \tag{43}$$

5.2.4.2. Term with $p_1 = \cdots = p_{n-1} = 2$ and $p_n = p_{n+1} = 4$ In this case we have that, according to the law of permutations, there must exist $n(n+1)/2$ terms of the form

$$\begin{aligned} &g_{B_1(2)\cdots B_{n-1}(2)B_n(4)B_{n+1}(4)} F^{B_1(2)} \wedge \cdots \wedge F^{B_{n-1}(2)} \wedge F^{B_n(4)} \wedge F^{B_{n+1}(4)} \\ &\quad = \langle F^{(2)} \wedge \cdots \wedge F^{(2)} \wedge F^{(4)} \wedge F^{(4)} \rangle = \langle [F^{(2)}]^{n-1}, [F^{(4)}]^2 \rangle, \end{aligned} \tag{44}$$

so that

$$\tilde{\mathcal{P}}_2 = \frac{n(n+1)}{2} \langle [F^{(2)}]^{n-1}, [F^{(4)}]^2 \rangle. \tag{45}$$

This means that the corresponding extended Chern–Pontryagin invariant is given by

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_1 + \tilde{\mathcal{P}}_2 = (n+1) \langle [F^{(2)}]^n, F^{(6)} \rangle + \frac{n(n+1)}{2} \langle [F^{(2)}]^{n-1}, [F^{(4)}]^2 \rangle, \tag{46}$$

which can be write as

$$\mathcal{P} = \langle [F^{(2)}]^n, F^{(6)} \rangle + \frac{n}{2} \langle [F^{(2)}]^{n-1}, [F^{(4)}]^2 \rangle. \tag{47}$$

Using the nomenclature used in Refs. [1–4] we can write

$$\mathcal{P}_{2n+6} = \langle F^n, F_6 \rangle + \frac{n}{2} \langle F^{n-1}, F_4^2 \rangle. \tag{48}$$

Now let us now prove that the expression (48) is, in addition to being gauge invariant, a closed form. The variation of $\mathcal{P}_{2n+6}$ is given by

$$\begin{aligned} \delta \mathcal{P}_{2n+6} &= n \langle F^{n-1}, \delta F, F_6 \rangle + \langle F^n, \delta F_6 \rangle \\ &\quad + \frac{n(n-1)}{2} \langle F^{n-2}, \delta F, F_4^2 \rangle + \frac{n}{2} \langle F^{n-1}, F_4, \delta F_4 \rangle. \end{aligned} \tag{49}$$

Introducing (28) into (49) we have

$$\begin{aligned} \delta \mathcal{P}_{2n+6} &= \langle [F_4, \lambda_2], F^n \rangle + \langle [F, \lambda_4], F^n \rangle + n \langle [F, \lambda_2], F_4, F^{n-1} \rangle \\ &= \{ \langle [F_4, \lambda_2], F^n \rangle + n \langle [F, \lambda_2], F_4, F^{n-1} \rangle \} + \langle [F, \lambda_4], F^n \rangle \\ &= 0. \end{aligned} \tag{50}$$

Now let us show that (48) is a closed form. Taking the exterior derivative of $\mathcal{P}_{2n+6}$ we have

$$\begin{aligned} \mathrm{d}\mathcal{P}_{2n+6} &= \langle \mathrm{D}F_6, F^n \rangle + n \langle F_6, \mathrm{D}F, F^{n-1} \rangle + n \langle \mathrm{D}F_4, F_4, F^{n-1} \rangle \\ &\quad + \frac{n(n-1)}{2} \langle F_4^2, \mathrm{D}F, F^{n-2} \rangle. \end{aligned} \tag{51}$$

Using (23) we have

$$\begin{aligned} \mathrm{d}\mathcal{P}_{2n+6} &= \langle [F_4, A_3], F^n \rangle + \langle [F, A_5], F^n \rangle + n \langle [F, A_3], F_4, F^{n-1} \rangle \\ &= \{ \langle [F_4, A_3], F^n \rangle + n \langle [F, A_3], F_4, F^{n-1} \rangle \} + \langle [F, A_5], F^n \rangle, \end{aligned} \tag{52}$$

and using the well known identity [16]

$$\sum_{i=1}^{n} (-1)^{(d_1 + \cdots + d_{i-1}) d_\Theta} \langle \Lambda_1, \ldots, [\Theta, \Lambda_i], \ldots, \Lambda_n \rangle = 0, \tag{53}$$

where each Λ_i is a d_i-form and Θ is an arbitrary d_Θ-form, we have

$$\mathrm{d}\mathcal{P}_{2n+6}=0, \tag{54}$$

which proves that the form $\mathcal{P}_{2n+6}$ is closed. This means that $\mathcal{P}_{2n+6}=\mathrm{d}\mathfrak{C}^{2n+5}$ where, following the usual procedure, we find

$$\mathfrak{C}^{(2n+5)}=\langle F^n, A_5\rangle+\frac{n}{2}\langle F_4, A_3, F^{n-1}\rangle+\mathrm{d}\varphi_{2n+4}, \tag{55}$$

5.2.5. Case $p_1+\cdots+p_{n+1}=2n+8$

In this case we will choice three combinations which will be analyze separately.

5.2.5.1. Term with $p_1=p_2=\cdots=p_{n-1}=2$ and $p_n=8$ Now we have according to the law of permutations, $n+1$ terms of the form

$$\begin{aligned} &g_{B_1(2)\cdots B_n(2)B_{n+1}(8)}F^{B_1(2)}\wedge\cdots\wedge F^{B_n(2)}\wedge F^{B_{n+1}(8)}\\ &\quad=\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(8)}\rangle=\langle\left[F^{(2)}\right]^n, F^{(8)}\rangle, \end{aligned} \tag{56}$$

so that

$$\tilde{\mathcal{P}}_1=(n+1)\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(8)}\rangle=(n+1)\langle\left[F^{(2)}\right]^n, F^{(8)}\rangle. \tag{57}$$

5.2.5.2. Term with $p_1=\cdots=p_{n-1}=2$, $p_n=4$ and $p_{n+1}=6$ According to the law of permutations, we have $n(n+1)$ terms of the form

$$\begin{aligned} &g_{B_1(2)\cdots B_{n-1}(2)B_n(4)B_{n+1}(6)}F^{B_1(2)}\wedge\cdots\wedge F^{B_{n-1}(2)}\wedge F^{B_n(4)}\wedge F^{B_{n+1}(6)}\\ &\quad=\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(4)}\wedge F^{(6)}\rangle=\langle\left[F^{(2)}\right]^{n-1}, F^{(4)}, F^{(6)}\rangle, \end{aligned} \tag{58}$$

so that

$$\tilde{\mathcal{P}}_2=n(n+1)\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(4)}\wedge F^{(6)}\rangle=n(n+1)\langle\left[F^{(2)}\right]^{n-1}, F^{(4)}, F^{(6)}\rangle. \tag{59}$$

