Naked singularity formation in $f(\mathcal{R})$ gravity

A. H. Ziaie · K. Atazadeh · S. M. M. Rasouli

Abstract We study the gravitational collapse of a star with barotropic equation of state $p = w\rho$ in the context of $f(\mathcal{R})$ theories of gravity. Utilizing the metric formalism, we rewrite the field equations as those of Brans-Dicke theory with vanishing coupling parameter. By choosing the functionality of Ricci scalar as $f(\mathcal{R}) = \alpha \mathcal{R}^m$, we show that for an appropriate initial value of the energy density, if α and m satisfy certain conditions, the resulting singularity would be naked, violating the cosmic censorship conjecture. These conditions are the ratio of the mass function to the area radius of the collapsing ball, negativity of the effective pressure, and the time behavior of the Kretschmann scalar. Also, as long as parameter α obeys certain conditions, the satisfaction of the weak energy condition is guaranteed by the collapsing configuration.

Keywords Naked singularity · $f(\mathcal{R})$ gravity

1 Introduction

Einstein's General theory of relativity is the classical theory of one of the four fundamental forces, gravity, which is the weakest but most dominant force of nature governing phenomena at large scales, and is described by a mathematically well-founded and elegant structure i.e., differential geometry of curved spacetime. The Einstein's field equations, a system of non-linear partial differential equations, relate

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the geometric property of spacetime to the four-momentum (energy density and linear momentum) of matter fields leading to precise predictions that have received considerable experimental confirmations with high accuracy such as solar system tests (see [1] and references therein). One of the most engrossing but open debates in general relativity is that of the final fate of gravitational collapse with possibility of the existence of spacetime singularities, the ultra-strong gravity regions where the densities and spacetime curvatures blow up, leading to a spacetime which is geodesically incomplete [2] and the structure of any classical theory of fields is vanquished. A star with a mass many times than that of the Sun would undergo a continual gravitational collapse due to its self-gravity without achieving an equilibrium state in contrast to a neutron star or a white dwarf. Then, according to singularity theorems established by Hawking and Penrose [3–5] a singularity is reached as the collapse endstate. Such a singularity may be a black hole hidden from external observers by an event horizon or visible to the outside Universe (naked singularity). In the latter collapse procedure, the information on super-dense regions can be transported via suitable non-spacelike trajectories to a distant observer. Although the occurrence of a spacetime singularity as the final outcome of a collapse scenario is proved by the singularity theorems, they do not specify the nature of such a singularity. The cosmic censorship conjecture first articulated by Penrose [6] states that a black hole is always formed in complete gravitational collapse of reasonable matter fields, or a physically reasonable spacetime contains no naked singularities. However, up to now many exact solutions of Einstein's field equations describing singularities, not hidden behind an event horizon of spacetime, are known. A remarkable study is the one by Shapiro and Teukolsky [7,8], who showed numerically that gravitational collapse of a spheroidal dust may end in a naked singularity. Also many exact solutions of Einstein's field equations with a variety of field-sources admitting naked singularities have been surveyed. The examples studied so far include gravitational collapse of a pressure-less matter [9–14], radiation [15–22], perfect fluids [23–25], imperfect fluids [26,27] and null strange quark fluids [28–30]. Beside general theory of relativity, there exist alternative theories of gravity explaining gravitational phenomena. Such theories have been studied for a long time [31–35]. From the theoretical standpoint there has been many attempts to correct the Einstein-Hilbert action in order to renormalize general relativity to build a quantum theory of gravity or at least some effective action (including the low-energy limit of string theories), or to quantize the scalar fields in curved spacetimes [36]. From the observational point of view, the discovery of current acceleration of the Universe using CMB Ia supernova [37–45] suggests that such acceleration may be explained within the framework of general relativity by assuming that 76% of energy content of the Universe is filled with a mysterious form of *dark energy* with equation of state $p \sim -\rho$ (where ρ and p are the energy density and pressure of the cosmic fluid, respectively). Another possibility is to include a cosmological constant Λ of a very small magnitude in Einstein's field equation, but encounters such difficulties as the well-known cosmological constant problem and the coincidence problem. Another alteration is $f(\mathcal{R})$ theories of gravity [46–50] where the Ricci scalar in the Einstein-Hilbert Lagrangian is replaced by a general function of it, providing alternative gravitational models for dark energy since the explanation of the cosmic acceleration comes back to the fact that we do not understand gravity at large scales. Such theories can describe the transition

from deceleration to acceleration in the evolution of the Universe [51]. Moreover, the coincidence problem may be solved simply in such theories by the Universe expansion. Also some models of modified theories of gravity are predicted by string/M-theory considerations [52]. Recently, it has been shown that the accelerated expansion of the Universe may be the result of a modification to the Einstein-Hilbert action in the context of higher order gravity theories [53–55]. Our purpose here is to consider the gravitational collapse of a star within the framework of $f(\mathcal{R})$ theories of gravity, whose matter content obeys the barotropic equation of state, $p = w\rho$. We investigate the conditions under which the resulting singularity may be naked or not. In Sect. 2 we apply the metric formalism to the action of $f(\mathcal{R})$ gravity [56–58] and rewrite it as that of the Brans-Dicke theory with $\omega_{BD} = 0$. Choosing $f(\mathcal{R}) = \alpha\mathcal{R}^m$ [59–61] in Sect.2 and fixing the corresponding potential, we find m as a function of initial energy density and α. In Sect. 4 we study the behavior of the expansion parameter which is the key factor in examining the formation or otherwise of trapped surfaces during the dynamical evolution of the collapse scenario. In Sect. 4 we examine the global features of the nakedness of the resulting singularity by investigating the behavior of the Kretschmann scalar as a function of time. In order to fully complete the model we utilize the Vaidya metric to match the interior spacetime to that of the exterior one.

