



Topological charged black holes in massive gravity's rainbow and their thermodynamical analysis through various approaches



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ABSTRACT

Violation of Lorentz invariance in the high energy quantum gravity motivates one to consider an energy dependent spacetime with massive deformation of standard general relativity. In this paper, we take into account an energy dependent metric in the context of a massive gravity model to obtain exact solutions. We investigate the geometry of black hole solutions and also calculate the conserved and thermodynamic quantities, which are fully reproduced by the analysis performed with the standard techniques. After examining the validity of the first law of thermodynamics, we conduct a study regarding the effects of different parameters on thermal stability of the solutions. In addition, we employ the relation between cosmological constant and thermodynamical pressure to study the possibility of phase transition. Interestingly, we will show that for the specific configuration considered in this paper, van der Waals like behavior is observed for different topology. In other words, for flat and hyperbolic horizons, similar to spherical horizon, a second order phase transition and van der Waals like behavior are observed. Furthermore, we use geometrical method to construct phase space and study phase transition and bound points for these black holes. Finally, we obtain critical values in extended phase space through the use of a new method.

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1. Introduction

It is arguable that Einstein gravity is an effective theory which is valid in infrared (IR) limit while in ultraviolet (UV) regime, it fails to produce accurate results. This shortcoming requires modification in order to incorporate the UV regime. It is believed that the Lorentz symmetry is an effective symmetry in IR limit of quantum gravitational processes. Since the standard energy–momentum dispersion relation depends on such symmetry, one expects to regard the modified energy–momentum dispersion relation in UV regime. Such modification motivates one to develop double special relativity [1,2], in which this theory has two upper bounds (speed of light (c) and the Planck energy (E_P)) [1]. In this theory, it is not possible for a particle to achieve velocity and energy larger than the speed of light and the Planck energy, respectively [3]. Generalization of this doubly special relativity to curved spacetime is gravity's rainbow [3]. On the other hand, if one considers the gravity as an emerging phenomenon due to quantum degrees of freedom, spacetime should be described with an energy dependent metric. Hence, the spacetime is affected by a particle probing it and since this particle can acquire a range of energies (E), a rainbow of energy is built. The gravity's rainbow can be constructed through the use of deforming the standard energy–momentum relation as $E^2 f^2(\varepsilon) - P^2 g^2(\varepsilon) = m^2$, in which $\varepsilon = E/E_P$, E is the energy of that test particle probing the geometry of spacetime and the functions $f^2(\varepsilon)$ and $g^2(\varepsilon)$ are called rainbow functions. It is notable that, in order to recover the standard energy–momentum relation in the IR limit, the rainbow functions satisfy the following relation

$$\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = \lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 1. \quad (1)$$

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Now, it is possible to define an energy dependent deformation of the metric \hat{g} with the following form [3]

$$\hat{g} = \eta^{\mu\nu} e_\mu(E) \otimes e_\nu(E), \quad (2)$$

where $e_0(E) = \frac{1}{f(\epsilon)} \hat{e}_0$ and $e_i(E) = \frac{1}{g(\epsilon)} \hat{e}_i$, in which the hatted quantities refer to the energy independent frame.

On the other hand, gravity's rainbow has specific properties which were highlighted in recent studies. At first, we point out that this theory enjoys a modification in energy–momentum dispersion relation. Such modification in the UV limit is examined in studies that were conducted in discrete spacetime [4], models based on string theory [5], spacetime foam [6], spin-network in loop quantum gravity (LQG) [7], non-commutative geometry [8], Horava–Lifshitz gravity [9,10] and also ghost condensation [11]. In addition, the observational evidences confirm that such modification could exist [12]. Second, it was pointed out that by treatment of the horizon radius of black holes as radial coordinate in this theory, the usual uncertainty principle stands [13,14]. It is possible to translate the uncertainty principle ($\Delta p \geq 1/\Delta x$) into a bound on the energy ($E \geq 1/\Delta x$) where E can be interpreted as the energy of a particle emitted in the Hawking radiation process. Also, it has been shown that the uncertainty in the position of a test particle in the vicinity of horizon should be equal to the event horizon radius (see Refs. [15–18] for more details) as $E \geq 1/\Delta x \approx 1/r_+$, in which E is the energy of a particle near the horizon that is bounded by the E_p and cannot increase to arbitrary values. Hence, this bound on the energy modifies the temperature and the entropy of black holes in gravity's rainbow [18]. Third, it was shown that the black hole thermodynamics in the presence of gravity's rainbow is modified. Such modification leads to results such as existence of remnant for black holes [18,19] which is proposed to be a solution to information paradox [20].

Recently, the theoretical aspects of gravity's rainbow have been investigated in various contexts [21–29]; different classes of black holes with different gauge fields have been studied in Refs. [18,30–32]. In addition, the hydrostatic equilibrium equation of stars and the effects of this generalization on neutron stars have been examined in Ref. [33]. Also, wormhole solutions in gravity's rainbow were obtained in Ref. [34]. Besides, the effects of rainbow functions on gravitational force are studied in Ref. [35].

The fundamental motivation of considering gravity's rainbow comes from the violation of Lorentz invariance (or diffeomorphism invariance) in the high energy regime. In this regime, one may regard a massive deformation of standard general relativity to obtain a Lorentz-invariant theory of massive gravity as well. It was shown that the Lorentz-breaking mass term of graviton leads to a physically viable ghost free model of gravity [36]. Although the graviton in general relativity is considered as a massless particle, there are some arguments regarding the existence of massive gravitons. The first attempt for building such theory was done by Fierz–Pauli [37] which Boulware and Deser have shown that it suffers the ghost instability in nonlinear extension [38]. There have been several reports regarding the interaction effects of nonlinear theory of massive gravity in the absence of ghost field [39–42]. Charged black holes in the presence of massive gravity were investigated in Refs. [43,44] (see [45–49] for more details regarding considering massive gravity). Moreover, phase transition and entanglement entropy of a specific massive theory were studied in Ref. [50]. On the other hand, de Rham, Gabadadze and Tolley (dRGT) proposed another theory of massive gravity [51] without Boulware–Deser ghost [52–54]. dRGT massive gravity employs a reference metric for constructing mass terms. After that, Vegh used a singular metric for constructing a dRGT like theory [55]. In his theory, the graviton behaves like a lattice in specific limits and a Drude peak was observed. It was shown that for an arbitrary singular metric the mentioned theory is ghost free [56] and enjoys stability which was addressed in Refs. [53,54]. Different classes of exact black hole solutions in the presence of this massive gravity and their thermodynamics, phase transition, geometrical thermodynamics and their thermal stability have been investigated [57–59]. In addition, the holographic superconductor–normal metal–superconductor Josephson junction for this specific massive gravity has been studied and it was shown that massive gravity has specific contributions to its properties [60].

It has been proposed that a consistent quantum theory of the gravity may be obtained through the use of black holes thermodynamics. The interpretation of geometrical aspect of black holes as thermodynamical variable provides a powerful viewpoint for constructing such theory. In addition, the recent advances in gauge/gravity duality emphasize on importance of black holes thermodynamics [61–74]. On the other hand, the pioneering work of Hawking and Page, which was based on the phase transition of asymptotically adS black holes [75], and also Witten's paper on the similar subject [76] highlighted the importance of black holes thermodynamics. In order to study black hole thermodynamics, we can use different approaches which are based on various ensembles. One of the thermodynamical aspects of black holes is investigation of thermal stability in the canonical ensemble. In the canonical ensemble, the sign of heat capacity determines thermal stability/instability of the black holes. In addition, the roots and divergencies of heat capacity are denoted as bound and phase transition point, respectively. Due to these reasons, black holes thermodynamics and their thermal stability have been investigated in literature [77–82].