5.2.5.3. Term with $p_1=\cdots=p_{n-2}=2$ and $p_{n-1}=p_n=p_{n+1}=4$ The permutations law tells us that there are $(n+1)n(n-1)/3!$ terms of the form

$$\begin{aligned} &g_{B_1(2)\cdots B_{n-2}(2)B_{n-1}(4)B_n(4)B_{n+1}(4)}F^{B_1(2)}\wedge\cdots\wedge F^{B_{n-2}(2)}\wedge F^{B_{n-1}(4)}\wedge F^{B_n(4)}\wedge F^{B_{n+1}(4)}\\ &\quad=\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(4)}\wedge F^{(4)}\wedge F^{(4)}\rangle=\langle\left[F^{(2)}\right]^{n-2}, \left[F^{(4)}\right]^3\rangle, \end{aligned} \tag{60}$$

so that

$$\begin{aligned} \tilde{\mathcal{P}}_3&=\frac{n(n+1)(n-1)}{3!}\langle F^{(2)}\wedge\cdots\wedge F^{(2)}\wedge F^{(4)}\wedge F^{(4)}\wedge F^{(4)}\rangle\\ &=\frac{n(n+1)(n-1)}{3!}\langle\left[F^{(2)}\right]^{n-2}, \left[F^{(4)}\right]^3\rangle. \end{aligned} \tag{61}$$

This means that the corresponding extended Chern–Pontryagin invariant is given by

$$\begin{aligned} \tilde{\mathcal{P}}&=\tilde{\mathcal{P}}_1+\tilde{\mathcal{P}}_2+\tilde{\mathcal{P}}_3\\ &=(n+1)\langle\left[F^{(2)}\right]^n, F^{(8)}\rangle+n(n+1)\langle\left[F^{(2)}\right]^{n-1}, F^{(4)}, F^{(6)}\rangle\\ &\quad+\frac{n(n+1)(n-1)}{3!}\langle\left[F^{(2)}\right]^{n-2}, \left[F^{(4)}\right]^3\rangle, \end{aligned} \tag{62}$$

which can be written as

$$\mathcal{P}_{2n+8} = \left\langle \left[F^{(2)}\right]^n, F^{(8)} \right\rangle + n\left\langle \left[F^{(2)}\right]^{n-1}, F^{(4)}, F^{(6)} \right\rangle + \frac{n(n-1)}{3!}\left\langle \left[F^{(2)}\right]^{n-2}, \left[F^{(4)}\right]^3 \right\rangle, \quad (63)$$

that coincides, except for two numerical coefficients, with the topological invariant Γ_{2n+8} found by Antoniadis and Savvidy in Refs. [1–3]. Using the nomenclature from these references, the equation (63) takes the form

$$\mathcal{P}_{2n+8} = \langle F_8, F^n \rangle + n\langle F_4, F_6, F^{n-1} \rangle + \frac{n(n-1)}{3!}\langle F_4^3, F^{n-2} \rangle. \quad (64)$$

Now let us prove that (64) is gauge invariant. The variation of $\mathcal{P}_{2n+8}$ is given by

$$\begin{aligned}\delta\mathcal{P}_{2n+8} = &\langle \delta F_8, F^n \rangle + n\langle F_8, \delta F, F^{n-1} \rangle + n\langle \delta F_4, F_6, F^{n-1} \rangle \\ &+ n\langle F_4, \delta F_6, F^{n-1} \rangle + n(n-1)\langle F_4, F_6, \delta F, F^{n-2} \rangle \\ &+ \frac{n(n-1)}{2}\langle \delta F_4, F_4^2, F^{n-2} \rangle + \frac{n(n-1)(n-2)}{3!}\langle F_4^3, \delta F, F^{n-3} \rangle. \end{aligned} \quad (65)$$

Introducing (28) into (65) we find

$$\begin{aligned}\delta\mathcal{P}_{2n+8} = &-\left\{\langle [F_6, \lambda_2], F^n \rangle + n\langle [F, \lambda_2], F_6, F^{n-1} \rangle\right\} \\ &- \frac{n}{2}\left\{2\langle F^{(4)}, [F_4, \lambda_2], F^{n-1} \rangle + (n-1)\langle [F_4, \lambda_2], F_4^2, F^{n-2} \rangle\right\} \\ &- \left\{\langle [F_4, \lambda_4], F^n \rangle + n\langle F_4, [F, \lambda_4], F^{n-1} \rangle\right\} + \langle [F, \lambda_6], F^n \rangle \\ = &\ 0. \end{aligned} \quad (66)$$

Let us now show that (64) is also a closed form. Taking the exterior derivative of $\mathcal{P}_{2n+8}$ we have

$$\begin{aligned}\mathrm{d}\mathcal{P}_{2n+8} = &\langle \mathrm{D}F_8, F^n \rangle + n\langle F_8, \mathrm{D}F, F^{n-1} \rangle + n\langle \mathrm{D}F_4, F_6, F^{n-1} \rangle \\ &+ n\langle F_4, \mathrm{D}F_6, F^{n-1} \rangle + n(n-1)\langle F_4, F_6, \mathrm{D}F, F^{n-2} \rangle \\ &+ \frac{3n(n-1)}{3!}\langle \mathrm{D}F_4, F_4^2, F^{n-2} \rangle + \frac{n(n-1)(n-2)}{3!}\langle F_4^3, \mathrm{D}F, F^{n-3} \rangle. \end{aligned} \quad (67)$$

Using (23) we find

$$\begin{aligned}\mathrm{d}\mathcal{P}_{2n+8} = &\langle [F_6, A_3], F^n \rangle + \langle [F_4, A_5], F^n \rangle + \langle [F, A_7], F^n \rangle \\ &+ n\langle [F, A_3], F_6, F^{n-1} \rangle + n\langle F_4, [F_4, A_3], F^{n-1} \rangle \\ &+ n\langle F_4, [F, A_5], F^{n-1} \rangle + \frac{n(n-1)}{2}\langle [F_4, A_3], F_4^2, F^{n-2} \rangle, \end{aligned} \quad (68)$$

or equivalently

$$\begin{aligned}\mathrm{d}\mathcal{P}_{2n+8} = &\left\{\langle [F_6, A_3], F^n \rangle + n\langle [F, A_3], F_6, F^{n-1} \rangle\right\} \\ &+ \frac{n}{2}\left\{2\langle F_4, [F_4, A_3], F^{n-1} \rangle + (n-1)\langle [F_4, A_3], F_4^2, F^{n-2} \rangle\right\} \\ &+ \left\{\langle [F_4, A_5], F^n \rangle + n\langle F_4, [F, A_5], F^{n-1} \rangle\right\} + \langle [F, A_7], F^n \rangle. \end{aligned} \quad (69)$$

