Classical String Solutions in AdS/CFT

by
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Preface

Integrability of both sides of AdS/CFT has enabled many quantitative checks of the correspondence. The Bethe ansatz determines the anomalous dimensions of gauge theory operators, which at strong coupling are also encoded in the energies of classical string configurations. The “giant magnon” is a particular limit that simplifies the spectrum on both sides of the correspondence. In this limit a general state can have any number of elementary excitations (magnons) and their bound states. In this dissertation we construct classical string solutions describing arbitrary superpositions of scattering and bound states of multi-charged giant magnons in various spaces including $AdS_5 \times S^5$ and $AdS_4 \times CP^3$. We use the sigma model dressing method to construct these solutions and analyze several of their properties, such as their scattering phase shift. We also use the inverse scattering and dressing methods to find various string solutions in AdS whose edges trace out complicated timelike curves on boundary. These solutions correspond to sinh-Gordon solitons and breathers and may be used to calculate certain Wilson loops via AdS/CFT. Our results provide important quantitative checks of the AdS/CFT correspondence.

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Contents

Chapter 1

Introduction

1.1 AdS/CFT

In 1974 't Hooft [1] realized that the perturbative expansion of $SU(N)$ gauge theory in the large N limit can be interpreted as a genus expansion of two dimensional surfaces built from field theory Feynman diagrams. In the case of $\mathcal{N}=4$ supersummetric Yang-Mills field theory (SYM) this expansion can be schematically written as

$$F=\underbrace{N^2(1+\lambda+\lambda^2+\ldots)}_{\text{planar}}+\underbrace{N^0(1+\lambda+\lambda^2+\ldots)}_{\text{genus 1}}+\underbrace{\frac{1}{N^2}(1+\lambda+\lambda^2+\ldots)}_{\text{genus 2}}+\ldots, \tag{1.1.0.1}$$

where $1/N$ counts the genus of the Feynman diagram, g_{YM} is the gauge theory coupling, $\lambda=g_{YM}^2N$ the 't Hooft coupling, and F the free energy. In the large N limit only planar diagrams contribute to the above expansion.

Ever since 1974 it has been widely suspected that there should exist a dual description of large N gauge theories in terms of string theories. The first concrete example was proposed more than 20 years later with the AdS/CFT correspondence [2, 3, 4] (see [5] for a review) that relates string theory on $AdS_5\times S^5$ background to $\mathcal{N}=4$ SYM gauge theory. AdS/CFT correspondence relates a four dimensional gauge theory to a higher dimensional string model, which is a manifestation of the holographic principle [6, 7] that indicates that the entire degrees of freedom of a quantum theory of gravity live on the boundary of the space-time region in question. In our case, the boundary of $AdS_5\times S^5$ is four dimensional and this is where the $\mathcal{N}=4$ theory lives.

The string model is controlled by two parameters, the string coupling constant g_s

and the string tension α', whereas the gauge theory is parameterized by the rank N of the gauge group and the coupling constant g_{YM} or equivalently the 't Hooft coupling $\lambda = g_{YM}^2 N$. According to the AdS/CFT those parameters should be related as

$$\frac{4\pi\lambda}{N} = g_s, \qquad \sqrt{\lambda} = \frac{R^2}{a'}, \tag{1.1.0.2}$$

where R is the common radius of AdS_5 and S^5.

The above equation (1.1.0.2) relates the coupling constants of the two theories, but there is also a dictionary that relates the string energy to suitable gauge theory operators according to

$$\langle \mathcal{O}_A(x)\, \mathcal{O}_B(y)\rangle = \frac{M\,\delta_{A,B}}{(x-y)^{2\,\Delta_A(\lambda,\frac{1}{N})}} \quad \Leftrightarrow \quad \mathcal{H}_{\text{string}}\,|\mathcal{O}_A\rangle = E_A(\frac{R^2}{\alpha'}, g_s)\,|\mathcal{O}_A\rangle, \tag{1.1.0.3}$$

where $|\mathcal{O}_A\rangle$ denotes a string eigenstate, A indicates suitable composite gauge theory operators that can be written as the trace of elementary fields of $\mathcal{N}=4$ in the adjoint representation (and their covariant derivatives), and Δ is the scaling dimension of the dual gauge theory operator and is determined by the two point function of the conformal field theory. AdS/CFT conjectures that

$$\Delta(\lambda, \frac{1}{N}) = E(\frac{R^2}{a'}, g_s). \tag{1.1.0.4}$$

A zeroth order test of the conjecture is the agreement of the underlying summetry supergroup $PSU(2,2|4)$ of the two theories.

String theory on $AdS_5 \times S^5$ is a complicated two dimensional field theory (even in its free version, $g_s = 0$) and the quantization in this background remains an open problem. Thus the string theory side of the correspondence could so far be addressed by its low energy effective description which is it terms of type IIB supergravity. In this approximation the curvature of the background is small compared to a string scale or in other words $\lambda \gg 1$. On the other hand perturbative calculations in $\mathcal{N} = 4$ require that $\lambda \ll 1$. In other words the duality relates a strongly coupled theory to a weakly coupled and vice versa. Because of the strong/weak nature of the correspondence, dynamical tests of the AdS/CFT in regimes which are not protected by the large amount of symmetry in the problem where very difficult.

1.2 BMN sector

The situation improved in 2002 with the work of Berenstein, Maldacena, and Nastase (BMN limit) [8] where it was shown that the two theories possess overlapping perturbative regimes. The key idea behind BMN limit is to consider a $U(1)$ charge J of $SO(6)$ and define a new effective coupling $\tilde{\lambda} \equiv \lambda/J^2$. Then if we take the quantum number J much larger than $\sqrt{\lambda}$ the effective coupling $\tilde{\lambda}$ can be very small even on the string theory side.

On the string theory side, the $U(1)$ charge corresponds to an angular momentum that the string carries on S^5, while on the gauge theory side it is a $U(1)$ R-charge of a local operator. In the large spin limit, the energies E of the string states as well as the conformal dimensions Δ of SYM operators are also very large. More precisely, the BMN limit is defined as

$$J, N \to \infty, \qquad \frac{J}{\sqrt{N}} = \text{fixed}, \qquad \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \text{fixed}, \qquad E - J = \text{fixed}. \tag{1.2.0.5}$$

1.3 Giant magnon limit

Another interesting limit of the AdS/CFT correspondence was considered by Hofman and Maldacena (HM) in [9], where they also considered operators where one of the $SO(6)$ charges, J, was considered to be very large.

According to the AdS/CFT dictionary states with $E-J=0$ correspond to a long chain of Z fields

$$E - J = 0 \quad \Leftrightarrow \quad \mathrm{tr}(\mathrm{Z}^{\mathrm{J}}). \tag{1.3.0.6}$$

One can also consider states with finite $E-J$ which would correspond to operators of the form

$$E - J = \text{finite} \quad \Leftrightarrow \quad \mathcal{O}_p = \sum_l e^{ipl}(\ldots ZZZ\underset{\underset{l}{\uparrow}}{W}ZZZ\ldots), \tag{1.3.0.7}$$

where an impurity W was inserted. p is the momentum the field W propagates along the chain of Z's. Since on the gauge theory side the problem of diagonalizing the planar Hamiltonian reduces to a type of spin chain [10, 11, 12], we call these excitation states magnons.

Using supersymmetry, Beisert has shown in [13] that these excitation states have a dispersion relation

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \tag{1.3.0.8}$$

that in the large coupling limit takes the form

$$E - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|. \tag{1.3.0.9}$$

Here the periodicity of p comes from the discreteness of the spin chain. In order to obtain (1.3.0.9) from the string point of view, HM identified p to be a geometric angle, thus recovering periodicity.

In order to take the HM limit, we first need to consider the usual 't Hooft limit. Then we pick up a $SO(6)$ generator J, and consider the limit where J is large. The states that are considered have finite $E-J$ and the 't Hooft coupling is kept fixed. Finally we consider the momentum of the excitation states p fixed. Summarizing the HM limit is

$$E, J \to \infty, \quad \lambda = g_{YM}^2 N = \text{fixed}, \quad p = \text{fixed}, \quad E - J = \text{finite}. \tag{1.3.0.10}$$

The HM limit differs from the BMN limit in two ways as summarized in the following table

HM	BMN
λ fixed	$\lambda \to \infty$
p fixed	pJ fixed

The HM limit has the nice feature that it decouples quantum effects characterized by the 't Hooft coupling λ from finite J effects. In this limit the spectrum on both sides can be analyzed in terms of asymptotic states and the S-matrix describing theirs scattering.

We now review the construction of the elementary giant magnon according to [9]. HM considered the Nambu-Goto action of the string model in $R \times S^2$ and they imposed the boundary conditions that the end points of the sting lie on the equator of S^2 moving with the speed of light. In spherical coordinates the sting action becomes

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt d\phi' \sqrt{\cos^2\theta \theta'^2 + \sin^2\theta} \tag{1.3.0.11}$$

and the equation of motion can be easily integrated to give the desired solution

$$\sin\theta = \frac{\sin\theta_0}{\cos\phi'}, \tag{1.3.0.12}$$

where θ_0 is the integration constant.

We can now compute the energy of the (1.3.0.12) to be

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\frac{\Delta\phi}{2}, \tag{1.3.0.13}$$

where $\Delta\phi$ is the angle between the end point of the string. In order to match (1.3.0.9) we need to identify $\Delta\phi$ with p, where p is the momentum of the excitation in the spin chain picture.

Let us also mention that giant magnons on $R\times S^2$ are in one to one correspondence with sine-Gordon solitons [14]. Some of the physical quantities appearing in these two theories like the time delay of a scattering process are the same, whereas other quantities like the speed or the phase shift are different according to the following table

sine-Gordon	giant magnon
$E_{\text{sG}} = \gamma$	$E_{\text{magnon}} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma}$
$\Delta T_{CM} = \frac{2}{\gamma v}\log v$	$\Delta T_{12} = \frac{2}{\gamma_1 v_1}\log v_{cm}$

$$v = \cos\frac{p}{2}, \quad \gamma^{-2} = 1 - v^2, \quad \text{phase shift} = \int dE_1 \Delta T_{12}. \tag{1.3.0.14}$$

1.4 Dyonic Giant Magnons

In addition to the elementary magnon, a spin chain can also contain an infinite number of elementary magnons as well as boundstates [15]. Magnons with polarizations in an $SU(2)$ subsector carry a second conserved $U(1)$ R-charge, J_2, and they can form boundstates with exact dispersion relation

$$E - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}. \tag{1.4.0.15}$$

that correspond to operators of the form

$$\mathcal{O}_p = \sum_l e^{ipl}(\ldots ZZZ\underset{\substack{\uparrow\\ l}}{W}{}^{J_1}ZZZ\ldots). \tag{1.4.0.16}$$

For $J_1 = 1$ we get the dispersion relation of the elementary magnon, where as for $J_1 = Q =$ integer we have a Q-magnon boundastate. These states should exist for all values of the 't Hooft coupling constant and we are free to consider states with $J_2 \sim \sqrt{\lambda}$. For such states we can consider the corresponding classical string carrying a second large angular momentum. We call these string states **dyonic giant magnons** and were first discussed in [16] where they exploited the equivalence of string theory on $R \times S^3$ with the complex sine-Gordon system.

1.5 Scattering amplitudes through AdS/CFT

One of the most interesting application of AdS/CFT is the work that was initiated by Alday and Maldacena [17] that gave a prescription on how to compute on-shell planar gluon scattering amplitudes at strong coupling in $\mathcal{N} = N$. The computation is based on finding a certain classical string configuration whose boundary conditions are determined by the gluon momenta. They also computed the leading order 4-point scattering amplitude and they showed agreement with the BDS ansatz [18] (the 8-point amplitude has been recently computed in [19]). The results in [17] are infrared divergent and, in order to regularize, AM introduced a gravity version of dimensional regularization.

1.6 ABJM theory

Initiated by the work of Bagger and Lambert [20], Aharony, Bergman, Jafferis, and Maldacena (ABJM) [21] constructed a $\mathcal{N} = 6$ superconformal Chern-Simons theory with $SU(N) \times SU(N)$ gauge symmetry at level k that is believed to be dual to M-theory on $AdS_4 \times S^7/Z_k$. Moreover, ABJM considered the large $N, k \to \infty$ limit keeping the 't Hooft coupling $\lambda = N/k$ fixed and they conjectured that the $\mathcal{N} = 6$ field theory is dual to type IIA string theory on $AdS_4 \times CP^3$. This new AdS_4/CFT_3 correspondence is also a strong-weak duality.

