Particle Accelerators
1982 Vol. 12 pp. 101–106
0031-2460/82/1202-0101$06.50/0

INTEGER RESONANCES IN THE MODIFIED BETATRON*

D. CHERNIN

Berkeley Research Associates, PO Box 852, Springfield, Virginia 22150 U.S.A.

P. SPRANGLE

Plasma Theory Branch, Plasma Physics Division, Naval Research Laboratory, Washington, DC 20375 U.S.A.

(Received February 5, 1982)

The integer resonances affecting beam motion in the presence of external field imperfections in the modified betatron are studied. An upper bound is obtained on the magnitude of field error that may be tolerated. A numerical example shows that for practical parameters the resulting bound is very restrictive. The effect of longitudinal temperature and other possible stabilizing effects are discussed.

I. INTRODUCTION

In a conventional betatron, low-order resonances between particle motion and field imperfections can be avoided by restricting the beam current so that the tune shift[1] remains sufficiently small. In a modified betatron, the addition of a strong toroidal magnetic field may allow large currents to be accelerated,[2,3] but resonances become much more difficult to avoid, especially if one contemplates removing the toroidal field before the beam is extracted. This paper examines the problem of integer resonances in the modified betatron[4] and obtains a condition bounding the rate of change of the fields; when the condition is satisfied the resonances are passed through with sufficient speed so that the beam is not significantly disturbed. We consider here only errors in the fields themselves, not in field gradients, so we discuss only integer, not half-integer resonances.

In what follows, we will first consider a "cold" beam, that is, one in which there is no spread in longitudinal energy or, therefore, in circulation frequency about the machine. The effect of orbital resonances on such a cold beam will be seen to place rather severe limits on the magnitude of the tolerable field imperfections. When the effects of temperature are taken into account, however, a numerical example below will illustrate a reduction of the effect of the resonance on the motion of the beam center of mass. An explanation of this temperature effect will be given.

II. ORBITAL RESONANCES FOR A COLD BEAM

We consider a beam of circular cross section and uniform density and current profiles, as shown in Fig. 1. The torus has a major radius r_0 and minor radius a; the chamber is assumed to be perfectly conducting as far as the rapidly varying part of the self-fields is concerned. The beam radius is r_b with center located at $r = r_0 + \Delta r$, $z = \Delta z$, as shown in the figure. If we define the displacement of a particle from the design orbit $r = r_0$, $z = 0$ as $r_1 = \Delta r + \delta r$, $z_1 = \Delta z + \delta z$ then the equations of motion for r_1 and z_1 are, to first order in the displacement from the design orbit

$$\ddot{r}_1 + \frac{\dot{\gamma}_0}{\gamma_0}\dot{r}_1 + \Omega_{z0}^2(1 - n - n_s)r_1 = \frac{e\dot{B}_{\theta 0}}{2m\gamma_0 c} z_1 + \Omega_{\theta 0}\dot{z}_1 - \frac{\omega_b{}^2}{2\gamma_0{}^2}\left(1 - \frac{r_b{}^2}{a^2}\right)\Delta r - \frac{e}{m\gamma_0}\left[\tilde{E}_r + \beta_0\tilde{B}_z + \Omega_{z0}\int_0^t dt'\tilde{E}_\theta(t')\right] \tag{1a}$$

* Supported by the Office of Naval Research.

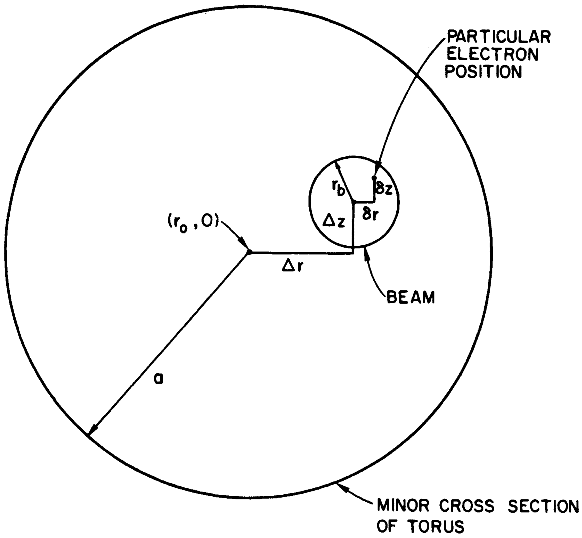

FIGURE 1 Minor cross section of modified betatron showing beam center and particle coordinates. The major radius of the device is r_0.