2 $f(\mathcal{R})$ field equations

We begin by the general action in modified theories of gravity given by

$$\mathcal{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{R}) + \mathcal{A}_{matter}(g_{\mu\nu}, \psi), \tag{1}$$

where $\kappa = 8\pi G$, G is the gravitational constant, g is the determinant of the metric, $\mathcal{R}$ represents the Ricci scalar and ψ collectively denotes the matter fields. Introducing an auxiliary field Ψ, one can write the dynamically equivalent action as (See [62] and references therein)

$$\mathcal{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[f(\Psi) + f'(\Psi)(\mathcal{R} - \Psi) \right] + \mathcal{A}_{matter}(g_{\mu\nu}, \psi), \tag{2}$$

where variation with respect to Ψ leads to the following equation as

$$f''(\Psi)(\mathcal{R} - \Psi) = 0. \tag{3}$$

If $f''(\Psi) \neq 0$, one can then recover action (1) by setting $\Psi = \mathcal{R}$. Redefining the field Ψ by $\phi = f'(\Psi)$ and setting

$$V(\phi) = \Psi(\phi)\phi - f(\Psi(\phi)), \tag{4}$$

action (2) will take the following form

$$\mathcal{A} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi\mathcal{R} - V(\phi)] + \mathcal{A}_{matter}(g_{\mu\nu}, \psi), \tag{5}$$

which corresponds to the Jordan frame representation of the action of Brans-Dicke theory with Brans-Dicke parameter $\omega_{BD} = 0$. Brans-Dicke theory with $\omega_{BD} = 0$ is sometimes called *massive dilaton gravity* [63] which was originally suggested in [64] in order to generate a Yukawa term in the Newtonian limit. Extremizing the action yields the following field equations as (we set $\kappa = 8\pi G = 1$ in the rest of this paper)

$$G_{\mu\nu} = T_{\mu\nu}^{(\text{eff})}, \tag{6}$$

and

$$3\Box\phi + 2V(\phi) - \phi\frac{dV(\phi)}{d\phi} = T^m, \tag{7}$$

where T^m stands for the trace of $T_{\mu\nu}^m$ and the subscript "m" refers to the matter fields (fields other than ϕ) and we have defined the effective stress-energy tensor as

$$T_{\mu\nu}^{(\text{eff})} = \frac{1}{\phi}\left(T_{\mu\nu}^{\text{m}} + T_{\mu\nu}^{\phi}\right), \tag{8}$$

with

$$T_{\mu\nu}^{\phi} = (\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) - \frac{1}{2}g_{\mu\nu}V(\phi), \tag{9}$$

and

$$T_\nu^{\mu m} = \text{diag}\left(\rho_m, p_m, p_m, p_m\right), \tag{10}$$

being the stress-energy tensors of the scalar field and a perfect fluid, respectively.

3 Gravitational collapse of a homogeneous cloud with $f(\mathcal{R}) = \alpha\mathcal{R}^m$

Let us now build and investigate a homogeneous class of collapsing models in $f(\mathcal{R})$ gravity with $m \neq 0$, where the trapping of light is avoided till the formation of singularity, allowing the singularity to be visible to outside observers. In order to achieve our purpose we examine a spherically symmetric homogeneous scalar field, $\Phi = \Phi(\tau)$ originating from geometry. Since the interior spacetime is a dynamical one, we parameterize its line element as follows

$$ds^2 = -d\tau^2 + a^2(\tau)\left(dr^2 + r^2 d\Omega^2\right), \tag{11}$$

where τ is the proper time of a free falling observer whose geodesic trajectories are distinguished by the comoving radial coordinate r and $d\Omega^2$ is the standard line element on the unit 2-sphere. It is worth mentioning that here we assume that starting from the homogeneous initial data, the collapsing configuration remains homogeneous till the singularity is formed. But as the collapse proceeds there may be some inhomogeneities

occurring throughout the collapse scenario, the existence of which can be investigated by perturbation theory, that is, imposing inhomogeneous perturbations on the energy density, scale factor and BD scalar field and then see whether the terms rising from inhomogeneity are dominant in the formation of the singularity or not. Here we do not deal with such an issue but for more details the reader may consult [95] and references there in. Since the presence of matter acting as a "seed" field prompts the collapse of the BD scalar field, we have considered perfect fluid models obeying barotropic equation of state as

$$p_m = w\rho_m. \tag{12}$$

Using the conservation equation for the matter ($\nabla^\alpha T^m_{\alpha\beta} = 0$) together with the use of above equation, one gets the following relations between ρ_m, p_m and the scale factor as

$$\rho_m = \rho_{0}m\, a^{-3(1+w)}; \quad p_m = w\rho_{0}m\, a^{-3(1+w)}, \tag{13}$$

where $\rho_{0}m = \rho_m(a = 1)$, is the initial value of energy density of matter on the collapsing volume. Making use of Eqs. (8) and (11) one finds the following equations for the effective stress-energy tensor as

$$\rho_{(\text{eff})} = -T^{\tau\,(\text{eff})}_{\ \tau} = \frac{1}{\phi}\left(\rho_m + \rho_\phi\right) = \frac{1}{\phi}\left[\rho_m - 3\frac{\dot{a}}{a}\dot{\phi} + \frac{V(\phi)}{2}\right], \tag{14}$$

and

$$P_{(\text{eff})} = T^{r\,(\text{eff})}_{\ r} = T^{\theta\,(\text{eff})}_{\ \theta} = T^{\varphi\,(\text{eff})}_{\ \varphi} = \frac{1}{\phi}\left(p_m + p_\phi\right) = \frac{1}{\phi}\left[p_m + 2\frac{\dot{a}}{a}\dot{\phi} + \ddot{\phi} - \frac{V(\phi)}{2}\right], \tag{15}$$

with all other off-diagonal terms being zero and the radial and tangential profiles of pressure are equal due to the homogeneity and isotropy. Substituting the line element (11) into Einstein's equation one gets the interior solution as

$$\rho_{(\text{eff})} = 3\frac{\dot{a}^2}{a^2} = \frac{\mathcal{M}'}{R^2 R'}; \quad P_{(\text{eff})} = -\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right] = -\frac{\dot{\mathcal{M}}}{R^2 \dot{R}}, \tag{16}$$

$$\dot{R}^2 = \frac{\mathcal{M}}{R}, \tag{17}$$

where $\mathcal{M}(\tau, r)$ rises as a free function from the integration of Einstein's field equation which can be interpreted physically as the total mass within the collapsing cloud at a coordinate radius r with $\mathcal{M} \geq 0$, and $R(\tau, r) = ra(\tau)$ is the physical area radius for the volume labeled by the comoving coordinate r. From Eq. (14) and first part of Eq. (16), one can solve for the mass function

$$\mathcal{M} = \frac{R^3}{3\phi}(\rho_\phi + \rho_m). \tag{18}$$

Using the above equation together with Eq. (17) we arrive at a relation between $\dot{a}$ and the effective energy density as follows