Besides, the interpretation of cosmological constant as a thermodynamical variable (pressure) has been recently employed in literatures. Such consideration will lead to enriching the thermodynamical behavior of black holes and observation of specific properties of usual thermodynamical systems such as van der Waals like behavior [83–95], reentrant of phase transition [96] and existence of triple point for black holes [97]. In addition, the consequences of adS/CFT correspondence [79,98–100], ensemble dependency of BTZ black holes [101] and two dimensional dilaton gravity [102] justify the consideration of cosmological constant not as a fixed parameter but as a thermodynamical variable. Phase transition of different classes of black holes in the presence of different gauge fields has been investigated by employing such proportionality between the cosmological constant and thermodynamical pressure [103–108]. In order to study phase diagrams for obtaining critical values, in Ref. [109] a new method for calculating these critical values was introduced. In this method, by using a relation between the cosmological constant and pressure with denominator of the heat capacity of black holes, one can obtain a relation for the pressure. The maximum of pressure in this method is critical pressure in which the phase transition takes place. Such method has been employed in several papers [58,59,109] and it was shown to be a successful approach toward calculating the critical pressure and volume of black holes in extended phase space.

Another approach for studying the thermodynamical phase transition of black holes is through the use of a geometrical way. In this method, one can use thermodynamical variables of the black holes to construct a thermodynamical metric. The information regarding bound and phase transition point of the corresponding black hole is within the behavior of Ricci scalar of the constructed thermodynamical metric. In other words, bound and phase transition point of black holes are presented as divergencies in the mentioned Ricci scalar.

A successful metric may cover bound and phase transition point of heat capacity by divergencies in its Ricci scalar without any extra (non-physical) divergency. Among various metrics that are proposed, one can point out Weinhold [110], Ruppeiner [111], Quevedo [112] and HPEM [113–115]. These metrics have been employed to study the phase transition of black holes in several papers [116–122]. In Refs. [113–115], it was shown that using Weinhold, Ruppeiner and Quevedo metrics for specific black holes may lead to inconsistent picture. In other words, bound and phase transition point may not be matched with divergencies of the Ricci scalar completely, or extra (non-physical) divergencies may arise. Interestingly, these shortcomings are avoided by using the HPEM approach [113–115]. Motivated by the points mentioned above, in this paper, we regard charged topological black holes in the presence of massive gravity's rainbow.

2. Basic equations and black hole solutions

The Lagrangian of massive gravity in the context of Einstein theory with a minimally coupled linear $U(1)$ gauge field can be written as [55]

$$\mathcal{L} = \mathcal{R} - 2\Lambda - \mathcal{F} + m^2 \sum_i^4 c_i \mathcal{U}_i(g, f), \quad (3)$$

where \mathcal{R} is the scalar curvature, Λ is the cosmological constant and f is a fixed symmetric tensor. Also, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is electromagnetic tensor field and A_μ is the gauge potential. It is notable that, in Eq. (3), c_i 's are constants and \mathcal{U}_i 's are symmetric polynomials of the eigenvalues of the $d \times d$ matrix $\mathcal{K}_\nu^\mu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$ which can be written as

$$\mathcal{U}_1 = [\mathcal{K}], \quad \mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \quad \mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4].$$

Taking into account Eq. (3) and using variational principle, one can obtain the following field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + 2 \left(F_{\mu\rho} F_\nu^\rho - \frac{1}{4} g_{\mu\nu} \mathcal{F} \right) + m^2 \chi_{\mu\nu} = 0, \quad (4)$$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0, \quad (5)$$

in which $G_{\mu\nu}$ is the Einstein tensor and $\chi_{\mu\nu}$ is

$$\begin{aligned} \chi_{\mu\nu} = & -\frac{c_1}{2} (\mathcal{U}_1 g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2} (\mathcal{U}_2 g_{\mu\nu} - 2\mathcal{U}_1 \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) - \frac{c_3}{2} (\mathcal{U}_3 g_{\mu\nu} - 3\mathcal{U}_2 \mathcal{K}_{\mu\nu} \\ & + 6\mathcal{U}_1 \mathcal{K}_{\mu\nu}^2 - 6\mathcal{K}_{\mu\nu}^3) - \frac{c_4}{2} (\mathcal{U}_4 g_{\mu\nu} - 4\mathcal{U}_3 \mathcal{K}_{\mu\nu} + 12\mathcal{U}_2 \mathcal{K}_{\mu\nu}^2 - 24\mathcal{U}_1 \mathcal{K}_{\mu\nu}^3 + 24\mathcal{K}_{\mu\nu}^4). \end{aligned} \quad (6)$$

The main goal is obtaining topological static charged black holes with adS asymptote in the massive gravity's rainbow. We consider the metric of d -dimensional spacetime with the following form

$$ds^2 = -\frac{\psi(r)}{f^2(\varepsilon)} dt^2 + \frac{1}{g^2(\varepsilon)} \left[\frac{dr^2}{\psi(r)} + r^2 h_{ij} dx_i dx_j \right], \quad i, j = 1, 2, 3, \dots, n, \quad (7)$$

where $h_{ij} dx_i dx_j$ is a $(d-2)$ -dimensional line element with constant curvature $d_2 d_3 k$ and volume V_{d_2} , in which $d_i = d - i$. We should note that the constant k , which indicates the boundary of $t = \text{constant}$ and $r = \text{constant}$, can be a positive (elliptic), zero (flat) or negative (hyperbolic) constant curvature hypersurface.

It is worth mentioning that due to generalization to gravity's rainbow, one has to modify the reference metric as $f_{\mu\nu} = \text{diag}(0, 0, \frac{c^2 h_{ij}}{g^2(\varepsilon)})$, where c is a positive constant. The obtained reference metric also indicates that massive graviton in massive gravity could acquire specific range of energies and the limitation of gravity's rainbow is enforced on gravitons as well. Now, we use the modified reference metric and obtain the \mathcal{U}_i as $\mathcal{U}_j = \frac{c^j}{r^j} \prod_{\chi=2}^{j+1} d_\chi$ [57,123].

Inserting the gauge potential ansatz $A_\mu = h(r) \delta_\mu^t$ in the Maxwell equation (5) and considering the metric (7), we obtain $h(r) = -\frac{q}{r^{d_3}}$ in which q is an integration constant and is related to the electric charge parameter. In addition, one may use the definition of electromagnetic field tensor to find its nonzero components as $F_{tr} = -F_{rt} = \frac{qd_3}{r^{d_2}}$.

Now, we are interested in obtaining the static black hole solutions for this gravity. In order to obtain the metric function $\psi(r)$, one may use the nonzero components of Eq. (4) which results into

$$\psi(r) = k - \frac{m_0}{r^{d_3}} - \frac{2\Lambda}{d_1 d_2 g^2(\varepsilon)} r^2 + \frac{2d_3 q^2 f^2(\varepsilon)}{d_2 r^{2d_3}} + \frac{m^2}{g^2(\varepsilon)} \left\{ \frac{cc_1}{d_2} r + c^2 c_2 + \frac{d_3 c^3 c_3}{r} + \frac{d_3 d_4 c^4 c_4}{r^2} \right\}, \quad (8)$$

where m_0 is an integration constant which is related to the total mass of solutions. It is notable that, the obtained metric function (8), satisfies all components of the Eq. (4).

In order to investigate the geometrical structure of the solutions, we first look for the essential singularity(ies). By calculating the Ricci and Kretschmann scalars, we find

$$\lim_{r \rightarrow 0} R \rightarrow \infty, \quad \lim_{r \rightarrow 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \rightarrow \infty, \quad (9)$$

$$\lim_{r \rightarrow \infty} R = \frac{2d}{d_2} \Lambda, \quad \lim_{r \rightarrow \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{8d}{d_1 d_2^2} \Lambda^2. \quad (10)$$

These scalars are finite for $r \neq 0$, and therefore, we conclude that there is only one curvature singularity located at the origin ($r = 0$). In addition, Eq. (10) confirms that the asymptotical behavior of the solutions is (a)dS. In order to study the geometrical effects of massive term and gravity's rainbow on our solutions, we refer the reader to Refs. [32,58].