$$\ddot{z}_1 + \frac{\dot{\gamma}_0}{\gamma_0}\dot{z}_1 + \Omega_{z0}^2(n - n_s)z_1$$
$$= -\frac{e\dot{B}_{\theta 0}}{2m\gamma_0 c} r_1 - \Omega_{\theta 0}\dot{r}_1$$
$$- \frac{\omega_b{}^2}{2\gamma_0{}^2}\left(1 - \frac{r_b{}^2}{a^2}\right)\Delta z$$
$$- \frac{e}{m\gamma_0}[\tilde{E}_z - \beta_0\tilde{B}_r], \quad (1b)$$

where γ_0 is the particle energy on the "ideal" design orbit (in units of mc^2), $\Omega_{z0} = eB_{z0}/m\gamma_0 c$, $\Omega_{\theta 0} = eB_{\theta 0}/m\gamma_0 c$, B_{z0} is the vertical magnetic field at the design orbit, $B_{\theta 0}$ is the toroidal magnetic field, n is the betatron field index (assumed constant here), $n_s = \omega_b{}^2/(2\gamma_0{}^2\Omega_{z0}^2)$ is the self-field index, $\omega_b = (4\pi n_0 e^2/m\gamma_0)^{1/2}$ is the beam plasma frequency and e, m are the magnitude of electron charge and rest mass, and where field components with a "wiggle" on the right hand sides of (1a, b) denote the value of the field imperfection which in general will depend on the value of θ, the azimuthal position of the particle. In deriving (1a, b) we have allowed all fields to depend on time; we have therefore included the inductive poloidal electric field ($\dot{B}_{\theta 0}$ terms). Also included are the effect of wall image charges and currents ($r_b{}^2/a^2$ terms), present when the beam is displaced from the center of the chamber.

We desire to have equations which describe only the motion of the beam center, $\Delta r(\theta, t)$, $\Delta z(\theta, t)$. To this end we define a distribution function f as

$$f(r, \theta, z, v_r, v_\theta, v_z, t) \equiv \sum_{\mathbf{r}^{(0)}, \mathbf{v}^{(0)}} g(\mathbf{r}^{(0)}, \mathbf{v}^{(0)}) \times \delta(r - \hat{r})\frac{\delta(\theta - \hat{\theta})}{r}\delta(z - \hat{z})\delta^{(3)}(\mathbf{v} - \hat{\mathbf{v}}), \quad (2)$$

where $\mathbf{r}^{(0)}$ and $\mathbf{v}^{(0)}$ are particle initial conditions, $\hat{r}$, $\hat{\theta}$, $\hat{z}$, and $\hat{\mathbf{v}}$ are the solutions for the particle trajectories as functions of initial position, velocity, and time, and where $g(\mathbf{r}^{(0)}, \mathbf{v}^{(0)})$ is a weighting function. We then have that

$$\Delta r(\theta, t) = \frac{\int r\, dr\, dz\, d\mathbf{v}(r - r_0)f}{\int r\, dr\, dz\, d\mathbf{v} f} = \frac{\sum g(\mathbf{r}^{(0)}, \mathbf{v}^{(0)})(\hat{r} - r_0)\delta(\theta - \hat{\theta})}{\sum g(\mathbf{r}^{(0)}, \mathbf{v}^{(0)})\delta(\theta - \hat{\theta})} \equiv \langle r_1 \rangle. \quad (3)$$

A similar expression holds for $\Delta z(\theta, t)$. It may be similarly shown that, for a cold beam,

$$\langle \dot{r}_1 \rangle = \left(\frac{\partial}{\partial t} + \Omega_{z0}\frac{\partial}{\partial \theta}\right)\Delta r \quad (4)$$

$$\langle \ddot{r}_1 \rangle = \left(\frac{\partial}{\partial t} + \Omega_{z0}\frac{\partial}{\partial \theta}\right)^2 \Delta r, \quad (5)$$

where, of course, analogous expressions hold for $\langle z_1 \rangle$, $\langle \dot{z}_1 \rangle$ and $\langle \ddot{z}_1 \rangle$. In Eqs. (4) and (5) we have assumed that all particles circulate the machine with $\dot{\hat{\theta}} = \Omega_{z0}$. This assumption will be relaxed in the next section where the effects of finite longitudinal temperature are considered.