$$\dot{a}^2 = \frac{a^2}{3\phi}(\rho_\phi + \rho_m).\tag{19}$$

Since we are concerned with a continual collapsing scenario, the time derivative of the scale factor should be negative ($\dot{a} < 0$) implying that the physical area radius of the collapsing volume for constant value of r decreases monotonically. The singularity arising as the final state of collapse at $\tau = \tau_s$ is given by $a(\tau_s) = 0$. On the other hand when the scale factor and physical area radius of all the collapsing shells vanish, the collapsing cloud has reached a singularity. A point at which the energy density and pressure blows up, the Kretschmann scalar $\mathcal{K} = R^{abcd}R_{abcd}$ diverges and the normal differentiability and manifold structures break down. In order to solve the field equations we proceed by substituting for $\rho_{(\text{eff})}$ and $p_{(\text{eff})}$ from Eq. (16) into Eqs. (14) and (15) and rewrite them as follows

$$3\frac{\dot{a}^2}{a^2} = \frac{1}{\alpha m \Phi}\left[\rho_0 m a^{-3(1+w)} - 3\alpha m \frac{\dot{a}}{a}\dot{\Phi} + \frac{V(\Phi)}{2}\right],\tag{20}$$

$$-\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right] = \frac{1}{\alpha m \Phi}\left[w\rho_0 m a^{-3(1+w)} + \alpha m \ddot{\Phi} + 2\alpha m \frac{\dot{a}}{a}\dot{\Phi} - \frac{V(\Phi)}{2}\right],\tag{21}$$

where for later convenience we have rescaled the scalar field as $\phi = \alpha m \Phi$ (α and m are real constants), and by the virtue of Eq. (4) the associated potential to $f(\mathcal{R}) = \alpha \mathcal{R}^m$ can be fixed as

$$V(\Phi) = \alpha(m-1)\Phi^{\frac{m}{m-1}}; \quad m \neq 1.\tag{22}$$

Since the scalar field must diverge at the singularity we examine its behavior by taking the following *ansatz* for the scalar field

$$\Phi(\tau) = a^\delta(\tau),\tag{23}$$

where δ is a constant whose sign decides the divergence of the scalar field. Substituting the first and second time derivatives of the scale factor from Eqs. (20) and (21) into Eq. (7) together with the use of Eq. (22) one finds

$$a^{-[\delta+3(1+w)]}\left\{\frac{2\rho_0 m(\delta+3w-1)}{3\alpha m \delta(2+\delta)}\right\} + a^{\left[\frac{\delta}{m-1}\right]}\left\{\frac{4+\delta(2m-1)-2m}{6m\delta+3m\delta^2}\right\} = 0.\tag{24}$$

Matching the powers of scale factor in Eq. (24) we arrive at the following expression for δ

$$\delta = \frac{3(1+w)(1-m)}{m},\tag{25}$$

whence by substituting the above equation into the pair of square brackets in Eq. (24) one gets the following equation to be satisfied by m

$$\alpha \Big\{ 3(1 + w) + m\left[-13 - 9w + m(8 + 6w)\right] \Big\} + 2\rho_{0m} \left[4m - 3(1 + w)\right] = 0. \quad (26)$$

Solving the above equation, we find m as a function of α and ρ_{0m} for w as

$$m_\pm = \frac{13\alpha - 8\rho_{0m} \pm \left[73\alpha^2 - 16\alpha\rho_{0m} + 64\rho_{0m}^2\right]^{\frac{1}{2}}}{16\alpha}, \quad (27)$$

for $w = 0$,

$$m_\pm = -\frac{-5\alpha + 4\rho_{0m} \pm \left[13\alpha^2 - 16\alpha\rho_{0m} + 16\rho_{0m}^2\right]^{\frac{1}{2}}}{6\alpha}, \quad (28)$$

for $w = -\frac{1}{3}$,

$$m_\pm = -\frac{-7\alpha + 8\rho_{0m} \pm \left[33\alpha^2 - 80\alpha\rho_{0m} + 64\rho_{0m}^2\right]^{\frac{1}{2}}}{8\alpha}, \quad (29)$$

for $w = -\frac{2}{3}$, and

$$m_\pm = \frac{4\alpha - 2\rho_{0m} \pm \sqrt{2}\left[3\alpha^2 + 2\alpha\rho_{0m} + 2\rho_{0m}^2\right]^{\frac{1}{2}}}{5\alpha}, \quad (30)$$

for $w = \frac{1}{3}$, where $\alpha \neq 0$.

4 Time behavior of the scale factor and singular epoch

One would like to study time behavior of the scale factor as the collapse evolves, considering matter fields. If at time $\tau = \tau^*$ (or equivalently for some $a = a^*$) the collapse begins, then by integrating Eq. (20) together with the use of Eqs. (22) and (23) near the singularity with respect to time one gets the time behavior of the scale factor as

$$a(\tau) = \left[a*^{\frac{1}{2}(\delta+3(1+w))} - \frac{1}{2}\sqrt{\frac{2\rho_{0m}+\alpha(m-1)}{6\alpha m(1+\delta)}}\left(\delta + 3(1+w)\right)\left(\tau - \tau_*\right) \right]^{\frac{2}{\delta+3(1+w)}},$$

(31)

and the corresponding singular epoch as

$$\tau_s = 2\sqrt{\frac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)}} \frac{a*^{\frac{1}{2}(\delta+3(1+w))}}{\delta + 3(1+w)} + \tau^*,$$

(32)

where the time τ_s corresponds to a vanishing scale factor. Thus the collapse reaches the singularity in a finite proper time. This result for the scale factor completes the interior solution within the collapsing cloud, providing us with the required construction.

5 The conditions

We are now in a position to investigate the nature of singularity as the endstate of a collapsing scenario. The singularity is called locally naked if it is only visible to an observer being in the neighborhood of it (such a singularity is necessarily covered by a spacetime event horizon) and is called globally naked if there exists a family of future directed non-spacelike geodesics reaching to the outside observers in the spacetime and terminating at the past in the singularity. To investigate the nature of spacetime singularity arising from the collapse procedure, we examine here whether such a singularity could be naked, or necessarily covered within a spacetime event horizon and if so under what conditions. Since α can be regarded as a free parameter, we seek for the appropriate values of it satisfying the following conditions

- The ratio $\mathcal{M}/R$ stays less than unity till the singular epoch which means that the singularity has been formed earlier than the formation of the apparent horizon or the formation of trapped surfaces have been failed till the singular epoch.
- $\delta < 0$ during the gravitational collapse scenario accompanied by divergence of the scalar field.
- For physical reason, weak energy condition, stated as $\rho_{(eff)} \geq 0$ and $\rho_{(eff)}+p_{(eff)} \geq 0$ must be satisfied during the dynamical evolution of the system.
- The effective energy density and effective pressure blow up in the vicinity of the singularity, the latter being negative during the evolution of the collapse process, since the absence of trapped surfaces is accompanied by a negative pressure. In fact the negativeness of the effective pressure implies that $\dot{\mathcal{M}} < 0$, that is, the mass contained in the collapsing volume with comoving coordinate r keeps decreasing leading to an outward energy flux during the gravitational collapse scenario.
- Kretschmann scalar diverges at the singular time and then converges to zero at late times.