There are different models of the rainbow functions with various motivations. In this paper, we try to present all analytical relations with general form of rainbow functions. This general form helps us to follow the trace of temporal ($f(\varepsilon)$) and spacial ($g(\varepsilon)$) rainbow functions, separately. But for plotting the diagrams and other numerical analysis we have to choose a class of rainbow functions. For numerical calculations, we are interested in a special class of rainbow functions which is based on the constancy of velocity of the light [124]

$$f(\varepsilon) = g(\varepsilon) = \frac{1}{1 + \lambda\varepsilon}, \tag{11}$$

where λ is an arbitrary parameter. To state the matter differently, hereafter, we keep the general form of rainbow functions in analytical relations and for numerical analysis, we use Eq. (11) as a typical form of rainbow functions.

3. Thermodynamics

Now, we should calculate the conserved and thermodynamic quantities of the solutions to check the first law of thermodynamics. We use the definition of Hawking temperature which is obtained through the concept of surface gravity on the outer horizon r_+ . We find

$$T = \frac{g(\varepsilon)}{4\pi f(\varepsilon)} \psi'(r) \Big|_{r=r_+} = \frac{kd_3 g(\varepsilon)}{4\pi f(\varepsilon)r_+} - \frac{r_+ \Lambda}{2\pi d_2 f(\varepsilon)g(\varepsilon)} - \frac{d_3^2 q^2 g(\varepsilon) f(\varepsilon)}{2\pi d_2 r_+^{2d_5/2}} + \frac{m^2}{4\pi f(\varepsilon)g(\varepsilon)r_+^3} \mathcal{A}, \tag{12}$$

where $\mathcal{A} = d_3 d_4 d_5 c_4 c^4 + d_3 d_4 c_3 c^3 r_+ + d_3 c_2 c^2 r_+^2 + c c_1 r_+^3$.

Here, we conduct a study regarding the sign of temperature. This study is necessary since the conditions regarding the positivity and negativity of temperature put restrictions on solutions being physical or non-physical.

By taking a closer look at the temperature, one can see that charge term (q^2 -term) is negative. Therefore, there is always an upper bound for the electric charge of the black hole solutions, similar to Reissner–Nordström and Kerr–Newman black holes. On the other hand, by considering $c_i > 0$, massive term (m^2 -term) is always positive which makes the effects of this term on the positivity of temperature. Unlike charge term, increasing energy functions leads to decreasing the effects of massive term. We point out that due to specific coupling of the energy functions with different terms, one can modify the effects of these terms accordingly.

Considering the same order of magnitude for energy functions, we find that for large values of them, charge term is dominant in the temperature. In other words, in high energy regime, electromagnetic field has dominant effect on the temperature of the black holes. On the other side, regarding small values for the energy functions, one finds that second and last terms of temperature are dominated. It is worthwhile to mention that for large black holes (with large event horizon radius) the cosmological constant is dominated for small values of energy functions. However, for small black holes (with small event horizon radius), massive term is more considerable. One may think about the possible relation between small horizon radius with short range effective action of massive gravitons as an intermediate particle.

In order to obtain the entropy of black holes in the Einstein gravity context, one can employ the area law of the black holes. It is a matter of calculation to show that entropy has the following form [125–127]

$$S = \frac{V_{d_2}}{4} \left(\frac{r_+}{g(\varepsilon)} \right)^{d_2}. \tag{13}$$

The electric charge, Q , can be found by calculating the flux of the electric field at infinity. We find

$$Q = \frac{d_3 V_{d_2} f(\varepsilon)}{4\pi g^{d_3}(\varepsilon)} q. \tag{14}$$

It was shown that by using the Hamiltonian approach, one can find the finite mass M of the black hole for massive gravity's rainbow as

$$M = \frac{d_2 V_{d_2}}{16\pi f(\varepsilon)g^{d_3}(\varepsilon)} m_0, \tag{15}$$

in which by evaluating metric function on the event horizon ($\psi(r = r_+) = 0$), one can obtain

$$M = \frac{d_2 V_{d_2}}{16\pi f(\varepsilon)g^{d_3}(\varepsilon)} \left(k r_+^{d_3} - \frac{2r_+^{d_1}}{d_1 d_2 g^2(\varepsilon)} \Lambda + \frac{2d_3 q^2 f^2(\varepsilon)}{d_2 r_+^{d_3}} + \frac{m^2 \mathcal{B} r_+^{d_5}}{d_2 g^2(\varepsilon)} \right), \tag{16}$$

where $\mathcal{B} = d_2 d_3 d_4 c_4 c^4 + d_2 d_3 c_3 c^3 r_+ + d_2 c_2 c^2 r_+^2 + c c_1 r_+^3$. It is notable that the electric potential, U , may be defined as the gauge potential at the event horizon with respect to the reference

$$U = A_\mu \chi^\mu \Big|_{r \rightarrow \infty} - A_\mu \chi^\mu \Big|_{r=r_+} = \frac{q}{r_+^{d_3}}. \tag{17}$$

Now, we are in a position to check the validity of the first law of thermodynamics for our solutions. It is straightforward to show that by using thermodynamic quantities such as entropy (13), charge (14) and mass (15), with the first law of black hole thermodynamics

$$dM = TdS + UdQ, \tag{18}$$

we can define the intensive parameters conjugate to S and Q . These quantities are the temperature and the electric potential

$$T = \left(\frac{\partial M}{\partial r_+} \right)_q \left(\frac{\partial r_+}{\partial S} \right)_q \quad \& \quad U = \left(\frac{\partial M}{\partial q} \right)_{r_+} \left(\frac{\partial q}{\partial Q} \right)_{r_+}, \quad (19)$$

which are coincidence with the ones calculated for the temperature and the electric potential which are calculated in Eqs. (12) and (17). As a result, the first law of thermodynamics is valid while some of quantities are modified in the presence of massive term and rainbow functions.

4. Thermal stability in canonical ensemble

Now, our subject of interest is studying thermal stability of the solutions in the canonical ensemble. This study is based on the behavior of heat capacity. The negativity of the heat capacity represents an unstable state which requires a phase transition. In the case of heat capacity, it is argued that its divergencies are indicated with phase transition point. On the other hand, the roots of heat capacity are the same with roots of the temperature, so they are marking bound points which identify physical solutions from non-physical ones. The heat capacity is given by

$$C_Q = \frac{T}{\left(\frac{\partial^2 M}{\partial S^2} \right)_Q} = \frac{T}{\left(\frac{\partial T}{\partial S} \right)_Q}, \quad (20)$$

which by using Eqs. (12), (13) and (15) leads to

$$C_Q = \frac{\left[\frac{2\Lambda r_+^{3d_1/3} - d_2 m^2 \mathcal{A} r_+^{3d_2/3}}{g^{d_2}(\varepsilon)} + \frac{d_3 (k d_2 r_+^{3d_1} - 2d_3 q^2 f^2(\varepsilon) r_+^{d-3})}{g^{d_4}(\varepsilon)} \right] r_+^3}{4d_3 \left[m^2 (3d_4 d_5 c_4 c^4 + 2d_4 c_3 c^3 r_+ + c_2 c^2 r_+^2) r_+^{2d} + g^2(\varepsilon) \left(k - \frac{d_3 d_5/2 q^2 f^2(\varepsilon)}{3d_2 r_+^{2d_3}} \right) + \frac{2\Lambda}{d_2 d_3} r_+^{2d-2} \right]}. \quad (21)$$

It is worthwhile to mention that for small value of energy functions, cosmological constant and massive terms are dominated in the heat capacity (the same behavior observed for the temperature). In other words, for small energy functions, we obtain

$$C_Q \Big|_{f(\varepsilon) \sim g(\varepsilon) \ll 1} = \frac{2\Lambda r_+^{3d-2/3} - d_2 m^2 \mathcal{A} r_+^{3d-1/3}}{4d_3 g^{d_2}(\varepsilon) \left[m^2 (3d_4 d_5 c_4 c^4 + 2d_4 c_3 c^3 r_+ + c_2 c^2 r_+^2) r_+^{2d} + \frac{2\Lambda}{d_2 d_3} r_+^{2d-2} \right]}, \quad (22)$$

while on the other side, like the temperature, charge term is more effective for large values of energy functions. It is easy to obtain

$$C_Q \Big|_{f(\varepsilon) \sim g(\varepsilon) \gg 1} = \frac{-2d_3^2 q^2 f^2(\varepsilon) r_+^{d-6}}{4d_3 g^{d_2}(\varepsilon) \left(k - \frac{d_3 d_5/2 q^2 f^2(\varepsilon)}{3d_2 r_+^{2d_3}} \right)}, \quad (23)$$

in which it is evident that for large values of energy functions, all black holes with flat and hyperbolic horizons are thermally stable, and for spherical horizon we can obtain stable solutions provided $q^2 > \frac{3d_2 r_+^{2d_3}}{d_3 d_5/2 f^2(\varepsilon)}$.