1.7 Useful tools

Here we review some useful relations and methods that will repeatedly be used in the course of this work.

sine-Gordon theory

The sine-Gordon (sG) equation of motion is

$$\phi_{\tau\tau} - \phi_{\sigma\sigma} = -\frac{1}{2}\sin 2\phi. \tag{1.7.0.17}$$

The fundamental soliton solutions of (1.7.0.17) on the infinite line are the single kink (+) and the single anti-kink (-) given by

$$\phi(\sigma,\tau) = 2\arctan e^{\pm\gamma(\sigma-\beta\tau)}, \tag{1.7.0.18}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. The solution describing the scattering of a kink and an anti-kink is given by

$$\phi_{s\bar{s}}(\sigma,\tau) = 2\arctan\frac{\sinh\gamma\beta\tau}{\beta\cosh\gamma\sigma}. \tag{1.7.0.19}$$

From this solution one can obtain the scattering of two kinks by the shift

$$\gamma\sigma \to \gamma\sigma + \frac{i\pi}{2}, \quad \gamma\beta\tau \to \gamma\beta\tau + \frac{i\pi}{2}, \tag{1.7.0.20}$$

and the result is

$$\phi_{ss}(\sigma,\tau) = 2\arctan\frac{\cosh\gamma\beta\tau}{\beta\sinh\gamma\sigma}. \tag{1.7.0.21}$$

The breather solution of (1.7.0.17) is obtained by analytically continuing the speed of the kink-antikink solution, $\beta \to ia$

$$\phi_{\text{br}}(\sigma,\tau) = 2\arctan\frac{\sin\gamma_a a\tau}{a\cosh\gamma_a\sigma}, \tag{1.7.0.22}$$

where $\gamma_a = \sqrt{1+a^2}$. The period of a breather is $T = \frac{2\pi}{a\gamma_a}$.

Periodic solution to the sG equation gives rise to strings with finite word-sheet [22, 23]. The fundamental periodic soliton solutions are given by the kink (+) and antikink (-) train

$$\phi(\alpha,\tau) = \frac{\pi}{2} + \text{am}(\pm(k\sigma - \omega\tau|m). \tag{1.7.0.23}$$

For $m < 1$ the above solution describes an infinite equally separated sequence of kinks and antikinks moving with constant velocity ω/k. The solution is quasi-periodic since every kink is a 2π step. For $m < 1$, on the other hand, we obtain an infinite sequence of kink-antikink, whereas for $m = 1$ we recover the periodic solutions. Periodic generalizations of scattering states are given in [24].

complex sine-Gordon

The complex sine-Gordon theory (CsG) consists of two real field ϕ and χ that together can be combined to a single complex field $\psi = \sin(\phi/2)\exp(i\chi/2)$ and in light cone coordinates obey the equation

$$\partial_+\partial_-\psi + \psi^*\frac{\partial_+\psi\partial_-\psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0. \tag{1.7.0.24}$$

The 1-soliton solution is given by

$$\psi = e^{i\mu}\frac{\cos a\exp(iT\sin a)}{\cosh((X-X_0)\cos a)}, \tag{1.7.0.25}$$

where

$$X = x\cosh\theta - t\sinh\theta, \quad T = t\cosh\theta - x\sinh\theta, \quad \mu = \text{constant} \tag{1.7.0.26}$$

and the equivalence of the CsG model to string on $R \times S^3$ have been used in the construction of dyonic giant magnons [16].

sinh-Gordon

In light cone coordinates the sinh-Gordon (shG) can be written in the standard form

$$\partial_+\partial_-\alpha - 4\sinh\alpha = 0 \tag{1.7.0.27}$$

and it admits solitonic solutions in a infinite line as well as periodic solutions that can be expressed as

$$\begin{aligned} \alpha_{\text{inf}} &= \pm\ln\left(\tan^2\gamma(\sigma - v\tau)\right), \\ \alpha_{\text{per}} &= \ln\left(k\ \text{sn}^2(\sigma/\sqrt{k}|k)\right). \end{aligned} \tag{1.7.0.28}$$

Solitons in shG models are related to classical string configurations moving in AdS through a Pohlemeyer map [14]. This equivalence has been successfully used (see for example [25, 26, 27].

The dressing method

The dressing method was introduced by Zakharov and Mikhailov in [28, 29]. It is a very general technique that allows for construction of solitonic classical solutions of integrable systems. One starts with any known solution of the unitary $N \times N$ matrix field $g(z, \bar{z})$ that satisfies the equation of motion

$$\bar{\partial}(\partial g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0 \tag{1.7.0.29}$$

and proceeds with the construction of a dressing factor χ such that $g' = \chi g$ is a new solution. The dressing factor χ depends on the model we are considering and the construction of it is given later in the text for the different physical problems we are considering.

The dressing method has several advantages and disadvantages compared to other methods for constructing of classical string solutions (for example the inverse scattering method). One of the advantages is that the dressing method reduces a second order differential equation that in principle can be difficult to solve to a system of two first order equations. As we said earlier we have to start with any known solution of our model that we either know or it is easy to guess like the vacuum of the theory in consideration. If we choose to start with the vacuum we would have to solve a much simpler system of first order differential equations.

One other advantage it that after we have constructed one solution we can use algebraic methods and find all other solutions of the problem without the need to solve any more differential equations. An example here can be the construction of he N-giant magnon solutions on $R \times S^3$.

Finally and most of the time we have the flexibility to approach the same problem with different kind of dressing methods.

The disadvantage is that it is not easy to see a priori that the dressing method can give us the solution we are interested in.

1.8 Outlook

In chapter 2 we consider the problem of constructing more general giant magnon solutions in $R \times S^5$, whereas the most general $N-$magnon solution on $R \times S^3$ is given in chapter 3. In view of the Alday-Maldacena problem we present in chapter 3 new solutions in Euclidian AdS_3 and AdS_5 spaces that we call giant gluons, whereas solitonic solutions in Minkowskian worldsheets are considered in chapter 4. Finally, in chapter 5 we attack the problem of classical solutions in CP^3 space.

Chapter 2

Dressing the Giant Magnon

2.1 Abstract

We extend earlier work by demonstrating how to construct classical string solutions describing arbitrary superpositions of scattering and bound states of dyonic giant magnons on S^5 using the dressing method for the $SU(4)/Sp(2)$ coset model. We present a particular scattering solution which generalizes solutions found in hep-th/0607009 and hep-th/0607044 to the case of arbitrary magnon momenta. We compute the classical time delay for the scattering of two dyonic magnons carrying angular momenta with arbitrary relative orientation on the S^5.

2.2 Introduction

The study of classical spinning string solutions in $AdS_5 \times S^5$ has provided a wealth of data for detailed study of the AdS/CFT correspondence. An interesting step forward was taken by Hofman and Maldacena [9], who found the classical string solution corresponding to a single magnon in the dual gauge theory. In this context the word magnon refers to an elementary excitation which can travel along a chain of Z's with some momentum p, i.e.

$$\mathcal{O}_p \sim \sum_l e^{ipl}(\cdots ZZZWZZZ\cdots), \tag{2.2.0.1}$$

where the magnon W is inserted at position l along the chain. The corresponding 'giant magnon' is an open string whose endpoints move at the speed of light along an equator of the S^5, separated in longitude by an angle p. This state carries an infinite amount of angular momentum J in the plane of the equator of the S^5 and is characterized by a finite value of $\Delta - J$. Work on giant magnons include [15, 16, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 22, 41, 42, 43].

In [34] the dressing method [28, 29, 44] was used to construct classical string solutions corresponding to various scattering and bound states of magnons. In particular, it was demonstrated how to obtain solutions representing superpositions of any number of elementary giant magnons (or bound states thereof) on $R \times S^5$, as well as any number of dyonic giant magnons on $R \times S^3$. The dyonic giant magnon, discovered in [15, 16], is a BPS bound state of many ($\mathcal{O}(\sqrt{\lambda})$) magnons which carries, in addition to an infinite amount of J in the equator of the S^3, a non-zero macroscopic amount of angular momentum J_1 in the orthogonal plane on S^3.

In this note we study scattering states of dyonic giant magnons on S^5, some special cases of which have appeared in [34, 35, 42]. We fill a gap in previous work by demonstrating how to construct classical string solutions describing general scattering states of dyonic giant magnons whose individual angular momenta J_i have arbitrary orientations in the directions transverse to the equator of the S^5.

After reviewing the basics of giant magnons in section 3, we explain in section 4 how to apply the dressing method for the $SU(4)/Sp(2) = S^5$ coset model, following the construction of [44]. This coset construction apparently has more flexibility than the $SO(6)/SO(5) = S^5$ coset construction employed in [34], since we have been unable to find the dyonic giant magnon solution via the latter dressing method. In [34] the $SU(2)$ principal chiral model was instead used to construct superpositions of dyonic magnons. That was sufficient for solutions living only on an $S^3 \subset S^5$, but the $SU(4)/Sp(2)$ coset used in this work allows us to construct solutions living on the full S^5. In section 5 we begin with a detailed analysis of the parameter space for a single soliton in the $SU(4)/Sp(2)$ coset model. We present in (2.5.2.33) a particular explicit solution for the scattering of two dyonic giant magnons with arbitrary momenta p_1, p_2 which carry angular momentum in orthogonal planes. This solution generalizes the special case $p_1 = -p_2 = \pi$ which was obtained in [34] and was generalized to $p_1 = -p_2 = p$ in [35]. Finally in section 6 we calculate the classical time delay for the scattering of two dyonic giant magnons with arbitrary relative orientations on the S^5. It would be interesting to compare the corresponding classical phase shift to a gauge theory analysis along the lines of [9, 39, 40].

2.3 Giant Magnons

We consider string theory on $R \times S^5$ in conformal gauge, writing the S^5 part of the theory in terms of three complex fields Z_i subject to the constraint

$$Z_i \bar{Z}_i = 1. \tag{2.3.0.2}$$

The equation of motion for Z_i can be written as

$$\bar{\partial}\partial Z_i + \frac{1}{2}(\partial Z_j \bar{\partial}\bar{Z}_j + \partial \bar{Z}_j \bar{\partial} Z_j) Z_i = 0, \tag{2.3.0.3}$$

where we use the worldsheet coordinates $z = \frac{1}{2}(x - t)$, $\bar{z} = \frac{1}{2}(x + t)$. The Virasoro constraints take the form

$$\partial Z_i \partial \bar{Z}_i = \bar{\partial} Z_i \bar{\partial} \bar{Z}_i = 1 \tag{2.3.0.4}$$

after setting the gauge $X^0 = t$ (X^0 is the time coordinate on $R \times S^5$).

We consider a giant magnon to be any open string whose endpoints move at the speed of light along an equator of the S^5, which we choose to lie in the Z_1 plane. The appropriate boundary conditions at fixed t are

$$\begin{aligned} Z_1(t, x \to \pm\infty) &= e^{i(t \pm p/2) + i\alpha}, \\ Z_i(t, x \to \pm\infty) &= 0, \qquad i = 2, 3, \end{aligned} \tag{2.3.0.5}$$

where $e^{i\alpha}$ is an arbitrary overall phase and p represents the difference in longitude between the endpoints of the string on the equator of the S^5. In the gauge theory picture, p is identified with the momentum of the magnon [9]. We may refer to p as the 'momentum' of a magnon, but it should be kept in mind that the worldsheet momentum of all of the solutions we consider is zero due to the Virasoro constraints (2.3.0.4).

The equations (2.3.0.2)–(2.3.0.5) have infinitely many distinct solutions, which can be partly classified by their conserved charges. The boundary conditions (2.3.0.5) explicitly break the $SO(6)$ symmetry of the S^5 down to $U(1) \times SO(4)$. The conserved charge associated with the $U(1)$ is

$$\Delta - J = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} dx \, \left(1 - \mathrm{Im}[\bar{Z}_1 \partial_t Z_1]\right), \tag{2.3.0.6}$$

where $\sqrt{\lambda}/2\pi$ is the string tension expressed in terms of the 't Hooft coupling λ of the dual gauge theory. The $SO(4)$ symmetry leads to conserved angular momentum matrix

$$J_{ab} = i\frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} dx \, \left(X_a \partial_t X_b - X_b \partial_t X_a\right), \qquad a, b = 1, \ldots, 4, \tag{2.3.0.7}$$

which we have written in terms of the real basis defined by $Z_2 = X_1 + iX_2$, $Z_3 = X_3 + iX_4$.