In order to determine whether the singularity is naked or not, one needs to investigate the formation of trapped surfaces during the collapse procedure. These surfaces are

defined as compact two-dimensional (smooth) spacelike surfaces such that both families of ingoing and outgoing null geodesics orthogonal to them necessarily converge or the expansion parameter Θ of the outgoing future-directed null geodesics is everywhere negative [65,66]. Consider a congruence of outgoing radial null geodesics having the tangent vector $(V^\tau, V^r, 0, 0)$, where $V^\tau = d\tau/dk$ and $V^r = dr/dk$ and k is an affine parameter along the geodesics. For the spacetime metric (11), the geodesic expansion parameter which is defined as the covariant divergence of the vector field V^ν is given by [67]

$$\Theta = \nabla_\nu V^\nu = \frac{2}{r} \left[1 - \sqrt{\frac{\mathcal{M}}{R}} \right] V^r. \tag{33}$$

If the null geodesics terminate at the singularity in the past with a definite tangent, then at the singularity we have $\Theta > 0$. If such family of curves do not exist and the event horizon forms earlier than the singularity, a black hole is formed. Utilizing Eq. (33), we now attempt to study the formation of trapped surfaces during the dynamical evolution of the gravitational collapse procedure. We show that physically, the formation of a black hole or a naked singularity as the final state for the dynamical evolution is governed by the rate of collapse and the presence of scalar field. It is seen that for a specified range of variation of α, the cosmic censorship conjecture may be violated for all cases of matter considered below. In the following subsections we consider first the four conditions mentioned in the beginning of this section and postpone the last one to the next section. We begin by calculating the ratio $\mathcal{M}/R$ in the general case which is considered for the four cases of matter, $w = \{0, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\}$ corresponding to dust, cosmic strings, domain walls and radiation, respectively. By the virtue of Eq. (17) we have

$$\frac{\mathcal{M}}{R} = r^2 \left[\frac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-(\delta+3w+1)}. \tag{34}$$

The weak energy condition which states that the energy density as measured by any local observer must be non-negative can be written for any timelike vector V^μ as follows

$$T_{\mu\nu} V^\mu V^\nu \geq 0, \tag{35}$$

whereby one gets the following conditions for the effective energy density ($\rho_{(\text{eff})} \geq 0$)

$$\rho_{(\text{eff})} = \left[\frac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+3(w+1))} \geq 0, \tag{36}$$

and the sum of effective energy density and pressure ($\rho_{(\text{eff})} + p_{(\text{eff})} \geq 0$) as

$$\rho_{(\text{eff})} + p_{(\text{eff})} = \left(1 + w + \frac{\delta}{3} \right) \left[\frac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+3(w+1))} \geq 0. \tag{37}$$

Finally for the rate of change of mass function with respect to time one has

$$\dot{\mathcal{M}} = r^3(\delta + 3w)\left[\frac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)}\right] a^{-(\delta+3w+1)} \mid \dot{a} \mid, \tag{38}$$

where the minus sign has been absorbed into $\dot{a}$. At the initial epoch where $a(\tau^*) = 1$ there should not be any trapping of light, then by assuming that $r = r_b$ is the boundary of the collapsing volume one may easily see that for a suitable initial value of energy density the ratio $\mathcal{M}/R$ is less than unity at the initial time. This fact is in accordance with the regularity conditions stating that if gravitationally collapsing massive stars are to be modeled, then the energy density, pressure, and other physical quantities must be finite and regular at the initial spacelike hyper-surface from which the collapse commences. For the case of homogeneous-density collapse the resulting singularity coincides with the curves $R(\tau_s, 0) = 0$ or $R(\tau_s, r \neq 0) = 0$, corresponding to a central or non-central singularity, respectively. In the next subsections we first consider the simpler case of non-central singularity and investigate formation or otherwise of trapped surfaces for different values of w.

5.1 Dust ($w = 0$)

For this case of matter we have the following relations (we set $\rho_{0m} = 1$ in the rest of this paper)

$$\begin{cases} \frac{\mathcal{M}}{R} = r^2\left[\frac{2\rho_{0m}+\alpha(m-1)}{6\alpha m(1+\delta)}\right]a^{-(\delta+1)}, \\ \rho_{(\text{eff})} + p_{(\text{eff})} \\ = \left(1 + \frac{\delta}{3}\right)\left[\frac{2\rho_{0m}+\alpha(m-1)}{2\alpha m(1+\delta)}\right]a^{-(\delta+3)}, \\ p_{(\text{eff})} = \frac{\delta}{3}\left[\frac{2\rho_{0m}+\alpha(m-1)}{2\alpha m(1+\delta)}\right]a^{-(\delta+3)}, \end{cases} \begin{cases} \rho_{(\text{eff})} = \left[\frac{2\rho_{0m}+\alpha(m-1)}{2\alpha m(1+\delta)}\right]a^{-(\delta+3)}, \\ \dot{\mathcal{M}} = r^3\delta\left[\frac{2\rho_{0m}+\alpha(m-1)}{6\alpha m(1+\delta)}\right]a^{-(\delta+1)} \mid \dot{a} \mid. \end{cases}$$

$$\tag{39}$$

If the interval for α is $-3.9 < \alpha < -0.1$ and by picking out m_- in Eq. (27), the corresponding parameter δ_- is always negative, its absolute value varies between $1 < \mid \delta_- \mid < 3$ and its minimum and maximum values are $\delta_- = -2.72484$ and $\delta_- = -1.00569$, respectively. It is seen that for such values of α and δ_-, the ratio $\mathcal{M}/R$ stays less than unity and then the expansion parameter is always positive up to the singularity, that is the singularity is formed earlier than the formation of apparent horizon which is the boundary of trapped surfaces. The negativeness of effective pressure for the allowed values of α ensures that $\dot{\mathcal{M}}$ is negative as collapse proceeds which means that the mass contained in the collapsing volume keeps waning. Then there exists an outward energy flux which may be visible to outside observers (for the case of globally naked singularity) since the trapped surfaces do not form early

enough to cover the singularity. In addition, the validity of the weak energy condition can be easily checked by the virtue of expressions obtained for $\rho_{(\text{eff})}$ and $\rho_{(\text{eff})} + p_{(\text{eff})}$.