It is not possible to obtain the root and divergencies of C_Q , analytically. Numerical calculations show that for small values of the horizon radius, a region of negative temperature and heat capacity exists which is ended as these quantities meet a bound point. For specific values of different parameters, only this bound exists which after it, system has positive heat capacity and is in stable state. For another set of different parameters, the phase structure is modified. In this case, the heat capacity has one bound point and two divergencies. The stable regions are observed between bound point and smaller divergency and also after larger divergency. Therefore, a phase transition takes place between these two stable states. The existence of divergencies depends on the massive parameter, energy ratio and dimensionality. Smaller bound point and smaller divergency are decreasing functions of massive term while the larger divergency is an increasing function of it. As final note, it is worth mentioning that in the absence of massive parameter and for special values of parameters, there is no phase transition. Also regarding Eq. (11), for large values of ε , hence high energy limit, similar behavior is observed and black holes have no phase transition. This shows that the mass of graviton and energy of particles probing the spacetime have profound contributions into thermal stability, phase transition and thermodynamical structure of the black holes.

5. Extended phase space

In this section, we intend to employ the analogy between the cosmological constant and thermodynamical pressure to study the critical behavior of the system. It should be pointed out that the usual relation between cosmological constant and pressure could be modified due to these generalization. By investigation of the energy–momentum tensor, we find that despite the modifications of thermodynamical properties, the proportionality between the cosmological constant and pressure remains unchanged ($P = -\frac{\Lambda}{8\pi}$). Now, by using Eq. (12) and $P = -\frac{\Lambda}{8\pi}$, one can find the following equation of state

$$P = \frac{d_2 (4\pi f(\varepsilon) g(\varepsilon) T r_+^3 - k d_3 g^2(\varepsilon) r_+^2 - m^2 \mathcal{A})}{16\pi r_+^4} + \frac{2d_3^2 q^2 f^2(\varepsilon) g^2(\varepsilon) r_+^{-2d_4}}{16\pi r_+^4} \quad (24)$$

The conjugating quantity of the pressure is thermodynamical volume which is obtained by derivation of the enthalpy with respect to pressure. The enthalpy of black holes in case of cosmological constant as pressure is total mass. The reason is that by such consideration, the cosmological constant is no longer a fixed parameter but a thermodynamical one. Therefore, the phase space is extended and the mass is regarded as enthalpy. Regarding this point, one can calculate the volume as

$$V = \left(\frac{\partial H}{\partial P} \right)_{S,Q} = \left(\frac{\partial M}{\partial P} \right)_{S,Q} = \frac{r_+^{d_1}}{d_1 f(\varepsilon) g^{d_1}(\varepsilon)}. \quad (25)$$

It is evident that the thermodynamical volume is modified due to the existence of gravity's rainbow and it is a decreasing function of the energy functions.

Due to modification in interpretation of the total mass of the black holes in extended phase space, the Gibbs free energy is given by

$$G = H - TS = M - TS, \quad (26)$$

which by using Eqs. (12), (13) and (15) with (26), one can obtain

$$G = \frac{4q^2 d_1 d_3 d_5/2 f^2(\varepsilon) g^d(\varepsilon) r_+^{-d_6} - d_1 g^{d_2}(\varepsilon) r_+^{d_2} [2kd_2^2 g^2(\varepsilon) r_+^2 + m^2 (d_2 \mathcal{A} + d_1 \mathcal{B})]}{16\pi d_1 d_2 r_+^3 f(\varepsilon) g^{2d_3/2}(\varepsilon)}. \quad (27)$$

Obtained relation for volume could be used to define a specific volume which is related to horizon radius. This relation enables us to use horizon radius instead of volume for studying the critical behavior of the system. In order to obtain critical values, one can use the following properties of inflection point in isothermal P - r_+ diagrams

$$\left(\frac{\partial P}{\partial r_+} \right)_T = \left(\frac{\partial^2 P}{\partial r_+^2} \right)_T = 0, \quad (28)$$

which for these specific class of charged black holes in the presence of massive gravity's rainbow will lead to

$$d_3 \left\{ -4d_3 d_5/2 q^2 f^2(\varepsilon) g^2(\varepsilon) r_+^{-2d_4} + kg^2(\varepsilon) r_+^2 + m^2 [6d_4 d_5 c_4 c^4 + 3d_4 c_3 c^3 r_+ + c^2 c_2 r_+^2] \right\} = 0. \quad (29)$$

In general, due to the contributions of massive term, it is not possible to obtain critical horizon radius analytically. Therefore, for the simplicity and in order to obtain critical values analytically, we exclude the effects of c_3 and c_4 . So, the critical horizon radius for arbitrary dimensions ($d \geq 4$) is

$$r_c = \left(\frac{4d_3 d_5/2 q^2 f^2(\varepsilon) g^2(\varepsilon)}{kg^2(\varepsilon) + m^2 c^2 c_2} \right)^{\frac{1}{2d_3}}. \quad (30)$$

Obtained relation confirms that, interestingly, one can find critical horizon for flat, spherical and hyperbolic horizons which highlight the effects of such configuration (charged Einsteinian black holes in the presence of massive gravity's rainbow). It is notable that such behavior and property was not observed for other type of black holes and it is a unique property of these black holes. Using Eq. (28) with equation of state, one can find critical temperature and pressure as

$$T_c = \frac{m^2 c c_1}{4\pi f(\varepsilon) g(\varepsilon)} + \frac{d_3}{\pi f(\varepsilon) g(\varepsilon) r_c} (kg^2(\varepsilon) + m^2 c^2 c_2) - \frac{d_3^2 q^2 f(\varepsilon) g(\varepsilon)}{\pi r_c^{2d_5/2}}, \quad (31)$$

$$P_c = \frac{d_3}{16\pi r_c^2} \left[d_2 (kg^2(\varepsilon) + m^2 c^2 c_2) - \frac{4d_3 d_5/2 q^2 f^2(\varepsilon) g^2(\varepsilon)}{r_c^{2d_2}} \right]. \quad (32)$$

The critical horizon radius is a decreasing function of the dimensionality and massive parameter and an increasing function of energy functions and electric charge (if the energy functions are considered the same). As for the critical temperature and pressure, they are increasing functions of the dimensionality and massive parameter and decreasing functions of energy functions and electric charge (the energy functions are considered the same). Using Eqs. (30)–(32) one can find following ratio

$$\frac{P_c r_c}{T_c} = \frac{d_3 f(\varepsilon) g(\varepsilon) \left[d_2 (kg^2(\varepsilon) + m^2 c^2 c_2) - 4d_3 d_5/2 q^2 f^2(\varepsilon) g^2(\varepsilon) r_c^{-2d_2} \right]}{16 \left[2d_3 (kg^2(\varepsilon) + m^2 c^2 c_2) + m^2 c c_1 r_c - 4d_3^2 q^2 f^2(\varepsilon) g^2(\varepsilon) r_c^{-2d_2} \right]}, \quad (33)$$

where for large values of energy functions, we obtain the following energy dependent ratio

$$\left. \frac{P_c r_c}{T_c} \right|_{f(\varepsilon) \sim g(\varepsilon) \gg 1} = \frac{f(\varepsilon) g(\varepsilon) (d_5/2)}{16}. \quad (34)$$

