# THE FORMALISM OF LIE GROUPS

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## 1. INTRODUCTION

Throughout the history of quantum theory, a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. In support of this claim, they of course, justifiably, point to the successes of that prince of amateurs in this field, Dirac, particularly with the spinor representations of the Lorentz group.

As an amateur myself, I strongly believe in the truth of the non-professionalist creed. I think perhaps there is not much one has to learn in the way of methodology from the group theorists except caution. But this does not mean one should not be aware of the riches which have been amassed over the course of years particularly in that most highly developed of all mathematical disciplines - the theory of Lie groups.

My lectures then are an amateur's attempt to gather some of the fascinating results for compact simple Lie groups which are likely to be of physical interest. I shall state theorems; and with a physicist's typical unconcern rarely, if ever, shall I prove these. Throughout, the emphasis will be to show the close similarity of these general groups with that most familiar of all groups, the group of rotations in three dimensions.

In 1951 I had the good fortune to listen to Prof. Racah lecture on Lie groups at Princeton. After attending these lectures I thought this is really too hard; I cannot learn this; one is hardly ever likely to need all this complicated matter. I was completely wrong. Eleven years later the wheel has gone full cycle and it is my turn to lecture on this subject. I am sure many of you will feel after these lectures that all this is too damned hard and unphysical. The only thing I can say is: I do very much hope and wish you do not have to learn this beautiful theory eleven years too late.

## 2. SOURCES

A word about the sources [1] and the scheme I wish to follow. The chief sources in this theory are the famous thesis of Cartan in which most of this subject was created Hermann Weyl and his classical text on "Classical Groups" and Racah's Princeton lectures [2]. However, I believe conceptually the most concise existing treatment of the subject is in the works of DYNKIN [3]. Dynkin's paper has a magnificent appendix which gives a review of the known results and this appendix is my major source. From the point of view of a physicist working on symmetry problems perhaps the best reference is to the review paper of BEHRENDS, LEE, FRONSDAL and DREITLEIN [4]. I have checked with Lee that apparently while these authors knew of Dynkin's work they did not have it accessible when they were writing their review. Thus their treatment of the fundamentals resembles Cartan and Racah more closely rather than Dynkin. Another excellent paper for physicists is SPEISER and TARSKI [5]. For a fuller exposition of Dynkin, reference may also be made to two Imperial College theses - those of NE'EMAN [6] and IONIDES [7].

## 3. DEFINITIONS

The general theory of Lie groups follows closely the pattern of the one group we are all thoroughly familiar with, the theory of the three-dimensional rotation group $O_3$. It is indeed a matter of deep regret that the elementary expositions of this familiar case do not employ the same terminology as that of the general theory. Half the conceptual difficulties of the subject would simply disappear if this had consistently been done in our undergraduate courses. To illustrate and to anticipate notation we summarize known facts about the rotation group $O_3$. (All statements made here will be formalized later.) We know that this group is completely determined by three infinitesimal generators:

$$J^{\pm} = 1/\sqrt{2}\,(J_1 \pm i\,J_2),\; J_3$$

and their commutation relations:

$$[J^+, J_3] = J^+,\; [J^-, J_3] = -J^-,\; [J^+, J^-] = J_3.$$

The commutation relations tell us that

(i) The number of operators (out of these three) which can be diagonalized is one ($J_3$). Call this number the "rank" of the group. Thus the rank of $O_3 = 1$.

(ii) Call the eigenvalues of $J_3$ (i.e. the magnetic quantum numbers) by the name "weights". The highest eigenvalues j of $J_3$ uniquely labels a representation. We shall call this "the highest weight".

(iii) The commutation relations tell us (from $[J^{\pm}, J_3] = \pm J_3$) that, irrespective of what the weights are, the difference of two consecutive weights is $\pm 1$. These numbers $\pm 1$ which are characteristic of the commutation relations of the group and not of any particular representation are called "roots". In the subsequent general study of Lie groups these three concepts, "rank" of the group, "roots" of the group and "weights" (and particularly the highest weight) will be generalized and will play crucial roles.

(iv) Another way of labelling the representations of $O_3$ is to use the operator $\underline{J}^2$. This operator commutes with all other operators and thus for a given representation equals a constant multiple of unity. If j is the highest weight, $\underline{J}^2 = j(j+1)\,\underline{I}$. This operator is called the "Casimir operator". We shall find that the concept of a general "Casimir operator" is not as highly developed, and for this reason we shall treat this concept at an early stage (section 5) and then not mention it at all later.

## 4. MATHEMATICAL PRELIMINARIES

4.1. *A group G is a set of elements a, b.... with* a composition law (multiplication) such that the following conditions are fulfilled:

(i) if a and b are elements of the set, then also the product $c = ab$ belongs to the set,

(ii) the composition is associative: $a(bc) = (ab)c$,

(iii) the set contains a unit element e such that $ae = ea = a$,

(iv) to any element a of the set, there exists one and only one element $a^{-1}$ of the set such that $a^{-1}a = aa^{-1} = e$.

The definition of a group does not imply that the two elements ab and ba are equal; i.e., the composition is not necessarily commutative. A group in which all elements commute is called abelian.

A sub-group H of a group G is a sub-set of elements of G, which again fulfils the group postulates. G and the group consisting of the unit element, e, are called trivial sub-groups of G. A sub-group N is called an invariant sub-group of G if for any element n of N ($n \in N$), $sns^{-1}$ is again an element of N where s is any element of G($s \in G$).

A group is called simple if it contains no non-trivial invariant sub-groups, except possibly discrete ones.

A group is called semi-simple if it contains no non-trivial invariant abelian sub-groups, except possibly discrete ones.

4.2. A representation of a group G is a mapping of the group into a set of linear transformations D of a vector space R such that

if $\qquad ab = c$

then $\qquad D(a)\,D(b) = D(c)$,

$$D(a^{-1}) = D^{-1}(a),$$

$$D(e) = I,$$

where I is the unit operator.

A representation is reducible if it leaves a sub-space of R invariant. Then every transformation matrix can be brought into form:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

A representation is fully reducible if every transformation matrix can be written as

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

4.3. A Lie group is a group whose elements form an analytic manifold in such a way that the composition $ab = c$ is an analytic mapping of the manifold $G \times G$ into G and the inverse $a \to a^{-1}$ is an analytic mapping of G into G. A Lie group can thus be viewed from an algebraic, topological or analytical point of view. The topological concepts of importance are connectedness, compactness and invariant integral on the group (see SPEISER and TARKSI [5]).

A group G is compact if every infinite sequence in G has a limit point in G. For a compact group one can define a finite total volume which is invariant under the group.

For example, the group of rotation in three dimensions $O_3$ without reflections is a connected and compact group. The proper Lorentz group is connected but not compact and the improper Lorentz group is neither connected nor compact.

The study of simple groups is important because every semi-simple connected group is essentially a direct product of simple groups, and any connected compact Lie group is essentially a product of a semi-simple and a one-parameter (abelian) compact group.

Ex. $O_4 \approx O_3 \times O_3$; $O_3$ simple; $O_4$ semi-simple.

The symbol $\approx$ means locally isomorphic. From now on we consider only simple compact Lie groups.

## 5. SIMPLE COMPACT LIE GROUPS

So far as a physicist is concerned, a Lie group is a group of transformation of variables which depend analytically on a finite set of N parameters. The fundamental idea of Lie was to consider not the whole group but that part of it which lies close to the identity consisting of the so-called infinitesimal transformations. To formalize this, we have Theorem I.

