Analytical investigations of electromagnetic cascades in photon gas

T. NAKATSUKA[1], A. IYONO[2], H. MATSUMOTO[2], K. OKEI[3], S. TSUJI[3], J. NISHIMURA[4]
[1]*Laboratory of Information Science, Okayama Shoka University, Japan*
[2]*Department of Fundamental Science, Okayama University of Science, Japan*
[3]*Department of Natural Sciences, Kawasaki Medical School, Japan*
[4]*The Institute of Space and Astronomical Science, JAXA, Japan*
nakatuka@osu.ac.jp
DOI: 10.7529/ICRC2011/V08/0970

Abstract: Exact analytical solutions of the diffusion equation are obtained by solving the differential-difference equations for electron-photon cascades developing in photon gas, assuming simplified cross-sections. The results are compared with those obtained by a numerical method, and analytical properties of cascades in photon gas are investigated and discussed.

Keywords: gamma-ray astronomy, electron-photon cascades, differential-difference equation

1 Introduction

Electron-photon cascades developing in astrophysical environments are closely investigated by numerical and Monte Carlo methods [1, 2], though analytical approaches to solve cascades in magnetic fields and photon gas are not yet succeeded. In those cases, the cross-sections are not expressed by the ratio of the primary and secondary energies, and Mellin transforms of the diffusion equation results in the differential-difference equation, so that the traditional Landau-Rumer method [3, 4] cannot solve the problem anymore. We have found a method to solve the differential-difference equation and found the exact analytical solution of cascades in the photon gas assuming simplified cross-sections valid in a certain range of particle energies. Analytical properties of the results are also investigated.

2 Diffusion equation for the cascades in photon gas

We proposed the diffusion equation for the electron-photon cascades in photon gas [5]:

$$\begin{aligned}\frac{\partial\pi(\kappa,t)}{\kappa_0\partial t} &= -\frac{\pi(\kappa)}{\kappa}\int_0^1\phi(\kappa,v)dv \\ &\quad+\int_\kappa^{\kappa_0}\phi(\kappa',1-\frac{\kappa}{\kappa'})\frac{\pi(\kappa')}{\kappa'}\frac{d\kappa'}{\kappa'} \\ &\quad+2\int_\kappa^{\kappa_0}\psi(\lambda',\frac{\kappa}{\lambda'})\frac{\gamma(\lambda')}{\lambda'}\frac{d\lambda'}{\lambda'}, \qquad (1)\\ \frac{\partial\gamma(\lambda,t)}{\kappa_0\partial t} &= \int_\lambda^{\kappa_0}\phi(\kappa',\frac{\lambda}{\kappa'})\frac{\pi(\kappa')}{\kappa'}\frac{d\kappa'}{\kappa'} \\ &\quad-\frac{\gamma(\lambda)}{\lambda}\int_0^1\psi(\lambda,u)du, \qquad (2)\end{aligned}$$

for the differential energy spectra of shower electrons $\pi(\kappa,t)d\kappa$ and photons $\gamma(\lambda,t)d\lambda$, with

$$\kappa\equiv\omega_0\varepsilon_{\rm e} \qquad \text{and} \qquad \lambda\equiv\omega_0\varepsilon_\gamma, \qquad (3)$$

where $\varepsilon_{\rm e}$, ε_γ, and ω_0 denote the energies of the shower electron, shower photon, and background photon in units of mc^2, respectively, and $u\equiv\varepsilon_{\rm e}/\varepsilon_\gamma$ and $v\equiv\varepsilon_\gamma/\varepsilon_{\rm e}$ denote fractional energies. t denotes the penetration depth of mono-energetic and isotropic photon gas, though we take the unit $X_*^{\rm (G)}$ twice of Aharonian and Plyasheshnikov's $X_0^{\rm (G)}$ [1] this time, for the sake of analytical simplicity:

$$X_*^{\rm (G)}\equiv 2X_0^{\rm (G)}=\left[2\pi n_0^{\rm (G)}r_0^2\right]^{-1}\kappa_0, \qquad (4)$$

where κ_0 denotes κ or λ of the incident particle. Then $\frac{dN_{\rm IC}(\varepsilon_\gamma)}{d\varepsilon_\gamma}\equiv\frac{3\sigma_{\rm T}}{4\kappa\varepsilon_e}\phi(\kappa,v)$ and $\frac{dN_{\rm PP}(\varepsilon_e)}{d\varepsilon_e}\equiv\frac{3\sigma_{\rm T}}{4\lambda\varepsilon_\gamma}\psi(\lambda,u)$ denote the cross-sections for the inverse Compton scattering and the photon-photon pair production, respectively, as described in Aharonian [6] and Zdziarski [7], with $\sigma_{\rm T}$ of the Thomson cross-section. They are