2.4 Dressing Method for $S^5 = SU(4)/Sp(2)$

In order to construct solutions of (2.3.0.2)–(2.3.0.5) we will apply the dressing method of Zakharov and Mikhailov [28, 29] for building soliton solutions of classically integrable equations, following the application of this method to the $SU(4)/Sp(2)$ coset model given in [44].

We will see that an elementary soliton of the $SU(4)/Sp(2)$ coset model is characterized by the choice of a complex parameter λ and a point w on $\mathbb{P}^3$. The most general solution obtainable[1] via the dressing method takes the form of a scattering state of any number of elementary solitons or bound states of them. Each individual soliton carries some 'momentum' p_i and a single non-zero $SO(4)$ angular momentum J_i (i.e., the eigenvalues of the matrix (2.3.0.7) for a single soliton are $\{+J_i, -J_i, 0, 0\}$). These two quantities are encoded in the parameter λ_i of the soliton, while the parameter w_i determines the plane of its angular momentum in the transverse $\mathbb{R}^4$ (i.e., the eigenvetors of (2.3.0.7)).

The simplest context in which the dressing method may be applied is the reduced [14] principal chiral model describing a unitary matrix $g(z, \bar{z})$ satisfying the equation of motion

$$\bar{\partial}(\partial g\, g^{-1}) + \partial(\bar{\partial} g\, g^{-1}) = 0 \tag{2.4.0.8}$$

subject to the Virasoro constraints

$$(ig^{-1}\partial g)^2 = 1, \qquad (ig^{-1}\bar{\partial} g)^2 = 1. \tag{2.4.0.9}$$

Given any solution $g(z, \bar{z})$ of these equations, the dressing method provides for the construction of an appropriate dressing matrix χ such that

$$g'(z, \bar{z}) = \chi(z, \bar{z}) g(z, \bar{z}) \tag{2.4.0.10}$$

is also solution of (2.4.0.8) and (2.4.0.9).

For the application to classical string theory on $\mathbb{R} \times S^5$ we are not interested in a principal chiral model but rather a coset model. In previous work [34] the

[1] It is not clear to us that all solutions may be obtained through the dressing method. For example, we have been unable to obtain the dyonic giant magnon solution via the dressing method in the $SO(6)/SO(5)$ coset model.

$S^5 = SO(6)/SO(5)$ was employed but for the present analysis it is more fruitful to use the coset $S^5 = SU(4)/Sp(2)$ following the analysis of [44]. We define this coset by imposing on $g \in SU(4)$ the constraint

$$g^{\mathrm{T}} = \mathcal{J} g \mathcal{J}^{-1}, \tag{2.4.0.11}$$

where $\mathcal{J}$ is the fixed antisymmetric matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.4.0.12}$$

A convenient parametrization of this coset, which allows us to immediately read off the S^5 coordinates Z_i from the matrix g, is given by

$$g = \begin{pmatrix} Z_1 & Z_2 & 0 & Z_3 \\ -\bar{Z}_2 & \bar{Z}_1 & -Z_3 & 0 \\ 0 & \bar{Z}_3 & Z_1 & -\bar{Z}_2 \\ -\bar{Z}_3 & 0 & Z_2 & \bar{Z}_1 \end{pmatrix}, \tag{2.4.0.13}$$

which is unitary and satisfies (2.4.0.11) precisely when (2.3.0.2) holds.

To apply the dressing method, we begin with a given solution g by solving the linear system

$$\partial\Psi = \frac{\partial g\, g^{-1}\Psi}{1-\lambda}, \qquad \bar{\partial}\Psi = \frac{\bar{\partial} g\, g^{-1}\Psi}{1+\lambda} \tag{2.4.0.14}$$

to find $\Psi(\lambda)$ as a function of the auxiliary complex parameter λ, subject to the initial condition

$$\Psi(0) = g, \tag{2.4.0.15}$$

the unitarity constraint

$$\left[\Psi(\bar{\lambda})\right]^{\dagger} \Psi(\lambda) = 1, \tag{2.4.0.16}$$

and the coset constraint

$$\Psi(\lambda) = \Psi(0)\mathcal{J}\overline{\Psi(1/\bar{\lambda})}\mathcal{J}^{-1}, \tag{2.4.0.17}$$

whose role is to ensure that the dressed solution g' we now construct will continue to satisfy the coset condition (2.4.0.11).

Once we know $\Psi(\lambda)$, the dressing factor for a single soliton may be written in terms of the parameters (λ_i, w_i) discussed above as [44]

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P + \frac{1/\bar{\lambda}_1 - 1/\lambda_1}{\lambda - 1/\bar{\lambda}_1} Q, \tag{2.4.0.18}$$

where P is the hermitian projection operator whose image is spanned by $\Psi(\bar{\lambda}_1)w_1$ for any constant four-component complex vector w_1 (the overall scale of w_1 clearly drops out so it parametrizes $\mathbb{P}^3$) and Q is the hermitian projection operator whose image is spanned by $\Psi(1/\lambda_1)\mathcal{J}\bar{w}_1$. Concretely,

$$P = \frac{\Psi(\bar{\lambda}_1)w_1 w_1^\dagger \left[\Psi(\bar{\lambda}_1)\right]^\dagger}{w_1^\dagger \left[\Psi(\bar{\lambda}_1)\right]^\dagger \Psi(\bar{\lambda}_1) w_1}, \qquad Q = \frac{\Psi(1/\lambda_1)\mathcal{J}\bar{w}_1 w_1^{\mathrm{T}}\mathcal{J}^{-1}\left[\Psi(1/\lambda_1)\right]^\dagger}{w_1^{\mathrm{T}}\mathcal{J}^{-1}\left[\Psi(1/\lambda_1)\right]^\dagger \Psi(1/\lambda_1)\mathcal{J}\bar{w}_1}. \quad (2.4.0.19)$$

Then

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda) \quad (2.4.0.20)$$

satisfies the constraints (2.4.0.16) and (2.4.0.17), and provides the desired one-soliton solution $g' = \Psi'(0)$ to the original equations (2.4.0.8) and (2.4.0.9). Unlike the $SO(6)/SO(5)$ coset considered in [34], in this case there are no restrictions on the complex polarization vector w_1. Repeated application of this procedure can be used to generate multi-soliton solutions.

2.5 Giant Magnons on $\mathbb{R} \times S^5$

To apply the dressing method we begin with the vacuum solution

$$\begin{aligned} Z_1 &= e^{it}, \\ Z_2 &= 0, \\ Z_3 &= 0, \end{aligned} \quad (2.5.0.21)$$

which describes a point-like string moving at the speed of light around the equator of the S^5. This state clearly has $\Delta - J = 0$. After embedding this solution into $SU(4)$ as in (2.4.0.13), a simple calculation reveals that the desired solution $\Psi(\lambda)$ to the linear system (2.4.0.14) subject to the constraints (2.4.0.15)–(2.4.0.17) is

$$\Psi(\lambda) = \mathrm{diag}(e^{+iZ(\lambda)}, e^{-iZ(\lambda)}, e^{+iZ(\lambda)}, e^{-iZ(\lambda)}), \qquad Z(\lambda) = \frac{z}{\lambda - 1} + \frac{\bar{z}}{\lambda + 1}. \quad (2.5.0.22)$$

2.5.1 A single dyonic giant magnon

Let us begin by applying the dressing method once to the vacuum (2.5.0.21). We will reproduce the dyonic giant magnon solution of [16]. The value of this exercise is to set some notation for subsequent solutions and also to illustrate the physical significance of the parameters λ_1 and w_1 which characterize the soliton. We parametrize the latter as

$$w_1 = \begin{pmatrix} +\, i e^{+y_1/2+i\psi_1/2+i\chi_1/2} \cos\alpha_1 \\ e^{-y_1/2-i\psi_1/2+i\chi_1/2} \cos\beta_1 \\ -i e^{+y_1/2-i\psi_1/2-i\chi_1/2} \sin\alpha_1 \\ e^{-y_1/2+i\psi_1/2-i\chi_1/2} \sin\beta_1 \end{pmatrix}, \tag{2.5.1.23}$$

where y_1 is complex and the remaining four angles are real. Here we have used the fact that the overall scale of w_1 drops out. Application of the dressing method gives the solution

$$\begin{aligned} Z_1 &= \frac{e^{+it}}{|\lambda_1|}\left[\frac{\bar\lambda_1 e^{-2iZ(\lambda_1)+\bar y_1}}{\mathcal{D}_1} + \frac{\lambda_1 e^{+2iZ(\lambda_1)-\bar y_1}}{\overline{\mathcal{D}}_1}\right], \\ Z_2 &= \frac{ie^{i\psi_1}(\bar\lambda_1-\lambda_1)}{|\lambda_1|}\left[\frac{e^{-it}\cos\alpha_1\cos\beta_1}{\mathcal{D}_1} + \frac{e^{+it}\sin\alpha_1\sin\beta_1}{\overline{\mathcal{D}}_1}\right], \\ Z_3 &= \frac{ie^{i\chi_1}(\bar\lambda_1-\lambda_1)}{|\lambda_1|}\left[\frac{e^{-it}\cos\alpha_1\sin\beta_1}{\mathcal{D}_1} - \frac{e^{+it}\sin\alpha_1\cos\beta_1}{\overline{\mathcal{D}}_1}\right], \end{aligned} \tag{2.5.1.24}$$

where

$$\mathcal{D}_1 = e^{-2iZ(\lambda_1)+\bar y_1} + e^{-2iZ(\bar\lambda_1)-y_1}. \tag{2.5.1.25}$$

The solution (2.5.1.24) carries $U(1)$ charge

$$\Delta - J = \frac{\sqrt{\lambda}}{4\pi}\left|\lambda_1 - \bar\lambda_1 - \frac{1}{\lambda_1} + \frac{1}{\bar\lambda_1}\right| \tag{2.5.1.26}$$

and one non-zero $SO(4)$ angular momentum

$$J_1 = \frac{\sqrt{\lambda}}{4\pi}\left|\lambda_1 - \bar\lambda_1 + \frac{1}{\lambda_1} - \frac{1}{\bar\lambda_1}\right|. \tag{2.5.1.27}$$

Note that $\Delta - J$ is always strictly positive, but we have defined J_1 to be positive by choice–the eigenvalues of (2.3.0.7) come in $\pm$ pairs. Furthermore, the value of p for this solution, which may be read off by comparing (2.5.1.24) to (2.3.0.5), is given by

$$e^{ip} = \frac{\lambda_1}{\bar\lambda_1}. \tag{2.5.1.28}$$

In fact, λ_1 and $\bar\lambda_1$ are (sometimes up to an author-dependent normalization factor) the quantities frequently referred to in the recent literature as x^+ and x^- (see in particular [13, 45]). From the worldsheet point of view, the solution (2.5.1.24) describes a wave which propagates with phase velocity (i.e., the waveform depends on $x - v_1 t$) given by

$$v_1 = \frac{\lambda_1 + \bar\lambda_1}{1+|\lambda_1|^2}. \tag{2.5.1.29}$$

Using (2.5.1.26), (2.5.1.27) and (2.5.1.28), we find that the dispersion relation takes the familiar form for the dyonic giant magnon [15, 16]

$$\Delta - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}. \tag{2.5.1.30}$$

Note that all of the parameters associated with the choice of the 'polarization' w_1 completely drop out of the expressions for the conserved charges and the dispersion relation.

We can be more explicit about the role of these parameters. The parameters α_1, β_1, ψ_1 and χ_1 determine the 'orientation' of the soliton in the transverse $\mathbb{R}^4$. Specifically, the angular momentum matrix (2.3.0.7) has eigenvalues $(+J_1, -J_1, 0, 0)$, so the soliton is characterized by an amount J_1 of angular momentum inside a certain 2-plane in the transverse $\mathbb{R}^4$. The four parameters α_1, β_1, ψ_1 and χ_1 label the particular plane (they are coordinates on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$).

The remaining complex parameter y_1 can be completely absorbed by making the translation

$$Z(\lambda_1) \to Z(\lambda_1) - \frac{i}{2}\bar{y}_1. \tag{2.5.1.31}$$

The real part of y_1 corresponds to a translation of the soliton in the x direction while the imaginary part of y_1 corresponds to a rotation inside the plane of the soliton's angular momentum.