5.2 Cosmic strings ($w = -\frac{1}{3}$)

Cosmic strings are the result of hypothetical 1-dimensional (spatially) topological defects which may have been constructed during a symmetry breaking phase transition at the early Universe. The possibility of their existence was first considered by Tom Kibble [68] in 1976. A fluid of cosmic strings may have an effective equation of state, $p_m = -\frac{1}{3}\rho_m$, so one has the following relations for this type of matter fluid as

$$
\begin{cases}
\dfrac{\mathcal{M}}{R} = r^2 \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-\delta}, \\[2ex]
\rho_{(\text{eff})} + p_{(\text{eff})} = \left(\dfrac{\delta+2}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+2)}, \\[2ex]
p_{(\text{eff})} = \left(\dfrac{\delta-1}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+2)},
\end{cases}
\qquad
\begin{cases}
\rho_{(\text{eff})} = \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+2)}, \\[2ex]
\dot{\mathcal{M}} = r^3(\delta-1) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-\delta} \, | \, \dot{a} \, | \, .
\end{cases}
\tag{40}
$$

Choosing m_+ in Eq. (28), it can be seen that the valid range of change for α is $-100 < \alpha < -0.1$ upon which the absolute value of parameter δ_+ is restricted to vary between $0 <| \delta_+ |< 2$ (in order to prevent the ratio $\mathcal{M}/R$ to be infinity, $\delta_+ = -1$ has to be excluded) and its minimum and maximum values are $\delta_+ = -1.86224$ and $\delta_+ = -0.61558$, respectively. It should be noted that one can choose the lower limit of α to be much less than -100, but such a choice does not affect considerably the value of δ_+ and its magnitude remains close to -0.6. Taking these values into account, the ratio $\mathcal{M}/R$ stays finite till the singular epoch and causes the expansion parameter to be positive up to the singularity, and if no trapped surfaces exist initially then none would form until the epoch $a(\tau_s) = 0$ which is consistent with the fact that there exist families of outgoing radial null geodesics emerging from the singularity. Also the weak energy condition is satisfied ($| \delta_+ |< 2$) and the effective pressure is negative (since $\delta_+ < 0$), consistent with the fact that time derivative of mass function is negative i.e., the mass incorporated in the region where the collapse procedure evolves keeps falling off. As a result, there exists an outward energy flux during the collapse scenario which may be visible to external Universe.

5.3 Domain walls ($w = -\frac{2}{3}$)

Domain walls are two-dimensional objects that form when a discrete symmetry is spontaneously broken at a phase transition [69]. It has been noticed that there exist a link between domain-walls and cosmologies such as brane cosmology [70]. The effective equation of state for a fluid of domain walls may be $p_m = -\frac{2}{3}\rho_m$ and the mentioned conditions for this case can be written as

$$
\begin{cases}
\dfrac{\mathcal{M}}{R} = r^2 \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-(\delta-1)}, \\[2ex]
\rho_{(\text{eff})} + p_{(\text{eff})} = \left(\dfrac{\delta+1}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+1)}, \\[2ex]
p_{(\text{eff})} = \left(\dfrac{\delta-2}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+1)},
\end{cases}
\qquad
\begin{cases}
\rho_{(\text{eff})} = \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+1)}, \\[2ex]
\dot{\mathcal{M}} = r^3 (\delta-2) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-(\delta-1)} \, |\, \dot{a} \,| \, .
\end{cases}
\tag{41}
$$

As long as α varies in the range $-100 < \alpha < -0.1$ (lower limit of α can be chosen much less than -100 but such a choice has no noticeable influence on the maximum value of δ), by choosing m_+ in Eq. (29), the parameter δ_+ is negative, its absolute value is less than unity and its minimum and maximum values are $\delta_+ = -0.953501$ and $\delta_+ = -0.379572$, respectively. One then may easily see that at initial epoch $(a(\tau^*) = 1)$, the regularity condition (there should be no trapped surfaces at the initial hyper-surface from which the collapse commences) is satisfied and the ratio of mass function to physical area radius of the collapsing volume is less than unity during the collapse procedure denoting that the expansion parameter being positive up to the singularity. In this case the collapse to a naked singularity may take place, where the trapped surfaces do not form early enough or are avoided in the spacetime. Also the mass contained in the collapsing ball reduces as the time advances due to the fact that the effective pressure stays negative till the singular epoch. It is obvious that for such values of α and δ_+ the weak energy condition is satisfied during the collapse scenario.

5.4 Radiation ($w = \frac{1}{3}$)

Finally, for this type of matter we have following relations for the said conditions

$$
\begin{cases}
\dfrac{\mathcal{M}}{R} = r^2 \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-(\delta+2)}, \\[2ex]
\rho_{(\text{eff})} + p_{(\text{eff})} = \left(\dfrac{\delta+4}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+4)}, \\[2ex]
p_{(\text{eff})} = \left(\dfrac{\delta+1}{3} \right) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+4)},
\end{cases}
\qquad
\begin{cases}
\rho_{(\text{eff})} = \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{2\alpha m(1+\delta)} \right] a^{-(\delta+4)}, \\[2ex]
\dot{\mathcal{M}} = r^3 (\delta+1) \left[\dfrac{2\rho_{0m} + \alpha(m-1)}{6\alpha m(1+\delta)} \right] a^{-(\delta+2)} \, |\, \dot{a} \,| \, .
\end{cases}
\tag{42}
$$

The suitable range of variation of α for m_- in Eq. (30) is $-0.66 < \alpha < -0.01$ which makes the corresponding values of δ_- to vary in the range $2 <|\, \delta_- \,|< 4$ with $\delta_- = -2.01007$ and $\delta_- = -3.95037$ are the maximum and minimum values of this parameter respectively. For such values of α and δ_- the ratio $\mathcal{M}/R$ stays less than unity as the collapse procedure ends. Then the expansion parameter remains positive up to the singularity denoting that the apparent horizon is failed to form. On the other hand, weak energy condition is satisfied for these values of α and δ_- and the effective pressure is negative till the singular epoch. It is worth noticing that if one chooses the

parameter α and the power of Ricci scalar in Eqs. (27)–(30) other than the ones determined in the above subsections, weak energy condition together with the conditions on effective pressure and the ratio $\mathcal{M}/R$ would be violated. The central singularity occurring at $R = 0, r = 0$ can be naked if we have any future-directed outgoing null geodesics terminating in the past at the singularity. In order to examine the possibility of existence of such families let us introduce a new variable $y = r^\gamma$ with $\gamma > 1$ defined in such away that the ratio $R'/r^{\gamma-1}$ is a unique finite quantity in the limit $r \to 0$. Now consider the outgoing radial null geodesic equation which is given by

$$\frac{d\tau}{dr} = a(\tau). \tag{43}$$

In terms of variables $y = r^\gamma$ and R the above equation reads

$$\frac{dR}{dy} = \frac{1}{\gamma r^{\gamma-1}} \left[\dot{R}\frac{d\tau}{dr} + R' \right], \tag{44}$$

whence using Eq. (17) we have

$$\frac{dR}{dy} = \frac{R'}{\gamma y^{\frac{\gamma-1}{\gamma}}} \left[1 - \sqrt{\frac{\mathcal{M}}{R}} \right]. \tag{45}$$

