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On the Universality of Einstein Equations

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It is proved that a Lagrangian field theory based on a linear connection in space-time is equivalent to Einstein's general relativity interacting with additional matter fields.

1. INTRODUCTION

It has been proved [3, 4] that Einstein's general relativity can be formulated in the so-called affine language. This means that the field equations of the theory can be derived from the affine Lagrangian

$$\mathscr{L}_a := \mathscr{L}_a(j^1\Gamma, j^1\varphi) \tag{1}$$

where Γ is a linear connection in space time, φ is a matter field, and $j^1 f$ denotes the first jet of the geometrical object f (the value of f and its derivatives). We assume that $\mathscr{L}_a$ is a scalar density which depends on $\partial_\sigma \Gamma^\lambda_{\mu\nu}$ via invariant objects, i.e., the Riemann tensor and the covariant derivatives of the torsion.

The following assumptions have to be imposed in order to obtain the theory equivalent to general relativity interacting with the matter field:

1. Γ is symmetric
2. $\mathscr{L}_a$ depends on first derivatives of Γ via the symmetric part of the Ricci tensor only, i.e.

$$\mathscr{L}_a(j^1\Gamma, j^1\varphi) = \mathscr{L}(K_{\mu\nu}, \Gamma^\lambda_{\mu\nu}, \varphi^A, \varphi^A{}_\sigma)$$

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where

$$K_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu\alpha} - \Gamma^{\alpha}_{\alpha(\mu\nu)} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\beta\alpha} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$$

$$\Gamma^{\alpha}_{\beta\mu\nu} := \partial_{\nu}\Gamma^{\alpha}_{\beta\mu} \qquad \text{and} \qquad \varphi^{A}{}_{\sigma} := \partial_{\sigma}\varphi^{A}$$

Dropping out conditions 1 and 2 leads to a theory which is a priori more general. Field equations involve the whole Riemann tensor

$$R^{\alpha}_{\beta\mu\nu} := \Gamma^{\alpha}_{\beta\nu\mu} - \Gamma^{\alpha}_{\beta\mu\nu} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu}$$

and not only the Ricci tensor as in general relativity.

In the present paper we prove that usually this is not a genuine generalization: for sufficiently "regular" Lagrangians all the dynamical effects due to the generalization of the Lagrangian may be implemented by the introduction of new matter fields. This way we prove that general relativity is really a universal framework for the description of interaction of the space-time geometry with matter.

It is well-known that the general linear connection $\mathbf{\Gamma}$ can be uniquely decomposed (cf., [7, 2])

$$\mathbf{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu} + \tfrac{1}{4}\delta^{\lambda}_{\mu}(A_{\nu} - \Gamma^{\alpha}_{\alpha\nu}) \tag{2}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is a symmetric connection, $\Delta^{\lambda}_{\mu\nu}$ is an antisymmetric traceless tensor, and A_{ν} is a linear connection in the bundle of scalar densities. Both the Riemann tensor of $\mathbf{\Gamma}$ and the covariant derivatives of the torsion can be expressed in terms of the Riemann tensor of Γ, $j^1\Delta$, and $j^1 A$ together with Γ itself. Therefore

$$\mathscr{L}_a(j^1\mathbf{\Gamma}, j^1\varphi) = \mathscr{L}(R^{\lambda}_{\mu\nu\sigma}(j^1\Gamma), \Gamma, j^1\Delta, j^1 A, j^1\varphi) \tag{3}$$

Treating Δ and A as additional matter fields, we have reduced the problem to the case of the symmetric connection Γ.

The "regularity" conditions which we have to impose on $\mathscr{L}$ in order to be able to perform our construction are the following

1. $\det\left(\dfrac{\partial\mathscr{L}}{\partial K_{\mu\nu}}\right) \neq 0$

2. $\det\left(\dfrac{\partial^2\mathscr{L}}{\partial K_{\mu\nu}\,\partial K_{\alpha\beta}}\right) \neq 0$

where the second derivative of $\mathscr{L}$ is treated as a 10×10 matrix.

Additional matter fields arising from our construction may of course be unphysical if they do not lead to positive total energy. In the present paper the energy-positivity condition is not given in terms of the original Lagrangian (1).