2.5.2 A scattering state of two dyonic magnons, with three spins on S^5

Having analyzed in detail the parameter space for a single soliton in the last section, we are now in a position to use the dressing method to construct multi-soliton scattering states. The general n-soliton solution is specified by n complex numbers λ_i which encode the energy (2.5.1.26) and angular momentum (2.5.1.27) of each soliton. Each soliton with non-zero angular momentum is also characterized by the choice of a 2-plane inside the transverse $\mathbb{R}^4$. Finally, the n-soliton solution has a non-obvious classical shift symmetry of the form (2.5.1.31) for each i. For n solitons this gives an additional $2n$ real moduli, but 2 linear combinations can be absorbed into overall t and x translations.

The procedure for constructing an n-soliton solution is therefore clear, but generic solutions are rather messy. We display here an explicit formula only for the special case of two dyonic giant magnons with completely orthogonal angular momenta, specifically, with soliton 1's angular momentum in the Z_2 plane and soliton 2's angular momentum in the Z_3 plane. To this end we pick the polarization vectors

$$w_1^{\mathrm{T}} = \begin{pmatrix} i & 1 & 0 & 0 \end{pmatrix}, \qquad w_2^{\mathrm{T}} = \begin{pmatrix} i & 0 & 0 & 1 \end{pmatrix}. \tag{2.5.2.32}$$

Applying the dressing method twice with parameters (λ_1, w_1) and then (λ_2, w_2) gives the solution

$$\begin{aligned}
Z_1 &= \frac{e^{+it}}{|\lambda_1\lambda_2|}\frac{\mathcal{N}_{12}}{\mathcal{D}_{12}}, \\
Z_2 &= \frac{ie^{-it}(\bar\lambda_1-\lambda_1)\lambda_2}{|\lambda_1\lambda_2|}\left[\frac{\lambda_1\bar\lambda_2-1}{\lambda_1\lambda_2-1}e^{2iZ(\lambda_2)}+\frac{\bar\lambda_1-\bar\lambda_2}{\bar\lambda_1-\lambda_2}e^{2iZ(\bar\lambda_2)}\right]\frac{e^{2i(Z(\lambda_1)+Z(\bar\lambda_1))}}{\mathcal{D}_{12}}, \\
Z_3 &= \frac{ie^{-it}(\bar\lambda_2-\lambda_2)\lambda_1}{|\lambda_1\lambda_2|}\left[\frac{\bar\lambda_1\lambda_2-1}{\lambda_1\lambda_2-1}e^{2iZ(\lambda_1)}+\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_1-\bar\lambda_2}e^{2iZ(\bar\lambda_1)}\right]\frac{e^{2i(Z(\lambda_2)+Z(\bar\lambda_2))}}{\mathcal{D}_{12}},
\end{aligned} \tag{2.5.2.33}$$

where

$$\begin{aligned}
\mathcal{N}_{12} &= \begin{pmatrix} \lambda_1 e^{2iZ(\lambda_1)} & \bar\lambda_1 e^{2iZ(\bar\lambda_1)} \end{pmatrix}\begin{pmatrix} \left|\frac{\lambda_1\bar\lambda_2-1}{\lambda_1\lambda_2-1}\right|^2 & 1 \\ 1 & \left|\frac{\lambda_1-\lambda_2}{\lambda_1-\bar\lambda_2}\right|^2 \end{pmatrix}\begin{pmatrix} \lambda_2 e^{2iZ(\lambda_2)} \\ \bar\lambda_2 e^{2iZ(\bar\lambda_2)} \end{pmatrix}, \\
\mathcal{D}_{12} &= \begin{pmatrix} e^{2iZ(\lambda_1)} & e^{2iZ(\bar\lambda_1)} \end{pmatrix}\begin{pmatrix} \left|\frac{\lambda_1\bar\lambda_2-1}{\lambda_1\lambda_2-1}\right|^2 & 1 \\ 1 & \left|\frac{\lambda_1-\lambda_2}{\lambda_1-\bar\lambda_2}\right|^2 \end{pmatrix}\begin{pmatrix} e^{2iZ(\lambda_2)} \\ e^{2iZ(\bar\lambda_2)} \end{pmatrix},
\end{aligned} \tag{2.5.2.34}$$

and $Z(\lambda)$ is defined in 2.5.0.22.

This solution carries $U(1)$ charge

$$\Delta - J = \frac{\sqrt{\lambda}}{4\pi}\sum_{i=1}^{2}\left|\lambda_i - \bar\lambda_i - \frac{1}{\lambda_i} + \frac{1}{\bar\lambda_i}\right| \tag{2.5.2.35}$$

and two independent angular momenta

$$J_i = \frac{\sqrt{\lambda}}{4\pi}\left|\lambda_i - \bar\lambda_i + \frac{1}{\lambda_i} - \frac{1}{\bar\lambda_i}\right|, \tag{2.5.2.36}$$

which are the eigenvalues of the angular momentum matrix (2.3.0.7) in the Z_2 and Z_3 planes respectively. The total momentum of this giant magnon is

$$e^{ip} = e^{i(p_1+p_2)} = \frac{\lambda_1}{\bar\lambda_1}\frac{\lambda_2}{\bar\lambda_2}, \tag{2.5.2.37}$$

and the dispersion relation can be written as

$$\Delta - J = \sqrt{J_1^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p_1}{2}} + \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p_2}{2}}. \tag{2.5.2.38}$$

Special cases of the solution (2.5.2.33) have appeared previously in the literature. The case $p_1 = -p_2 = \pi$ was presented in [34], and a generalization to $p_1 = -p_2$ was given in [35]. Making direct contact with equation (5.19) of the former requires taking the shift parameters c_i in that work to be

$$\tanh c_1 = -\tanh c_2 = -\left|\frac{\lambda_2}{\lambda_1}\right| \frac{|\lambda_1|^2 - 1}{|\lambda_2|^2 - 1}. \tag{2.5.2.39}$$

2.5.3 A scattering state of two HM magnons, with arbitrary positions on the transverse S^3

In the previous subsection we chose the particular polarization vectors (2.5.2.32) in order to avoid too much clutter in (2.5.2.33). An interesting limit in which the formulas simplify is when $|\lambda_i| \to 1$. Taking λ_i onto the unit circle sets the angular momentum of each soliton to zero–the dyonic giant magnon reduces to the elementary Hofman-Maldacena magnon [9]. Each such magnon is characterized by a momentum p and a unit vector n^a in the transverse $\mathbb{R}^4$ which specifies the polarization of its fluctuation away from the equator of the S^5. A giant magnon with polarization n^a describes a scalar field impurity ϕ^a, $a = 1, 2, 3, 4$ in the dual gauge theory. Using the real basis defined under (2.3.0.7), the solution for such a scattering state can be written as

$$\begin{aligned} Z_1 &= e^{it} + \frac{e^{it}}{\mathcal{D}_{12}} \left[\cos\tfrac{p_1}{2} - \cos\tfrac{p_2}{2} + i\sin\tfrac{p_1}{2}\tanh u_1 - i\sin\tfrac{p_2}{2}\tanh u_2\right], \\ X^a &= \frac{1}{\mathcal{D}_{12}} \left[n_1^a \sin\tfrac{p_1}{2}\,\mathrm{sech}\, u_1 - n_2^a \sin\tfrac{p_2}{2}\,\mathrm{sech}\, u_2\right], \qquad a = 1, 2, 3, 4, \end{aligned} \tag{2.5.3.40}$$

where

$$u_i = i(Z(\lambda_i) - Z(\bar{\lambda}_i)) = (x - t\cos\tfrac{p_i}{2})\csc\tfrac{p_i}{2} \tag{2.5.3.41}$$

and now

$$\mathcal{D}_{12} = \frac{1 - \cos\frac{p_1}{2}\cos\frac{p_2}{2} - \sin\frac{p_1}{2}\sin\frac{p_2}{2}\left[\tanh u_1 \tanh u_2 + (n_1 \cdot n_2)\,\mathrm{sech}\, u_1\,\mathrm{sech}\, u_2\right]}{\cos\frac{p_2}{2} - \cos\frac{p_1}{2}}. \tag{2.5.3.42}$$

It is also straightforward to obtain this solution via the Bäcklund transformation (see [46] in particular). The conserved charges and dispersion relation of this solution do not depend on the polarization vectors n_i^a.

2.6 Classical Time Delay for Scattering of Dyonic Magnons

With explicit formulas for the scattering solutions in hand, it is a simple matter to read off the classical time delay for soliton scattering. To find the time delay as soliton 1 passes soliton 2 (let us take $v_1 > v_2 > 0$ with the velocities v_i given by (2.5.1.29) we set $x = v_1(t - \delta t)$ and compare the solution at $t \to \pm\infty$ to the single soliton solution. The total time delay is then $\Delta T_{12} = \delta t_+ - \delta t_-$.

We are particularly interested in seeing the dependence of the time delay on the relative orientations of the angular momenta of the two scattering solitons. Without loss of generality we can take the polarization of w_1 as in (2.5.2.32), but we keep w_2 arbitrary as in (2.5.1.23). We find

$$\Delta T_{12} = \frac{i}{2}\frac{|1-\lambda_1|^2|1+\lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2} \log\left[(A\cos^2\alpha_2 + B\sin^2\alpha_2)(A\cos^2\beta_2 + B\sin^2\beta_2)\right], \tag{2.6.0.43}$$

where

$$A = \frac{|\lambda_1 - \lambda_2|^2}{|\lambda_1 - \bar{\lambda}_2|^2}, \qquad B = \frac{|\lambda_1 - 1/\bar{\lambda}_2|^2}{|\lambda_1 - 1/\lambda_2|^2}. \tag{2.6.0.44}$$

It would be interesting to evaluate the corresponding classical phase shift δ_{12} (i.e., the S-matrix element $e^{i\delta_{12}}$), which may be obtained by integrating ΔT_{12} with respect to the energy of soliton 1 (2.5.1.26) while holding the angular momentum (2.5.1.27) fixed, and to compare the result with a corresponding gauge theory calculation along the lines of [39, 40].

We can subject 2.6.0.43 to some consistency checks by comparing special cases of the formula to known results. First of all, we can recover the scattering of two HM magnons by taking $\lambda_i = e^{ip_i/2}$ on the unit circle. In this case we obtain

$$\Delta T_{12} = \tan\frac{p_1}{2}\log\left[\frac{1-\cos\frac{1}{2}(p_1 - p_2)}{1-\cos\frac{1}{2}(p_1+p_2)}\right], \tag{2.6.0.45}$$

in complete agreement with the result of [9]. Note that this result is independent of the positions of the two magnons on the transverse S^3, in accord with the expectation of [9]. The result (2.6.0.45) can also be read off directly from the solution (2.5.3.40).

Another check is obtained by setting $\alpha_2 = \beta_2 = 0$ so that we have two dyonic giant magnons whose angular momenta both lie within the Z_2 plane, leading to

$$\Delta T_{12} = 2i\frac{|1-\lambda_1|^2|1+\lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2}\log\left|\frac{\lambda_1 - \lambda_2}{\lambda_1 - \bar{\lambda}_2}\right| \tag{2.6.0.46}$$

in complete agreement with [39, 40] where this case was studied.

As a final consistency check, we can take $\alpha_2 = \beta_2 = \pi/2$, which leads to

$$\Delta T_{12} = 2i \frac{|1-\lambda_1|^2|1+\lambda_1|^2}{\lambda_1^2 - \bar{\lambda}_1^2} \log\left|\frac{\lambda_1 - 1/\bar{\lambda}_2}{\lambda_1 - 1/\lambda_2}\right| . \tag{2.6.0.47}$$

From equation (2.5.1.24) it is evident that this choice of orientation simply reverses the sign of the angular momentum of the second soliton relative to $\alpha = \beta = 0$. But this is completely equivalent to changing $\lambda_2 \to 1/\bar{\lambda}_2$, which is indeed precisely the transformation between (2.6.0.46) and (2.6.0.47).

Chapter 3

Exact solutions for N-magnon scattering

3.1 Abstract

Giant magnon solutions play an important role in various aspects of the AdS/CFT correspondence. We apply the dressing method to construct an explicit formula for scattering states of an arbitrary number N of magnons on $\mathbb{R} \times S^3$. The solution can be written in Hirota form and in terms of determinants of $N \times N$ matrices. Such a representation may prove useful for the construction of an effective particle Hamiltonian describing magnon dynamics.