Theorem 1

Every representation of a compact Lie group is equivalent to a unitary representation and is fully reducible (RACAH, WEYL [2]). Thus, since the matrices D(g) can be taken as unitary, they can be put into the form:

$$D = \exp(i\epsilon^{\alpha} X_{\alpha}),$$

where $X_\alpha$ are constant hermitian matrices ($X_\alpha^+ = X_\alpha$), which are called infinitesimal generators of the group. $\epsilon^\alpha (\alpha = 1, 2..., N)$ are N real parameters on which the set of transformations D depend.

The group is called unimodular if for any D(S), det[D(s)] = 1.
Then tr X = 0.

Theorem 2

Fundamental Theorem of Lie

The local structure of a Lie group is completely specified by the commutation relations between the operators $X_\alpha$:

$$[X_\alpha, X_\beta] = C^{\gamma}_{\alpha\beta} X_\gamma \; ; \; \alpha, \beta, \gamma = 1, 2, \ldots, N, \qquad (5.1)$$

where the coefficients $C^{\gamma}_{\alpha\beta}$ which are independent of the representations of the group are numbers (called the structure constants of the group). These numbers satisfy two requirements:

(a) antisymmetry in the two lower indices

$$C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha},$$

(b)

$$C^{\delta}_{\alpha\beta}\, C^{\epsilon}_{\delta\gamma} + C^{\delta}_{\gamma\alpha}\, C^{\epsilon}_{\delta\beta} + C^{\delta}_{\beta\gamma}\, C^{\epsilon}_{\delta\alpha} = 0 .$$

Note that conditions (a) and (b) are equivalent to the antisymmetry of the Commutator bracket $[X_\alpha, X_\beta]$ and the Jacobi identity:

$$[[X_\alpha, X_\beta], X_\gamma] + [[X_\gamma, X_\alpha], X_\beta] + [[X_\beta, X_\gamma], X_\alpha] = 0.$$

Rewrite (b) in the form:

$$(C_\alpha)^{\epsilon}_{\delta}\,(C_\beta)^{\delta}_{\gamma} - (C_\beta)^{\epsilon}_{\delta}\,(C_\alpha)^{\delta}_{\gamma} = C^{\delta}_{\alpha\beta}\,(C_\delta)^{\epsilon}_{\gamma} .$$

Thus, we have shown the following:

Theorem 3

The N matrices $C_\alpha$ with matrix elements $(C_\alpha)^{\epsilon}_{\gamma}$ form the so-called <u>regular</u> or <u>adjoint</u> representation of the Lie algebra*.

The problem of classification of Lie groups is the problem of finding the numbers c's which satisfy (a) and (b) and then of finding N constant matrices which satisfy the fundamental commutation relation of Theorem 1. This problem was completely solved by Cartan in 1913. Before however we state Cartan's results, we first wish to recast the fundamental commutation relation (5.1) in a "canonical" form and also get over a number of auxiliary results connected with Casimir operators.

## 6. CASIMIR OPERATORS

From the structure constants we can define a metric tensor:

$$g_{\mu\nu} = C^{\beta}_{\mu\alpha}\, C^{\alpha}_{\nu\beta} .$$

Theorem 4

The necessary and sufficient condition for a Lie group to be semi-simple is that

* The set of N matrices $X_\alpha$ span a linear vector space over the field of complex numbers and define a Lie Algebra; the sum of two matrices is an element of the algebra and so is their commutator. Lie algebras and Lie groups possess a one-one correspondence, and it is possible to go freely from Lie groups to Lie algebras. The study of Lie algebras (first introduced by Weyl) is in effect the study of the infinitesimal aspect of Lie group theory. Even though it is galling to bring in a new concept (of a Lie algebra) at this stage, this apparently improves the mathematical rigour of the statements made in these lectures!

$$\det [g_{\mu\nu}] \neq 0 \quad \text{(Cartan)}.$$

Thus for a semi-simple group we can define an inverse metric $g^{\mu\nu}$ such that

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho},$$

and we can use the metric tensors for raising and lowering indices.

Now define an operator $F = g_{\alpha_1 \alpha_2} X^{\alpha_1} X^{\alpha_2}$. This is called the Casimir operator and has the property that it commutes with all the generators of the group:

$$[F, X_\alpha] = 0.$$

The proof of the result is trivial. The significance of the Casimir operator lies in recalling that by Schur's Lemma any operator which commutes with all the generators of the group must be a multiple of the identity.

For $O_3$ this operator is the total angular momentum $\underline{J}^2$. One can define generalized Casimir operators:

$$F^n = C^{\beta_2}_{\alpha_1 \beta_1} C^{\beta_3}_{\alpha_2 \beta_2} \ldots\ldots C^{\beta_1}_{\alpha_m \beta_n} X^{\alpha_1} X^{\alpha_2} \ldots\ldots X^{\alpha_n}.$$

It is easy to see that all these commute with $X^\alpha$.

For $O_3$ all inequivalent irreducible representations can be characterized by giving different values of $\lambda$ where $\lambda I = J^2$. The question arises if this is true in general. Racah gives the following partial answer: Write the set $\{\lambda^k\}$ defined by $\lambda^k I = F^k$. For simple groups if the representation D and $(D^{-1})^T$ are equivalent representations, then the set $\{\lambda^k\}$ gives an unequivocal characterization of all the inequivalent representations.

## 7. CANONICAL FORMS OF THE COMMUTATION RELATIONS AND RANK OF A GROUP

Theorem 6 (P. Ionides)

By a suitable choice of linear combination of the X's, the $C^\alpha_{\beta\gamma}$ can be made antisymmetric in all three indices and pure imaginary; i.e. one can write the commutation relations in the form:

$$[X_\alpha, X_\beta] = i f_{\alpha\beta\gamma} X_\gamma,$$

with $f_{\alpha\beta\gamma}$ purely antisymmetric and real.

In the usual theory of angular momentum, the first step is to rewrite (the Ionides type of) commutation relations,

$$[J_\alpha, J_\beta] = i\, \epsilon_{\alpha\beta\gamma} J_\gamma \quad , \quad \alpha, \beta, \gamma = 1, 2, 3, \tag{7.1}$$

in the so-called "canonical form". Defining the non-hermitian operators,

$$J_\pm = (J_1 \pm i J_2)/\sqrt{2},$$

we rewrite (7.1) as

$$[J_\pm, J_3] = \pm J_\pm ,$$
$$[J_+, J_-] = J_3 . \qquad (7.2)$$

There are two virtues of this canonical form:

(1) If $J_3$ is diagonalized $(J_3|m) = m|m)$, we infer from (7.2) that the operators $J_\pm$ act as "creation" and "annihilation" operators.

(2) (7.2) shows that the consecutive eigenvalues m of $J_3$ differ by $\pm 1$.

Our first task is to cast the commutation relations (5.1) in the "canonical form".

Assume that among the N generators, there are $\ell$ which mutually commute and can thus be simultaneously diagonalized. This number $\ell$ is called the rank, and we shall designate these $\ell$ (hermitian) operators as $H_1$, $H_2$... ... $H_\ell$. (For $O_3$, $\ell = 1$). These operators have a direct physical meaning since their eigenvalues for any representation provide us the quantum numbers.