Using this averaging procedure on Eqs. (1a, b), one obtains equations for the beam-center motion. Though these may be solved in general, the special choice $n = \frac{1}{2}$ (which is consistent with our assumption of a circular beam) simplifies the analysis. With $n = \frac{1}{2}$ and defining

$$\Delta\psi = \Delta r + i\Delta z \equiv \sum_{l=-\infty}^{\infty} \overline{\Delta\psi_l} e^{il\theta}, \quad (6)$$

the equation for $\overline{\Delta\psi_l}$ is

$$\frac{\partial^2}{\partial t^2}\overline{\Delta\psi_l} + \left[\frac{\dot{\gamma}_0}{\gamma_0} + i\Omega_{\theta 0} + 2il\Omega_{z0}\right]\frac{\partial\overline{\Delta\psi_l}}{\partial t} + \left[\Omega_{z0}^2\left(\frac{1}{2} - \frac{r_b{}^2}{a^2}n_s - l^2\right) + il\Omega_{z0}\frac{\dot{\gamma}_0}{\gamma_0} + i\frac{e\dot{B}_{\theta 0}}{2m\gamma_0 c} - l\Omega_{\theta 0}\Omega_{z0} + il\dot{\Omega}_{z0}\right]\overline{\Delta\psi_l} = F_l \tag{7}$$

where F_l is the l-th Fourier component of

$$-\frac{e}{m\gamma_0}\left[\tilde{E}_r + \beta_0\tilde{B}_z + i(\tilde{E}_z - \beta_0\tilde{B}_r) + \Omega_{z0}\left\langle\int_0^t dt'\tilde{E}_\theta(t')\right\rangle\right].$$

Equation (7) may be solved, assuming the functions multiplying derivatives of $\overline{\Delta\psi_l}$ are slowly varying over the period of a betatron oscillation:

$$\overline{\Delta\psi_l} \approx (\gamma_0\omega_0)^{-1/2}\int^t dt' e^{-i\int^t dt''[\frac{1}{2}\Omega_{\theta 0} + l\Omega_{z0}]} \times \left[\frac{\gamma_0(t')}{\omega_0(t')}\right]^{1/2}\sin\left[\int_{t'}^t dt''\omega_0\right]F_l(t'), \tag{8}$$

where

$$\omega_0(t) \equiv \left[\Omega_{z0}^2\left(\frac{1}{2} - \frac{r_b{}^2}{a^2}n_s\right) + \frac{1}{4}\Omega_{\theta 0}^2\right]^{1/2}. \tag{9}$$

For long times (many betatron periods), the integral in (8) may be evaluated by the method of stationary phase. The points of stationary phase (resonance points) occur when

$$\Omega_l{}^{\pm} \equiv -\tfrac{1}{2}\Omega_{\theta 0} - l\Omega_{z0} \pm \omega_0 = 0 \tag{10}$$

for a given l. This is just the condition that the betatron frequency be l times the fundamental cyclotron frequency, Ω_{z0}. Condition (10) may also be written

$$B_{\theta 0} = -\frac{1}{l}\left(l^2 + \frac{r_b{}^2}{a^2}n_s - \frac{1}{2}\right)B_{z0}. \tag{11}$$

For positive $B_{\theta 0}$ and B_{z0}, Eq. (11) may be satisfied only by negative l and for such l, Eq. (10) may be satisfied only for the lower sign (fast mode resonance). Evaluating Eq. (8) then gives

$$\overline{\Delta\psi_l} \sim i\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\gamma_0(t_-)}{\gamma_0(t)\omega_0(t)\omega_0(t_-)}\right]^{1/2} \times \frac{F_l(t_-)}{|\dot{\Omega}_l{}^-(t_-)|^{1/2}} e^{i\int_{t_-}^t \Omega_l{}^- dt'' \pm i\pi/4}, \tag{12}$$

where t_- is the time at which $\Omega_l{}^- = 0$ and where the + or − sign is used in the exponent according as $\dot{\Omega}_l{}^-(t_-) > 0$ or < 0 respectively. If we neglect the possibility of cancellation due to different phases as we pass through different resonances and if we interpret F_l generically as $[-(e/m\gamma_0)\delta f_l]$ where δf_l is the l-th Fourier component of any field error, we may obtain a lower bound on $|\dot{\Omega}_l{}^-|$ by requiring