$$\begin{aligned}\phi(\kappa,v) &= \frac{1}{2}\left(1-v+\frac{1}{1-v}\right) \\ &\quad+\frac{v}{8\kappa(1-v)}\left(3+v-\frac{1}{1-v}+4\ln\frac{v}{4\kappa(1-v)}\right) \\ &\quad-\frac{v^2}{8\kappa^2(1-v)^2} \quad \text{with} \quad v<1/\left(1+\frac{1}{4\kappa}\right), \qquad (5)\\ \psi(\lambda,u) &= \frac{1}{2}\left(\frac{1-u}{u}+\frac{u}{1-u}\right)+\frac{1}{8\lambda^2u^2(1-u)^2}\end{aligned}$$

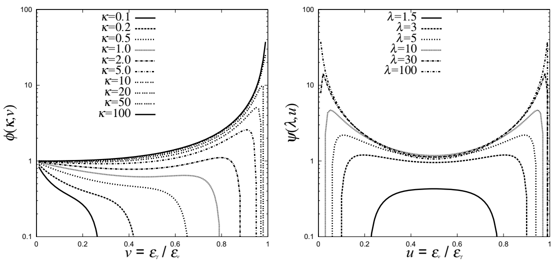

Figure 1: The normalized cross-sections for the inverse Compton scattering $\phi(\kappa, v) \equiv \frac{dN_{\rm IC}(\varepsilon_\gamma)}{d\varepsilon_\gamma} / \frac{3\sigma_{\rm T}}{4\kappa\varepsilon_e}$, where $\kappa = .1, .2, .5, \cdots, 100$, from bottom to top (left), and the photon-photon pair production $\psi(\lambda, u) \equiv \frac{dN_{\rm PP}(\varepsilon_e)}{d\varepsilon_e} / \frac{3\sigma_{\rm T}}{4\lambda\varepsilon_\gamma}$, where $\lambda = 1.5, 3, 5, 10, 30, 100$, from bottom to top (right).

$$- \frac{1}{8\lambda u(1-u)} \left(4 + \frac{1-u}{u} + \frac{u}{1-u} - 4 \ln \{4\lambda u(1-u)\} \right) \qquad \text{with}$$

$$\frac{1}{2} \left(1 - \sqrt{1 - \frac{1}{\lambda}} \right) < u < \frac{1}{2} \left(1 + \sqrt{1 - \frac{1}{\lambda}} \right), \quad (6)$$

as indicated in Fig. 1.

3 Analytical investigations of electromagnetic cascade processes in photon gas

3.1 Simplified cross-sections

We assume

$$\phi(\kappa, v) \simeq 1, \qquad \text{and} \qquad \psi(\lambda, u) \simeq 1, \qquad (7)$$

or simply approximate the cross-sections to be

$$\frac{dN_{\rm IC}(\varepsilon_\gamma)}{d\varepsilon_\gamma} \simeq \frac{3\sigma_{\rm T}}{4\kappa\varepsilon_e}, \quad \text{and} \quad \frac{dN_{\rm PP}(\varepsilon_e)}{d\varepsilon_e} \simeq \frac{3\sigma_{\rm T}}{4\lambda\varepsilon_\gamma}, \qquad (8)$$

which are valid in energy ranges of κ and λ of Eq. (3), from a few to ten.

3.2 Analytical solutions for shower equation with the simplified cross-sections

The diffusion equations (1) and (2) under the simplified cross-sections (7) are described as[1]

$$\frac{\partial}{\kappa_0 \partial t} \pi(\kappa, t) = -\frac{\pi(\kappa, t)}{\kappa} + \int_\kappa^{\kappa_0} \frac{\pi(\kappa', t)}{\kappa'^2} d\kappa' + 2 \int_\kappa^{\kappa_0} \frac{\gamma(\kappa', t)}{\kappa'^2} d\kappa', \qquad (9)$$

$$\frac{\partial}{\kappa_0 \partial t} \gamma(\kappa, t) = \int_\kappa^{\kappa_0} \frac{\pi(\kappa', t)}{\kappa'^2} d\kappa' - \frac{\gamma(\kappa, t)}{\kappa}. \qquad (10)$$