3.2 Introduction

Classical string solutions in $AdS_5 \times S^5$ play an important role in understanding various aspects of the AdS/CFT correspondence (see [47, 48, 49, 50] for review). Integrability [51] is a powerful computational tool which has enabled many quantitative checks of the correspondence. A lot of work has been done exploring both string theory and gauge theory sides of the correspondence, culminating in the proposal for an exact S-matrix for planar $\mathcal{N} = 4$ Yang-Mills theory [52].

Magnons are building blocks of the spectrum in the spin chain description of AdS/CFT. The Hofman-Maldacena elementary magnon corresponds to a particular string configuration moving on an $\mathbb{R} \times S^2$ subspace of $AdS_5 \times S^5$ [9]. String theory on $\mathbb{R} \times S^2$ (or $\mathbb{R} \times S^3$) is classically equivalent to sine-Gordon theory (or complex sine-Gordon theory) via Pohlmeyer reduction [14, 53] (see [54] for AdS case). Giant one-magnon solutions on $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^3$ map to one-soliton solutions in sine-Gordon and complex sine-Gordon respectively [9, 16]. Using this map, the scattering phase of two magnons was computed in [9] and shown to match that of [55]. Moreover, a sine-Gordon-like action has been proposed for the full Green-Schwarz superstring on $AdS_5 \times S^5$ [56, 57].

In sine-Gordon theory, the dynamics of N-solitons is captured by the Ruijsenaars-Schneider model [58, 59]. Specifically, the eigenvalues of a particular $N \times N$ matrix entering into the description of the N-soliton solution (or τ-function) of sine-Gordon evolve according to the Ruijsenaars-Schneider Hamiltonian. Positions and momenta in the Hamiltonian are related to the positions and rapidities of the solitons, and the phase shift for soliton scattering can be calculated from the quantum mechanical model. It is natural to wonder what the analagous Hamiltonian in the case of complex sine-Gordon and giant magnons is. Explicit N-soliton solutions (in τ-function form) serve as a useful starting point in deriving the Ruijsenaars-Schneider model from the sine-Gordon theory, and it is likely that a similar technique may prove useful for complex sine-Gordon and giant magnons as well.

Interest for an effective particle description of giant magnon scattering emerged through the work of Dorey, Hofman and Maldacena [60], where they illuminated the nature of double poles appearing in the proposed S-matrix of planar $\mathcal{N} = 4$ Yang-Mills [52]. They were able to interpret these double poles as occurring from the exchange of pairs of particles, and in particular to precisely match their position on the complex domain with the prediction of [52], under the assumption that the exchanged particles are BPS magnon boundstates [15]. By studying the quantum mechanical problem corresponding to an effective particle Hamiltonian describing the scattering of two magnons with very small relative velocity, one should obtain an S-matrix whose double poles compare to the aforementioned results in the appropriate limit.

Superposing magnons is a difficult problem because of the nonlinear equations of motion they satisfy. Integrability allows the use of algebraic methods such as dressing to construct solutions of nonlinear equations of motion [28, 29]. Indeed, the dressing method was used to describe the scattering of two magnons and spikes on $\mathbb{R} \times S^5$ (and various subsectors) as well as spikes in AdS_3 [34, 61, 62, 63, 25, 26]. However, it is a tedious process to obtain even the three-magnon solution. In this work we will present an explicit string solution on $\mathbb{R}\times S^3$ describing scattering of an arbitrary number N of magnons by solving the recursive formula following from the dressing the $(N-1)$-magnon.

This work is organized as follows. In section 3 we review the dressing method for $\mathbb{R}\times S^3$ and derive a recursive formula for the N-magnon solution in terms of $(N-1)$-magnons. In section 4 we solve this recursion and present the N-magnon solution. The solution can be presented in various ways, we find useful Hirota and determinental forms. As a consistency check we verify that our solution separates asymptotically into a linear sum of N well-separated single magnon solutions and demonstrate that the only nontrivial effect of the N-magnon interaction is the expected sum of two-magnon time delays. The appendix clarifies the rules to construct the N-magnon solution and some examples are presented.

3.3 Giant magnons on $\mathbb{R}\times S^3$

The classical action for bosonic strings on $\mathbb{R}\times S^3$ can be written as

$$S=-\frac{1}{2}\int dt\,dx\,\Big[\partial^a X^\mu \partial_a X_\mu+\Lambda(X_i\cdot X_i-1)\Big], \tag{3.3.0.1}$$

where μ runs from 0 to 4 and i from 1 to 4. The X_i are embedding coordinates on $\mathbb{R}^4$ and the Lagrange multiplier Λ constrains them on S^3.

After we impose the gauge $X^0(t,x)=t$, eliminate Λ in terms of the embedding coordinates and switch to light-cone worldsheet coordinates $z=(x-t)/2$, $\bar z=(x+t)/2$, the equations of motion and Virasoro constraints become

$$\bar\partial\partial Z_i+\frac{1}{2}(\partial Z_j\bar\partial\bar Z_j+\partial\bar Z_j\bar\partial Z_j)Z_i=0,\quad Z_i\bar Z_i=1, \tag{3.3.0.2}$$

and

$$\partial Z_i\partial\bar Z_i=\bar\partial Z_i\bar\partial\bar Z_i=1, \tag{3.3.0.3}$$

where we have used the parametrization

$$Z_1=X_1+iX_2,\quad Z_2=X_3+iX_4. \tag{3.3.0.4}$$

Giant magnons on $\mathbb{R}\times S^3$ are defined as solutions to the above system of equations, obeying the boundary conditions

$$\begin{aligned} Z_1(t,x\to\pm\infty) &= e^{it\pm ip/2+i\alpha}\,, \\ Z_2(t,x\to\pm\infty) &= 0. \end{aligned} \tag{3.3.0.5}$$

The physical meaning of the boundary conditions is that the endpoints of the string lie on the equator of the S^3 on the Z_1 plane moving at the speed of light, and the quantity p called total momentum represents the angular distance between them. Finally, α can be any real constant.

3.3.1 Review of the dressing method

The dressing method is a general procedure for constructing soliton solutions to integrable differential equations first developed by Zakharov and Mikhailov [28, 29]. It was applied in the context of giant magnons [34, 61], providing classical solutions for a variety of backgrounds. In what follows, we will review the basic steps of the method as they apply to the particular case of $\mathbb{R} \times S^3$.

We start by defining the matrix-valued field

$$g(z,\bar{z}) \equiv \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2) \tag{3.3.1.6}$$

and recasting (3.3.0.2) into

$$\partial A + \bar{\partial} B = 0, \tag{3.3.1.7}$$

where the currents A and B are given by

$$A = i\bar{\partial} g g^{-1}, \qquad B = i\partial g g^{-1}. \tag{3.3.1.8}$$

The Virasoro constraints (3.3.0.3) can be also written as

$$\mathrm{Tr}A^2 = \mathrm{Tr}B^2 = 2. \tag{3.3.1.9}$$

The nonlinear second order equation for g in (3.3.1.7) is equivalent to a system of linear first order equations for auxiliary field $\Psi(z,\bar{z},\lambda)$

$$i\partial\Psi = \frac{A\Psi}{1-\lambda}, \qquad i\bar{\partial}\Psi = \frac{B\Psi}{1+\lambda} \tag{3.3.1.10}$$

provided (3.3.1.10) holds for any value of the new complex variable λ called the spectral parameter, with A and B independent of λ.

Given any known solution g, we can determine A, B and solve (3.3.1.10) to find $\Psi(\lambda)$ subject to the condition

$$\Psi(\lambda = 0) = g. \tag{3.3.1.11}$$

Any ambiguity on factors that don't depend on $z, \bar{z}$ is removed by also imposing the unitarity condition

$$\left[\Psi(\bar{\lambda})\right]^{\dagger}\Psi(\lambda) = I. \tag{3.3.1.12}$$

It is easy to show that the equations of motion for the auxiliary field (3.3.1.10) are covariant under the following transformation with a λ-dependent parameter $\chi(\lambda)$,

$$\begin{array}{ccccl} \Psi(\lambda) & \to & \Psi'(\lambda) & = & \chi\Psi(\lambda), \\ A & \to & A' & = & \chi A\chi^{-1} + i(1+\lambda)\bar{\partial}\chi\chi^{-1}, \\ B & \to & B' & = & \chi B\chi^{-1} + i(1-\lambda)\bar{\partial}\chi\chi^{-1}, \end{array} \tag{3.3.1.13}$$

under the condition that A', B' remain independent of λ. Thus, performing the above transformation to the known solution $(\Psi(\lambda), A, B)$ produces a new solution to (3.3.1.7) with $g' = \Psi'(\lambda = 0)$.

The condition (3.3.1.12) implies that $\chi(\lambda)$ must obey

$$\left[\chi(\bar{\lambda})\right]^{\dagger}\chi(\lambda) = I, \tag{3.3.1.14}$$

whereas the demand that A', B' are independent of λ can be translated as further constraints on the analytic properties of $\chi(\lambda)$. For the $\mathbb{R}\times S^3$ case it turns out [34] that the dressing factor $\chi(\lambda)$ is

$$\chi(\lambda) = I + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1}P, \tag{3.3.1.15}$$

where λ_1 is an arbitrary complex number and the hermitian projection operator P is given by

$$P = \frac{\upsilon_1\upsilon_1^{\dagger}}{\upsilon_1^{\dagger}\upsilon_1}, \quad \upsilon_1 = \Psi(\bar{\lambda}_1)e, \tag{3.3.1.16}$$

where e is an arbitrary vector with constant complex entries called the polarization vector. The projector P does not depend on the length of the e vector.

The determinant of $\chi(\lambda)$ is

$$\det\,\chi(\lambda) = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \tag{3.3.1.17}$$

and if we want our dressed solution $\chi(0)\Psi(0)$ to sit in $SU(2)$ we should rescale it by the compensating factor $\sqrt{\lambda_1/\bar{\lambda}_1}$.

Putting everything together, the new solution $g' = \Psi'(\lambda = 0)$ to the system (3.3.1.7) is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}}\left(I + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1}P\right)g. \tag{3.3.1.18}$$

3.3.2 Application and recursion

This procedure can be repeated with g' as the solution we begin with, in order to obtain another new solution. In fact, once we have solved the differential equation (3.3.1.10) for $\Psi(\lambda)$ the first time, we no longer need to repeat this step for $\Psi'(\lambda)$, as we have that information already. Thus, from this point the method proceeds iteratively in a purely algebraic manner.

More specifically, we can show that the auxiliary field $\Psi^N(\lambda)$ that is constructed after N iterations is related to the auxiliary field $\Psi^{N-1}(\lambda)$ occuring after $N-1$ iterations through

$$\Psi^N(\lambda) = \sqrt{\frac{\lambda_N}{\bar{\lambda}_N}}\,\frac{1}{(\lambda-\lambda_N)(ab-cd)}\begin{pmatrix}\psi^N_{11} & \psi^N_{12}\\ \psi^N_{21} & \psi^N_{22}\end{pmatrix}, \qquad (3.3.2.19)$$

where

$$\begin{aligned}
\psi^N_{11} &= (-cd(\lambda-\lambda_N)+ab(\lambda-\bar{\lambda}_N))\Psi^{N-1}_{11}(\lambda) - ac(\lambda_N-\bar{\lambda}_N)\Psi^{N-1}_{21}(\lambda),\\
\psi^N_{12} &= (-cd(\lambda-\lambda_N)+ab(\lambda-\bar{\lambda}_N))\Psi^{N-1}_{12}(\lambda) - ac(\lambda_N-\bar{\lambda}_N)\Psi^{N-1}_{22}(\lambda),\\
\psi^N_{21} &= (ab(\lambda-\lambda_N)-cd(\lambda-\bar{\lambda}_N))\Psi^{N-1}_{21}(\lambda) + bd(\lambda_N-\bar{\lambda}_N)\Psi^{N-1}_{11}(\lambda),\\
\psi^N_{22} &= (ab(\lambda-\lambda_N)-cd(\lambda-\bar{\lambda}_N))\Psi^{N-1}_{22}(\lambda) + bd(\lambda_N-\bar{\lambda}_N)\Psi^{N-1}_{12}(\lambda),
\end{aligned} \qquad (3.3.2.20)$$

and

$$\begin{aligned}
a &= \Psi^{N-1}_{11}(\bar{\lambda}_N) + \Psi^{N-1}_{12}(\bar{\lambda}_N),\\
b &= \Psi^{N-1}_{21}(\lambda_N) - \Psi^{N-1}_{22}(\lambda_N),\\
c &= \Psi^{N-1}_{11}(\lambda_N) - \Psi^{N-1}_{12}(\lambda_N),\\
d &= \Psi^{N-1}_{21}(\bar{\lambda}_N) + \Psi^{N-1}_{22}(\bar{\lambda}_N).
\end{aligned} \qquad (3.3.2.21)$$