If there exist outgoing radial null geodesics in the past at the central singularity which occurs at $\tau = \tau_s$, then along such geodesics we have $R \to 0$ as $r \to 0$, or in terms of the variables y and R, the point $y = 0, R = 0$ is a singularity of the above first order differential equation. For such a congruence of geodesics dR/dy must be positive up to the singularity. Then as long as α, ρ_{0m}, and m satisfy the values which was determined previously, trapped surfaces would fail to form as the collapse evolves.

6 Nakedness of the singularity

The final fate of a continual gravitational collapse of a matter cloud ends in either a black hole or a naked singularity where in the former there exists an event horizon of spacetime developing earlier than the formation of the singularity to cover it. Thus the regions of extreme curvatures and densities are concealed from the outside observers. The event horizon or surface of a black hole is defined as the boundary of the spacetime that is causally connected to future null infinity [71], or in other word the boundary between events from which light rays emitted inside this boundary surface can not escape to future infinity while those emitted outside in a suitable direction can. Both event and apparent horizons coincide in stationary spacetimes, however, this case is not generally true in dynamical ones. Although the existence of an apparent horizon predicates the existence of spacetime event horizon, the converse is not always true and the event horizon may veil the singularity even if apparent horizon does not emerge on a spatial slice. So far we have discussed the conditions under which the collapse scenario ends in a locally naked singularity, that is, the singularity is visible to an observer being

in the neighborhood of it. In this case, the trajectories coming out of the singularity do not actually come out to a distant observer but fall back into the singularity again at a later time without going out of the boundary of the star. Thus the locally naked singularity could still be covered by the event horizon and only strong version of cosmic censorship conjecture is violated but the week form of it is intact [72]. However, if the event horizon is delayed to form or the singularity forms early enough before the formation of event horizon, then it would be visible to external observers and thus a globally naked singularity would born as the endstate of collapse rather than a black hole. Therefore in such a situation curvature invariants namely the Kretschmann scalar should increase near the singularity, diverge at the singular epoch and then converge to zero at late times [74]. In previous section we showed that for suitable values of α, trapping of light can be avoided which means that the apparent horizon is failed to form. But since the failure of formation of an apparent horizon does not necessarily bode the absence of an event horizon, we investigate the nakedness of the singularity in spherically symmetric collapse of a fluid by considering the behavior of Kretschmann scalar with respect to time. For the line element (11) this quantity is given by

$$\mathcal{K} \equiv R^{abcd} R_{abcd} = \frac{12}{a^4} \left[a^2 \ddot{a}^2 + \dot{a}^4 \right]. \tag{46}$$

By the virtue of Eq. (31) one can easily obtain this quantity as a function of time for $w = \left\{ 0, -\frac{1}{3}, -\frac{2}{3}, \frac{1}{3} \right\}$ and the results are sketched in Figures 1, 2, 3 and 4. It is seen that Kretschmann scalar diverges at singular time and then tends to zero at late times. In order to better understand the situation one should resort to critical behavior at the black hole threshold. Such a behavior was discovered in gravitational collapse of a spherically symmetric massless scalar field [75], axisymmetric gravitational waves [76], spherical system of a radiation fluid [77] and spherical system of a perfect fluid obeying the equation of state $p = w\rho$ [78]. Consider a type II critical collapse in which the black hole mass is scaled as $M_{BH} \propto \mid p - p_* \mid^\gamma$, where p parameterizes a family of initial data sets evolving through Einstein's equations, p_* is a critical value and γ is a positive constant which is called a critical exponent [74,79,80]. For a sufficiently large value of p the collapse procedure develops to a black hole and for a sufficiently small one it evolves to a dispersion [79, 80]. The boundary between these two regimes is the black hole threshold. Now consider the limit from supercritical collapse ($p > p_*$) to a critical collapse, i.e., $p \to p_*$. In such a limit, the black hole mass tends to zero and the maximum value of curvature diverges just outside the event horizon. Since we have arbitrarily strong curvature outside the event horizon by fine-tuning, the black hole threshold can be regarded as a globally naked singularity [79,80]. Let us now consider the geometry of the exterior spacetime. In order to complete the model we need to match the interior spacetime of the dynamical collapse to a suitable exterior geometry. The Schwarzschild solution is a useful model to describe the spacetime outside stars but the spacetime outside such a star may be filled with radiated energy from the star in the form of electromagnetic radiation. The Schwarzschild solution does not describe this properly as it deals with an empty spacetime given by $T_{ab} = 0$. The spacetime outside a spherically symmetric star surrounded by radiation emitted from the
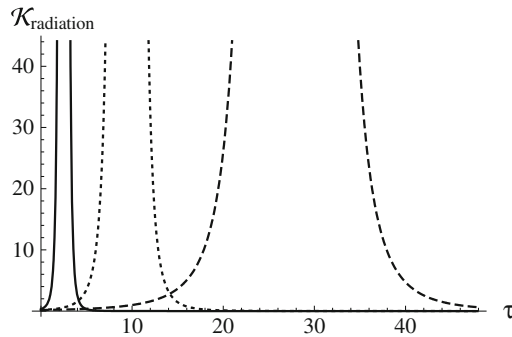
Fig. 1 The behavior of Kretschmann scalar (in units of s^{-4}) as a function of proper time for $w = \frac{1}{3}$ and different values of α and δ_-. $\alpha = -0.65, m_- = 2.02575$ and $\delta_- = -2.02542$, *Solid curve*, $\alpha = -0.15, m_- = 5.97355$ and $\delta_- = -3.33038$, *Dotted curve* and $\alpha = -0.05, m_- = 6.6128$ and $\delta_- = -3.75922$, *Dashed curve*. For the initial energy density, scale factor, and proper time we have adopted the values $\rho_{0m} = 1, a^* = 1$, and $\tau^* = 0$, respectively. The corresponding singular epochs are, $\tau_s = 2.49674$ for *Solid curve*, $\tau_s = 9.44081$ for *Dotted curve*, and $\tau_s = 27.8945$ for *Dashed curve*
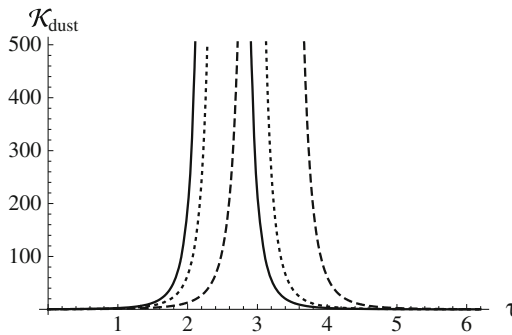