2. THE CASE OF THE SYMMETRIC CONNECTION

Therefore, let the connection Γ in (1) be symmetric and

$$\mathscr{L}_a(j^1\Gamma, j^1\varphi) = \mathscr{L}(R^{\lambda}_{\mu\nu\sigma}, \Gamma^{\lambda}_{\mu\nu}, \varphi^A, \varphi^A{}_\sigma) \tag{4}$$

Introduce momenta canonically conjugate to both Γ and φ

$$\pi^{\mu\nu\sigma}_{\lambda} := \frac{\partial \mathscr{L}_a}{\partial \Gamma^{\lambda}_{\mu\nu\sigma}} = \frac{\partial \mathscr{L}}{\partial R^{\alpha}_{\beta\gamma\delta}} \frac{\partial R^{\alpha}_{\beta\gamma\delta}}{\partial \Gamma^{\lambda}_{\mu\nu\sigma}} \tag{5}$$

$$p_A{}^\sigma := \frac{\partial \mathscr{L}_a}{\partial \varphi^A{}_\sigma} = \frac{\partial \mathscr{L}}{\partial \varphi^A{}_\sigma} \tag{6}$$

The Riemann tensor R fulfils the following identities

$$\begin{gathered} R^{\lambda}_{\sigma\mu\nu} + R^{\lambda}_{\sigma\nu\mu} = 0 \\ R^{\lambda}_{\sigma\mu\nu} + R^{\lambda}_{\mu\nu\sigma} + R^{\lambda}_{\nu\sigma\mu} = 0 \end{gathered} \tag{7}$$

The number of independent components of R is 80. In order to define uniquely the derivatives (5), we impose the symmetries of $\pi^{\mu\nu\sigma}_{\lambda}$ corresponding to the symmetries of $\Gamma^{\lambda}_{\mu\nu}$ and $R^{\lambda}_{\mu\nu\sigma}$

$$\pi^{\mu\nu\sigma}_{\lambda} - \pi^{\nu\mu\sigma}_{\lambda} = 0 \tag{8a}$$

$$\pi^{\mu\nu\sigma}_{\lambda} + \pi^{\nu\sigma\mu}_{\lambda} + \pi^{\sigma\mu\nu}_{\lambda} = 0 \tag{8b}$$

The number of independent components of $\pi^{\mu\nu\sigma}_{\lambda}$ is 80. The Euler–Lagrange equations

$$\frac{\delta \mathscr{L}_a}{\delta \Gamma^{\lambda}_{\mu\nu}} = 0 \quad . \quad \frac{\delta \mathscr{L}_a}{\delta \varphi^A} = 0 \tag{9}$$

together with the definitions (5) and (6) of the momenta can be rewritten in the form

$$\begin{aligned} d\mathscr{L}_a &= \partial_\sigma(\pi^{\mu\nu\sigma}_{\lambda} d\Gamma^{\lambda}_{\mu\nu}) + \partial_\sigma(p_A{}^\sigma\, d\varphi^A) \\ &= \pi^{\mu\nu\sigma}_{\lambda}\, d\Gamma^{\lambda}_{\mu\nu\sigma} + \partial_\sigma \pi^{\mu\nu\sigma}_{\lambda}\, d\Gamma^{\lambda}_{\mu\nu} + p_A{}^\sigma\, d\varphi^A{}_\sigma + \partial_\sigma p_A{}^\sigma\, d\varphi^A \end{aligned} \tag{10}$$

Let

$$K_{\mu\nu} = \tfrac{1}{2}(R^{\lambda}_{\mu\lambda\nu} + R^{\lambda}_{\nu\lambda\mu})$$

and

$$P_{\mu\nu} = R^{\lambda}_{\lambda\mu\nu}$$

$\frac{1}{2}P_{\mu\nu}$ is the antisymmetric part of the Ricci tensor of Γ. Now, the Riemann tensor R can be uniquely decomposed

$$R^{\alpha}_{\beta\mu\nu} = \tfrac{1}{3}(\delta^{\alpha}_{\mu}K_{\beta\nu} - \delta^{\alpha}_{\nu}K_{\beta\mu}) + \tfrac{1}{10}(\delta^{\alpha}_{\mu}P_{\beta\nu} - \delta^{\alpha}_{\nu}P_{\beta\mu} + 2\delta^{\alpha}_{\beta}P_{\mu\nu}) + W^{\alpha}_{\beta\mu\nu} \quad (11)$$