Let us consider $H_1$, $H_2$.... $H_\ell$ as the components of an $\ell$-dimensional operator-valued vector $\underline{H}$. The components of $\underline{H}$ clearly satisfy the commutation relations:

$$[H_i, H_j] = 0 \qquad \text{for } i, j = 1, 2, \ldots, \ell .$$

If the dimension of the algebra is N (i.e. the number of parameters of the corresponding group is N), we still need (N - $\ell$) elements to complete a basis of the algebra. A suitable choice of these is provided by the following:

Theorem 7

There exists a basis of the Lie algebra consisting of the elements $H_1$, $H_2 \ldots, H_\ell$; $E_{\pm 1}$, $E_{\pm 2} \ldots E_{\pm(N-\ell)/2}$ such that the following commutation relations hold:

$$[\underline{H}, E_\alpha] = \underline{r}(\alpha) E_\alpha, \qquad (7.3)$$

$$[E_\alpha, E_{-\alpha}] = \underline{r}(\alpha) \underline{H} , \qquad (7.4)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_\gamma \text{ for } \alpha \neq -\beta , \qquad (7.5)$$

with $\alpha$, $\beta = \pm 1, \pm 2, \ldots \pm(N-\ell)/2$. E's are non-hermitian matrices and $\underline{r}(\alpha)$ are real vectors in an $\ell$-dimensional space. The $\underline{r}$'s are called roots of the algebra; they have the property that

$$\underline{r}(\alpha) = -\underline{r}(-\alpha). \qquad (7.6)$$

Clearly the total number of the roots is (N - $\ell$).

The scalar product appearing in (7.4) is the usual Euclidean scalar product provided the H's are chosen in such a way that the following normalization conditions hold:

$$\sum_\alpha r_i(\alpha)\, r_j(\alpha) = R\, \delta_{ij} ; \; ij = 1, 2 \ldots \ell , \qquad (7.7)$$

with an arbitrary scale constant. Finally, $N_{\alpha\beta}$ are real numbers which are different from zero if and only if $\underline{r}(\alpha) + \underline{r}(\beta)$ is also a root.

The roots, being essentially our old friends the structure constants, specify completely the group (at least in the local sense). They possess a twin role in the theory. First, as may be inferred from (7.3), the roots are the differences of the eigenvalues of $\underline{H}$. Second and more important for our present purposes, the roots allow us to classify Lie groups. In terms of the roots we can state Cartan's solution of the problem of finding all simple Lie groups. The crucial theorem here is Theorem 8 which lists further properties of the roots and in terms of these gives a complete classification of Lie groups.

## 8. CLASSIFICATION OF LIE GROUPS

A root is said to be <u>positive</u> if its first non-vanishing component (in an arbitrary basis) is positive. A root is called <u>simple</u> if it is a positive root and in addition it cannot be decomposed into the sum of two positive roots.

Theorem 8

(i) For a simple group of rank $\ell$ there exist $\ell$ simple roots and they are all linearly independent. (We shall call the set of simple roots the $\pi$-system.)

(ii) Every positive non-simple root can be expressed as a linear combination $\sum_{\underline{r}(\alpha)\in\pi} R_\alpha \underline{r}(\alpha)$ where $R_\alpha$ are non-negative integers.

(iii) If $\underline{r}(\alpha)$ and $\underline{r}(\beta)$ are two simple roots, the angle $\theta_{\alpha\beta}$ between these can take only the following values:

$$90^\circ \qquad 120^\circ \qquad 135^\circ \qquad \text{and} \qquad 150^\circ,$$

so that $2\underline{r}(\alpha)\cdot\underline{r}(\beta)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$ and $2\underline{r}(\alpha)\cdot\underline{r}(\beta)/\underline{r}(\beta)\cdot\underline{r}(\beta)$ are both integers.

(iv) For every simple group, all the simple roots either have the same length or their length ratios assume simple values. More explicitly one has

$$\frac{|r(\alpha)|^2}{|r(\beta)|^2} = \begin{cases} 1 & \text{if } \theta_{\alpha\beta} = 120^\circ \\ 2 & \text{if } \theta_{\alpha\beta} = 135^\circ \\ 3 & \text{if } \theta_{\alpha\beta} = 150^\circ. \end{cases}$$

If $\theta_{\alpha\beta} = 90^\circ$, the ratio of lengths is undetermined.

Dynkin diagrams

As we shall see in a moment, the geometrical properties of the simple roots in the $\pi$-system characterize in a unique manner the corresponding Lie groups. Therefore it is most convenient to incorporate them in a schematic diagram. These diagrams (the so-called Schouten-Dynkin diagrams) are drawn in Fig. 1.

From Theorem 8, the lengths of the simple roots of a given simple Lie group can assume at most two different values. This fact together with the

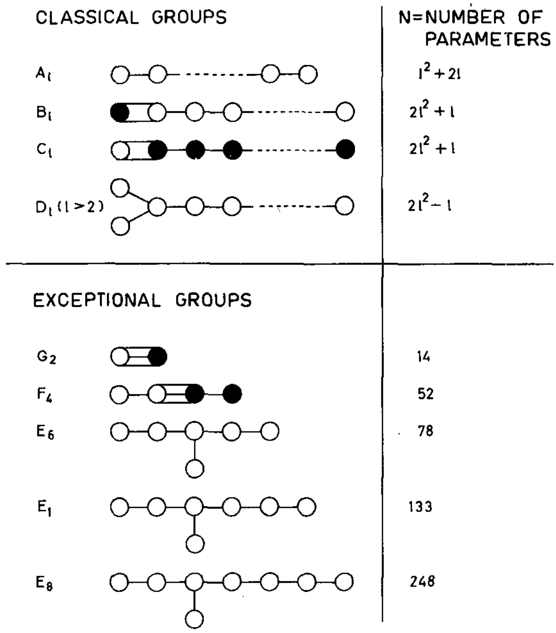



Fig. 1

Cartan solution of all possible single Lie groups.

properties about the angles enumerated above can be symbolically described by associating with each simple root a small circle. For the roots of greatest length the circle is marked in black. If the angle between two consecutive simple roots is equal to 120°, 135° or 150°, the corresponding circles are joined by simple, double or triple lines respectively. If the angle is 90°, the circles are not joined. For a group of rank $\ell$ there are $\ell$ simple roots and therefore $\ell$ circles (black or white).

In terms of these diagrams we give now the Cartan solution of all possible simple Lie groups. Broadly these fall into two categories: the so-called "classical groups" and the five "exceptional groups".

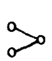
To anticipate we shall find that the classical Lie groups are some of the well known objects:

$A_\ell$ is the group of unitary unimodular matrices in complex space of $(\ell + 1)$ dimensions ($SU_{\ell+1}$).

$B_\ell$ and $D_\ell$ are groups of orthogonal transformations (rotations) in real spaces of $2\ell + 1$ and $2\ell$ dimensions respectively ($O_{2\ell+1}$ and $O_{2\ell}$).

$C_\ell$ is the group of unitary matrices U in complex space of $2\ell$ dimensions which fulfil the condition $U^T J U = J$ where J is a non-singular antisymmetric matrix (the symplectic group)*.

* Note from the Dynkin diagrams:

(i) $D_3$ =  $\approx$ ↓ isomorphic  $= A_3$

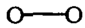
Also i.e. $O_6 \approx SU_4$.

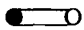
(ii) $C_2$ =  $\approx$  $= B_2$

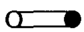
i.e. $O_5 \approx C_2$.

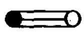
To take simple examples of root structures:

For $\ell = 1$ (i.e. group $O_3$) there is just one simple root + 1. The space spanned by simple roots (the $\pi$-space) is $\{1\}$. For $\ell = 2$, the space is a plane, the relevant groups being

$A_2$:  Two simple roots of equal length, and the angle between them is 120°.

$B_2$: 

$C_2$:  Two simple roots. Their length ratio is 2. The angle between them is 135°.

$G_2$:  Two simple roots with length ratio equal to 3, and angle 150°.