The new solution of (3.3.1.7) follows from (3.3.2.19) when taking $\lambda = 0$. Due to (3.3.1.6) we can then read off the relation between the Z_i coordinates of the two solutions as

$$\begin{aligned}
Z^N_1 &= \frac{1}{|\lambda_N|(ab-cd)}\left[(ab\bar{\lambda}_N - cd\lambda_N)Z^{N-1}_1 + ac(\lambda_N-\bar{\lambda}_N)(-i\bar{Z}^{N-1}_2)\right],\\
Z^N_2 &= \frac{i}{|\lambda_N|(ab-cd)}\left[(ab\bar{\lambda}_N - cd\lambda_N)(-iZ^{N-1}_2) + ac(\lambda_N-\bar{\lambda}_N)\bar{Z}^{N-1}_1\right]
\end{aligned} \qquad (3.3.2.22)$$

Starting with the simple 'vacuum' solution representing a point particle rotating around the equator in the Z_1 plane,

$$\begin{aligned}
Z_1 &= e^{it},\\
Z_2 &= 0,
\end{aligned} \qquad (3.3.2.23)$$

and using the polarization vector $e = (1,\ 1)$ the dressing method yields [34] the single magnon solution on $\mathbb{R} \times S^3$ first obtained in [16] as a generalization of the original Hofman-Maldacena giant magnon solution on $\mathbb{R} \times S^2$. Applying the method once more using the same polarization vector as before then gives a solution which asymptotically reduces to a sum of two single magnon solutions, and whose conserved charges are sums of the respective charges of two single magnon solutions. Hence it can be interpreted as a scattering state of two single magnons.

From the above considerations, it is natural to expect that the N-times dressed solution will correspond to a scattering state of N magnons. The quantities λ_i are parameters of the N-magnon solution which we can more conventionally express as $\lambda_i = r_i e^{ip_i/2}$, with p_i the momentum of each constituent magnon and r_i a quantity associated to its $U(1)$ charge.

3.4 The N-magnon solution

Successive application of the dressing method suggests a compact closed form for the N-magnon solution, which can be written as follows

$$\begin{aligned} Z_1 &= \frac{e^{it}}{\prod_{l=1}^{N}|\lambda_l|}\frac{N_1}{D}\,, \\ Z_2 &= -i\frac{e^{-it}}{\prod_{l=1}^{N}|\lambda_l|}\frac{N_2}{D}, \end{aligned} \tag{3.4.0.24}$$

with

$$\begin{aligned} D &= \sum_{\mu_i=0,1}\exp\left[\sum_{i<j}^{2N} B_{ij}[\mu_i\mu_j + (\mu_i-1)(\mu_j-1)] + \sum_{i=1}^{2N}\mu_i(2i\mathcal{Z}_i)\right], \\ N_1 &= \sum_{\mu_i=0,1}\exp\left[\sum_{i<j}^{2N} B_{ij}[\mu_i\mu_j + (\mu_i-1)(\mu_j-1)] + \sum_{i=1}^{2N}\mu_i(2i\mathcal{Z}_i + C_i)\right], \\ N_2 &= \sum_{\mu_i=0,1}\exp\left[\sum_{i<j}^{2N} B_{ij}[\mu_i\mu_j + (\mu_i-1)(\mu_j-1)] + \sum_{i=1}^{2N}[\mu_i(2i\mathcal{Z}_i) + (\mu_i-1)C_i]\right], \end{aligned} \tag{3.4.0.25}$$

where

$$\begin{aligned} \mathcal{Z}_i &= \frac{z}{\lambda_i - 1} + \frac{\bar{z}}{\lambda_i + 1}, \\ e^{B_{ij}} &= \lambda_i - \lambda_j, \\ e^{C_i} &= \lambda_i, \end{aligned} \tag{3.4.0.26}$$

and N is the number of magnons.

In the above formula the indices i, j take the $2N$ values $(1, \bar{1}, 2, \bar{2}, ..., \bar{N})$, $i < j$ implies this particular ordering, and we identify $\lambda_{\bar{k}} \equiv \bar{\lambda}_k$, $\mathcal{Z}_{\bar{k}} \equiv \bar{\mathcal{Z}}_k$ [1]. The symbol $\sum_{\mu_i=0,1}$ implies the summation over all possible combinations of $\mu_1 = 0, 1$, $\mu_{\bar{1}} = 0, 1, \ldots,$ $\mu_{\bar{N}} = 0, 1$ under the conditions

$$\sum_{i=1}^{2N} \mu_i = \begin{cases} N, & \text{for } N_1,\ D, \\ N+1, & \text{for } N_2. \end{cases} \tag{3.4.0.27}$$

This description makes contact with a variety of N-soliton expressions of other integrable systems (for example see [64]).

We have numerically checked (3.4.0.25) for high number of magnons, whereas in Fig. (3.1) we plot $|Z_2|$ for the first 4 magnons. At the end of the chapter we give some examples.

Our $\mathbb{R} \times S^3$ N-magnon solution is reduced to the $\mathbb{R} \times S^2$ one if we let the spectral parameters λ_l lie on a unit circle, $|\lambda_l| = 1$.

3.4.1 Hirota form of the solution

It is possible to write Z_1, Z_2 of (3.4.0.24) in an equivalent form similar to Hirota's [65], where N_1, N_2, D are given by

$$\begin{aligned} D &= \sum_{{}_{2N}C_N} d(i_1, i_2, \ldots, i_N) \exp\left[2i(\mathcal{Z}_{i_1} + \mathcal{Z}_{i_2} + \cdots + \mathcal{Z}_{i_N})\right], \\ N_1 &= \sum_{{}_{2N}C_N} n_1(i_1, i_2, \ldots, i_N) \exp\left[2i(\mathcal{Z}_{i_1} + \mathcal{Z}_{i_2} + \cdots + \mathcal{Z}_{i_N})\right], \\ N_2 &= \sum_{{}_{2N}C_{N+1}} n_2(i_1, i_2, \ldots, i_{N+1}) \exp\left[2i(\mathcal{Z}_{i_1} + \mathcal{Z}_{i_2} + \cdots + \mathcal{Z}_{i_{N+1}})\right], \end{aligned} \tag{3.4.1.28}$$

and

$$\begin{aligned} d(i_1, i_2, \ldots, i_N) &= \prod_{k<l\le N}^{(N)} \lambda_{i_k i_l} \prod_{N<m<n}^{(N)} \lambda_{i_m i_n}, \\ n_1(i_1, i_2, \ldots, i_N) &= \prod_{j=1}^{N} \lambda_{i_j} \prod_{k<l\le N}^{(N)} \lambda_{i_k i_l} \prod_{N<m<n}^{(N)} \lambda_{i_m i_n}, \\ n_2(i_1, i_2, \ldots, i_{N+1}) &= \prod_{j=N+1}^{2N} \lambda_{i_j} \prod_{k<l\le N+1}^{(N+1)} \lambda_{i_k i_l} \prod_{N+1<m<n}^{(N-1)} \lambda_{i_m i_n}, \end{aligned} \tag{3.4.1.29}$$

[1] Alternatively we may define new quantities ρ_k such that $\rho_{2l-1} = \lambda_l$ and $\rho_{2l} = \bar{\lambda}_l$, and similarly for $\mathcal{Z}_l$. These will take values $1, 2...2N$ as usual.

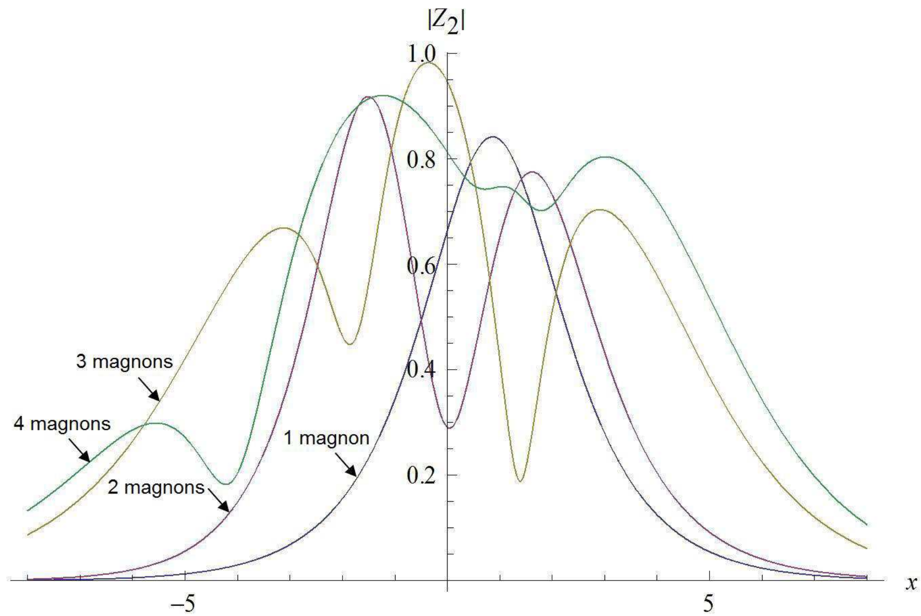

Figure 3.1: Plot of $|Z_2|$ for the first 4 magnons on $\mathbb{R} \times S^3$ at time t=2 as a function of the worldsheet coordinate x. The chosen spectral parameters are $\lambda_1 = 2e^i,\ \lambda_2 = e^{2i},\ \lambda_3 = 3e^{2i},\ \lambda_4 = e^{4i}$.

where N is the number of magnons, ${}_NC_n$ indicates summation over all possible combinations of n elements taken from N, $\prod^{(n)}$ indicates the product of all possible combinations of the n elements, and $\lambda_{ij} = \lambda_i - \lambda_j$. Finally, we have arranged our $2N$ elements $\mathcal{Z}_i$ as $\{\mathcal{Z}_1, \bar{\mathcal{Z}}_1, \ldots, \bar{\mathcal{Z}}_N\}$ and our $2N$ λ's as $\{\lambda_1, \bar{\lambda}_1, \ldots, \bar{\lambda}_N\}$. We always assume that $i_1 < \ldots < i_N$.

Finally, we should mention that we can get a more symmetric yet complicated-looking version of our N-magnon expressions, by factoring out the terms

$$\begin{cases} \prod_{l=1}^N \lambda_l \exp\left(2i \sum_{l=1}^N \mathcal{Z}_l\right) & \text{from } N_1, \\ \prod_{l=1}^N \bar{\lambda}_l \exp\left(2i \sum_{l=1}^N \mathcal{Z}_l\right) & \text{from } N_2, \\ \exp\left(2i \sum_{l=1}^N \mathcal{Z}_l\right) & \text{from } D. \end{cases} \tag{3.4.1.30}$$

Written in this way, D has the nice feature of being real. More importantly, and as we will see in the following sections, this form of the N-magnon solution is useful for analyzing its asymptotic behavior and demonstrates the symmetry that will allow us to write it in a determinant form.

3.4.2 Determinant form for Z_1

It is known that for the (complex) sine-Gordon equation and several other integrable equations, the N-soliton expressions similar to (3.4.0.24)-(3.4.0.27) and (3.4.1.28)-(3.4.1.29) can also be rewritten in a form involving determinants of $N \times N$ matrices [66]. It is precisely expressions of this type that become particularly useful when extracting the effective particle description of the soliton problem [59]. Motivated by the same goal for the case of giant magnons, we haven been able to find a determinant formula for Z_1. In particular, we may write

$$Z_1 = e^{it} \prod_{l=1}^{N} \left(\frac{\lambda_l}{\bar{\lambda}_l}\right)^{1/2} \frac{\det(I + \Lambda^{-1} F \bar{\Lambda} \bar{F})}{\det(I + F\bar{F})}, \tag{3.4.2.31}$$

where Λ, F are $N \times N$ matrices[2] with elements

$$\begin{aligned} \Lambda_{kl} &= \delta_{kl}\lambda_l, \\ F_{kl} &= e^{-2i\mathcal{Z}_k} G_{kl}, \\ G_{kl} &= \prod_{m \neq l} \frac{\lambda_{k\bar{m}}}{\lambda_{\bar{l}\bar{m}}}, \end{aligned} \tag{3.4.2.32}$$

$k, l = 1, 2, \ldots, N$, and I the identity matrix. Interestingly, the matrix G can further be expressed as $G = H\left(\bar{H}\right)^{-1}$ where H is a matrix with elements $H_{kl} = (\lambda_k)^{l-1}$. The determinant of H is what is known in the literature as the Vandermonde determinant, given by the simple formula

$$\det H = \prod_{k<l} (\lambda_l - \lambda_k). \tag{3.4.2.33}$$

This decomposition in terms of H also reveals the property of G, that $\bar{G} = G^{-1}$. Finally, one may use the property that two square matrices related by a similarity transformation $A' = SAS^{-1}$ obey $\det(I + A') = \det(I + A)$ to regroup the matrix products of (3.4.2.31) in a different manner if desired.