This ratio is an increasing function of the dimensionality and energy functions while it is a decreasing function of the massive parameter and electric charge. The mentioned effects are observed for spherical horizon. The formation of swallow-tail in G - T diagrams and subcritical isobar for T - r_+ and isothermal behavior in P - r_+ diagrams, confirm that for hyperbolic (continuous lines of Fig. 1), flat (dashed-dotted lines of Fig. 1) and spherical (dashed lines of Fig. 1) horizons, a second order phase transition takes place. Observed behavior for all three

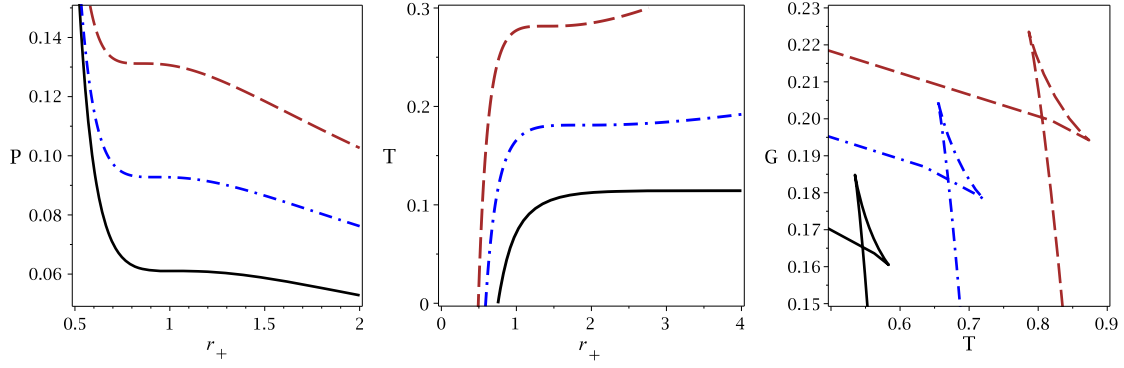


Fig. 1. Phase diagrams for $\varepsilon = 0.5$, $\lambda = 0.3$, $q = m = c = c_1 = c_2 = 1$, $c_3 = c_4 = 0$ and $d = 4$. $k = -1$ (continuous line), $k = 0$ (dashed-dotted line) and $k = 1$ (dashed line). Left panel: $P-r_+$ for $T = T_c$; Middle panel: $T-r_+$ for $P = P_c$; Right panel: $G-T$ for $P = 0.5P_c$.

Table 1

For $q = c = c_1 = c_2 = c_3 = c_4 = 1$, $\lambda = 0.3$ and $d = 6$. Left: variation of m ; $\varepsilon = 0.5$. Right: variation of ε ; $m = 1$.

k	m	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$	k	ε	r_c	T_c	P_c	$\frac{P_c r_c}{T_c}$
-1	0.9	1.12051	1.81518	0.71931	0.44403	-1	0.1	1.18122	1.58593	0.73735	0.54919
0	0.9	1.10542	2.24425	0.86508	0.42610	0	0.1	1.16385	1.99321	0.90108	0.52615
1	0.9	1.09189	2.67889	1.01467	0.41357	1	0.1	1.14843	2.40625	1.06948	0.51043
-1	1	1.06499	2.61700	1.08930	0.44329	-1	0.6	1.03965	2.93556	1.19082	0.42173
0	1	1.05428	3.06763	1.25010	0.42963	0	0.6	1.03007	3.39697	1.35094	0.40965
1	1	1.04441	3.52267	1.41406	0.41924	1	0.6	1.02120	3.86252	1.51395	0.40027
-1	1.1	1.01777	3.60572	1.57104	0.44345	-1	1.0	0.94982	4.50073	1.65745	0.34978
0	1.1	1.00989	4.07669	1.74668	0.43269	0	1.0	0.94356	5.00510	1.81508	0.34218
1	1.1	1.00250	4.55123	1.92499	0.42402	1	1.0	0.93764	5.51273	1.97475	0.33587

horizons is van der Waals like behavior. Gibbs free energy of each phases (right panel of Fig. 1), critical pressure (left panel of Fig. 1) and temperature (middle panel of Fig. 1) are sensitive functions of topological factor, k . (See also Table 1.)

Next, we include the effects of c_3 and c_4 . Since it is not possible to obtain critical values analytically, we employ numerical methods. It is evident that the largest and smallest critical horizon radii belong to hyperbolic and spherical horizons, respectively. On the contrary, the largest critical temperature and pressure are obtained in spherical horizon while the smallest ones are found for hyperbolic horizons. For the ratio $\frac{P_c r_c}{T_c}$, the smallest is derived for spherical horizon while the largest one belongs to the black holes with hyperbolic horizons.

It is also notable that the critical values are highly sensitive to variation of massive parameters and energy ratio in this case. In addition, depending on being in high energy limit (large ε) or low energy limit (small ε), the critical point of the system would be different. This shows that consideration of the gravity's rainbow is necessary to have a better picture regarding critical behavior of the black holes.

6. Geometrical thermodynamics

In this section, we employ geometrical approach toward phase transition of the black holes in the canonical ensemble. The idea is to build the phase space of black holes through the use of thermodynamical variables. There are several approaches with different metric constructions toward this matter which include Weinhold [110], Ruppeiner [111], Quevedo [112] and HPEM [113–115]. In these approaches, the phase transition and bound points should be represented as divergencies of the calculated Ricci scalar. In previous studies, it was pointed out that the Ricci scalars of the Weinhold, Ruppeiner and Quevedo metrics may lead to extra divergencies which are not matched with bound points and phase transition [113–115]. Therefore, they may result into misleading conclusions. In order to avoid this problem, another metric was proposed before [113–115]. This metric contains information which enables one to determine the type of phase transition and distinguish divergencies related to phase transition and those correspond to bound points. The HPEM metric has the following structure [113–115]

$$ds_{HPEM}^2 = \frac{SM_S}{\left(\prod_{i=2}^n \frac{\partial^2 M}{\partial \chi_i^2}\right)^3} \left(-M_{SS} dS^2 + \sum_{i=2}^n \left(\frac{\partial^2 M}{\partial \chi_i^2} \right) d\chi_i^2 \right), \quad (35)$$

where $M_S = \partial M / \partial S$, $M_{SS} = \partial^2 M / \partial S^2$ and χ_i ($\chi_i \neq S$) are extensive parameters. Here, our solutions are static charged black holes. Therefore, the other extensive parameter will be total electric charge. By using Eqs. (13), (14) and (15) with (35), one can construct phase space and calculate its Ricci scalar. For the economical reasons, we will not present the calculated Ricci scalar, instead by employing the values that are used for plotting heat capacity diagrams, we plot some diagrams (Fig. 2). A simple comparison between plotted diagrams for the heat capacity and thermodynamical Ricci scalar shows that employed approach provides satisfactory results. In addition, the signs of divergencies of the Ricci scalar are different, which are depending on their matching on the phase transition or bound points.