Then using (53) we can see

$$\mathrm{d}\mathcal{P}_{2n+8} = 0, \quad (70)$$

which proves that the form $\mathcal{P}_{2n+8}$ is a closed form. This means that $\mathcal{P}_{2n+8} = \mathrm{d}\mathfrak{C}^{2n+7}$, where $\mathfrak{C}^{2n+7}$ is given by

$$\mathfrak{C}^{(2n+7)} = \langle F^n, A_7\rangle + \frac{n(n-1)}{3!}\langle F_4, F_4, A_3, F^{n-2}\rangle + \frac{n}{3}\langle F_6, A_3, F^{n-1}\rangle + \frac{2n}{3}\langle F_4, A_5, F^{n-1}\rangle + d\varphi_{2n+6}. \tag{71}$$

The equations (48), (64), (55), (71) match those found in Refs. [1–4] after an appropriate redefinition of the gauge fields (see Appendix F).

6. Concluding remarks

In this article we have shown that the so-called ChSAS invariants [1–4] can be constructed from an algebraic structure known as gauged free differential algebra. The series of exact $(2n+p)$-forms is given by

$$\mathcal{P}_{2n+3} = d\mathfrak{C}^{(2n+2)} = \langle F^n, F_3\rangle,$$
$$\mathcal{P}_{2n+4} = d\mathfrak{C}^{(2n+3)} = \langle F^n, F_4\rangle,$$
$$\mathcal{P}_{2n+6} = d\mathfrak{C}^{(2n+5)} = \langle F^n, F_6\rangle + \frac{n}{2}\langle F^{n-1}, F_4^2\rangle,$$
$$\mathcal{P}_{2n+8} = d\mathfrak{C}^{(2n+7)} = \langle F^n, F_8\rangle + n\langle F^{n-1}, F_4, F_6\rangle + \frac{n(n-1)}{3!}\langle F^{n-2}, F_4^3\rangle,$$

where each F_{q+1} is a $(q+1)$-form field-strength for the rank-q gauge field A_q which depends also on other gauge fields A_r with $r < q$. The corresponding secondary $(2n+p)$-form $\mathfrak{C}^{(2n+p)}$ are also defined in terms of such gauge fields in the following way

$$\mathfrak{C}^{(2n+2)} = \langle F^n, A_2\rangle + d\varphi_{2n+1},$$
$$\mathfrak{C}^{(2n+3)} = \langle F^n, A_3\rangle + d\varphi_{2n+2},$$
$$\mathfrak{C}^{(2n+5)} = \langle F^n, A_5\rangle + \frac{n}{2}\langle F_4, A_3, F^{n-1}\rangle + d\varphi_{2n+4},$$
$$\mathfrak{C}^{(2n+7)} = \langle F^n, A_7\rangle + \frac{n(n-1)}{3!}\langle F_4, F_4, A_3, F^{n-2}\rangle + \frac{n}{3}\langle F_6, A_3, F^{n-1}\rangle + \frac{2n}{3}\langle F_4, A_5, F^{n-1}\rangle + d\varphi_{2n+6}.$$

If we consider the $n = 2$ case in the definition of $\mathfrak{C}^{(2n+7)}$, we find that the 11-dimensional ChSAS form is given by

$$\mathfrak{C}^{11} = \langle F^2, A_7\rangle + \frac{1}{3}\langle F_4, F_4, A_3\rangle + \frac{2}{3}\langle F_6, A_3, F\rangle + \frac{4}{3}\langle F_4, A_5, F\rangle. \tag{72}$$

From here we can see that the second term has the same form as a term that appears in the CJS supergravity [17], whose action is given by

$$S_{11} = \int_{M_{11}} L_{11} = \int_{M_{11}} -\frac{1}{4}R^{ab}\Sigma_{ab} + \frac{i}{2}\bar{\psi}\Gamma_{(8)}D\psi + \frac{i}{8}\left(T^a - \frac{i}{4}\bar{\psi}\Gamma^a\psi\right)e_a\bar{\psi}\Gamma_{(6)}\psi - \frac{1}{2}F_4 * F_4 + (*F_4 + b)(F - a) + \frac{1}{2}ab - \frac{1}{3}A_3F_4F_4, \tag{73}$$

where

$$\Sigma_{a_1\cdots a_r} := \frac{1}{(D-r)!}\varepsilon_{a_1\cdots a_r a_{r+1}\cdots a_D} e^{a_{r+1}} \cdots e^{a_D}, \tag{74}$$

$$\Gamma_{(n)} := \frac{1}{n!}\Gamma_{a_1\cdots a_n} e^{a_1} \cdots e^{a_n}, \; a := \frac{i}{4}\bar{\psi}\Gamma_{(2)}\psi, \tag{75}$$

$$b := \frac{i}{4}\bar{\psi}\Gamma_{(5)}\psi, \; F_4 = \mathrm{d}A_3, \tag{76}$$

and the $*$ symbol denotes the Hodge operator. In fact, if one sets the metric and gravitino field to zero, 11 dimensional supergravity [17] is reduced to a Chern–Simons like theory based on a three form A whose action is

$$S = \int\limits_{M_{11}} A_3 F_4 F_4, \tag{77}$$

where F_4 is a 4-form and M_{11} is an eleven dimensional manifold. This result allows us to conjecture that it would be possible to construct a theory of 11-dimensional Chern–Simons supergravity using a procedure similar to that shown in Ref. [8], which contains or ends at some limit in standard 11-dimensional supergravity theory [17].

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Appendix A

Consider now the explicit form of the equation (20) for $n = 2$ and $p = 1, 2, 3, 5, 7, 9$

$$\begin{aligned}
F^{A(2)} &= \mathrm{d}A^{A(1)} + \frac{1}{2}C^{A(1)}_{B_1(1)B_2(1)}A^{B_1(1)}A^{B_2(1)}, \\
F^{A(3)} &= \mathrm{d}A^{A(2)} + C^{A(2)}_{B_1(1)B_2(2)}A^{B_1(1)}A^{B_2(2)}, \\
F^{A(4)} &= \mathrm{d}A^{A(3)} + C^{A(3)}_{B_1(1)B_2(3)}A^{B_1(1)}A^{B_2(3)}, \\
F^{A(6)} &= \mathrm{d}A^{A(5)} + C^{A(5)}_{B_1(1)B_2(5)}A^{B_1(1)}A^{B_2(5)} + \frac{1}{2}C^{A(5)}_{B_1(3)B_2(3)}A^{B_1(3)}A^{B_2(3)}, \\
F^{A(8)} &= \mathrm{d}A^{A(7)} + C^{A(7)}_{B_1(1)B_2(7)}A^{B_1(1)}A^{B_2(7)} + C^{A(7)}_{B_1(3)B_2(5)}A^{B_1(3)}A^{B_2(5)}, \\
F^{A(10)} &= \mathrm{d}A^{A(9)} + C^{A(9)}_{B_1(1)B_2(9)}A^{B_1(1)}A^{B_2(9)} + C^{A(9)}_{B_1(3)B_2(7)}A^{B_1(3)}A^{B_2(7)} \\
&\quad + \frac{1}{2}C^{A(9)}_{B_1(5)B_2(5)}A^{B_1(5)}A^{B_2(5)},
\end{aligned} \tag{78}$$