$$|\overline{\Delta\psi_l}| \ll a, \tag{13}$$

which gives

$$|\dot{\Omega}_l{}^-| \gg \frac{\pi}{2}\left[\frac{e\delta f_l}{m\gamma_0\omega_0 a}\right]^2, \tag{14}$$

which is our basic result. For γ_0 large enough that we may neglect $\dot{\Omega}_{z0}$ compared with $\dot{\Omega}_{\theta 0}$ and $r_b{}^2 n_s/a^2$ compared with 1/2, this constraint may be rewritten, using the relations

$$\dot{\Omega}_l{}^- = -\frac{l^2}{l^2 + 1/2}\dot{\Omega}_{\theta 0} \tag{15}$$

and

$$\omega_0 = \frac{l^2 + 1/2}{2|l|}\Omega_{z0}, \tag{16}$$

as

$$|\dot{\Omega}_{\theta 0}| \gg \frac{2\pi}{l^2 + 1/2}\left[\frac{\delta f_l}{B_{z0}}\frac{c}{a}\right]^2. \tag{17}$$

As an example, we consider the problem of passing through the $l = -1$ resonance. We consider a hypothetical experiment ($r_0 = 1$ m, $a = 10$ cm, $r_b = 1$ cm) in which γ_0 is increased linearly in time from an initial (injection) value of 7 to a final value ($t_{\rm final} = 1$ millisecond) of 100, while simultaneously $B_{\theta 0}$ is decreased from 1.5 kG to 0. The $l = -1$ resonance will occur at $t = 627$ μsec, at which time $B_{z0} = 1120$ G, $B_{\theta 0} = 560$ G and $\gamma_0 = 65.3$. At resonance, $\dot{\Omega}_{\theta 0} = -6.2 \times$

10^{11} sec^{-2}. Substituting in the expression (17) we obtain an upper bound on the allowable field error

$$\frac{\delta f_{-1}}{B_{z0}} \ll 1.3 \times 10^{-4},$$

a rather severe requirement.

We conclude that, at least for the case of a cold beam, it may not be desirable to remove the toroidal field and pass through these resonances. Perhaps the toroidal field may be reduced somewhat from its initial value, assuming the high-l resonances are not too important and can be passed through easily. It may then be possible, by the use of an intentionally introduced field perturbation, to use a low-l resonance in a controlled way to extract the beam before B_θ is completely removed.

It should be noted that it is possible, at least in principle, to avoid the integer resonances altogether by raising both B_{z0} and $B_{\theta 0}$ proportionately and in such a way that condition (11) is never satisfied for any l. At the end of such an acceleration cycle, however, one will have a very large toroidal magnetic field in the device, possibly complicating the extraction process.

The above results apply to a beam all of whose particles are traveling to lowest order at the same azimuthal angular velocity. All particles are then in resonance at precisely the same moment and receive the same periodic perturbations to their orbits. In the next section we relax this assumption and examine the behavior of a beam, the particles of which possess a spread in energy.

III. EFFECT OF FINITE BEAM TEMPERATURE ON RESONANCES

To calculate the effect of beam temperature on beam behavior near a resonance, we consider an ensemble of beams, each cold and each consisting of particles traveling with a zero-order angular frequency $\dot{\theta}_0$ given by

$$\dot{\theta}_0 = \Omega_{z0} - kP, \tag{18}$$

where P is the canonical angular momentum of a particle, which is related to the difference in energy between the particle under consideration and the (reference) particle maintained at the design orbit $r = r_0$, $z = 0$ by

$$P = \frac{\Delta\gamma mc^2}{\Omega_{z0}}, \tag{19}$$

and where, in Eq. (18),

$$-k \equiv \left(\frac{1}{\gamma_0{}^2} - \frac{1}{1/2 - n_s}\right) \Big/ \gamma_0 m r_0{}^2. \tag{20}$$

For each cold beam, the relations (4) and (5) are then modified by the replacement

$$\Omega_{z0} \rightarrow \Omega_{z0} - kP, \tag{21}$$

and therefore we may obtain the solution for each cold beam by making the replacement, in Eq. (8),

$$l\Omega_{z0} \rightarrow l(\Omega_{z0} - kP). \tag{22}$$

The behavior of the actual warm beam will then be given by

$$\overline{\Delta\psi_l} \approx \left\langle (\gamma_0\omega_0)^{-1/2} \int^t dt' e^{-i\int_{t'}^{t} dt''[\frac{1}{2}\Omega_{\theta 0} + l(\Omega_{z0} - kP)]} \times \left[\frac{\gamma_0(t')}{\omega_0(t')}\right]^{1/2} \sin\left[\int_{t'}^{t} dt''\omega_0\right] F_l(t')\right\rangle_P , \tag{23}$$

where the average is defined over some normalized distribution function in P, i.e.