where the Weyl tensor W fulfils identities (7) and additionally

$$W^{\lambda}_{\lambda\mu\nu} = W^{\lambda}_{\mu\lambda\nu} = 0$$

Similarly (cf. [1, 6]) the symmetric conection Γ can be split into two independent objects

$$\Gamma^{\lambda}_{\mu\nu} = \Sigma^{\lambda}_{\mu\nu} + \tfrac{2}{5}\delta^{\lambda}_{(\mu}\alpha_{\nu)} \quad (12)$$

The object $\alpha_{\mu} := \Gamma^{\lambda}_{\lambda\mu}$ is a connection in the bundle of scalar densities and $\Sigma^{\lambda}_{\mu\nu}$ is a projective connection ($\Sigma^{\lambda}_{\lambda\mu} = 0$). $P_{\mu\nu}$ is a curvature form of α

$$P_{\mu\nu} = \alpha_{\nu\mu} - \alpha_{\mu\nu}$$

where as usual

$$\alpha_{\nu\mu} = \partial_{\mu}\alpha_{\nu}$$

The Weyl tensor is "the curvature" of Σ

$$W^{\alpha}_{\beta\mu\nu}(j^1\Sigma) = \partial_{\mu}\Sigma^{\alpha}_{\beta\nu} - \partial_{\nu}\Sigma^{\alpha}_{\beta\mu} + \tfrac{1}{3}(\delta^{\alpha}_{\nu}\partial_{\lambda}\Sigma^{\lambda}_{\beta\mu} - \delta^{\alpha}_{\mu}\partial_{\lambda}\Sigma^{\lambda}_{\beta\nu})$$
$$+ \Sigma^{\alpha}_{\lambda\mu}\Sigma^{\lambda}_{\beta\nu} - \Sigma^{\alpha}_{\lambda\nu}\Sigma^{\lambda}_{\beta\mu} + \tfrac{1}{3}(\delta^{\alpha}_{\mu}\Sigma^{\gamma}_{\lambda\nu} - \delta^{\alpha}_{\nu}\Sigma^{\gamma}_{\lambda\mu})\,\Sigma^{\lambda}_{\beta\gamma}$$

The decomposition (11) induces the dual decomposition in the space of momenta

$$\pi^{\mu\nu\sigma}_{\lambda} = \mathring{\pi}^{\mu\nu\sigma}_{\lambda} - 2\Omega^{(\mu\nu)\sigma}_{\lambda} - 2\delta^{(\mu}_{\lambda}\rho^{\nu)\sigma} \quad (13)$$

where

$$\mathring{\pi}^{\mu\nu\sigma}_{\lambda} = \delta^{\sigma}_{\lambda}\pi^{\mu\nu} - \delta^{(\nu}_{\lambda}\pi^{\mu)\sigma}$$
$$\pi^{\mu\nu} = \tfrac{1}{3}\pi^{\mu\nu\lambda}_{\lambda}$$
$$\rho^{\mu\nu} = -\tfrac{1}{5}\pi^{\lambda[\mu\nu]}_{\lambda}$$

The tensor density Ω fulfils identities (8b) and additionally

$$\Omega_{\lambda}^{\lambda\mu\nu} = \Omega_{\lambda}^{\mu\lambda\nu} = 0$$

$$\Omega_{\lambda}^{\mu\nu\sigma} + \Omega_{\lambda}^{\mu\sigma\nu} = 0$$

The formula (5) splits into three independent parts

$$\pi^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial K_{\mu\nu}} \tag{14}$$

$$\rho^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial P_{\mu\nu}} \tag{15}$$

$$\Omega_{\lambda}^{\mu\nu\sigma} = \frac{\partial \mathscr{L}}{\partial W^{\lambda}_{\mu\nu\sigma}} \tag{16}$$