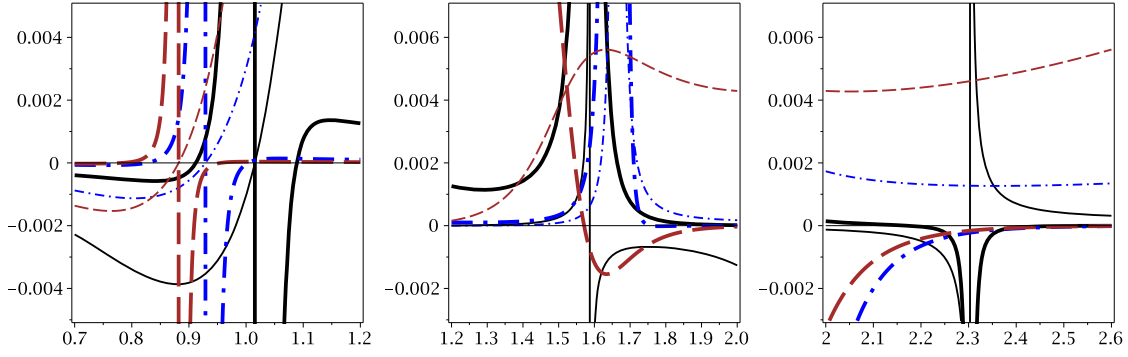


Fig. 2. For different scale: \mathcal{R} (bold lines) and C_Q versus r_+ for $q = 1$, $c = 1$, $c_1 = c_2 = c_3 = c_4 = 0.1$, $k = 1$, $\Lambda = -1$, $\lambda = 0.5$, $m = 0.3$, $d = 6$, $\varepsilon = 0.1$ (continuous line), $\varepsilon = 0.639$ (dashed-dotted line) and $\varepsilon = 1$ (dashed line).

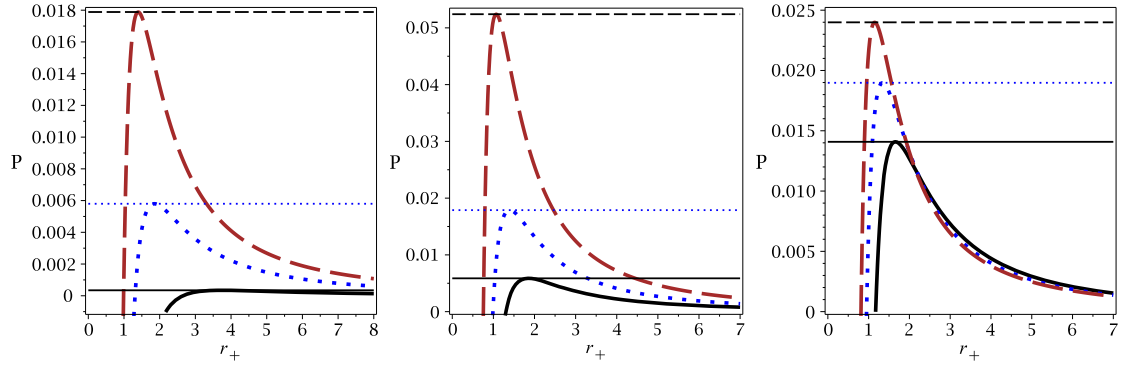


Fig. 3. P versus r_+ diagrams for $\lambda = 0.3$, $q = c = c_1 = c_2 = 1$, $c_3 = c_4 = 0$ and $d = 4$. *Left panel:* $m = 1$, $\varepsilon = 0.5$, $k = -1$ (bold continuous line), $P = 0.00034$ (continuous line), $k = 0$ (bold dotted line), $P = 0.00579$ (dotted line), $k = 1$ (bold dashed line), $P = 0.01788$ (dashed line). *Middle panel:* $k = 1$, $\varepsilon = 0.5$, $m = 0.5$ (bold continuous line), $P = 0.00587$ (continuous line), $m = 1$ (bold dotted line), $P = 0.017885$ (dotted line), $m = 1.5$ (bold dashed line), $P = 0.05240$ (dashed line). *Right panel:* $k = 1$, $m = 1$, $\varepsilon = 0.1$ (bold continuous line), $P = 0.01408$ (continuous line), $\varepsilon = 0.6$ (bold dotted line), $P = 0.01897$ (dotted line), $\varepsilon = 1$ (bold dashed line), $P = 0.02399$ (dashed line).

7. Maximum of pressure

Our final study will be related to obtaining critical pressure through the use of denominator of heat capacity. Previously, it was shown that for finding critical pressure in extended phase space, one can use the denominator of heat capacity [58,59,109]. To do so, one should solve the denominator of heat capacity with respect to pressure. This method leads to an independent relation for the pressure. The existence of maximum for pressure marks the possibility of second order phase transition. The maximum of pressure corresponds to the critical pressure which is observed in phase diagrams. For pressures smaller than critical pressure, a phase transition is observed whereas for pressures larger than critical pressure no phase transition is observed. This behavior is consistent with what is observed in $T-V$ ($T-r_+$) diagrams.

Now, using Eq. (21) with $P = -\frac{\Lambda}{8\pi}$, one can rewrite the heat capacity with dependency on thermodynamical pressure. Solving the denominator of heat capacity with respect to pressure, we obtain

$$P = \frac{d_3 \left[d_2 (kg^2(\varepsilon)r_+^2 + m^2 [c^2c_2r_+^2 + 2d_4c^3c_3r_+ + 3d_4d_5c^4c_4]) - 4d_3d_5/2q^2f^2(\varepsilon)g^2(\varepsilon)r_+^{-2d_4} \right]}{16\pi r_+^4}. \quad (36)$$

In order to elaborate the efficiency of this method, we have considered obtained critical values in the previous section to plot different diagrams for the new method (Fig. 3). It is evident that the maximum of pressure for considered values is the same as critical pressure which could be obtained in the extended phase space through the use of phase diagrams. Once more, we point out that for this specific behavior a second order phase transition is observed for non-spherical horizon. In other words, the topological black holes irrespective of their horizon curvatures, enjoy the existence of second order phase transition and van der Waals like behavior in their phase diagrams. This is a key property which was not observed before for other types of black holes.

8. Closing remarks

In this paper, we have considered massive gravity in an energy dependent spacetime. At first, we have obtained exact charged black hole solutions and investigated their geometrical properties. We have found that considering massive and rainbow configurations do not change the asymptotical adS behavior of the solutions. In addition, we have obtained the conserved and thermodynamic quantities and their modifications are observed. Then, we have checked the validity of the first law of thermodynamics and found that it holds for the obtained black holes.

Next, we have investigated thermodynamical behavior of solutions and their stability criteria. It was shown that based on the behavior of temperature, and depending on the choices of parameters, the physical solutions may not exist or it could be limited to a region.

Regarding thermal stability, we have shown that adS black holes may have only one bound point or one bound point and a phase transition. It was shown that number of divergencies and bound points are functions of massive parameter and gravity's rainbow. We observed that the energy of particles probing spacetime, hence high/low energy limit affects thermodynamical structure of the black holes, specially their phase transition. This highlights the effects of gravity's rainbow on the structure of solutions and emphasizes on the necessity of such consideration.

The most interesting result of this paper was found in study that was conducted in the context of extended phase space. Using an analytical approach, we have shown that for these specific black holes, a second order phase transition exists not only for spherical black holes but also for black holes with hyperbolic and flat horizons. Such behavior was not observed for other black holes and is a unique property of these black holes. This shows that considering all black holes as thermodynamical systems, the generalization to massive gravity is necessary to have a second order phase transition and van der Waals like behavior irrespective of their horizons structure. Such property enables one to preform specific studies in the context of black holes with hyperbolic and flat horizons that were not possible before due to the absence of second order phase transition and van der Waals like behavior. In addition, the holographic phase transition and renormalization group flow that were studied in adS/CFT context only for spherical black holes, could be investigated through the use of these new results. In addition, the van der Waals like behavior was also observed for plotted $T-r_+$ diagram without the use of proportionality between thermodynamical pressure and cosmological constant. In other words, the subcritical isobar was observed.

Next, we have employed geometrical method for studying critical behavior of the system in the context of canonical ensemble. We have indicated that employed metric provides divergencies in its thermodynamical Ricci scalar which coincided with bound and phase transition point of these black holes. Our final study was conducted in regard of finding critical pressure and horizon radius in the extended phase space through the use of denominator of the heat capacity. We have shown that the maximum of obtained relation for pressure using the denominator of heat capacity coincides with critical pressure that was obtained in studying phase diagrams. Once more, it was shown that second order phase transition is obtainable for black holes with hyperbolic and flat horizons.