$D_2$: o o is semi-simple, $D_2 \approx A_1 \times A_1$

Summarizing this section then, from the Dynkin diagrams we read off immediately the rank $\ell$ of the group, the lengths of the simple roots and their mutual angles (and of course the dimensionality of the Euclidean space ($\pi$) spanned by these $\ell$ independent vectors)*,**. The simple roots $\underline{r}(1)$, $\underline{r}(2), \ldots, \underline{r}(\ell)$, are given by the following formulae:

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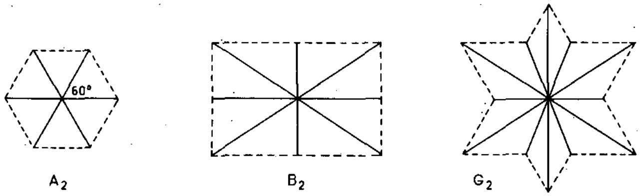
* It is perhaps worthwhile to make the reminder at this stage that not all roots are simple. In fact the total number of roots is $(N-\ell)$, the distinct ones being $(N-\ell)/2$ in virtue of $\underline{r}(\alpha) = -\underline{r}(-\alpha)$, $\alpha = 1, 2, \ldots, (N-\ell)/2$. The remaining $(N-3\ell)/2$ distinct non-simple roots can easily be constructed, and in Footnote ** we give a complete ansatz for drawing a complete root diagram (for $\ell = 2$ for example in a plane; for $\ell = 3$ in $\{3\}$ space and so on). Personnally, I consider these diagrams pointless. However, to satisfy current prejudice the root diagrams for $A_2$, $B_2$ and $G_2$ are reproduced in Fig. 2.




$A_2$ $N=8$ $B_2$ $N=10$ $G_2$ $N=14$

Fig. 2

Root diagrams for $A_2, B_2$ and $G_2$

** The following scheme incorporates all the requirements about angles and lengths of simples roots specified by the diagrams.

For $A_\ell$ define the following vectors:

$$\underline{\lambda}_1, \underline{\lambda}_2, \ldots \underline{\lambda}_{\ell+1}$$

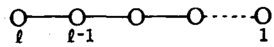
by the conditions

$$\underline{\lambda}_1 + \underline{\lambda}_2 + \ldots + \underline{\lambda}_{\ell+1} = 0,$$

$$\underline{\lambda}_1^2 = \underline{\lambda}_2^2 = \ldots = \underline{\lambda}_{\ell+1}^2 = \ell A,$$

$$\underline{\lambda}_p \cdot \underline{\lambda}_q = -A, \quad p \neq q = 1, 2, \ldots, \ell+1.$$

$$\begin{aligned} \underline{r}(\ell) &= \underline{\lambda}_\ell - \underline{\lambda}_{\ell+1}, \\ \underline{r}(\ell-1) &= \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell, \\ &\vdots \\ \underline{r}(1) &= \underline{\lambda}_1 - \lambda_2. \end{aligned} \tag{8.1}$$

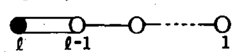



For $B_\ell$ : the simple root structure is as follows:

$$\begin{aligned} \underline{r}(\ell) &= \underline{\lambda}_\ell, \text{ (This is the smallest root)} \\ \underline{r}(\ell-1) &= \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell, \\ \underline{r}(1) &= \underline{\lambda}_1 - \underline{\lambda}_2, \end{aligned} \tag{8.2}$$

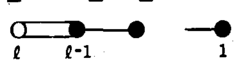
where

$$\begin{aligned} \underline{\lambda}_1^2 = \underline{\lambda}_2^2 &= \ldots\ldots \underline{\lambda}_\ell^2 = A, \\ \underline{\lambda}_p \underline{\lambda}_q &= 0, \; p \neq q, \end{aligned} \tag{8.3}$$




For $C_\ell$ : the simple roots are given by:

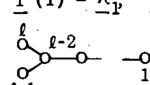
$$\begin{aligned} \underline{r}(\ell) &= 2\,\underline{\lambda}_\ell, \text{ (This is the greatest root.)} \\ \underline{r}(\ell-1) &= \underline{\lambda}_{\ell-1} - \underline{\lambda}_{\ell-2}, \\ &\vdots \\ \underline{r}(1) &= \underline{\lambda}_2 - \underline{\lambda}_1, \end{aligned} \tag{8.4}$$




where the $\underline{\lambda}$'s satisfy (8.3).

For $D_\ell$ : the simple roots are given by:

$$\begin{aligned} \underline{r}(\ell) &= \underline{\lambda}_{\ell-1} + \underline{\lambda}_\ell, \\ \underline{r}(\ell-1) &= \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell \\ &\vdots \\ \underline{r}(1) &= \underline{\lambda}_1 - \underline{\lambda}_2 \end{aligned} \tag{8.5}$$




The λ's satisfy (8.3). So much for simple roots. All roots are given for the classical groups by the following expressions:

$A_\ell$ : $(\underline{\lambda}_p - \underline{\lambda}_q)$ ; p, q = 1, 2, ....., $\ell + 1$

$B_\ell$ : $\pm \underline{\lambda}_p$, $\pm\underline{\lambda}_p \pm\underline{\lambda}_q$ ; p, q = 1, 2, ....., $\ell$
$C_\ell$ : $\pm 2\underline{\lambda}_p$, $\pm\underline{\lambda}_p \pm\underline{\lambda}_q$ ; p, q = 1, 2,....., $\ell$
D : $\pm \underline{\lambda}_p \pm\lambda_q$ ; p, q = 1, 2,....., $\ell$.

The ± signs are to be taken in arbitrary combinations.

Similar expressions can be given for the exceptional groups. Also one can give a full correspondence between the "canonical" expressions for the commutation relations and the more familiar manner in which one writes the commutation relations for the orthogonal, symplectic groups, etc.

Thus, for the orthogonal group in $(2\ell + 1)$ dimensions which leaves invariant the quadratic form

$$\sum_{p=-\ell}^{\ell} X^p X^{-p}$$

one may write the infinitesimal operators:

$$X_{pq} = -X_{qp} = X^p \frac{\delta}{\delta X^{-q}} - X^q \frac{\delta}{\delta X^{-p}},$$

with the commutation relations:

$$[X_{ik}, X_{mn}] = \delta_{k+m} X_{in} - \delta_{k+n} X_{im} - \delta_{\ell+m} X_{kn} - \delta_{\ell+n} X_{km}$$

where $\delta q = 1$ if $q = 0$ and zero otherwise. These operators correspond to the E's and the H's of $B_\ell$ if we make the following identifications:

$$X_{p-p} \equiv H_p, \; X_{\pm p \pm q} \equiv E_{\pm\lambda p \pm \lambda q}, \; X_{0\pm p} = E_{\pm\lambda p} \; ; \; p, q > 0.$$

Similar correspondence can be stated for $A_\ell$, $C_\ell$, $D_\ell$ etc. (Racah's notes).

## 9. REPRESENTATIONS OF LIE GROUPS: WEIGHTS

9.1. Now we come to physically the most important problem of all - the problem of finding representations of the group, i.e. the matrices corresponding to $\underset{\sim}{H}$ and $E_\alpha$.

Consider a representation of dimension (or degree) d. Since $H_1, H_2, \ldots, H_\ell$ are hermitian matrices, and since they commute with each other, we can simultaneously diagonalize these. Let $|m\rangle$ be a simultaneous eigenket:

$$\underset{\sim}{H} \, |m\rangle = \underline{m} \, |m\rangle. \tag{9.1}$$

Since H's are d × d matrices, the total number of such eigenkets $|m\rangle$ is d.