Equation (10) can now be rewritten in the following way

$$d\mathscr{L}_a = \partial_\sigma(\mathring{\pi}_{\lambda}^{\mu\nu\sigma}\, d\Gamma^{\lambda}_{\mu\nu} - 2\rho^{\mu\sigma}\, d\alpha_\mu - 2\Omega_{\lambda}^{\mu\nu\sigma}\, d\Sigma^{\lambda}_{\mu\nu} + p_A{}^{\sigma}\, d\varphi^A) \tag{17}$$

The formula (17) could be interpreted as a variational formula for four fields Γ, α, Σ, and φ with the Lagrangian

$$\mathscr{L}_a = \mathscr{L}[K_{\mu\nu}(j^1\Gamma), P_{\mu\nu}(j^1\alpha), W^{\lambda}_{\mu\nu\sigma}(j^1\Sigma), \Gamma, \alpha, \Sigma, j^1\varphi]$$

and with the Lagrangian constraint (12).

Now following the method introduced in [3] we perform the complete Legendre transformation between Γ and $\mathring{\pi}$. We define the new Lagrangian U by the formula

$$\begin{aligned} U := {} & -\partial_\sigma(\Gamma^{\lambda}_{\mu\nu}\mathring{\pi}_{\lambda}^{\mu\nu\sigma}) + \mathscr{L} = \partial_\sigma(B^{\sigma}_{\mu\nu}\pi^{\mu\nu}) + \mathscr{L} \\ & = \pi^{\mu\nu}{}_{\sigma}(\Gamma^{\lambda}_{\lambda\nu}\delta^{\sigma}_{\mu} - \Gamma^{\sigma}_{\mu\nu}) + \pi^{\mu\nu}(\Gamma^{\lambda}_{\lambda\nu\mu} - \Gamma^{\lambda}_{\mu\nu\lambda}) + \mathscr{L} \\ & = \pi^{\mu\nu}{}_{\sigma}(\Gamma^{\lambda}_{\lambda\nu}\delta^{\sigma}_{\mu} - \Gamma^{\sigma}_{\mu\nu}) + \pi^{\mu\nu}[\Gamma^{\lambda}_{\mu\nu}\Gamma^{\beta}_{\lambda\beta} - \Gamma^{\lambda}_{\beta\mu}\Gamma^{\beta}_{\lambda\nu} - K_{\mu\nu}(j^1\Gamma)] \\ & \quad + \mathscr{L}(K, P, W, \alpha, \Sigma, j^1\varphi) \end{aligned} \tag{18}$$

where

$$\begin{aligned} B^{\sigma}_{\mu\nu} &:= \Gamma^{\lambda}_{\lambda(\nu}\delta^{\sigma}_{\mu)} - \Gamma^{\sigma}_{\mu\nu} \\ \pi^{\mu\nu}{}_{\sigma} &:= \partial_\sigma \pi^{\mu\nu} \end{aligned} \tag{19}$$

The formula (17) together with the definition of U implies

$$dU = \partial_\sigma(B^{\sigma}_{\mu\nu}\, d\pi^{\mu\nu} - 2\rho^{\mu\sigma}\, d\alpha_\mu - 2\Omega_{\lambda}^{\mu\nu\sigma}\, d\Sigma^{\lambda}_{\mu\nu} + p_A{}^{\sigma}\, d\varphi^A) \tag{20}$$

Due to the assumption we made in the introduction the above Legendre transformation is regular; i.e., 10 equations (14) can be solved with respect to $K_{\mu\nu}$

$$K_{\mu\nu} = \mathscr{K}_{\mu\nu}(\pi, P, W, \alpha, \Sigma, j^1\varphi) \tag{21}$$

This enables us to express U in terms of $j^1\pi$, $j^1\alpha$, $j^1\Sigma$, and $j^1\varphi$. More precisely, (18), (12), and (21) give us

$$\begin{aligned} U &= U(j^1\pi, j^1\alpha, j^1\varphi) \\ &= \mathscr{L}[\mathscr{K}(\pi, P, W, \alpha, \Sigma, j^1\varphi),\ P, W, \alpha, \Sigma, j^1\varphi] \\ &\quad + \pi^{\mu\nu}{}_\sigma(\tfrac{3}{5}\delta^\sigma_\mu\alpha_\nu - \Gamma^\sigma_{\mu\nu}) + \pi^{\mu\nu}[-\Sigma^\lambda_{\beta\mu}\Sigma^\beta_{\lambda\nu} + \tfrac{3}{5}\Sigma^\beta_{\mu\nu}\alpha_\beta + \tfrac{3}{25}\alpha_\mu\alpha_\nu \\ &\quad - \mathscr{K}_{\mu\nu}(\pi, P, W, \alpha, \Sigma, j^1\varphi)] \end{aligned} \tag{22}$$