The fact that the exponents in N_2 contain $N + 1$ $\mathcal{Z}_i$ terms complicates the derivation of a determinant formula for Z_2.

3.4.3 Asymptotic behavior

In this section we will examine how our solution behaves for $x \to \pm\infty$ and $t \to \pm\infty$ respectively. Since the dependence of our solutions on the worldsheet coordinates is

[2] The matrix Λ is not to be confused with the Lagrange multiplier of (3.3.0.1).

encoded in the factors $2i\mathcal{Z}_i$, the asymptotic behavior of the N-magnon solution will be determined by their respective real parts.

Using notation similar to [34], we define

$$\begin{aligned} u_l &\equiv i(\mathcal{Z}_l - \bar{\mathcal{Z}}_l) = \kappa_l x - \nu_l t, \\ w_l &\equiv \mathcal{Z}_l + \bar{\mathcal{Z}}_l, \\ v_l &\equiv w_l - t, \end{aligned} \tag{3.4.3.34}$$

with

$$\begin{aligned} \kappa_l &= -i\frac{(\lambda_l - \bar{\lambda}_l)(1+|\lambda_l|^2)}{|1-\lambda_l|^2\,|1+\lambda_l|^2} = \frac{2(1+r_l^2)r_l \sin\frac{p_l}{2}}{1+r_l^4-2r_l^2\cos p_l}, \\ \nu_l &= \frac{-i(\lambda_l^2 - \bar{\lambda}_l^2)}{|1-\lambda_l|^2\,|1+\lambda_l|^2} = \frac{2r_l \sin p_l}{1+r_l^4-2r_l^2\cos p_l}, \end{aligned} \tag{3.4.3.35}$$

and in the second equality we have also employed the usual parametrization $\lambda_l = r_l e^{ip_l/2}$ for the spectral parameters. Additionally, the relations (3.4.3.35) imply

$$2i\mathcal{Z}_l = u_l + iw_l, \quad 2i\bar{\mathcal{Z}}_l = -u_l + iw_l. \tag{3.4.3.36}$$

The parameter range for a single dyonic magnon is $r \in (0,\infty)$ and $p \in [0, 2\pi)$, with $p \sim p + 2\pi$ for any other p. We can use the same restrictions for our parameters r_l, p_l of the N-magnon solution, in which case the κ_l are clearly positive. From the formulas (3.4.1.28)-(3.4.1.29) after we factor out (3.4.1.30), it is then easy to see that the our solution has its boundaries on the equator of S^3 on the Z_1 plane. Namely, for $x \to \pm\infty$ the boundary conditions (3.3.0.5) are satisfied, with $p = \sum_{l=1}^N p_l$ as expected.

Next, we proceed to determine the behavior of the solution for $t \to \pm\infty$ and large magnon separation. Without loss of generality, we can assume that the magnons are ordered such that their velocities $\frac{\nu_k}{\kappa_k}$ obey

$$\frac{\nu_1}{\kappa_1} > \frac{\nu_2}{\kappa_2} > ... > \frac{\nu_N}{\kappa_N}. \tag{3.4.3.37}$$

In order to focus on the k-th magnon, we keep u_k fixed as $t \to \pm\infty$. This means that x should scale as $x = \frac{\nu_k}{\kappa_k}t + \frac{u_k}{\kappa_k}$ and in total the u_l will behave as

$$u_l = \kappa_l \left(\frac{\nu_k}{\kappa_k} - \frac{\nu_l}{\kappa_l}\right) t + \kappa_l \frac{u_k}{\kappa_k}. \tag{3.4.3.38}$$

In particular, the limit $t \to -\infty$ under the aforementioned ordering and scaling implies

$$\begin{aligned} & u_1, u_2, \ldots, u_{k-1} \to +\infty, \\ & u_k \ \text{ finite}, \\ & u_{k+1}, u_{k+2}, \ldots, u_N \to -\infty. \end{aligned} \tag{3.4.3.39}$$

Thus, it is easy to see from (3.4.0.25)-(3.4.0.27) that the terms which dominate in the limit have $\mu_i = 1$ for $i \in \{1, \ldots, k-1, k, \overline{k+1}, \ldots, \overline{N}\}$ and $i \in \{1, \ldots, k-1, \overline{k}, \overline{k+1}, \ldots, \overline{N}\}$ in the case of N_1, D, and $i \in \{1, \ldots, k-1, k, \overline{k}, \overline{k+1}, \ldots, \overline{N}\}$ in the case of N_2, with the rest of the μ's being zero.

Up to common factors that will eventually cancel out (including the divergent terms), we can express the limiting values of N_1, N_2 and D as

$$\begin{aligned} D &\sim \left(f_+ e^{u_k} + f_- e^{-u_k}\right) e^{iw_k}, \\ N_1 &\sim \prod_{l=1}^{k-1} \lambda_l \prod_{l=k+1}^{N} \bar{\lambda}_l \ \left(\lambda_k \, f_+ \, e^{u_k} + \bar{\lambda}_k \, f_- \, e^{-u_k}\right) e^{iw_k}, \\ N_2 &\sim \prod_{l=1}^{k-1} \bar{\lambda}_l \prod_{l=k+1}^{N} \lambda_l \ \lambda_{2\bar{2}} \ h \ e^{2iw_k}, \end{aligned} \tag{3.4.3.40}$$

where $f_+, \ f_-, \ h$ are functions of the spectral parameters λ_i given by

$$\begin{aligned} f_+ &= \prod_{l=1}^{k-1} |\lambda_k - \lambda_l|^2 \prod_{l=k+1}^{N} |\bar{\lambda}_k - \lambda_l|^2, \\ f_- &= \prod_{l=1}^{k-1} |\bar{\lambda}_k - \lambda_l|^2 \prod_{l=k+1}^{N} |\lambda_k - \lambda_l|^2, \\ h &= \prod_{l=1}^{k-1} (\lambda_k - \lambda_l)(\bar{\lambda}_k - \lambda_l) \prod_{l=k+1}^{N} (\lambda_k - \bar{\lambda}_l)(\bar{\lambda}_k - \bar{\lambda}_l). \end{aligned} \tag{3.4.3.41}$$

Noticing that $|h|^2 = f_+ f_-$, and with the help of (3.4.0.24), (3.4.3.40) and (3.4.3.41), we can write the $t \to -\infty$ limit of the N-magnon solution as

$$\begin{aligned} Z_1 &= e^{i\theta_1} e^{it} \left[\cos \frac{p_k}{2} + i \sin \frac{p_k}{2} \tanh(u_k + \delta u_-(k))\right], \\ Z_2 &= e^{i\theta_2} e^{iv_k} \frac{\sin \frac{p_k}{2}}{\cosh\left[u_k + \delta u_-(k)\right]}, \end{aligned} \tag{3.4.3.42}$$

where[3]

$$\delta u_-(k) = \frac{1}{2}\log\frac{f_+}{f_-} = \sum_{l=1}^{k-1}\delta u_{k,l} - \sum_{l=k+1}^{N}\delta u_{k,l} \tag{3.4.3.43}$$

with

$$\delta u_{k,l} = \log\left|\frac{\lambda_k - \lambda_l}{\bar{\lambda}_k - \lambda_l}\right|, \tag{3.4.3.44}$$

and the phase factors $e^{i\theta_1}$, $e^{i\theta_2}$ are independent of x and t. For completeness, we can write them explicitly as

$$\begin{aligned} e^{i\theta_1} &= \prod_{l=1}^{k-1}\left(\frac{\lambda_l}{\bar{\lambda}_l}\right)^{1/2}\prod_{l=k+1}^{N}\left(\frac{\bar{\lambda}_l}{\lambda_l}\right)^{1/2} = \exp\left[\frac{i}{2}\left(\sum_{l=1}^{k-1}p_l - \sum_{l=k+1}^{N}p_l\right)\right], \\ e^{i\theta_2} &= e^{i\zeta}e^{-i\theta_1} = \left(\frac{h}{\bar{h}}\right)^{1/2} e^{-i\theta_1}. \end{aligned} \tag{3.4.3.45}$$

Equation (3.4.3.42) is precisely the single magnon solution on $\mathbb{R}\times S^3$ [16, 34], up to a pure phase and a shift in u_k, which reflects the additional freedom of the solution.

The case $t\to\infty$ can be treated in a similar manner, yielding (3.4.3.42) with

$$\begin{aligned} \delta u_-(k) &\to \delta u_+(k) = -\delta u_-(k), \\ \theta_1 &\to -\theta_1, \\ \zeta &\to -\zeta. \end{aligned} \tag{3.4.3.46}$$

Since k is arbitrary, we have in fact proven that asymptotically our N-magnon solution splits into N single magnon solutions. Each magnon retains its shape after scattering with the rest of the magnons, with the effect of the interaction being encoded only in a relative shift in u_k,

$$\delta u(k) \equiv \delta u_+(k) - \delta u_-(k) = -2\delta u_-(k). \tag{3.4.3.47}$$

Because of (3.4.3.35), the shift in u_k is usually interpreted as a time delay [67],

$$\delta t(k) \equiv \frac{\delta u(k)}{\nu_k} = -\sum_{l=1}^{k-1}\delta t_{k,l} + \sum_{l=k+1}^{N}\delta t_{k,l}, \tag{3.4.3.48}$$

where

$$\delta t_{k,l} \equiv \frac{2\delta u_{k,l}}{\nu_k} = 2i\frac{|1-\lambda_k|^2\,|1+\lambda_k|^2}{\lambda_k^2 - \bar{\lambda}_k^2}\log\left|\frac{\lambda_k - \lambda_l}{\bar{\lambda}_k - \lambda_l}\right| \tag{3.4.3.49}$$

[3]The signs of $\delta u_\pm(k)$ are chosen for compatibility with the most standard method of determining time delays, whereby one performs the ansatz $u_k = -\nu_k\delta t_\pm(k)$ and solves for the position of the magnon's peak, given by $-\nu_k\delta t_\pm(k) + \delta u_\pm(k) = 0$. Note the agreement with the definition (3.4.3.48) below.

is the time delay that occurs because of the interaction of the k-th with the l-th magnon, namely two-magnon scattering.

Hence, our N-magnon solution exhibits the property of factorized scattering, as expected by the integrability of the $\mathbb{R} \times S^3$ σ-model and its classical equivalence to the complex sine-Gordon system. Finally, the dyonic two-magnon time-delay we retrieved in (3.4.3.49) is in complete agreement with [39, 40, 61].

3.5 Construction rules - Examples

To help clarify the meaning of the formulas (3.4.0.24)-(3.4.0.27) and (3.4.1.28)-(3.4.1.29), we reduce them to a simple set of rules for the construction of $N_1,\ N_2,\ D$. These rules may also facilitate computer code for generating N-magnon solutions.

The N-magnon solution can be written as

$$\begin{aligned} Z_1 &= \frac{e^{it}}{\prod_{l=1}^{N}|\lambda_l|}\frac{N_1}{D}\,, \\ Z_2 &= -i\frac{e^{-it}}{\prod_{l=1}^{N}|\lambda_l|}\frac{N_2}{D}. \end{aligned} \tag{3.5.0.50}$$

and it contains N spectral parameters λ_i along with their conjugates $\bar{\lambda}_i$ that we can arrange as the set $A = \{\lambda_1,\ \bar{\lambda}_1,\ \lambda_2, \ldots,\ \lambda_N,\ \bar{\lambda}_N\}$.