where from $p = 3$ we have considered only odd-order gauge fields. Note that we have considered a FDA whose structure constants satisfy the condition $C^{A(q+r-1)}_{B(q)C(r)}$ for any $r < q$. These equations can be written as

$$
\begin{aligned}
F_2^A &= \mathrm{d}A_1^A + \frac{1}{2} C_{BC}^A A_1^B A_1^C, \\
F_3^A &= \mathrm{d}A_2^A + C_{BC}^A A_1^B A_2^C, \\
F_4^A &= \mathrm{d}A_3^A + C_{BC}^A A_1^B A_3^C, \\
F_6^A &= \mathrm{d}A_5^A + C_{BC}^A A_1^B A_5^C + \frac{1}{2} C_{BC}^A A_3^B A_3^C, \\
F_8^A &= \mathrm{d}A_7^A + C_{BC}^A A_1^B A_7^C + C_{BC}^A A_3^B A_5^C, \\
F_{10}^A &= \mathrm{d}A_9^A + C_{BC}^A A_1^B A_9^C + C_{BC}^A A_3^B A_7^C + \frac{1}{2} C_{BC}^A A_5^B A_5^C.
\end{aligned} \tag{79}
$$

Appendix B

The explicit form of the equation (21) for $n = 2$ and $p = 1, 2, 3, 5, 7, 9$ is

$$
\begin{aligned}
\mathrm{d}F^{A(2)} &= -C_{B_1(1)B_2(1)}^{A(1)} F^{B_1(2)} A^{B_2(1)}, \\
\mathrm{d}F^{A(3)} &= -C_{B_1(1)B_2(2)}^{A(2)} F^{B_1(2)} A^{B_2(2)} - C_{B_1(2)B_2(1)}^{A(2)} F^{B_1(3)} A^{B_2(1)}, \\
\mathrm{d}F^{A(4)} &= -C_{B_1(1)B_2(3)}^{A(3)} F^{B_1(2)} A^{B_2(3)} - C_{B_1(3)B_2(1)}^{A(3)} F^{B_1(4)} A^{B_2(1)}, \\
\mathrm{d}F^{A(6)} &= -C_{B_1(1)B_2(5)}^{A(5)} F^{B_1(2)} A^{B_2(5)} - C_{B_1(3)B_2(3)}^{A(5)} F^{B_1(4)} A^{B_2(3)} \\
&\quad - C_{B_1(5)B_2(1)}^{A(5)} F^{B_1(6)} A^{B_2(1)}, \\
\mathrm{d}F^{A(8)} &= -C_{B_1(1)B_2(7)}^{A(7)} F^{B_1(2)} A^{B_2(7)} - C_{B_1(3)B_2(5)}^{A(7)} F^{B_1(4)} A^{B_2(5)} \\
&\quad - C_{B_1(5)B_2(3)}^{A(7)} F^{B_1(6)} A^{B_2(3)} - C_{B_1(7)B_2(1)}^{A(7)} F^{B_1(8)} A^{B_2(1)}, \\
\mathrm{d}F^{A(10)} &= -C_{B_1(1)B_2(9)}^{A(9)} F^{B_1(2)} A^{B_2(9)} - C_{B_1(3)B_2(7)}^{A(9)} F^{B_1(4)} A^{B_2(7)} \\
&\quad - C_{B_1(5)B_2(5)}^{A(9)} F^{B_1(6)} A^{B_2(5)} - C_{B_1(7)B_2(3)}^{A(9)} F^{B_1(8)} A^{B_2(3)} \\
&\quad - C_{B_1(9)B_2(1)}^{A(9)} F^{B_1(10)} A^{B_2(1)},
\end{aligned} \tag{80}
$$

where from $p = 3$ we have considered only odd-order gauge fields. These equations can be written as

$$
\begin{aligned}
\mathrm{d}F_2^A &= -C_{BC}^A A_1^B F_2^C, \\
\mathrm{d}F_3^A &= -C_{BC}^A A_2^B F_2^C - C_{BC}^A A_1^B F_3^C, \\
\mathrm{d}F_4^A &= -C_{BC}^A A_3^B F_2^C - C_{BC}^A A_1^B F_4^C, \\
\mathrm{d}F_6^A &= -C_{BC}^A A_5^B F_2^C - C_{BC}^A A_3^B F_4^C - C_{BC}^A A_1^B F_6^C, \\
\mathrm{d}F_8^A &= -C_{BC}^A A_7^B F_2^C - C_{BC}^A A_5^B F_4^C - C_{BC}^A A_3^B F_6^C - C_{BC}^A A_1^B F_8^C, \\
\mathrm{d}F_{10}^A &= -C_{BC}^A A_9^B F_2^C - C_{BC}^A A_7^B F_4^C - C_{BC}^A A_5^B F_6^C - C_{BC}^A A_3^B F_8^C \\
&\quad - C_{BC}^A A_1^B F_{10}^C.
\end{aligned} \tag{81}
$$

Appendix C

The explicit form of the equation (24) for $n = 2$ and $p = 1, 2, 3, 5, 7, 9$ is given by