The formula (20) gives us the Euler–Lagrange equations

$$\frac{\delta U}{\delta\pi} = 0, \qquad \frac{\delta U}{\delta\alpha} = 0, \qquad \frac{\delta U}{\delta\Sigma} = 0, \qquad \frac{\delta U}{\delta\varphi} = 0 \tag{23}$$

together with the definition of corresponding momenta (e.g., Eq. 19 follows from the derivation of U with respect to $\pi^{\mu\nu}{}_\sigma$). The above equations are obviously equivalent to our original equations (9). No Lagrangian constraints are left. The Lagrangian (22) is coordinate-dependent, as is usual in the case of the first-order Einstein Lagrangians. We introduce the auxiliary symmetric connection γ such that

$$\overset{\gamma}{\nabla}_\lambda \pi^{\mu\nu} = \partial_\lambda \pi^{\mu\nu} + \pi^{\sigma\nu}\gamma^\mu_{\sigma\lambda} + \pi^{\sigma\mu}\gamma^\nu_{\sigma\lambda} - \pi^{\mu\nu}\gamma^\sigma_{\sigma\lambda} = 0$$

If $\pi^{\mu\nu}$ is nondegenerate $[\det(\pi^{\mu\nu}) \neq 0]$ then γ is defined uniquely by $j^1\pi$ (Christoffel symbols). We add to the Lagrangian U the term

$$\partial_\sigma(\gamma^\lambda_{\mu\nu}\mathring{\pi}^{\mu\nu\sigma}_\lambda) \tag{24}$$

We obtain an invariant Lagrangian which is second-order with respect to $\pi^{\mu\nu}$

$$\begin{aligned} L_H = L_H(j^2\pi, P, W, \alpha, \Sigma, j^1\varphi) &:= \partial_\sigma(\gamma^\lambda_{\mu\nu}\mathring{\pi}^{\mu\nu\sigma}_\lambda) + U \\ &= \partial_\sigma[(\gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu})\,\mathring{\pi}^{\mu\nu\sigma}_\lambda] + \mathscr{L} \end{aligned} \tag{25}$$

Easy calculations show that the only term in L_H which contains second-order derivatives of π equals

$$L_G = L_G(j^2\pi) = k_{\mu\nu}\pi^{\mu\nu} \tag{26}$$

where $k_{\mu\nu}$ is a Ricci tensor of γ. We denote $L_M = L_M(j^1\pi, P, W, \alpha, \Sigma, j^1\varphi)$ the remaining part of L_H

$$L_H = L_G + L_M$$

Both L_G and L_M are now invariant scalar densities. We see that our theory can be interpreted as a standard Einstein theory (interacting with three matter fields α, Σ, φ, and the matter Lagrangian L_M) provided (26) is equal to the standard gravitational Lagrangian proposed by Hilbert. This is true if we introduce the metric tensor g by the formula

$$\pi^{\mu\nu} = -\frac{1}{2\kappa}(-g)^{1/2} g^{\mu\nu}$$

κ is the gravitational constant; in geometric system of units $\kappa = 8\pi$. Similarly, as in the standard general relativity, only those solutions have physical meaning which correspond to the correct signature $(-, +, +, +)$ of the tensor $\pi^{\mu\nu}$. Now γ is the Levi-Civita metric connection of g and

$$L_G = -\frac{1}{2\kappa}(-g)^{1/2} r$$

where $r = g^{\mu\nu}k_{\mu\nu}$ is the scalar curvature of γ. It is easy to calculate the matter Lagrangian L_M