It is interesting to extend our results to higher derivative gravity in higher dimensions as well as other gravitational models. In addition, we can examine the causal structure of the solutions, energy conditions and their applications with cosmological point of view. Also, one can use the perturbation equations for such solutions to examine the well known Gregory–Laflamme type instability [128–131]. We left these subjects for the future works.

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References

- [1] J. Magueijo, L. Smolin, *Class. Quantum Gravity* 21 (2004) 1725.
- [2] X. Calmet, S. Hossenfelder, R. Percacci, *Phys. Rev. D* 82 (2010) 124024.
- [3] J.J. Peng, S.Q. Wu, *Gen. Relativ. Gravit.* 40 (2008) 2619.
- [4] G. 'tHooft, *Class. Quantum Gravity* 13 (1996) 1023.
- [5] V.A. Kostelecky, S. Samuel, *Phys. Rev. D* 39 (1999) 683.
- [6] G. Amelino-Camelia, J.R. Ellis, N.E. Mavromatos, D.V. Nanopoulos, S. Sarkar, *Nature* 393 (1998) 763.
- [7] R. Gambini, J. Pullin, *Phys. Rev. D* 59 (1999) 124021.
- [8] S.M. Carroll, J.A. Harvey, V.A. Kostelecky, C.D. Lane, T. Okamoto, *Phys. Rev. Lett.* 87 (2001) 141601.
- [9] P. Horava, *Phys. Rev. Lett.* 102 (2009) 161301.
- [10] P. Horava, *Phys. Rev. D* 79 (2009) 084008.
- [11] M. Faizal, *J. Phys. A* 44 (2011) 402001.
- [12] J. Abraham, et al., *Phys. Lett. B* 685 (2010) 239.
- [13] Y. Ling, X. Li, H. Zhang, *Mod. Phys. Lett. A* 22 (2007) 2749.
- [14] H. Li, Y. Ling, X. Han, *Class. Quantum Gravity* 26 (2009) 065004.
- [15] R.J. Adler, P. Chen, D.I. Santiago, *Gen. Relativ. Gravit.* 33 (2001) 2101.
- [16] M. Cavaglia, S. Das, R. Maartens, *Class. Quantum Gravity* 20 (2003) L205.
- [17] G. Amelino-Camelia, M. Arzano, A. Procaccini, *Phys. Rev. D* 70 (2004) 107501.
- [18] A.F. Ali, *Phys. Rev. D* 89 (2014) 104040.
- [19] A.F. Ali, M. Faizal, M.M. Khalil, *Nucl. Phys. B* 894 (2015) 341.
- [20] A.F. Ali, M. Faizal, B. Majumder, *Europhys. Lett.* 109 (2015) 20001.
- [21] C. Leiva, J. Saavedra, J. Villanueva, *Mod. Phys. Lett. A* 24 (2009) 1443.
- [22] R. Garattini, B. Majumder, *Nucl. Phys. B* 884 (2014) 125.
- [23] A.F. Ali, M. Faizal, M.M. Khalil, *J. High Energy Phys.* 12 (2014) 159.
- [24] R. Garattini, F.S.N. Lobo, *Eur. Phys. J. C* 74 (2014) 2884.
- [25] S.H. Hendi, M. Faizal, *Phys. Rev. D* 92 (2015) 044027.
- [26] Z. Chang, S. Wang, *Eur. Phys. J. C* 75 (2015) 259.
- [27] G. Santos, G. Gubitosi, G. Amelino-Camelia, *J. Cosmol. Astropart. Phys.* 08 (2015) 005.
- [28] S.H. Hendi, M. Faizal, B. Eslam Panah, S. Panahiyan, *Eur. Phys. J. C* 76 (2016) 296.
- [29] G.G. Carvalho, I.P. Lobo, E. Bittencourt, *Phys. Rev. D* 93 (2016) 044005.
- [30] P. Galan, G.A. Mena Marugan, *Phys. Rev. D* 74 (2006) 044035.
- [31] Y. Gim, W. Kim, *J. Cosmol. Astropart. Phys.* 05 (2015) 002.
- [32] S.H. Hendi, S. Panahiyan, B. Eslam Panah, M. Momennia, *Eur. Phys. J. C* 76 (2016) 150.
- [33] S.H. Hendi, G.H. Bordbar, B. Eslam Panah, S. Panahiyan, *J. Cosmol. Astropart. Phys.* 09 (2016) 013.
- [34] R. Garattini, F.S.N. Lobo, arXiv:1512.04470.
- [35] A.S. Sefiedgar, arXiv:1512.08372.
- [36] V. Rubakov, arXiv:hep-th/0407104.
- [37] M. Fierz, W. Pauli, *Proc. R. Soc. Lond. A* 173 (1939) 211.
- [38] D.G. Boulware, S. Deser, *Phys. Rev. D* 6 (1972) 3368.
- [39] P. Minjoo, *Class. Quantum Gravity* 28 (2011) 105012.