$$\begin{aligned}
\delta A^{A(1)} &= \mathrm{d}\lambda^{A(0)} + C^{A(0)}_{B_1(1)B_2(0)} A^{B_1(1)} \lambda^{B_2(0)}, \\
\delta A^{A(2)} &= \mathrm{d}\lambda^{A(1)} + C^{A(1)}_{B_1(1)B_2(1)} A^{B_1(1)} \lambda^{B_2(1)} + C^{A(1)}_{B_1(2)B_2(0)} A^{B_1(2)} \lambda^{B_2(0)}, \\
\delta A^{A(3)} &= \mathrm{d}\lambda^{A(2)} + C^{A(2)}_{B_1(1)B_2(2)} A^{B_1(1)} \lambda^{B_2(2)} + C^{A(2)}_{B_1(3)B_2(0)} A^{B_1(3)} \lambda^{B_2(0)}, \\
\delta A^{A(5)} &= \mathrm{d}\lambda^{A(4)} + C^{A(4)}_{B_1(1)B_2(4)} A^{B_1(1)} \lambda^{B_2(4)} + C^{A(4)}_{B_1(3)B_2(2)} A^{B_1(3)} \lambda^{B_2(2)} \\
&\quad + C^{A(4)}_{B_1(5)B_2(0)} A^{B_1(5)} \lambda^{B_2(0)}, \\
\delta A^{A(7)} &= \mathrm{d}\lambda^{A(6)} + C^{A(6)}_{B_1(1)B_2(6)} A^{B_1(1)} \lambda^{B_2(76)} + C^{A(6)}_{B_1(3)B_2(4)} A^{B_1(3)} \lambda^{B_2(4)} \\
&\quad + C^{A(6)}_{B_1(5)B_2(2)} A^{B_1(5)} \lambda^{B_2(2)} + C^{A(6)}_{B_1(7)B_2(0)} A^{B_1(7)} \lambda^{B_2(0)}, \\
\delta A^{A(9)} &= \mathrm{d}\lambda^{A(8)} + C^{A(8)}_{B_1(1)B_2(8)} A^{B_1(1)} \lambda^{B_2(8)} + C^{A(8)}_{B_1(3)B_2(6)} A^{B_1(3)} \lambda^{B_2(6)} \\
&\quad + C^{A(8)}_{B_1(5)B_2(4)} A^{B_1(5)} \lambda^{B_2(4)} + C^{A(8)}_{B_1(7)B_2(2)} A^{B_1(7)} \lambda^{B_2(2)} + C^{A(8)}_{B_1(9)B_2(0)} A^{B_1(9)} \lambda^{B_2(0)},
\end{aligned} \tag{82}$$

where from $p = 3$ we have considered only odd-order gauge fields. These equations can be written as

$$\begin{aligned}
\delta A^A_1 &= \mathrm{d}\lambda^A_0 + C^A_{BC} A^B_1 \lambda^C_0, \\
\delta A^A_2 &= \mathrm{d}\lambda^A_1 + C^A_{BC} A^B_1 \lambda^C_1 + C^A_{BC} A^B_2 \lambda^C_0, \\
\delta A^A_3 &= \mathrm{d}\lambda^A_2 + C^A_{BC} A^B_1 \lambda^C_2 + C^A_{BC} A^B_3 \lambda^C_0, \\
\delta A^A_5 &= \mathrm{d}\lambda^A_4 + C^A_{BC} A^B_1 \lambda^C_4 + C^A_{BC} A^B_3 \lambda^C_2 + C^A_{BC} A^B_5 \lambda^C_0, \\
\delta A^A_7 &= \mathrm{d}\lambda^A_6 + C^A_{BC} A^B_1 \lambda^C_6 + C^A_{BC} A^B_3 \lambda^C_4 + C^A_{BC} A^B_5 \lambda^C_2 + C^A_{BC} A^B_7 \lambda^C_0, \\
\delta A^A_9 &= \mathrm{d}\lambda^A_8 + C^A_{BC} A^B_1 \lambda^C_8 + C^A_{BC} A^B_3 \lambda^C_6 + C^A_{BC} A^B_5 \lambda^C_4 + C^A_{BC} A^B_7 \lambda^C_2 \\
&\quad + C^A_{BC} A^B_9 \lambda^C_0.
\end{aligned} \tag{83}$$

Appendix D

The explicit form of the equation (27) for $n = 2$ and $p = 1, 2, 3, 5, 7, 9$ is given by

$$\begin{aligned}
\delta F^{A(2)} &= \nabla\left(\delta A^{A(1)}\right) = \mathrm{d}\left(\delta A^{A(1)}\right) + C^{A(1)}_{B_1(1)B_2(1)} \delta A^{B_1(1)} A^{B_2(1)}, \\
\delta F^{A(4)} &= \nabla\left(\delta A^{A(3)}\right) = \mathrm{d}\left(\delta A^{A(3)}\right) + C^{A(3)}_{B_1(1)B_2(3)} \delta A^{B_1(1)} A^{B_2(3)} \\
&\quad + C^{A(3)}_{B_1(3)B_2(1)} \delta A^{B_1(3)} A^{B_2(1)}, \\
\delta F^{A(6)} &= \nabla\left(\delta A^{A(5)}\right) = \mathrm{d}\left(\delta A^{A(5)}\right) + C^{A(5)}_{B_1(1)B_2(5)} \delta A^{B_1(1)} A^{B_2(5)} \\
&\quad + C^{A(5)}_{B_1(3)B_2(3)} \delta A^{B_1(3)} A^{B_2(3)} + C^{A(5)}_{B_1(5)B_2(1)} \delta A^{B_1(5)} A^{B_2(1)}, \\
\delta F^{A(8)} &= \nabla\left(\delta A^{A(7)}\right) = \mathrm{d}\left(\delta A^{A(7)}\right) + C^{A(7)}_{B_1(1)B_2(7)} \delta A^{B_1(1)} A^{B_2(7)} \\
&\quad + C^{A(7)}_{B_1(3)B_2(5)} \delta A^{B_1(3)} A^{B_2(5)} + C^{A(7)}_{B_1(5)B_2(3)} \delta A^{B_1(5)} A^{B_2(3)} \\
&\quad + C^{A(7)}_{B_1(7)B_2(1)} \delta A^{B_1(7)} A^{B_2(1)},
\end{aligned}$$

$$\begin{aligned}\delta F^{A(10)} = \nabla\left(\delta A^{A(9)}\right) = \mathrm{d}\left(\delta A^{A(9)}\right) &+ C^{A(9)}_{B_1(1)B_2(9)}\delta A^{B_1(1)}A^{B_2(9)} \\ &+ C^{A(9)}_{B_1(3)B_2(7)}\delta A^{B_1(3)}A^{B_2(7)} \\ &+ C^{A(9)}_{B_1(5)B_2(5)}\delta A^{B_1(5)}A^{B_2(5)} + C^{A(9)}_{B_1(7)B_2(3)}\delta A^{B_1(7)}A^{B_2(3)} \\ &+ C^{A(9)}_{B_1(9)B_2(1)}\delta A^{B_1(9)}A^{B_2(1)}, \end{aligned} \tag{84}$$

where, from $p = 3$ we have considered only odd-order gauge fields. These equations can be written as

$$\begin{aligned}\delta F_2^A &= \mathrm{d}\left(\delta A_1^A\right) + C^A_{BC}A_1^B\delta A_1^C, \\ \delta F_4^A &= \mathrm{d}\left(\delta A_3^A\right) + C^A_{BC}A_3^B\delta A_1^C + C^A_{BC}A_1^B\delta A_3^C, \\ \delta F_6^A &= \mathrm{d}\left(\delta A_5^A\right) + C^A_{BC}A_5^B\delta A_1^C + C^A_{BC}A_3^B\delta A_3^C + C^A_{BC}A_1^B\delta A_5^C, \\ \delta F_8^A &= \mathrm{d}\left(\delta A_7^A\right) + C^A_{BC}A_7^B\delta A_1^C + C^A_{BC}A_5^B\delta A_3^C + C^A_{BC}A_3^B\delta A_5^C \\ &\quad + C^A_{BC}A_1^B\delta A_7^C, \\ \delta F_{10}^A &= \mathrm{d}\left(\delta A_9^A\right) + C^A_{BC}A_9^B\delta A_1^C + C^A_{BC}A_7^B\delta A_3^C + C^A_{BC}A_5^B\delta A_5^C \\ &\quad + C^A_{BC}A_3^B\delta A_7^C + C^A_{BC}A_1^B\delta A_9^C. \end{aligned} \tag{85}$$

Appendix E

In this appendix we show that (85) correspond to homogeneous transformations.