We solve the problem for the shower initiated by photon with the incident energy of κ_0;

$$\pi(\kappa, 0) = 0, \quad \text{and} \quad \gamma(\kappa, 0) = \delta(\kappa - \kappa_0). \qquad (11)$$

We apply Mellin transforms

$$\begin{pmatrix} \mathcal{M}(s,t) \\ \mathcal{N}(s,t) \end{pmatrix} \equiv \int_0^\infty d\kappa \left(\frac{\kappa}{\kappa_0} \right)^s \begin{pmatrix} \pi(\kappa,t) \\ \gamma(\kappa,t) \end{pmatrix}, \qquad (12)$$

then the diffusion equations become the differential-difference equations,

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathcal{M}(s,t) \\ \mathcal{N}(s,t) \end{pmatrix} = R(s) \begin{pmatrix} \mathcal{M}(s-1,t) \\ \mathcal{N}(s-1,t) \end{pmatrix} \qquad (13)$$

with

$$R(s) \equiv \begin{pmatrix} -s/(s+1) & 2/(s+1) \\ 1/(s+1) & -1 \end{pmatrix}, \qquad (14)$$

and the differential spectra of shower particles become

$$\begin{pmatrix} \pi(\kappa,t) \\ \gamma(\kappa,t) \end{pmatrix} \equiv \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds \frac{\kappa_0^s}{\kappa^{s+1}} \begin{pmatrix} \mathcal{M}(s,t) \\ \mathcal{N}(s,t) \end{pmatrix}. \qquad (15)$$

Initial conditions of (11) correspond to

$$\mathcal{M}(s, 0) = 0, \quad \text{and} \quad \mathcal{N}(s, 0) = 1. \qquad (16)$$

We derive the approximated solution for Eq. (13) by dividing t with n equal stepsizes, $\Delta t \equiv t/n$, and solve

$$\begin{pmatrix} \mathcal{M}_k(s) \\ \mathcal{N}_k(s) \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{k-1}(s) \\ \mathcal{N}_{k-1}(s) \end{pmatrix} + R(s) \begin{pmatrix} \mathcal{M}_{k-1}(s-1) \\ \mathcal{N}_{k-1}(s-1) \end{pmatrix} \Delta t, \quad k = 1, 2, \cdots, n, \qquad (17)$$

with $\mathcal{M}_0(s) = 0$ and $\mathcal{N}_0(s) = 1$. Then we have

$$\begin{pmatrix} \mathcal{M}_n(s) \\ \mathcal{N}_n(s) \end{pmatrix} = \sum_{k=0}^n {}_nC_k \left(\frac{t}{n} \right)^k R^{[k]}(s) \begin{pmatrix} \mathcal{M}_0(s-k) \\ \mathcal{N}_0(s-k) \end{pmatrix}, \qquad (18)$$

where

$$R^{[0]}(s) \equiv 1 \quad \text{and} \quad R^{[k]}(s) \equiv R(s) \cdot R(s-1) \cdots R(s-k+1). \qquad (19)$$

Applying the inverse Mellin transforms, we have the approximated differential electron spectrum $\pi_n(\kappa, t)$ as

$$\kappa \pi_n(\kappa, t) = \sum_{k=0}^n {}_nC_k \frac{t^k}{n^k} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{\kappa_0}{\kappa} \right)^s R_{1,2}^{[k]}(s) ds, \qquad (20)$$