The $\underline{m}$'s in Eq. (9.1) are real numbers and are called "weights". They form $\ell$-dimensional vectors in a Euclidean space for whose basis one may take the $\pi$-space of the group (the space spanned by the $\ell$ simple roots). Summarizing, for the case of a group of rank $\ell$ and for a given representation of dimensionality d, there are

$\ell$ : simple root vectors

$(N-3\ell)/2$ : distinct non-simple root vectors

d : weight vectors (provided we count each weight vector as many times as its multiplicity indicates, the multiplicity being defined as the number of independent eigenkets $|m\rangle$ corresponding to a given weight $\underset{\sim}{m}$).

Note that root vectors are characteristic of the group. They are really the structure constants. The weight vectors on the other hand are characteristic of the representation. There are only $\ell$ linearly independent roots (simple roots). There are also only $\ell$ linearly independent weight vectors. The simplest (oblique axis) basis for the weight vectors is that provided by the simple root vectors.

All this intertwining of weights and roots is exciting enough, but still further and the more exciting result comes when we look for the analogue of the result in $O_3$ that all weights are either integers or half-integers. The analogous result is Theorem 9, which gives the "component" of any weight-vector along a simple root-vector.

Theorem 9

For every weight $\underline{m}$, the number $\underline{m}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$, where $\underline{r}(\alpha)\in\pi$, is an integer or a half-integer, $\gtrless 0$.

Theorem 9 provides the justification for Dynkin's insistence on simple roots as the primary entities on which all conceptual emphasis should be placed. Dynkin cares neither for the non-simple roots nor for the weight vectors. Given the simple roots, Theorem 9 tells us what the weights look like through the simplest possible generalization of the familiar results for the {3} rotation group*. In this insistence on simple roots possibly lies the superiority of Dynkin's presentation of Lie group theory.

## 10. IRREDUCIBLE REPRESENTATIONS AND THEIR DIMENSIONALITY

Definition: A weight $\underline{m}$ is said to be higher than $\underline{m}'$ if $\underline{m}-\underline{m}'$ has a positive number for its first non-vanishing component in an arbitrary basis. The weight $\underline{\Lambda}$ which is higher than all the others is called the highest (or greatest) weight.

Theorem 10

A representation is uniquely characterized by its highest weight $\underline{\Lambda}$, and the highest weight always has multiplicity one.

---

* Earlier it was mentioned that roots are differences of weights. The formal result is: If $|\underline{m}\rangle$ is an eigenket of $\underset{\sim}{H}$ corresponding to a weight $\underline{m}$, $E_\alpha|\underline{m}\rangle$ is also an eigenket with weight $\underline{m}+\underline{r}(\alpha)$. The result follows from

$$[E_\alpha, \underset{\sim}{H}] = \underline{r}(\alpha)E_\alpha .$$

Note the role of $E_\alpha$ as a creation operator.

Theorem 11

In order that a vector $\underline{\Lambda}$ be the highest weight of some irreducible representation, it is necessary and sufficient that $j_\alpha$, defined as $j_\alpha = \underline{\Lambda}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot r(\alpha)$, is a non-negative integer or half-integer.

Thus to get the irreducible representations of any Lie group, we should mark each circle in the Dynkin diagram with a non-negative integer or half-integer $j_\alpha$. These numbers characterize uniquely the irreducible representation with $\underline{\Lambda}$ as its highest weight, the "components" $\underline{\Lambda}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$ of $\underline{\Lambda}$ being just $(j_1, j_2 \ldots\ldots)$. The dimensionality of this representation is given by the following theorem of Weyl:

Weyl's Theorem: Theorem 12

Let $\Sigma_+$ be the system of all positive roots of a semi-simple Lie algebra, and let an irreducible representation be uniquely characterized by the highest weight $\underline{\Lambda}$. Then its dimensionality d is given by the formula:

$$d = \prod_{\underline{r}(\alpha)\in\Sigma_+}\left[1 + \underline{\Lambda}\cdot\underline{r}(\alpha)\right]/\underline{g}\cdot\underline{r}(\alpha),$$

where

$$\underline{g} = \frac{1}{2}\sum_{\underline{r}(\beta)\in\Sigma_+}\underline{r}(\beta).$$

If one writes the vectors $\underline{\Lambda}$ and $\underline{g}$ in terms of the auxiliary quantities $\lambda$'s previously introduced in the third footnote of section 8,

$$\underline{\Lambda} = \Sigma f_i \underline{\lambda}_i,$$
$$\underline{g} = \Sigma g_i \underline{\lambda}_i.$$

The Weyl formula above gives the explicit expressions listed in Table I.

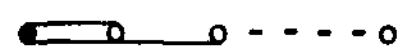
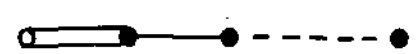
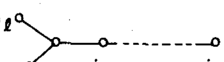
As examples consider some of the interesting physical cases, namely, the case of rank $\ell = 2$. In this case the number of commuting matrices in the algebra is two, and we can associate them, for example, with the third component of the isotopic spin and the hypercharge. The only simple compact Lie groups of rank 2 are $A_2$, $B_2$, $C_2$ and $G_2$. Any irreducible representation of these groups can be labelled by means of two non-negative integers $j_1$, $j_2$. The formulae for the dimensionality given in Table I can be written explicitly in a simple way and is shown in Table II.

For instance, for the simplest choices of the arrays $j_1$, $j_2$ one gets the following dimensions:

| | | | | | |
|---|---|---|---|---|---|
| $A_2$ : | $d(0,0) = 1$ | $B_2(\approx C_2 \approx O_5)$ : | $d(0,0) = 1$ | $G_2$ : | $d(0,0) = 1$ |
| | $d(1,0) = 3$ | | $d(\frac{1}{2},0) = 4$ | | $d(\frac{1}{2},0) = 7$ |
| | $d(0,\frac{1}{2}) = 3$ | | $d(0,\frac{1}{2}) = 5$ | | $d(0,\frac{1}{2}) = 14$ |
| | $d(1,0) = 6$ | | $d(1,0) = 10$ | | $d(1,0) = 27$ |
| | $d(\frac{1}{2},\frac{1}{2}) = 8$ | | $d(0,1) = 14$ | | |
| | $d(1,\frac{1}{2}) = 15$ | | $d(\frac{1}{2},\frac{1}{2}) = 16$ | | |
| | $d(1,1) = 27$ | | | | |

TABLE I

($j_i$'s are non-negative integers or half-integers)

| Group | N number of parameters | Dynkin diagrams | Dimension of the irred. represent. | | Expressions for f and g |
|---|---|---|---|---|---|
| $A_\ell$ | $\ell^2 + 2\ell$ | $j_\ell$ $j_{\ell-1}$ $j_1$  | $\prod_{p,q}(1+\alpha_{pq})$ | where $\alpha_{pq} = \frac{f_p - f_q}{g_p - g_q}$ $\beta_{pq} = \frac{f_p + f_q}{g_p + g_q}$ $\gamma_p = f_p/g_p$ | $f_k = f_{\ell+1} + 2\sum_{i=k}^{\ell} j_i$ $f_{\ell+1} = \frac{-4}{\ell+1}\sum_{i\neq 1}^{\ell} i\, j_i$ $g_k = \frac{-\ell}{2} + (\ell-k+1)$ $g_{\ell+1} = -\ell/2$ Note $f_1 + \ldots + f_{\ell+1} = 0$ |
| $B_\ell$ | $2\ell^2 + \ell$ |  $j_\ell$ $j_{\ell-1}$ $j_1$ | $\prod_{p,q}(1+\gamma_p)(1+\alpha_{pq})(1+\beta_{pq})$ | " | $f_k = j_\ell + 2\sum_{i=k}^{\ell-1} j_i$ $g_k = (\ell-k+\frac{1}{2})$ |
| $C_\ell$ | $2\ell^2 + \ell$ |  $j_\ell$ $j_{\ell-1}$ $j_1$ | " | " | $f_k = 2\sum_{i=k}^{\ell} j_i$ $g_k = (\ell-k+\frac{1}{2})$ |
| $D_\ell$ | $2\ell^2 - \ell$ | $j_\ell$  $j_{\ell-1}$ $j_{\ell-2}$ $j_1$ | $\prod_{p,q}(1+\gamma_p)(1+\alpha_{pq})$ | " | $f_k = j_{\ell-1} + j_\ell + 2\sum_{k}^{\ell-2} j_i$ $f_\ell = j_{\ell-1} - j_\ell$, $g_\ell = 0$, $g_k = \ell-k$ for $k \leq \ell-1$ |

The products here range over all possible values of p and q; the indices denoted by distinct letters must have distinct values, and of all sets of values obtained from one another by permutations of indices only one must be chosen.