E.1. Gauge transformation of the rank-2 field strength tensor F_2

Introducing the first equations of (83) in the first equation of (85) we have

$$\begin{aligned}\delta F_2^A &= C^A_{BC}\mathrm{d}A_1^B\lambda_0^C - C^A_{BC}A_1^B\mathrm{d}\lambda_0^C + C^A_{BC}A_1^B\mathrm{d}\lambda_0^C + C^A_{BC}A_1^B C^C_{EF}A_1^E\lambda_0^F \\ &= C^A_{BC}\mathrm{d}A_1^B\lambda_0^C + C^A_{BC}A_1^B C^C_{EF}A_1^E\lambda_0^F. \end{aligned} \tag{86}$$

Using the nomenclature of Refs. [1–4], this equation takes the form

$$\delta F_2 = [\mathrm{d}A_1 + A_1A_1, \lambda_0] = [F_2, \lambda_0]. \tag{87}$$

E.2. Gauge transformation of the rank-4 field strength tensor F_4

Introducing the first and third equations of (83) in the second equation of (85), we have

$$\begin{aligned}\delta F_4^A &= C^A_{BC}\mathrm{d}A_1^B\lambda_2^C - C^A_{BC}A_1^B\mathrm{d}\lambda_2^C + C^A_{BC}\mathrm{d}A_3^B\lambda_0^C - C^A_{BC}A_3^B\mathrm{d}\lambda_0^C \\ &\quad + C^A_{BC}A_3^B d\lambda_0^C + C^A_{BC}A_3^B C^C_{EF}A_1^E\lambda_0^F + C^A_{BC}A_1^B d\lambda_2^C \\ &\quad + C^A_{BC}A_1^B C^C_{EF}A_1^E\lambda_2^F + C^A_{BC}A_1^B C^C_{EF}A_3^E\lambda_0^F \\ &= C^A_{BC}\mathrm{d}A_1^B\lambda_2^C + C^A_{BC}\mathrm{d}A_3^B\lambda_0^C + C^A_{BC}A_3^B C^C_{EF}A_1^E\lambda_0^F \\ &\quad + C^A_{BC}A_1^B C^C_{EF}A_1^E\lambda_2^F + C^A_{BC}A_1^B C^C_{EF}A_3^E\lambda_0^F. \end{aligned} \tag{88}$$

Using the nomenclature of Refs. [1–4], this equation takes the form

$$\begin{aligned}\delta F_4 &= [\mathrm{d}A_1, \lambda_2] + [\mathrm{d}A_3, \lambda_0] + \left[A_3, [A_1, \lambda_0]\right] + \left[A_1, [A_1, \lambda_2]\right] + \left[A_1, [A_3, \lambda_0]\right] \\ &= [\mathrm{d}A_1 + A_1A_1, \lambda_2] + \left[\mathrm{d}A_3 + \{A_1A_3\}, \lambda_0\right] \\ &= [F_2, \lambda_2] + [F_4, \lambda_0]. \end{aligned} \tag{89}$$

E.3. Gauge transformation of the rank-4 field strength tensor F_6

Introducing the first, third and fifth equations of (83) in the third equation of (85), we have

$$\begin{aligned}
\delta F_6^A &= C_{BC}^A \mathrm{d}A_1^B \lambda_4^C - C_{BC}^A A_1^B \mathrm{d}\lambda_4^C + C_{BC}^A \mathrm{d}A_3^B \lambda_2^C - C_{BC}^A A_3^B \mathrm{d}\lambda_2^C \\
&\quad + C_{BC}^A \mathrm{d}A_5^B \lambda_0^C - C_{BC}^A A_5^B \mathrm{d}\lambda_0^C + C_{BC}^A A_5^B d\lambda_0^C + C_{BC}^A A_5^B C_{EF}^C A_1^E \lambda_0^F \\
&\quad + C_{BC}^A A_3^B d\lambda_2^C + C_{BC}^A A_3^B C_{EF}^C A_1^E \lambda_2^F + C_{BC}^A A_3^B C_{EF}^C A_3^E \lambda_0^F \\
&\quad + C_{BC}^A A_1^B d\lambda_4^C + C_{BC}^A A_1^B C_{EF}^C A_1^E \lambda_4^F + C_{BC}^A A_1^B C_{EF}^C A_3^E \lambda_2^F \\
&\quad + C_{BC}^A A_1^B C_{EF}^C A_5^E \lambda_0^F \\
&= C_{BC}^A \mathrm{d}A_1^B \lambda_4^C + C_{BC}^A \mathrm{d}A_3^B \lambda_2^C + C_{BC}^A \mathrm{d}A_5^B \lambda_0^C + C_{BC}^A A_5^B C_{EF}^C A_1^E \lambda_0^F \\
&\quad + C_{BC}^A A_3^B C_{EF}^C A_1^E \lambda_2^F + C_{BC}^A A_3^B C_{EF}^C A_3^E \lambda_0^F + C_{BC}^A A_1^B C_{EF}^C A_1^E \lambda_4^F \\
&\quad + C_{BC}^A A_1^B C_{EF}^C A_3^E \lambda_2^F + C_{BC}^A A_1^B C_{EF}^C A_5^E \lambda_0^F .
\end{aligned}$$

Using the nomenclature of Refs. [1–4] this equation takes the form

$$\begin{aligned}
\delta F_6 &= [\mathrm{d}A_1, \lambda_4] + [\mathrm{d}A_3, \lambda_2] + [\mathrm{d}A_5, \lambda_0] + \big[A_5, [A_1, \lambda_0]\big] + \big[A_3, [A_1, \lambda_2]\big] \\
&\quad + \big[A_3, [A_3, \lambda_0]\big] + \big[A_1, [A_1, \lambda_4]\big] + \big[A_1, [A_3, \lambda_2]\big] + \big[A_1, [A_5, \lambda_0]\big],
\end{aligned}$$

so that

$$\begin{aligned}
\delta F_6 &= [\mathrm{d}A_1 + A_1 A_1, \lambda_4] + \big[\mathrm{d}A_3 + [A_1, A_3], \lambda_2\big] + \left[\mathrm{d}A_5 + [A_1, A_5] + \frac{1}{2}[A_3, A_3], \lambda_0\right] \\
&= [F_6, \lambda_0] + [F_4, \lambda_2] + [F_2, \lambda_4].
\end{aligned}$$