$$\begin{aligned}
L_M &= L_M(j^1 g, j^1\alpha, j^1\Sigma, j^1\varphi) \\
&= \mathscr{L}(\mathscr{K}, P, W, \alpha, \Sigma, j^1\varphi) \\
&\quad + \frac{1}{2\kappa}(-g)^{1/2} g^{\mu\nu}\Big[\mathscr{K}_{\mu\nu}(g, P, W, \alpha, \Sigma, j^1\varphi) + \gamma^{\alpha}_{\beta\mu}\gamma^{\beta}_{\alpha\nu} - \gamma^{\alpha}_{\mu\nu}\gamma^{\beta}_{\alpha\beta} + \Sigma^{\alpha}_{\beta\mu}\Sigma^{\beta}_{\alpha\nu} \\
&\quad - \frac{3}{5}\Sigma^{\beta}_{\mu\nu}\alpha_\beta - \frac{3}{25}\alpha_\mu\alpha_\nu - 2\Sigma^{\beta}_{\alpha\nu}\gamma^{\alpha}_{\beta\mu} + \Sigma^{\alpha}_{\mu\nu}\gamma^{\beta}_{\beta\alpha} + \frac{3}{5}\alpha_\beta\gamma^{\beta}_{\mu\nu}\Big]
\end{aligned} \tag{27}$$

The gravitational field enters via standard Riemannian geometry represented by the metric tensor g and its Levi-Civita connection γ. Field equations

$$\frac{\delta L_H}{\delta g} = 0, \qquad \frac{\delta L_H}{\delta \alpha} = 0, \qquad \frac{\delta L_H}{\delta \Sigma} = 0, \qquad \frac{\delta L_H}{\delta \varphi} = 0$$

are obviously equivalent to (23), i.e., to (9) since U and L_H differ by the full divergence term (24). The Einstein equations $\delta L_H/\delta g = 0$ are equivalent to

$$K_{\mu\nu}(j^1\Gamma) - \mathscr{K}_{\mu\nu}(g, P, W, \alpha, \Sigma, j^1\varphi) = 0$$

i.e., to (21), which in turn are equivalent to (14) in the affine formulation of the theory.

The trace $\gamma^{\lambda}_{\lambda\nu}$ of the Christoffel symbols defines another connection in the bundle of scalar densities. One can used it to rescale α

$$a_{\mu} := \alpha_{\mu} - \gamma^{\lambda}_{\lambda\mu} \tag{28}$$

The new object a is a covector field. Since

$$\gamma^{\lambda}_{\lambda\mu} = \partial_{\mu}(\ln(-g)^{1/2})$$

we have

$$P_{\mu\nu} = a_{\nu\mu} - a_{\mu\nu}$$

where again

$$a_{\nu\mu} = \partial_{\mu} a_{\nu}$$

Therefore we can choose a, Σ, and φ as independent matter fields

$$L_M = \tilde{L}_M[j^1 g, P(j^1 a), W(j_1 \Sigma), a, \Sigma, j^1\varphi]$$

The formulas (3), (25), and (26) prove that any affine theory based on the Lagrangian (1) can be formulated as a theory of the original matter field φ and additional matter fields: Δ, A, a, Σ minimally coupled to the Einstein general relativity. Given a solution of field equations of the latter, we can reconstruct the original connection $\mathbf{\Gamma}$ due to the formulas (2), (12), and (28)

$$\begin{aligned}\mathbf{\Gamma}^{\lambda}_{\mu\nu} = {} & \Sigma^{\lambda}_{\mu\nu} + \Delta^{\lambda}_{\mu\nu} + \tfrac{1}{4}\delta^{\lambda}_{\mu} A_{\nu} + \tfrac{1}{5}\delta^{\lambda}_{\nu} a_{\mu} - \tfrac{1}{20}\delta^{\lambda}_{\mu} a_{\nu} + \tfrac{1}{5}\delta^{\lambda}_{\nu}\partial_{\mu}(\ln(-g)^{1/2}) \\ & - \tfrac{1}{20}\delta^{\lambda}_{\mu}\partial_{\nu}(\ln(-g)^{1/2})\end{aligned}$$

It is worthwhile to notice that the derivatives of both fields A and a enter into the Lagrangian L_M via their curls only

$$F_{\mu\nu} := R^{\lambda}_{\lambda\mu\nu}(\mathbf{\Gamma}) = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

and

$$P_{\mu\nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu}$$

However, the field A itself does not appear in the Lagrangian. Therefore, A is a "Maxwell-like" field and a is a "Proca-like" field.

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