Fig. 2 The behavior of Kretschmann scalar (in units of s^{-4}) as a function of proper time for $w = 0$ and different values of α and δ_-. $\alpha = -3.5, m_- = 1.52406$ and $\delta_- = -1.03175$, *Solid curve*, $\alpha = -2.5, m_- = 1.60424$ and $\delta_- = -1.12996$, *Dotted curve* and $\alpha = -1.5, m_- = 1.8076$ and $\delta_- = -1.34034$, *Dashed curve*. For the initial energy density, scale factor, and proper time we have adopted the values $\rho_{0m} = 1, a^* = 1$, and $\tau^* = 0$, respectively. The corresponding singular epochs are, $\tau_s = 2.5083$ for *Solid curve*, $\tau_s = 2.70354$ for *Dotted curve*, and $\tau_s = 3.19309$ for *Dashed curve*

star is described by the Vaidya metric [81–83] which can be given in the following form as

$$ds^2_{out} = -\left[1 - \frac{2M(r_v, v)}{r_v}\right]dv^2 - 2dvdr_v + r_v^2 d\Omega^2, \qquad (47)$$

where v is the retarded null coordinate, r_v and $M(r_v, v)$ are the Vaidya radius and Vaidya mass, respectively. In what follows, we use the Isreal-Darmois junction conditions to match the interior spacetime to a Vaidya exterior geometry at the boundary hyper-surface Σ given by $r = r_b$. The spacetime metric just inside Σ is given by

$$ds^2_{in} = -d\tau^2 + a^2(\tau)\left[dr^2 + r_b^2 d\Omega^2\right], \qquad (48)$$

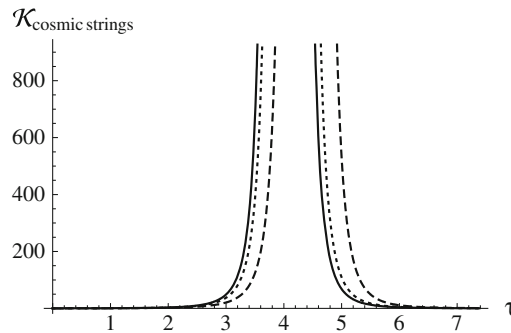**Fig. 3** The behavior of Kretschmann scalar (in units of s^{-4}) as a function of proper time for $w = -\frac{1}{3}$ and different values of α and δ_+. $\alpha = -90$, $m_+ = 1.44581$ and $\delta_+ = -0.616689$, *Solid curve*, $\alpha = -30$, $m_+ = 1.46909$ and $\delta_+ = -0.638609$, *Dotted curve* and $\alpha = -10$, $m_+ = 1.54031$ and $\delta_+ = -0.701562$, *Dashed curve*. For the initial energy density, scale factor, and proper time we have adopted the values $\rho_{0m} = 1$, $a^* = 1$, and $\tau^* = 0$, respectively. The corresponding singular epochs are, $\tau_s = 4.04176$ for *Solid curve*, $\tau_s = 4.13327$ for *Dotted curve*, and $\tau_s = 4.38509$ for *Dashed curve*
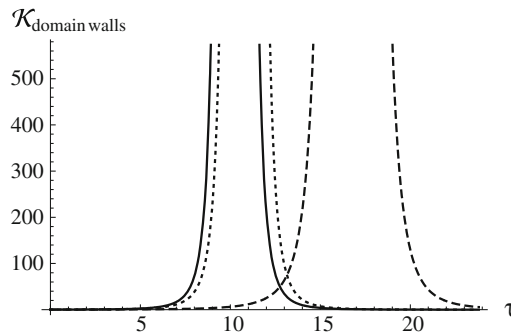

Fig. 4 The behavior of Kretschmann scalar (in units of s^{-4}) as a function of proper time for $w = -\frac{2}{3}$ and different values of α and δ_+. $\alpha = -100$, $m_+ = 1.61179$ and $\delta_+ = -0.379572$, *Solid curve*, $\alpha = -20$, $m_+ = 1.68699$ and $\delta_+ = -0.407227$, *Dotted curve* and $\alpha = -2$, $m_+ = 2.55425$ and $\delta_+ = -0.608495$, *Dashed curve*. For the initial energy density, scale factor, and proper time we have adopted the values $\rho_{0m} = 1$, $a^* = 1$, and $\tau^* = 0$, respectively. The corresponding singular epochs are, $\tau_s = 10.2643$ for *Solid curve*, $\tau_s = 10.787$ for *Dotted curve*, and $\tau_s = 16.808$ for *Dashed curve*

whereby matching the area radius at the boundary one gets

$$R(r_b, \tau) = r_v(v). \tag{49}$$

One then gets the interior and exterior metrics on the hyper-surface Σ as follows

$$ds^2_{\Sigma in} = -d\tau^2 + a^2(\tau) r_b^2 d\Omega^2, \tag{50}$$

$$ds^2_{\Sigma out} = -\left[1 - \frac{2M(r_v, v)}{r_v} + 2\frac{dr_v}{dv}\right] dv^2 + r_v^2 d\Omega^2, \tag{51}$$

where matching the first fundamental form gives

$$\left[\frac{dv}{d\tau}\right]_\Sigma = \frac{1}{\left[1 - \frac{2M(r_v,v)}{r_v} + 2\frac{dr_v}{dv}\right]^{\frac{1}{2}}}, \quad (r_v)_\Sigma = r_b a(\tau). \tag{52}$$