$$\langle\cdots\rangle_P = \int_{-\infty}^{\infty} dPG(P). \; . \; . \; . \tag{24}$$

In Eq. (23) we can immediately anticipate the effect of temperature on the behavior of the beam; the entire effect is included in the phase factor, in the term kP. Such a term, when averaged over any reasonable momentum distribution, will give a reduction in amplitude of the average as the "width" of $G(P)$ is increased. Physically this means that the various particles of different energies within the beam receive, when passing through resonance, displacements in slightly different directions. The net effect on the motion of the beam center is therefore reduced. (Though our linearized treatment here necessarily includes a fixed beam size, it may in fact be the case that a warm beam will just expand

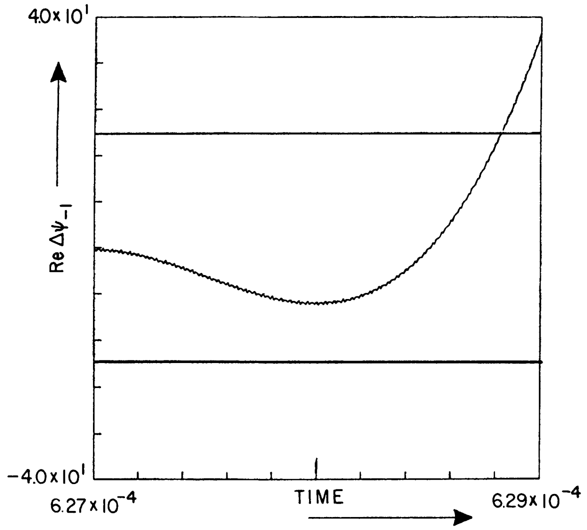

FIGURE 2 Re ($\overline{\Delta\psi_{-1}}$) vs. time for $T_L = 0$.

slightly while passing through resonance while the motion of the beam center remains relatively undisturbed.)

As an example, we consider a beam made up of particles having the energy distribution

$$G_E(\Delta\gamma) = \begin{cases} 1/T_L & |\Delta\gamma| < T_L/2 \\ 0 & |\Delta\gamma| > T_L/2 \end{cases}, \tag{25}$$

where T_L is a measure of the longitudinal tem-

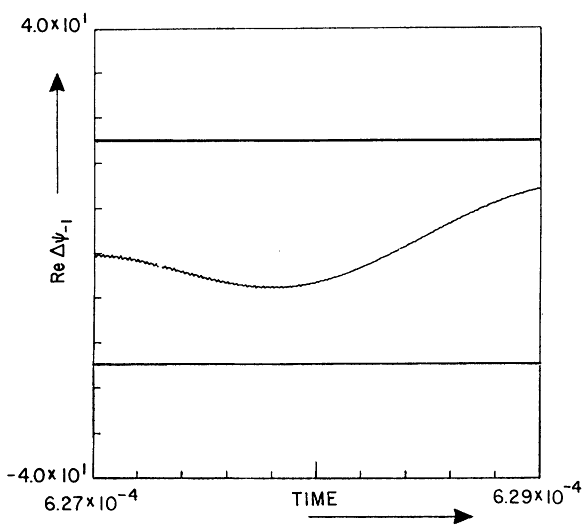

FIGURE 3 Re ($\overline{\Delta\psi_{-1}}$) vs. time for $T_L = 0.5$.

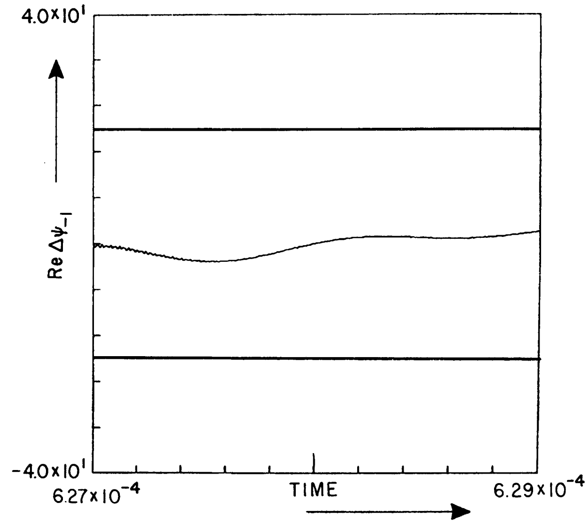

FIGURE 4 Re ($\overline{\Delta\psi_{-1}}$) vs. time for $T_L = 1.0$.