where $R_{1,2}^{[k]}(s)$ denotes the 1,2 element of $R^{[k]}(s)$,

$$(-1)^0 R_{1,2}^{[0]}(s) = 0,$$

$$(-1)^1 R_{1,2}^{[1]}(s) = -\frac{2}{s+1},$$

$$(-1)^2 R_{1,2}^{[2]}(s) = -\frac{4}{s+1},$$

$$(-1)^3 R_{1,2}^{[3]}(s) = -\frac{2(3s^2 - 3s + 2)}{s(s+1)(s-1)},$$

$$(-1)^4 R_{1,2}^{[4]}(s) = -\frac{8(s^2 - 2s + 2)}{s(s+1)(s-2)}, \; \cdots.$$

1. The energy parameter λ of photon is also described as κ, hereafter.

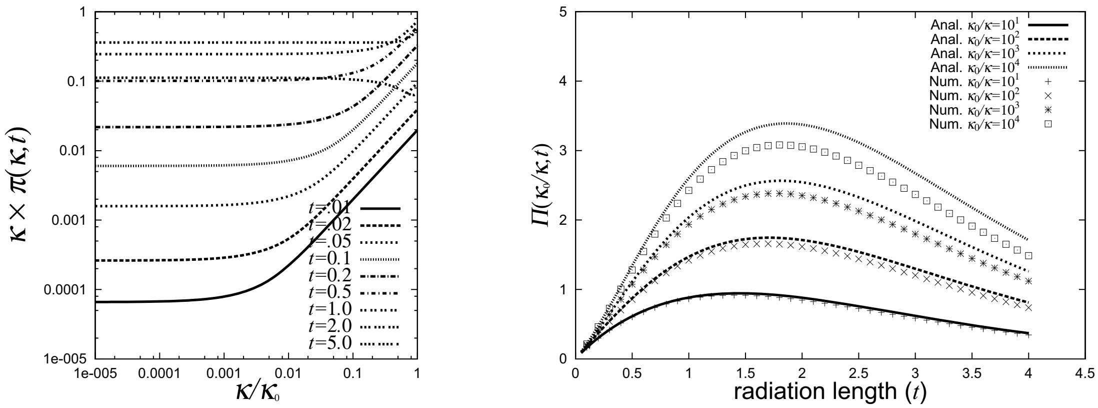

Figure 2: Exact differential spectra (left) and transition curves (right, lines) of electrons in photon-initiated electron-photon cascades in photon gas under the simplified cross-sections (7). Transition curves obtained by a numerical method under the same conditions are also plotted (right, points).

$R_{1,2}^{[k]}(s)$ have poles of the first order at $s=-1$ for $k=1,2$ and at $s=-1,0,k-2$ for $k\geq 3$, so we can derive $\kappa\pi_n(\kappa,t)$ as

$$\begin{aligned}
&\kappa\pi_n(\kappa,t)\\
=\ &\sum_{k=1}^{n} {}_nC_k \frac{t^k}{n^k}\left\{\left(\frac{\kappa_0}{\kappa}\right)^{k-2}[(s-k+2)R_{1,2}^{[k]}(s)]_{s\to k-2}\right.\\
&\left.+[sR_{1,2}^{[k]}(s)]_{s\to 0}+\frac{\kappa}{\kappa_0}[(s+1)R_{1,2}^{[k]}(s)]_{s\to -1}\right\}, \quad (21)
\end{aligned}$$

using residues at the poles. Residues for $k\geq 3$ are expressed by the empirical formulae

$$(-1)^k[(s-k+2)R_{1,2}^{[k]}(s)]_{s\to k-2}=-2k/3, \quad (22)$$

$$(-1)^k[sR_{1,2}^{[k]}(s)]_{s\to 0}=2k(k-1)/3, \quad (23)$$

$$(-1)^k[(s+1)R_{1,2}^{[k]}(s)]_{s\to -1}=-2k(k+1)/3. \quad (24)$$

The residues are $0,0,-2$ for $k=1$ and $0,0,-4$ for $k=2$ and are not expressed by these formulae, nevertheless the formulae give accurate results as a whole to the summation at $k=1$ and $k=2$. Thus we have

$$\begin{aligned}
&\kappa\pi_n(\kappa,t)\\
=\ &-\frac{2}{3}\sum_{k=1}^{n} {}_nC_k\left(-\frac{t}{n}\right)^k\\
&\times\left\{k\left(\frac{\kappa_0}{\kappa}\right)^{k-2}-k(k-1)+\frac{k(k+1)}{\kappa_0/\kappa}\right\}\\
=\ &-\frac{2}{3}\frac{\kappa^2}{\kappa_0^2}\left(-\frac{\kappa_0 t}{n\kappa}\right)n\left(1-\frac{\kappa_0 t}{n\kappa}\right)^{n-1}\\
&+\frac{2}{3}\frac{t^2}{n^2}n(n-1)\left(1-\frac{t}{n}\right)^{n-2}\\
&-\frac{2}{3}\frac{\kappa}{\kappa_0}\left(-\frac{t}{n}\right)n\left(1-\frac{t}{n}\right)^{n-2}\left(2-t-\frac{t}{n}\right)\\
=\ &\frac{2}{3}t^2\left(1-\frac{1}{n}\right)\left(1-\frac{t}{n}\right)^{n-2}\\
&+\frac{2}{3}\frac{\kappa t}{\kappa_0}\left\{\left(1-\frac{\kappa_0 t}{n\kappa}\right)^{n-1}\right.\\
&\left.+\left(1-\frac{t}{n}\right)^{n-2}\left(2-t-\frac{t}{n}\right)\right\}. \quad (25)
\end{aligned}$$