TABLE II

| Group | Number of parameters N | Dimension of the irr. rep. |
|---|---|---|
| $A_2$ | 8 | $\frac{1}{2!}(J_1)(J_2)[J_1+J_2]$ |
| $B_2$, $C_2$ | 10 | $\frac{1}{3!}(J_1)(J_2)[J_1+J_2](2J_2+J_1)$ |
| $G_2$ | 14 | $\frac{1}{5!}(J_1)(J_2)(J_1+J_2)[2J_2+J_1]\times[3J_2+J_1][3J_2+2J_1]$ |

{ Note: Here $J_1 = (2j_1+1)$ and $J_2 = (2j_2+1)$ }

These numbers $d(j_1, j_2)$* represent the number of particles which can be accomodated in any given multiplet in physical applications.

The adjoint (or regular) representation R plays a very important role in vector meson theories. For the case of $\ell = 2$, these representations are the following:

$$
\begin{aligned}
A_2 &: \quad d_R = d(\tfrac{1}{2},\tfrac{1}{2}) = 8,\\
B_2\,(C_2) &: \quad d_R = d(1,0) = 10,\\
G_2 &: \quad d_R = d(0,\tfrac{1}{2}) = 14.
\end{aligned}
$$

These groups, therefore, can accommodate 8, 10 and 14 vector gauge mesons respectively if these mesons correspond to the adjoint representation.

## 11. COMPUTATION OF ALL WEIGHTS OF A GIVEN IRREDUCIBLE REPRESENTATION

Notwithstanding the fact that the greatest weight uniquely characterizes an irreducible representation, it is important for physical applications to be able to compute all the weights of an irreducible representation. Later we shall construct weight diagrams for some irreducible representation of low dimensionality for the case of rank 2 groups ($A_2$, $B_2$, $C_2$, $G_2$). In contrast to the root diagrams, the weight diagrams are directly of physical interest.

An explicit method to calculate all the weights in terms of the highest weight and the simple roots is given by the next theorem. We have learnt earlier that the roots equal differences of weights.

---

* I have introduced a small change of notation in the labelling of representations. Dynkin and Behrends et al. label irreducible representations with numbers $a_1, a_2, \ldots, a_\ell$ where $a_i$ are (non-negative) integers. I have used for labelling the numbers $j_1, j_2, \ldots, j_\ell$ where the j's are (non-negative) integers or half-integers. The new notation possibly brings out still more the fact that a general Lie group of rank $\ell$ is a simple "generalization" of $O_3$ and has $\ell$ distinct "angular momenta" $j_1, j_2, \ldots, j_\ell$ rather than just one ($j_1$).

Let $\underline{\Lambda}$ and W be the highest weight and the set of all weights respectively of a given irreducible representation.

An element $\underline{m} \in W$ is said to belong to the layer $\Delta^{(k)}$ if it can be obtained by subtracting K simple roots from $\underline{\Lambda}$. Clearly $\Delta^{(0)}$ consists only of $\underline{\Lambda}$, and

$$W = \Delta^{(0)} \cup \Delta^{(1)} \cup \Delta^{(2)} \ldots\ldots$$

Note that all the layers are disjointed.

Theorem 13

Every element $\underline{m}^{(k)} \in \Delta^{(k)}$ can be expressed as

$$\underline{m}^{(k)} = \underline{m}^{(k-1)} - \underline{r}\,(\alpha),$$

where

$$\underline{m}^{(k-1)} \in \Delta^{(k-1)}$$

and

$$\underline{r}\,(\alpha) \in \pi .$$

However, if $\underline{m}^{(k-1)}$ belongs to $\Delta^{(k)}$ and $\underline{r}(\alpha)$ is an arbitrary simple root, the difference $\underline{m}^{(k-1)} - \underline{r}\,(\alpha) \in \Delta^{(k)}$ if and only if the following condition is satisfied:

$$2\, \underline{m}^{(k-1)} \cdot \underline{r}\,(\alpha)/\underline{r}\,(\alpha) \cdot \underline{r}\,(\alpha) + Q > 0,$$

where the number Q is defined by the requirements:

$$\underline{m}^{(k-1)} + q\, \underline{r}\,(\alpha) \quad \in W \quad \text{for} \quad q \leqslant Q,$$

$$m^{(k-1)} + q\, \underline{r}\,(\alpha) \quad \in W \quad \text{for} \quad q = Q + 1.$$

Example:

Perhaps the best way to show that the theorem is actually quite harmless and simple in practice is to construct the weights for a specific case. Consider the group $A_2 \approx SU_3$ for which $\ell = 2$. The Dynkin diagram is O—O. The $\pi$-space is two-dimensional; and if we call the roots $\alpha$ and $\beta$, the diagram tells us that their lengths are equal ($|\alpha|^2 = |\beta|^2$) and the angle between them is 120° so that

$$\underline{\alpha} \cdot \underline{\beta} / \underline{\alpha} \cdot \underline{\alpha} = -\tfrac{1}{2}.$$

Consider now the regular representation $(\frac{1}{2}, \frac{1}{2})$. The dimensionality in this case is d = 8, so that the representation could accommodate 8 particles. The "components" of the highest weight $\underline{\Lambda}$ (ie)$j_\alpha$, $j_\beta$ are given by

$$j_\alpha = \underline{\Lambda} \cdot \underline{\alpha} \,/\, \underline{\alpha} \cdot \underline{\alpha} = \tfrac{1}{2}, \qquad (11.1)$$

$$j_\beta = \underline{\Lambda} \cdot \underline{\beta} \,/\, \underline{\beta} \cdot \underline{\beta} = \tfrac{1}{2}. \qquad (11.2)$$

Noticing that $\underline{\alpha}$ and $\underline{\beta}$ do not form an orthogonal basis, we find from (11.1) and (11.2) that

$$\underline{\Lambda} = \underline{\alpha} + \underline{\beta}.$$

Now using Theorem 13, if we are given an arbitrary weight M and we wish to know whether $\underline{M}-\underline{\alpha}$ is a possible weight or not, we proceed as follows:

Write the series $\underline{M}$, $\underline{M} + \underline{\alpha}$, $\underline{M} + 2\underline{\alpha}, \ldots \underline{M} + (Q+1)\underline{\alpha}$ where Q is an integer. The series terminates for a Q defined by the requirement that while $\underline{M}$, $\underline{M} + \underline{\alpha}, \ldots . \underline{M} + Q\underline{\alpha}$ are weights, $\underline{M} + (Q+1)\underline{\alpha}$ is not a weight. Now compute the number,

$$Q + M_\alpha \quad \text{where} \quad M_\alpha = 2\,\underline{M}\cdot\underline{\alpha}/\underline{\alpha}\cdot\underline{\alpha}.$$

If $M_\alpha + Q > 0$, then $\underline{M}-\underline{\alpha}$ is a weight; otherwise it is not. In starting this procedure the crucial point to remember is that $\underline{\Lambda} + \underline{\alpha}$ where $\underline{\alpha}$ is a simple root is never a possible weight.