E.4. Gauge transformation of the field strength tensor F_8

Introducing the first, third, fifth and seventh equations of (83) in the fourth equation of (85), we have

$$\begin{aligned}
\delta F_8^A &= C_{BC}^A \mathrm{d}A_1^B \lambda_6^C + C_{BC}^A \mathrm{d}A_3^B \lambda_4^C + C_{BC}^A \mathrm{d}A_5^B \lambda_2^C + C_{BC}^A \mathrm{d}A_7^B \lambda_0^C \\
&\quad + C_{BC}^A A_7^B C_{EF}^C A_1^E \lambda_0^F + C_{BC}^A A_5^B C_{EF}^C A_1^E \lambda_2^F + C_{BC}^A A_5^B C_{EF}^C A_3^E \lambda_0^F \\
&\quad + C_{BC}^A A_3^B C_{EF}^C A_1^E \lambda_4^F + C_{BC}^A A_3^B C_{EF}^C A_3^E \lambda_2^F + C_{BC}^A A_3^B C_{EF}^C A_5^E \lambda_0^F \\
&\quad + C_{BC}^A A_1^B C_{EF}^C A_1^E \lambda_6^F + C_{BC}^A A_1^B C_{EF}^C A_3^E \lambda_4^F + C_{BC}^A A_1^B C_{EF}^C A_5^E \lambda_2^F \\
&\quad + C_{BC}^A A_1^B C_{EF}^C A_7^E \lambda_0^F .
\end{aligned}$$

Using the nomenclature of Refs. [1–4], this equation takes the form

$$\begin{aligned}
\delta F_8 &= [\mathrm{d}A_1, \lambda_6] + [\mathrm{d}A_3, \lambda_4] + [\mathrm{d}A_5, \lambda_2] + [\mathrm{d}A_7, \lambda_0] + \big[A_7, [A_1, \lambda_0]\big] \\
&\quad + \big[A_5, [A_1, \lambda_2]\big] + \big[A_5, [A_3, \lambda_0]\big] + \big[A_3, [A_1, \lambda_4]\big] + \big[A_3, [A_3, \lambda_2]\big] \\
&\quad + \big[A_3, [A_5, \lambda_0]\big] + \big[A_1, [A_1, \lambda_6]\big] + \big[A_1, [A_3, \lambda_4]\big] + \big[A_1, [A_5, \lambda_2]\big] \\
&\quad + \big[A_1, [A_7, \lambda_0]\big],
\end{aligned}$$

so that

$$\begin{aligned}
\delta F_8 &= [\mathrm{d}A_1 + A_1 A_1, \lambda_6] + \big[\mathrm{d}A_3 + [A_1, A_3], \lambda_4\big] + \left[\mathrm{d}A_5 + [A_1, A_5] + \frac{1}{2}[A_3, A_3], \lambda_2\right] \\
&\quad + \big[\mathrm{d}A_7 + [A_3, A_5] + [A_1, A_7], \lambda_0\big] = [F_8, \lambda] + [F_6, \lambda_2] + [F_4, \lambda_4] + [F, \lambda_6].
\end{aligned}$$

E.5. Gauge transformation of the field strength tensor F_{10}

Introducing the first, third, fifth, seventh and ninth equations from (83) in the fifth equation of (85) we have

$$\begin{aligned}\delta F_{10}^A = {} & C_{BC}^A \mathrm{d}A_1^B \lambda_8^C + C_{BC}^A \mathrm{d}A_3^B \lambda_6^C + C_{BC}^A \mathrm{d}A_5^B \lambda_4^C + C_{BC}^A \mathrm{d}A_7^B \lambda_2^C \\ & + C_{BC}^A \mathrm{d}A_9^B \lambda_0^C + C_{BC}^A A_9^B C_{EF}^C A_1^E \lambda_0^F + C_{BC}^A A_7^B C_{EF}^C A_1^E \lambda_2^F \\ & + C_{BC}^A A_7^B C_{EF}^C A_3^E \lambda_0^F + C_{BC}^A A_5^B C_{EF}^C A_1^E \lambda_4^F + C_{BC}^A A_5^B C_{EF}^C A_3^E \lambda_2^F \\ & + C_{BC}^A A_5^B C_{EF}^C A_5^E \lambda_0^F + C_{BC}^A A_3^B C_{EF}^C A_1^E \lambda_6^F + C_{BC}^A A_3^B C_{EF}^C A_3^E \lambda_4^F \\ & + C_{BC}^A A_3^B C_{EF}^C A_5^E \lambda_2^F + C_{BC}^A A_3^B C_{EF}^C A_7^E \lambda_0^F + C_{BC}^A A_1^B C_{EF}^C A_1^E \lambda_8^F \\ & + C_{BC}^A A_1^B C_{EF}^C A_3^E \lambda_6^F + C_{BC}^A A_1^B C_{EF}^C A_5^E \lambda_4^F + C_{BC}^A A_1^B C_{EF}^C A_7^E \lambda_2^F \\ & + C_{BC}^A A_1^B C_{EF}^C A_9^E \lambda_0^F.\end{aligned}$$

Using the nomenclature of Refs. [1–4], this equation takes the form

$$\begin{aligned}\delta F_{10} = {} & [\mathrm{d}A_1, \lambda_8] + [\mathrm{d}A_3, \lambda_6] + [\mathrm{d}A_5, \lambda_4] + [\mathrm{d}A_7, \lambda_2] + [\mathrm{d}A_9, \lambda_0] \\ & + \big[A_9, [A_1, \lambda_0]\big] + \big[A_7, [A_1, \lambda_2]\big] + \big[A_7, [A_3, \lambda_0]\big] + \big[A_5, [A_1, \lambda_4]\big] \\ & + \big[A_5, [A_3, \lambda_2]\big] + \big[A_5, [A_5, \lambda_0]\big] + \big[A_3, [A_1, \lambda_6]\big] + \big[A_3, [A_3, \lambda_4]\big] \\ & + \big[A_3, [A_5, \lambda_2]\big] + \big[A_3, [A_7, \lambda_0]\big] + \big[A_1, [A_1, \lambda_8]\big] + \big[A_1, [A_3, \lambda_6]\big] \\ & + \big[A_1, [A_5, \lambda_4]\big] + \big[A_1, [A_7, \lambda_2]\big] + \big[A_1, [A_9, \lambda_0]\big],\end{aligned}$$

so that

$$\begin{aligned}\delta F_{10} = {} & [\mathrm{d}A_1 + A_1 A_1, \lambda_8] + \big[\mathrm{d}A_3 + [A_1, A_3], \lambda_6\big] + \left[\mathrm{d}A_5 + [A_1, A_5] + \frac{1}{2}[A_3, A_3], \lambda_4\right] \\ & + \big[\mathrm{d}A_7 + [A_3, A_5] + [A_1, A_7], \lambda_2\big] + \left[\mathrm{d}A_9 + [A_1, A_9] + [A_3, A_7]\right. \\ & \left. + \frac{1}{2}[A_5, A_5], \lambda_0\right] \\ = {} & [F_{10}, \lambda_0] + [F_8, \lambda_2] + [F_6, \lambda_4] + [F_4, \lambda_6] + [F_2, \lambda_8].\end{aligned}$$