perature and where $\Delta\gamma$ is related to P by Eq. (19). We consider again the hypothetical experiment described in the preceding section. For $\delta f_{-1}/B_{z0} = 5 \times 10^{-3}$ the results of a numerical evaluation of Eq. (23) are shown in Figs. 2-5, which correspond to T_L = (0., 0.5, 1.0, 2.0). In each figure, the real part of $\overline{\Delta\psi_{-1}}$ in centimeters is plotted versus time in seconds. The resonance condition, Eq. (11), is satisfied at the center of the time axis. Total elapsed time is 2.1 μsec. The

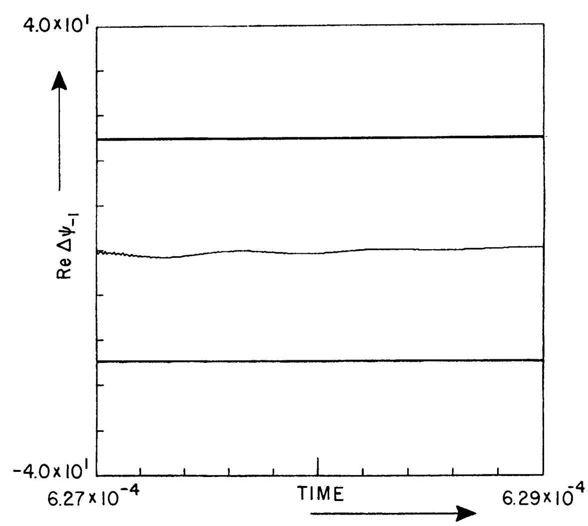

FIGURE 5 Re ($\overline{\Delta\psi_{-1}}$) vs. time for $T_L = 2.0$.

chamber diameter, $2a = 20$ cm is indicated by solid horizontal lines on each plot. We observe that for this example, $T_L = 1.0$, or a 0.5-MeV energy spread, is adequate to smooth out the effect of the resonance. This is the same order of magnitude of spread needed to damp the negative-mass/kink instability in this device[5].

IV. CONCLUSIONS

We have obtained a bound on the magnitude of field errors that can be tolerated in a modified betatron in order that certain integer resonances may be safely passed through. We have found that for practical parameters the bound is extremely restrictive. The basic difficulty stems from the fact that unless the external parameters of the system are changed very quickly, the orbits remain in or near resonance for many betatron oscillations, allowing the displacements to grow to large levels. Such a result suggests that nonlinear effects may play an important role in beam behavior near a resonance. For example, one may ask whether the radial dependence of B_θ would be sufficient to "detune" the resonance as the beam moves a finite but small distance from its equilibrium position. This possibility is receiving further study.

We have also shown that a finite longitudinal beam temperature acts to reduce the effect of the resonance on the motion of the center of the beam. The temperature spread required appears to be comparable, in a specific example, to that needed to stabilize certain microinstabilities.[5] It remains unresolved in this analysis whether the beam expands when passing through a resonance. Such behavior, of course, if severe, could be as unacceptable as large whole-beam displacement.

Should it be possible to achieve significantly lower field errors than those used in our example (0.5%), or if it is possible experimentally to detect and correct by some feedback mechanism the sudden, resonant displacement of the beam, then perhaps lower toroidal fields may be employed initially and be removed either during or following acceleration. The effects of passage through the low-l resonances may thereby be reduced to a tolerably small level.

REFERENCES

1. L. J. Laslett in "Proc. of the 1963 Summer Study on Storage Rings, Accelerators and Experimentation at Super-High Energies" BNL-7534.
2. P. Sprangle and C. A. Kapetanakos, J. Appl. Phys. *49*, 1 (1978).
3. N. Rostoker, Bull. APS. *25*, 854 (1980).
4. Laslett (ERAN-51, Jan 1970 (unpublished)) has discussed certain aspects of the resonance problem in the ERA with a toroidal field. He has derived explicit expressions for ν.
5. P. Sprangle and J. L. Vomvoridis, NRL Memorandum Report 4688 (to be published).