Consider now the case when $\underline{M} = \underline{\Lambda}$. Since $\underline{\Lambda} + \underline{\alpha}$ is not a weight, $Q = 0$. Since

$$\Lambda_\alpha = \underline{\Lambda}\cdot\underline{\alpha}\,/\,\underline{\alpha}\cdot\underline{\alpha} = j_\alpha > 0, \tag{11.3}$$

we see from (11.3) that $\underline{\Lambda} - \underline{\alpha}$ is indeed a weight. Likewise, since $j_\beta > 0$, $\underline{\Lambda} - \underline{\beta}$ is also a weight.

We can now start with $(\underline{\Lambda} - \underline{\alpha})$ and test if $(\underline{\Lambda} - \underline{\alpha}) - \underline{\alpha}$ and $(\underline{\Lambda} - \underline{\alpha}) - \underline{\beta}$ are possible weights or not. It is easy to see that $\underline{\Lambda} - 2\underline{\alpha}$ is not a weight, but $\underline{\Lambda}-\underline{\alpha}-\underline{\beta}$ is. Proceeding in this fashion, we find that all possible weights are given by the diagram shown in Fig. 3.

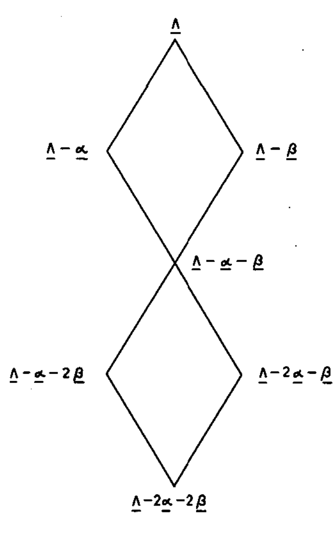



Fig. 3

Notice that the weight $\underline{\Lambda}-\underline{\alpha}-\underline{\beta}$ is of multiplicity two. The diagram does not further fan out, and we obtain a totality of eight weights. Writing $\Lambda = \underline{\alpha}+\underline{\beta}$, we have the following system of weights:

$$\underline{\alpha}+\underline{\beta},\ \underline{\alpha},\ \underline{\beta},\ 0,\ 0,\ -\underline{\beta},\ -\underline{\alpha},\ -(\underline{\alpha}+\underline{\beta}). \qquad (11.4)$$

The multiplicities are spindle-shaped: they increase, come to a maximum and decrease again. (The weight zero has multiplicity two.) This is a general result which will not be discussed further.

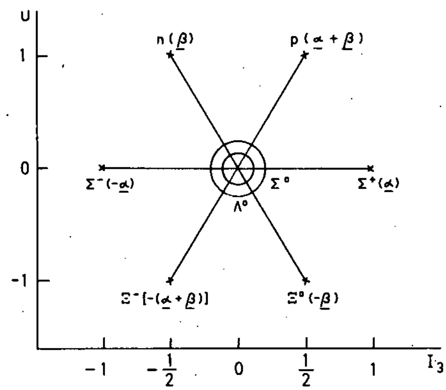



Fig. 4

Euclidean diagrams

Fig.4 gives the Euclidean diagram of these weights. The two rings in the centre indicate the two zero weights. A tentative identification of the stable baryons with the appropriate weights has also been made in the figure, provided we identify

$$m_1 = I_3,$$

$$m_2 = (2/\sqrt{3})U,$$

where $\underline{m} = \binom{m_1}{m_2}$ in a Euclidean basis.

For illustrative purposes, here are some more weight diagrams corresponding to the representations [4] shown in Fig. 5.

Before concluding this section we state one important theorem and make one final remark.

Theorem 14

For the adjoint representation, the root vectors and the non-zero weight vectors coincide. The weight zero occurs with a multiplicity equal to the rank of the group.

An illustration of this theorem is given by the weight diagram of the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU_3$ computed earlier in this section. Because of

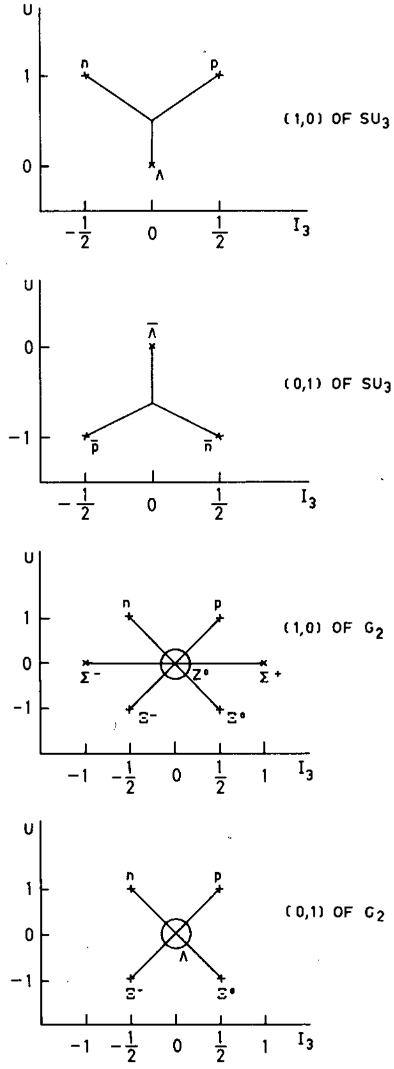



Fig. 5

this rather remarkable property clearly the adjoint representation has a greater claim to attention than any other.

Remark

In $O_3$, the eigenvalues of $J_3$ (the weights) are non-degenerate for any given representation and hence suffice to label the representation. For general Lie groups, except for the highest weight, all others may possess multiplicities of > 1 (compare the weight (0, 0) for $SU_3$ which has multiplicity 2). If the multiplicity is > 1 we need additional operators all commuting with each other and with the $\underset{\sim}{H}$'s, whose eigenvalues will enable us to remove the degeneracy and label uniquely the eigenvectors of the $\underset{\sim}{H}$'s, belonging to the same given weight. (A Casimir operator which has the same eigenvalue for all vectors of a given representation is clearly useless for this purpose.) The number of extra operators needed can be shown to equal $(N-\ell)/2-\ell = (N-3\ell)/2$. For $O_3$, $N = 3$, $\ell = 1$ so that no extra operator is needed to characterize all the eigenkets of $J_3$ in a representation specified (uniquely) by the highest weight j. For $SU_3$, however, $N = 8$, $\ell = 2$ so that we need one more operator besides $I_3$ and U to label uniquely the eigenkets of $I_3$ and U. It is not hard to show that in this case such an operator is given by $\underset{\sim}{I}^2$. For $C_2$, $(N-3\ell)/2 = 2$. Thus, even additional to $\underset{\sim}{I}^2$ (and U and $I_3$ ), one more quantum number is needed to form a complete set of commuting observables. For $G_2$, $(N-3\ell)/2 = 4$.

## 12. REDUCIBLE REPRESENTATIONS

Let us take stock of the situation. For a physicist working in symmetry problems, the information necessary for progress is the following:

(i) Classification of irreducible representation for a group of rank $\ell$. We possess a complete solution of this problem.

(ii) The eigenvalues of the commuting operators $H_1, \ldots, H_\ell$. This is the same problem as the problem of determination of weights. Again we possess a complete solution of this.