Appendix F

It is interesting to note that the difference between the coefficients that accompany the terms of equations (22), (23), (25), (28) of this article and the coefficients of the corresponding equations of Refs. [1–4] can be understood as follows. Consider the FDA given by Eq. (16), which leads to the definition of curvature given by Eq. (20). This last equation was restricted to the case where the only nonzero structure constants are those with only two low indices. This means that equations (16) and (20) take the form

$$\mathrm{d}\Theta^{A(p)} + \frac{1}{2} C^{A(p)}_{B_1(p_1) B_2(p_2)} \Theta^{B_1(p_1)} \wedge \Theta^{B_2(p_2)} = 0, \tag{90}$$

$$F^{A(p+1)} = \mathrm{d}A^{A(p)} + \frac{1}{2} C^{A(p)}_{B_1(p_1) B_2(p_2)} A^{B_1(p_1)} \wedge A^{B_2(p_2)}. \tag{91}$$

The next step is to consider that all the structure constants of the FDA (91) can be written in terms of the structure constants $C^{A}_{B_1B_2}$ of a Lie algebra. This allows us to write the Eq. (91) in the form shown in equations (80) and (22).

We have seen that: (i) the generalized field strength tensors transform homogeneously and (ii) the generalized Chern–Pontryagin invariants are polynomials in the fields strength tensors and are invariant under gauged and diffeomorphism transformations. Since this invariance is maintained under a linear redefinition of the tensor gauge fields it is direct to prove that the fields strength tensor found in the Eqs. (80) and (22) can be mapped into the field strength tensors defined in Refs. [1–4]. In fact, defining the extended gauged fields of the following form

$$\begin{aligned} &A \longrightarrow \bar{A} = A;\ A_3 \longrightarrow \bar{A}_3 = aA_3;\ A_5 \longrightarrow \bar{A}_5 = 2aA_5,\\ &A_7 \longrightarrow \bar{A}_7 = 6a^3A_7;\ A_9 \longrightarrow \bar{A}_9 = 24a^4A_9, \end{aligned} \tag{92}$$

where a is an arbitrary number, we found that Eq. (80) takes the form

$$\begin{aligned} \bar{F}^A_2 &= \mathrm{d}\bar{A}^A_1 + \frac{1}{2}C^A_{BC}\bar{A}^B_1\bar{A}^C_1,\\ \bar{F}^A_3 &= \mathrm{d}\bar{A}^A_2 + \frac{1}{2}C^A_{BC}\bar{A}^B_1\bar{A}^C_2,\\ \bar{F}^A_4 &= aF^A_4 = \mathrm{d}A^A_3 + C^A_{BC}\bar{A}^B_1\bar{A}^C_3,\\ \bar{F}^A_6 &= 2a^2F^A_6 = \mathrm{d}\bar{A}^A_5 + C^A_{BC}\bar{A}^B_1\bar{A}^C_5 + C^A_{BC}\bar{A}^B_3\bar{A}^C_3,\\ \bar{F}^A_8 &= 6a^3F^A_8 = \mathrm{d}\bar{A}^A_7 + C^A_{BC}\bar{A}^B_1\bar{A}^C_7 + 3C^A_{BC}\bar{A}^B_3\bar{A}^C_5,\\ \bar{F}^A_{10} &= 24a^4F^A_{10} = \mathrm{d}\bar{A}^A_9 + C^A_{BC}\bar{A}^B_1\bar{A}^C_9 + 4C^A_{BC}\bar{A}^B_3\bar{A}^C_7 + 3C^A_{BC}\bar{A}^B_5\bar{A}^C_5, \end{aligned} \tag{93}$$

which can be written in the form

$$\begin{aligned} \bar{F} &= \mathrm{d}\bar{A} + \bar{A}^2,\\ \bar{F}_3 &= \mathrm{d}\bar{A}_2 + [\bar{A}, \bar{A}_2],\\ \bar{F}_4 &= \mathrm{d}\bar{A}_3 + [\bar{A}, \bar{A}_3],\\ \bar{F}_6 &= \mathrm{d}\bar{A}_5 + [\bar{A}, \bar{A}_5] + [\bar{A}_3, \bar{A}_3],\\ \bar{F}_8 &= \mathrm{d}\bar{A}_7 + [\bar{A}, \bar{A}_7] + 3[\bar{A}_3, \bar{A}_5],\\ \bar{F}_{10} &= \mathrm{d}\bar{A}_9 + [\bar{A}, \bar{A}_9] + 4[\bar{A}_3, \bar{A}_7] + 3[\bar{A}_5, \bar{A}_5]. \end{aligned} \tag{94}$$

These equations coincide exactly with the equations (A2) of Ref. [3].

The equations (A5), (A1), and (A4) of Ref. [3] can be obtained in an analogous way. In fact, taking into account that the transformations (92) induce in the field strengths and in the gauge parameters the transformations

$$\begin{aligned} &\bar{F} = F,\ \bar{F}_4 = aF_4,\ \bar{F}_6 = 2a^2F_6,\\ &\bar{F}_8 = 6a^3F_8,\ \bar{F}_{10} = 24a^4F_{10}, \end{aligned} \tag{95}$$

$$\begin{aligned} &\bar{\lambda} = \lambda,\ \bar{\lambda}_4 = a\lambda_4, \bar{\lambda}_6 = 2a^2\lambda_6,\\ &\bar{\lambda}_8 = 6a^3\lambda_8,\ \bar{\lambda}_{10} = 24a^4\lambda_{10}, \end{aligned} \tag{96}$$

it is straightforward to find that the equations (23), (25), (28) take the form from Eqs. (A5), (A1), and (A4) of Ref. [3]. In the same way we can see that, after using the Eqs. (92), (95), (96), the Eqs. (48), (64), (55), (71) take the form from Eqs. (1.7), (1.8), (2.15) and (3.10) in Ref. [3].

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