- [40] C. de Rham, G. Gabadadze, *Phys. Rev. D* 82 (2010) 044020.
- [41] C. de Rham, G. Gabadadze, A.J. Tolley, *Phys. Rev. Lett.* 106 (2011) 231101.
- [42] K. Hinterbichler, *Rev. Mod. Phys.* 84 (2012) 671.
- [43] Y.F. Cai, D.A. Easson, C. Gao, E.N. Saridakis, *Phys. Rev. D* 87 (2013) 064001.
- [44] E. Babichev, A. Fabbri, *J. High Energy Phys.* 07 (2014) 016.
- [45] E. Babichev, C. Deffayet, R. Ziour, *Phys. Rev. Lett.* 103 (2009) 201102.
- [46] L. Alberte, A.H. Chamseddine, V. Mukhanov, *J. High Energy Phys.* 12 (2010) 023.
- [47] K. Koyama, G. Niz, G. Tasinato, *Phys. Rev. Lett.* 107 (2011) 131101.
- [48] T.M. Nieuwenhuizen, *Phys. Rev. D* 84 (2011) 024038.
- [49] M.S. Volkov, *Class. Quantum Gravity* 30 (2013) 184009.
- [50] M. Ghodrati, A. Naseh, *Class. Quantum Gravity* 34 (2017) 075009.
- [51] C. de Rham, G. Gabadadze, A. Tolley, *Phys. Lett. B* 711 (2012) 190.
- [52] C. de Rham, *Living Rev. Relativ.* 17 (2014) 7.
- [53] S.F. Hassan, R.A. Rosen, *Phys. Rev. Lett.* 108 (2012) 041101.
- [54] S.F. Hassan, R.A. Rosen, A. Schmidt-May, *J. High Energy Phys.* 02 (2012) 026.
- [55] D. Vegh, [arXiv:1301.0537](https://arxiv.org/abs/1301.0537).
- [56] H. Zhang, X.Z. Li, *Phys. Rev. D* 93 (2016) 124039.
- [57] R.G. Cai, Y.P. Hu, Q.Y. Pan, Y.L. Zhang, *Phys. Rev. D* 91 (2015) 024032.
- [58] S.H. Hendi, B. Eslam Panah, S. Panahiyan, *J. High Energy Phys.* 11 (2015) 157.
- [59] S.H. Hendi, S. Panahiyan, B. Eslam Panah, *J. High Energy Phys.* 01 (2016) 129.
- [60] Y.P. Hu, H.F. Li, H.B. Zeng, H.Q. Zhang, *Phys. Rev. D* 93 (2016) 104009.
- [61] E. Witten, *Adv. Theor. Math. Phys.* 2 (1998) 253.
- [62] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, *Phys. Rep.* 323 (2000) 183.
- [63] O. Aharony, O. Bergman, D.L. Jafferis, J.M. Maldacena, *J. High Energy Phys.* 10 (2008) 091.
- [64] D.A. Lowe, *Phys. Rev. D* 79 (2009) 106008.
- [65] J. Jing, S. Chen, *Phys. Lett. B* 686 (2010) 68.
- [66] Y.P. Hu, P. Sun, J.H. Zhang, *Phys. Rev. D* 83 (2011) 126003.
- [67] X.O. Camanho, J.D. Edelstein, *J. High Energy Phys.* 04 (2010) 007.
- [68] Y.P. Hu, H.F. Li, Z.Y. Nie, *J. High Energy Phys.* 01 (2011) 123.
- [69] J. Jing, Q. Pan, S. Chen, *J. High Energy Phys.* 11 (2011) 045.
- [70] D. Bazeia, L. Losano, G.J. Olmo, D. Rubiera-Garcia, *Phys. Rev. D* 90 (2014) 044011.
- [71] D. Kabat, G. Lifschytz, *J. High Energy Phys.* 09 (2014) 077.
- [72] J. Polchinski, [arXiv:1010.6134](https://arxiv.org/abs/1010.6134).
- [73] I.R. Klebanov, [arXiv:hep-th/9901018](https://arxiv.org/abs/hep-th/9901018).
- [74] D.T. Son, A.O. Starinets, *Annu. Rev. Nucl. Part. Sci.* 57 (2007) 95.
- [75] S.W. Hawking, D.N. Page, *Commun. Math. Phys.* 87 (1983) 577.
- [76] E. Witten, *Adv. Theor. Math. Phys.* 2 (1998) 505.
- [77] Y.S. Myung, *Phys. Rev. D* 77 (2008) 104007.
- [78] B.M.N. Carter, I.P. Neupane, *Phys. Rev. D* 72 (2005) 043534.
- [79] D. Kastor, S. Ray, J. Traschen, *Class. Quantum Gravity* 26 (2009) 195011.
- [80] F. Capela, G. Nardini, *Phys. Rev. D* 86 (2012) 024030.
- [81] S.H. Hendi, S. Panahiyan, *Phys. Rev. D* 90 (2014) 124008.
- [82] A. Perez, M. Riquelme, D. Tempo, R. Troncoso, *J. High Energy Phys.* 10 (2015) 161.
- [83] B.P. Dolan, *Class. Quantum Gravity* 28 (2011) 125020.
- [84] D. Kubiznak, R.B. Mann, *J. High Energy Phys.* 07 (2012) 033.
- [85] R.G. Cai, L.M. Cao, L. Li, R.Q. Yang, *J. High Energy Phys.* 09 (2013) 005.
- [86] M.B.J. Poshteh, B. Mirza, Z. Sherkatghanad, *Phys. Rev. D* 88 (2013) 024005.
- [87] S. Chen, X. Liu, C. Liu, J. Jing, *Chin. Phys. Lett.* 30 (2013) 060401.
- [88] S.H. Hendi, M.H. Vahidinia, *Phys. Rev. D* 88 (2013) 084045.
- [89] J.X. Mo, W.B. Liu, *Eur. Phys. J. C* 74 (2014) 2836.
- [90] D.C. Zou, S.J. Zhang, B. Wang, *Phys. Rev. D* 89 (2014) 044002.
- [91] W. Xu, L. Zhao, *Phys. Lett. B* 736 (2014) 214.
- [92] S.H. Hendi, S. Panahiyan, B. Eslam Panah, *Prog. Theor. Exp. Phys.* 2015 (2015) 103E01.
- [93] X.X. Zeng, L.F. Li, *Phys. Lett. B* 764 (2017) 100.
- [94] M.H. Dehghani, A. Sheykhi, Z. Dayyani, *Phys. Rev. D* 93 (2016) 024022.
- [95] S.Q. Lan, J.X. Mo, W.B. Liu, *Eur. Phys. J. C* 75 (2015) 419.
- [96] N. Altamirano, D. Kubiznak, R.B. Mann, *Phys. Rev. D* 88 (2013) 101502.
- [97] S.W. Wei, Y.X. Liu, *Phys. Rev. D* 90 (2014) 044057.
- [98] C.V. Johnson, *Class. Quantum Gravity* 31 (2014) 205002.
- [99] B.P. Dolan, *J. High Energy Phys.* 10 (2014) 179.
- [100] E. Caceres, P.H. Nguyen, J.F. Pedraza, *J. High Energy Phys.* 09 (2015) 184.
- [101] S.H. Hendi, S. Panahiyan, R. Mamasani, *Gen. Relativ. Gravit.* 47 (2015) 91.
- [102] D. Grumiller, R. McNees, J. Salzer, *Phys. Rev. D* 90 (2014) 044032.
- [103] J. Creighton, R.B. Mann, *Phys. Rev. D* 52 (1995) 4569.
- [104] G.W. Gibbons, R. Kallosh, B. Kol, *Phys. Rev. Lett.* 77 (1996) 4992.
- [105] J. Xu, L.M. Cao, Y.P. Hu, *Phys. Rev. D* 91 (2015) 124033.
- [106] S.W. Wei, P. Cheng, Y.X. Liu, *Phys. Rev. D* 93 (2016) 084015.
- [107] J.X. Mo, G.Q. Li, X.B. Xu, *Phys. Rev. D* 93 (2016) 084041.
- [108] X.X. Zeng, X.M. Liu, L.F. Li, [arXiv:1601.01160](https://arxiv.org/abs/1601.01160).
- [109] S.H. Hendi, S. Panahiyan, B. Eslam Panah, *Int. J. Mod. Phys. D* 25 (2016) 1650010.
- [110] F. Weinhold, *J. Chem. Phys.* 63 (1975) 2479.
- [111] G. Ruppeiner, *Phys. Rev. A* 20 (1979) 1608.
- [112] H. Quevedo, A. Sanchez, *J. High Energy Phys.* 09 (2008) 034.
- [113] S.H. Hendi, S. Panahiyan, B. Eslam Panah, M. Momennia, *Eur. Phys. J. C* 75 (2015) 507.
- [114] S.H. Hendi, S. Panahiyan, B. Eslam Panah, *Adv. High Energy Phys.* 2015 (2015) 743086.
- [115] S.H. Hendi, A. Sheykhi, S. Panahiyan, B. Eslam Panah, *Phys. Rev. D* 92 (2015) 064028.
- [116] Y.W. Han, G. Chen, *Phys. Lett. B* 714 (2012) 127.
- [117] A. Bravetti, D. Momeni, R. Myrzakulov, A. Altaibayeva, *Adv. High Energy Phys.* 2013 (2013) 549808.
- [118] M.S. Ma, *Phys. Lett. B* 735 (2014) 45.

- [119] M.A. García-Ariza, M. Montesinos, G.F.T.d. Castillo, *Entropy* 16 (2014) 6515.
- [120] J.L. Zhang, R.G. Cai, H. Yu, *J. High Energy Phys.* 02 (2015) 143.
- [121] S. Basak, P. Chaturvedi, P. Nandi, G. Sengupta, *Phys. Lett. B* 753 (2016) 493.
- [122] J.X. Mo, G.Q. Li, Y.C. Wu, *J. Cosmol. Astropart. Phys.* 04 (2016) 045.
- [123] S.H. Hendi, R.B. Mann, S. Panahyan, B. Eslam Panah, *Phys. Rev. D* 95 (2017) 021501 (R), <https://doi.org/10.1103/PhysRevD.95.021501>.
- [124] J. Magueijo, L. Smolin, *Phys. Rev. Lett.* 88 (2002) 190403.
- [125] J.D. Beckenstein, *Phys. Rev. D* 7 (1973) 2333.
- [126] C.J. Hunter, *Phys. Rev. D* 59 (1999) 024009.
- [127] S.W. Hawking, C.J. Hunter, D.N. Page, *Phys. Rev. D* 59 (1999) 044033.
- [128] R. Gregory, R. Laflamme, *Phys. Rev. Lett.* 70 (1993) 2837.
- [129] R. Gregory, R. Laflamme, *Nucl. Phys. B* 428 (1994) 399.
- [130] M. Mirbabayi, A. Gruzinov, *Phys. Rev. D* 88 (2013) 064008.
- [131] E. Babichev, A. Fabbri, *Class. Quantum Gravity* 30 (2013) 152001.