At the limit of $n\to\infty$, we have the solution $\pi(\kappa,t)$ of Eq. (9),

$$\kappa\pi(\kappa,t)=\frac{2}{3}\left(t-\frac{t-2}{\kappa_0/\kappa}\right)te^{-t}+\frac{2}{3}\frac{\kappa t}{\kappa_0}e^{-\kappa_0 t/\kappa}. \quad (26)$$

Likewise we have the approximated differential spectra of photon components,

$$\begin{aligned}
\kappa\gamma_n(\kappa,t)\ =\ &\kappa\delta(\kappa-\kappa_0)\left(1-\frac{t}{n}\right)^n\\
+\frac{t^2}{3}\left(1-\frac{1}{n}\right)&\left(1-\frac{t}{n}\right)^{n-2}-\frac{1}{3}\frac{\kappa t}{\kappa_0}\left\{2\left(1-\frac{\kappa_0 t}{n\kappa}\right)^{n-1}\right.\\
&\left.-\left(1-\frac{t}{n}\right)^{n-2}\left(2-t-\frac{t}{n}\right)\right\}, \quad (27)
\end{aligned}$$

thus at $n\to\infty$, we have the solution $\gamma(\kappa,t)$ of Eq. (10),

$$\begin{aligned}
\kappa\gamma(\kappa,t)\ =\ &\kappa\delta(\kappa-\kappa_0)e^{-t}\\
&+\frac{t}{3}\left(t-\frac{t-2}{\kappa_0/\kappa}\right)e^{-t}-\frac{2}{3}\frac{\kappa t}{\kappa_0}e^{-\kappa_0 t/\kappa}. \quad (28)
\end{aligned}$$

The results are indicated in Figs. 2 and 3. It should be noted that the solutions (26) and (28) are exact as they satisfy the diffusion equations (9), (10), and the initial conditions (11). Limiting values of

$$\kappa\pi(\kappa,t)\to(2/3)t^2e^{-t},\quad \kappa\gamma(\kappa,t)\to(1/3)t^2e^{-t} \quad (29)$$

for $\kappa/\kappa_0\ll 1$ well explain the differential spectra at the corresponding energies in Fig. 2 and 3, and limiting value

$$\kappa\pi(\kappa,t)\to(2\kappa/\kappa_0)te^{-t} \quad (30)$$

for $\kappa/\kappa_0\to 1$ well explains the differential spectra of electron in Fig. 2 at the higher energies, especially at $t\ll 1$.

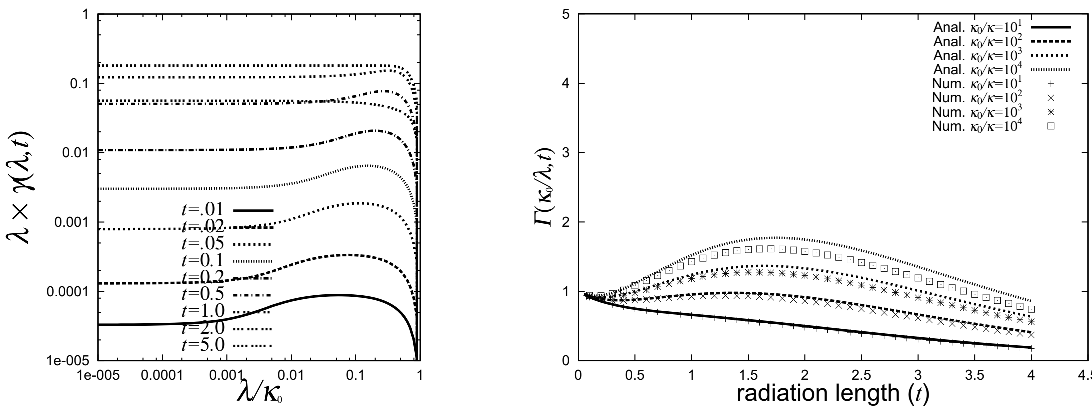

Figure 3: Exact differential spectra (left) and transition curves (right, lines) of photons in photon-initiated electron-photon cascades in photon gas under the simplified cross-sections (7). Transition curves obtained by a numerical method under the same conditions are also plotted (right, points).