In order to match the second fundamental form (extrinsic curvature) for interior and exterior spacetimes we need to find the unit vector field normal to the hyper-surface Σ. We then proceed by taking into account the fact that any spacetime metric can be written locally in the following form as

$$ds^2 = -\left(\alpha^2 - \beta_i\beta^i\right)d\tau^2 - 2\beta_i dx^i d\tau + h_{ij}dx^i dx^j, \tag{53}$$

where α, β^i, and h_{ij} are the lapse function, shift vector, and induced metric, respectively and i, j run in $\{1, 2, 3\}$. Comparing the above equation with Eqs. (47) and (48) together with using the following normalization condition for n^v and n^{r_v}

$$n^v n_v + n^{r_v} n_{r_v} = 1, \tag{54}$$

one gets the normal vector fields for the interior and exterior spacetimes as

$$n_{in}^a = [0, a(\tau)^{-1}, 0, 0], \tag{55}$$

$$n^v = -\frac{1}{\left[1 - \frac{2M(r_v,v)}{r_v} + 2\frac{dr_v}{dv}\right]^{\frac{1}{2}}}, \quad n^{r_v} = \frac{1 - \frac{2M(r_v,v)}{r_v} + \frac{dr_v}{dv}}{\left[1 - \frac{2M(r_v,v)}{r_v} + 2\frac{dr_v}{dv}\right]^{\frac{1}{2}}}. \tag{56}$$

The extrinsic curvature of the hyper-surface Σ is defined as the Lie derivative of the metric tensor with respect to the normal vector n, given by the following relation as

$$K_{ab} = \frac{1}{2}\mathcal{L}_n g_{ab} = \frac{1}{2}\left[g_{ab,c}n^c + g_{cb}n^c_{,a} + g_{ac}n^c_{,b}\right], \tag{57}$$

whereby the nonzero θ components of the extrinsic curvature read

$$K_{\theta\theta}^{in} = r_b a(\tau), \quad K_{\theta\theta}^{out} = r_v \frac{1 - \frac{2M(r_v,v)}{r_v} + \frac{dr_v}{dv}}{\left[1 - \frac{2M(r_v,v)}{r_v} + 2\frac{dr_v}{dv}\right]^{\frac{1}{2}}}. \tag{58}$$

Setting $\left[K_{\theta\theta}^{in} - K_{\theta\theta}^{out}\right]_\Sigma = 0$ on the hyper-surface Σ, together by using Eqs. (17) and (52) one gets the following relation between the mass function and Vaidya mass on the boundary as

$$\mathcal{M}(\tau, r_b) = 2M(r_v, v). \tag{59}$$

From Eqs. (59) and (18) it is seen that the behavior of Vaidya mass is decided by the allowed values of ρ_{0_m}, α and m which prompt the gravitational collapse scenario to end in the formation of a naked singularity. In order to get a relation describing the rate of change of Vaidya mass with respect to r_v one has to match the τ component of the extrinsic curvature on the hyper-surface Σ. Setting $\left[K_{\tau\tau}^{in} - K_{\tau\tau}^{out} \right] = 0$ we have

$$M(r_v, v)_{,r_v} = \frac{\mathcal{M}}{2r_v} + r_b^2 a\ddot{a}. \tag{60}$$

Now it can be seen that at the singular time, $\tau = \tau_s$ the ratio $2M(r_v, v)/r_v$ tends to zero. Thus the exterior spacetime at the singular epoch reads

$$ds^2 = -dv^2 - 2dvdr_v + r_v^2 d\Omega^2, \tag{61}$$

which describes a Minkowski spacetime in retarded null coordinates. Hence, the exterior generalized Vaidya metric at singular time can be smoothly extended to the Minkowski spacetime as the collapse completes [84]. The occurrence of a naked singularity as the final fate of a collapse scenario depends on the existence of families of non-spacelike trajectories reaching faraway observers and terminating in the past at the singularity. In order to show this we begin by Eq. (52) and after using Eq. (59) we get

$$\left[\frac{dv}{d\tau} \right]_\Sigma = \frac{1 - r_b \dot{a}}{1 - \frac{\mathcal{M}(\tau, r_b)}{r_v}}. \tag{62}$$

It is seen that imposing the null condition on the Vaidya metric leads to the same relation as the above. This means that null geodesics can come out from the singularity and reach distant observers before it evaporates into the free space. On the other hand, since for the allowed values of α, ρ_{0_m} and m formation of trapped surfaces in spacetime is avoided and from another side the singularity emerges outside of the event horizon, such a congruence of trajectories can be detected by the outside observer.

7 Conclusion and outlook

One of the physical motivations for discussing naked singularities is that these objects provide a useful laboratory for quantum gravity, since in such ultra-strong gravity regions the length and time scales are comparable to the Planck length and time. In other words, quantum effects occurring in such super-dense regimes are no longer covered by the spacetime event horizon and the chance to observe such effects in the universe is provided. An example is quantum particle creation due to the formation of a naked singularity, which has been studied in the literature [85–91]. During the past 20 years, cosmic censorship conjecture has been extensively investigated in spherical models of gravitational collapse of physically reasonable matter. The simplest of those which has been scrutinized in detail and has been shown that both black holes and naked singularities form from generic initial conditions is gravitational collapse of a dust fluid. Dwivedi and Joshi in [92] and Waugh and Lake in [93] showed that a naked

strong curvature singularity can be formed in an inhomogeneous dust collapse and a self-similar one, respectively. Also some examples of naked singularity formation in gravitational collapse of a scalar field is given in [94]. In this work we have studied the process of gravitational collapse of a star where the matter fluid obeys the barotropic equation of state $p = w\rho$, in the context of $f(\mathcal{R})$ theories of gravity. Making use of metric formalism we wrote the action of $f(\mathcal{R})$ gravity as the Brans-Dicke one with vanishing coupling parameter. Having solved the resulting field equations by taking the *ansatz* (23) for scalar field we arrived at the expressions (27)–(30) for the exponent of Ricci scalar as a function of α and initial energy density. In Sect. 5 we imposed five conditions on the effective energy density and pressure, the ratio $\mathcal{M}/R$, parameter δ, time behavior of the mass function, and Kretschmann scalar, the validity of which depends on determining appropriate values of α. As long as these conditions are fulfilled the resulting singularity can be globally naked, i.e., ultra-dense regions are no longer covered by a spacetime event horizon and physical effects are allowed to be shared by the external Universe. It is worth mentioning that there are future finite-time singularities in the dark energy universe coming from modified gravity as well as in other dark energy theories. Considering $f(\mathcal{R})$ gravity models that satisfy cosmological viability conditions (chameleon mechanism), it is possible to show that finite-time singularities emerge in several cases. Such singularities can be classified according to the values of the scale factor $a(t)$, the density ρ, and the pressure p [96].

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