(iii) Determination of the extra $(N-3\ell)/2$ operators to enable a unique labelling of the eigenkets of $H_1, \ldots, H_\ell$. For groups like $A_2$, $B_2$, $C_2$, $D_2$ we know how to construct such operators but a general systematic procedure apparently is not known.

(iv) The reduction of a reducible representation into the direct sum of irreducible representations. There are two parts of this problem: first, finding out which irreducible representations make their appearance in this direct sum; second, to find the Clebsch-Gordon coefficients. Theorem 15 will give the procedure for solving the first problem. The second problem will be dealt with by Ruegg and Goldberg in their lectures for some special (fortunately for the physicist, extremely important) cases. No general solution however exists.

First, some obvious definitions:

### Kronecker products

If $R_1$, $R_2$, $R_3$ are three linear spaces of dimensions m, n and mn respectively, we shall say $R_3$ is the Kronecker product of $R_1$ and $R_2$ ($R_3 = R_1 \times R_2$) provided to every vector $|\xi_1\rangle \in R_1$, $|\xi_2\rangle \in R_2$, there corresponds a vector $|\xi_3\rangle \in R_3$ (notation $|\xi_3\rangle = |\xi_1\rangle \times |\xi_2\rangle$) such that:

(i) The operation $|\xi_1\rangle \times |\xi_2\rangle$ is linear in each argument;

(ii) $R_3$ is spanned by vectors of the form $|\xi_1\rangle \times |\xi_2\rangle$.

If $\phi_1$ and $\phi_2$ are linear representations of a Lie algebra operating in $R_1$ and $R_2$, the representation $\phi_3$ defined in $R_1 \times R_2$ by the formula,

$$\phi_3 \{|\xi_1\rangle \times |\xi_2\rangle\} = \{\phi_1 |\xi_1\rangle\} \times |\xi_2\rangle + |\xi_1\rangle \times \{\phi_2 |\xi_2\rangle\},$$

is called the Kronecker product of $\phi_1$ and $\phi_2$ and will be denoted as

$$\phi_3 = \phi_1 \times \phi_2 .$$

Theorem 15

(i) Addition of weights
If $\Delta_{\phi_1}$ is the weight space of $\phi_1$ and $\Delta_{\phi_2}$ is the weight space of the representation $\phi_2$, then $\Delta_{\phi_3} = \Delta_{\phi_1} + \Delta_{\phi_2}$.

(ii) If $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$ are the greatest weights of $\phi_1$ and $\phi_2$, the greatest weight of $\phi_3$ is $\underline{\Lambda}_1 + \underline{\Lambda}_2$.

This theorem is an obvious generalization of the addition theorem for angular momenta in $O_3$ which we consider in detail. If $j_1$ and $j_2$ are the highest weights of two irreducible representations $\phi(j_1)$ and $\phi(j_2)$, the (reducible) product representation has the highest weight $j_1 + j_2$. Also the totality of its weights is given by

| Weight → | $j_1+j_2$, | $j_1+j_2-1$, | $j_1+j_2-2$, | ..., | $-j_1-j_2$ |
|---|---|---|---|---|---|
| multiplicity → | 1 , | 2 , | 3 | , ..., | 1 |

The multiplicities are easily deduced. For example, $j_1+j_2-1$ arises in two ways: either as the sum $j_1+(j_2-1)$ or equally as the sum of the weights $(j_1-1)+j_2$. The usual procedure to find the irreducible representations contained in $\phi(j_1)\times\phi(j_2)$ can be stated thus: Take away from the totality of weights those which belong to the representation $\phi(j_1+j_2)$. Among the remaining weights occurs the weight $j_1+j_2-1$ with unit multiplicity. Clearly this must be the highest weight of the representation $\phi(j_1+j_2-1)$ which therefore must also be contained in $\phi(j_1)\times\phi(j_2)$. Taking away all the weights belonging to $\phi(j_1+j_2-1)$, we next identify the occurrence of $\phi(j_1+j_2-2)$ in the direct sum from the fact that the highest weight left is $(j_1+j_2-2)$. This procedure is continued till we reach $\phi(|j_1-j_2|)$. At this stage all weights are exhausted, leading to the inference that

$$\phi(j_1)\times\phi(j_2) = \phi(j_1+j_2) + \phi(j_1+j_2-1) + \dots + \phi(|j_1-j_2|).$$

The procedure is obviously completely general. Its only drawback is that in order to apply it we need to know all the weights. A simpler version has been developed by Racah, Speiser and Ruegg where, if $j_1 \geqslant j_2$, one adds all weights belonging to the representation $\phi(j_2)$ (i.e. $j_2, j_2-1, \dots, -j_2$) to the highest weight $j_1$ of $\phi(j_1)$. For $O_3$, the resulting weights are clearly the highest weights of the irreducible representations contained in $\phi(j_1)\times\phi(j_2)$. For the more general cases this sum may lead to a certain number of negative weights which certainly cannot qualify as highest weights. These then have to be excluded, and the procedure for this is explained in Ruegg's lecture.

Cartan composition

If $\phi_1$ and $\phi_2$ are two irreducible representations, the Kronecker product $\phi_1\times\phi_2$ is in general a reducible representation. Consider its greatest component, $\overline{\phi_1 \times \phi_2}$. This is an irreducible representation with the highest weight $\underline{\Lambda}_1 + \underline{\Lambda}_2$. The operation of Kronecker multiplication of two irreducible representations followed by the operation of isolating the greatest component lead to the formation of a new irreducible representation $\overline{(\phi_1 \times \phi_2)}$ and is called the cartan composition of irreducible representations.

Those irreducible representations of an algebra which cannot be obtained from other irreducible representations are called basic representations by Cartan. These representations are characterized by the fact that their highest weights cannot be split into the sums of two elements that are themselves highest weights. Clearly a representation $\phi$ is basic if, and only if, all the labelling numbers $j_1, j_2, \ldots, j_\ell$ are zero except one which equals $\frac{1}{2}$. Thus every simple algebra of rank $\ell$ has $\ell$ basic representations.

One can go further and show that all basic representations themselves can be constituted from a few so-called elementary representations by Kronecker multiplications followed by an antisymmetrization procedure which is somewhat familiar in ordinary tensor theory and will not be described here in detail. For $A_\ell$ and $B_\ell$ there are just two elementary representations. $C_\ell$ has one elementary representation and $D_\ell$ has three. One of the elementary representations $\phi$ of $A_\ell$ is realized as the group $SL(\ell + 1)$ of all matrices of order $\ell + 1$ with determinant + 1, the other being given by

$$\phi' = - [\phi_1]^T.$$

For $B_\ell$, one of the elementary representations is obtained by considering the group $O(2\ell + 1)$ of all unimodular orthogonal transformations of the $(2\ell + 1)$ dimensional space, while the second elementary representation is the so-called spinor representation. The realization of the group $C_\ell$ in the form of the group $Sp(2n)$ of the symplectic matrices of order $2\ell$ gives its elementary representation , while for $D_\ell$ ($\ell \geqslant 5$) one elementary representation is given by the group of unimodular orthogonal matrices of order $2\ell$ and in addition there are two distinct spinor representations. For the elementary representations of the exceptional groups reference may be made to Dynkin.

This brief description of the results in representation theory does not even touch the practical problem of reduction of representation in the manner the physicist wants it solved. For this we must fall back on our amateur methods, multiplying matrices, symmetrizing and antisymmetrizing tensor indices, though perhaps somewhat emboldened by the knowledge that this is also the entire, and when I say entire - I mean entire, stock-in-trade of the professional group theorist.

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