3.3 Integrated spectra of shower particles, or transition curves

Integrating the spectra (26) and (28), we have the exact integrated spectra or transition curves of electron and photon, $\Pi(\kappa,t) \equiv \int_{\kappa}^{\kappa_0} \pi(\kappa,t)d\kappa$ and $\Gamma(\kappa,t) \equiv \int_{\kappa}^{\kappa_0} \gamma(\kappa,t)d\kappa$, as

$$\begin{aligned}\Pi(\kappa,t) &= \frac{2}{3}t\left\{(2-t)(1-\frac{\kappa}{\kappa_0})e^{-t} + te^{-t}\ln\frac{\kappa_0}{\kappa}\right.\\ &\quad\left. +E_2(t) - \frac{\kappa}{\kappa_0}E_2(\frac{\kappa_0 t}{\kappa})\right\}, \qquad (31)\end{aligned}$$

$$\begin{aligned}\Gamma(\kappa,t) &= e^{-t} + \frac{t}{3}\left\{(2-t)(1-\frac{\kappa}{\kappa_0})e^{-t}\right.\\ &\quad\left. +te^{-t}\ln\frac{\kappa_0}{\kappa} + 2E_2(t) - \frac{2\kappa}{\kappa_0}E_2(\frac{\kappa_0 t}{\kappa})\right\}, \qquad (32)\end{aligned}$$

where

$$E_2(z) \equiv \int_1^\infty \frac{e^{-zt}}{t^2}dt \qquad (33)$$

denotes the exponential integral function. Results are compared in Figs. 2 and 3 with those obtained by a numerical method [5] with tentative converging conditions, assuming the same cross-sections of (7). Both agree well in case of small incident energies, though some improved conditions should be required for the numerical method to agree with the analytical solutions in case of large incident energies.

Limiting values of

$$\begin{aligned}\Pi(\kappa,t) &\to 2(1-\kappa/\kappa_0)te^{-t}, \qquad (34)\\ \Gamma(\kappa,t) &\to e^{-t} + (4/3)(1-\kappa/\kappa_0)te^{-t} \qquad (35)\end{aligned}$$

for $\kappa_0/\kappa \to 1$ well explain the shape of transition curves of low incident energies in Fig. 2 and 3. It shows the peak position t_{m} of transition curve in Fig. 2 approaches to 1 for small incident energies, though it approaches to 2 for large incident energies due to the asymptotic attribute

$$\Pi(\kappa,t) \sim (2/3)t^2e^{-t}\ln(\kappa_0/\kappa) \qquad (36)$$

of Eq. (31) for $\kappa_0/\kappa \gg 1$, which gives an analytical confirmation of Aharonian and Plyasheshnikov's numerical prediction [1], $t_{\mathrm{m}} \sim$ 1–2. So that the peak value of $\Pi(\kappa,t)$ increases almost as $(8/3e^2)\ln(\kappa_0/\kappa)$ with increase of κ_0/κ.

4 Conclusions

The diffusion equation for the electron-photon cascades developing in photon gas is solved analytically, by solving differential-difference equations assuming simplified cross-sections. Analytical properties of the differential energy spectra and the transition curves are investigated. The solutions will valuable for examining the reliability of other numerical and Monte Carlo methods by comparing results derived under the same approximated cross-sections. The method will also be applicable to other cascade equations which the traditional Landau-Rumer method cannot solve.

References

[1] F.A. Aharonian, and A.V. Plyasheshnikov, Astropart. Phys., 2003, **19**: 525-548.

[2] V. Anguelov and H. Vankov, J. Phys. G, 1999, **25**: 1755-1764.

[3] B. Rossi and K. Greisen, Rev. Mod. Phys. , 1941, **27**, 240-309.

[4] J. Nishimura, in *Handbuch der Physik, Band 46*, edited by S. Flügge (Springer, Berlin, 1967), Teil **2**, p. 1.

[5] T. Nakatsuka, A. Iyono, and J. Nishimura, Proc. of *22nd European Cosmic Ray Symposium*, 2010, Turku, to be published.

[6] F.A. Aharonian: 2004, *Very High Energy Cosmic Gamma Radiations*, World Scientific Publishing Co. Pte. Ltd., Singapore.

[7] A.A. Zdziarski, ApJ, 1988, **335**: 786-802.