In order to write the denominator D we take all the possible subsets of N numbers of the set A. There are $(2N)!/N!^2$ such subsets. For each subset we form a product and then D is the sum of all those products. Let us see how to form the product for a specific subset B. The product contains

a) an exponential with exponent $2i\sum_i \mathcal{Z}(\lambda_i) \equiv 2i\sum_i \mathcal{Z}_i$, where λ_i are all the λ's that belong to B,

b) all the possible differences $\lambda_i - \lambda_j,\ i<j$, where $\lambda_i,\ \lambda_j$ all belong to the subset B and

c) finally all the possible differences $\lambda_i - \lambda_j,\ i<j$, where $\lambda_i,\ \lambda_j$ all belong to the complement subset of B.

The rules for N_1 are the same as D except that now the product contains in addition all the λ's that belong to the subset B.

The rules for N_2 are the same as the rules for N_1, but now all the subsets B should have $N+1$ elements instead of N and the product contains all the λ's that belong to the complement subset of B instead of the B itself.

As an example let us write $N_1,\ N_2,\ D$ in the case of 1, 2 and 3-magnons. For 1-magnon we have [16]

$$
\begin{aligned}
D &= e^{2i\mathcal{Z}_1}+e^{2i\bar{\mathcal{Z}}_1},\\
N_1 &= \lambda_1 e^{2i\mathcal{Z}_1}+\bar{\lambda}_1 e^{2i\bar{\mathcal{Z}}_1},\\
N_2 &= \lambda_{1\bar{1}} e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1)}.
\end{aligned}
\tag{3.5.0.51}
$$

For 2-magnons we have [34]

$$
\begin{aligned}
D &= \lambda_{1\bar{1}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1)}+\lambda_{12}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2)}+\lambda_{1\bar{2}}\lambda_{\bar{1}2}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2)}\\
&\quad+\lambda_{\bar{1}2}\lambda_{1\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2)}+\lambda_{\bar{1}\bar{2}}\lambda_{12}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2)}+\lambda_{2\bar{2}}\lambda_{1\bar{1}}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2)},\\
N_1 &= \lambda_1\bar{\lambda}_1\lambda_{1\bar{1}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1)}+\lambda_1\lambda_2\lambda_{12}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2)}+\lambda_1\bar{\lambda}_2\lambda_{1\bar{2}}\lambda_{\bar{1}2}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2)}\\
&\quad+\bar{\lambda}_1\lambda_2\lambda_{\bar{1}2}\lambda_{1\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2)}+\bar{\lambda}_1\bar{\lambda}_2\lambda_{\bar{1}\bar{2}}\lambda_{12}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2)}+\lambda_2\bar{\lambda}_2\lambda_{2\bar{2}}\lambda_{1\bar{1}}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2)},\\
N_2 &= \bar{\lambda}_2\lambda_{1\bar{1}}\lambda_{12}\lambda_{\bar{1}2}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2)}+\lambda_2\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2)}\\
&\quad+\bar{\lambda}_1\lambda_{12}\lambda_{1\bar{2}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)}+\lambda_1\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)}.
\end{aligned}
\tag{3.5.0.52}
$$

For 3-magnons we have

$$
\begin{aligned}
D = {}&\lambda_{1\bar{1}}\lambda_{12}\lambda_{\bar{1}2}\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2)}+\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{\bar{1}\bar{2}}\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2)}\\
&+\lambda_{1\bar{1}}\lambda_{13}\lambda_{\bar{1}3}\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_3)}+\lambda_{1\bar{1}}\lambda_{1\bar{3}}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_3)}\\
&+\lambda_{12}\lambda_{1\bar{2}}\lambda_{2\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)}+\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\mathcal{Z}_3)}\\
&+\lambda_{12}\lambda_{1\bar{3}}\lambda_{2\bar{3}}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{2}3}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_3)}+\lambda_{1\bar{2}}\lambda_{13}\lambda_{\bar{2}3}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)}\\
&+\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{23}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)}+\lambda_{13}\lambda_{1\bar{3}}\lambda_{3\bar{3}}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)}\\
&+\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}\lambda_{13}\lambda_{1\bar{3}}\lambda_{3\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)}+\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{23}\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\mathcal{Z}_3)}\\
&+\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{3}}\lambda_{1\bar{2}}\lambda_{13}\lambda_{\bar{2}3}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_3)}+\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{2}3}\lambda_{12}\lambda_{1\bar{3}}\lambda_{2\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)}\\
&+\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{12}\lambda_{13}\lambda_{23}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)}+\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{3\bar{3}}\lambda_{12}\lambda_{1\bar{2}}\lambda_{2\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)}\\
&+\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}\lambda_{1\bar{1}}\lambda_{1\bar{3}}\lambda_{\bar{1}\bar{3}}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)}+\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{1\bar{1}}\lambda_{13}\lambda_{\bar{1}3}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)}\\
&+\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)}+\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{1}}\lambda_{12}\lambda_{\bar{1}2}e^{2i(\bar{\mathcal{Z}}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)},
\end{aligned}
$$

$$
\begin{aligned}
N_1 = {} & \lambda_1\bar{\lambda}_1\lambda_2\lambda_{1\bar{1}}\lambda_{12}\lambda_{\bar{1}2}\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2)} + \lambda_1\bar{\lambda}_1\bar{\lambda}_2\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{\bar{1}\bar{2}}\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2)} \\
& + \lambda_1\bar{\lambda}_1\lambda_3\lambda_{1\bar{1}}\lambda_{13}\lambda_{\bar{1}3}\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_3)} + \lambda_1\bar{\lambda}_1\bar{\lambda}_3\lambda_{1\bar{1}}\lambda_{1\bar{3}}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_3)} \\
& + \lambda_1\lambda_2\bar{\lambda}_2\lambda_{12}\lambda_{1\bar{2}}\lambda_{2\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)} + \lambda_1\lambda_2\lambda_3\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\mathcal{Z}_3)} \\
& + \lambda_1\lambda_2\bar{\lambda}_3\lambda_{12}\lambda_{1\bar{3}}\lambda_{2\bar{3}}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{2}3}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_3)} + \lambda_1\bar{\lambda}_2\lambda_3\lambda_{1\bar{2}}\lambda_{13}\lambda_{\bar{2}3}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} \\
& + \lambda_1\bar{\lambda}_2\bar{\lambda}_3\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{23}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} + \lambda_1\lambda_3\bar{\lambda}_3\lambda_{13}\lambda_{1\bar{3}}\lambda_{3\bar{3}}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} \\
& + \bar{\lambda}_1\lambda_2\bar{\lambda}_2\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}\lambda_{13}\lambda_{1\bar{3}}\lambda_{3\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)} + \bar{\lambda}_1\lambda_2\lambda_3\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{23}\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{\bar{2}\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\mathcal{Z}_3)} \\
& + \bar{\lambda}_1\lambda_2\bar{\lambda}_3\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{3}}\lambda_{1\bar{2}}\lambda_{13}\lambda_{\bar{2}3}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_3)} + \bar{\lambda}_1\bar{\lambda}_2\lambda_3\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{2}3}\lambda_{12}\lambda_{1\bar{3}}\lambda_{2\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} \\
& + \bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{12}\lambda_{13}\lambda_{23}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} + \bar{\lambda}_1\lambda_3\bar{\lambda}_3\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{3\bar{3}}\lambda_{12}\lambda_{1\bar{2}}\lambda_{2\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} \\
& + \lambda_2\bar{\lambda}_2\lambda_3\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}\lambda_{1\bar{1}}\lambda_{1\bar{3}}\lambda_{\bar{1}\bar{3}}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} + \lambda_2\bar{\lambda}_2\bar{\lambda}_3\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{1\bar{1}}\lambda_{13}\lambda_{\bar{1}3}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} \\
& + \lambda_2\lambda_3\bar{\lambda}_3\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} + \bar{\lambda}_2\lambda_3\bar{\lambda}_3\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{1}}\lambda_{12}\lambda_{\bar{1}2}e^{2i(\bar{\mathcal{Z}}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)},
\end{aligned}
$$

$$
\begin{aligned}
N_2 = {} & \lambda_3\bar{\lambda}_3\lambda_{1\bar{1}}\lambda_{12}\lambda_{1\bar{2}}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{2\bar{2}}\lambda_{3\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2)} + \bar{\lambda}_2\bar{\lambda}_3\lambda_{1\bar{1}}\lambda_{12}\lambda_{13}\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{23}\lambda_{\bar{2}\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\mathcal{Z}_3)} \\
& + \bar{\lambda}_2\lambda_3\lambda_{1\bar{1}}\lambda_{12}\lambda_{1\bar{3}}\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{3}}\lambda_{\bar{2}3}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_3)} + \lambda_2\bar{\lambda}_3\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{13}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{2}3}\lambda_{2\bar{3}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} \\
& + \lambda_2\lambda_3\lambda_{1\bar{1}}\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{23}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} + \lambda_2\bar{\lambda}_2\lambda_{1\bar{1}}\lambda_{13}\lambda_{1\bar{3}}\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{3\bar{3}}\lambda_{2\bar{2}}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_1+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} \\
& + \bar{\lambda}_1\bar{\lambda}_3\lambda_{12}\lambda_{1\bar{2}}\lambda_{13}\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}\lambda_{\bar{1}\bar{3}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} + \bar{\lambda}_1\lambda_3\lambda_{12}\lambda_{1\bar{2}}\lambda_{1\bar{3}}\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{\bar{1}3}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} \\
& + \bar{\lambda}_1\bar{\lambda}_2\lambda_{12}\lambda_{13}\lambda_{1\bar{3}}\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}\lambda_{\bar{1}\bar{2}}e^{2i(\mathcal{Z}_1+\mathcal{Z}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} + \bar{\lambda}_1\lambda_2\lambda_{1\bar{2}}\lambda_{13}\lambda_{1\bar{3}}\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}\lambda_{\bar{1}2}e^{2i(\mathcal{Z}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} \\
& + \lambda_1\bar{\lambda}_3\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{2\bar{2}}\lambda_{23}\lambda_{\bar{2}3}\lambda_{1\bar{3}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\mathcal{Z}_3)} + \lambda_1\lambda_3\lambda_{\bar{1}2}\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}\bar{3}}\lambda_{2\bar{2}}\lambda_{2\bar{3}}\lambda_{\bar{2}\bar{3}}\lambda_{13}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\bar{\mathcal{Z}}_3)} \\
& + \lambda_1\bar{\lambda}_2\lambda_{\bar{1}2}\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{23}\lambda_{2\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{2}}e^{2i(\bar{\mathcal{Z}}_1+\mathcal{Z}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} + \lambda_1\lambda_2\lambda_{\bar{1}\bar{2}}\lambda_{\bar{1}3}\lambda_{\bar{1}\bar{3}}\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}\lambda_{12}e^{2i(\bar{\mathcal{Z}}_1+\bar{\mathcal{Z}}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)} \\
& + \lambda_1\bar{\lambda}_1\lambda_{2\bar{2}}\lambda_{23}\lambda_{2\bar{3}}\lambda_{\bar{2}3}\lambda_{\bar{2}\bar{3}}\lambda_{3\bar{3}}\lambda_{1\bar{1}}e^{2i(\mathcal{Z}_2+\bar{\mathcal{Z}}_2+\mathcal{Z}_3+\bar{\mathcal{Z}}_3)},
\end{aligned}
\tag{3.5.0.53}
$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$, and $\mathcal{Z}_i = z/(\lambda_i - 1) + \bar{z}/(\lambda_i + 1)$.

Chapter 4

Dressing the Giant Gluon

4.1 Abstract

We demonstrate the applicability of the dressing method to the problem of constructing new classical solutions for Euclidean worldsheets in anti-de Sitter space. The motivation stems from the work of Alday and Maldacena, who studied gluon scattering amplitudes at strong coupling using a generalization of a particular worldsheet found by Kruczenski whose edge traces a path composed of light-like segments on the boundary of AdS. We dress this 'giant gluon' to find new solutions in AdS_3 and AdS_5 whose edges trace out more complicated, timelike curves on the boundary. These solutions may be used to calculate certain Wilson loops via AdS/CFT.

4.2 Introduction

Classical string solutions play an important role in exploring the AdS/CFT correspondence (see [47] and [48, 49, 50] for reviews). Generally speaking such solutions fall into two categories. On the one hand there are closed string energy eigenstates in AdS, which are in correspondence with gauge invariant operators of definite scaling dimension in the dual gauge theory. On the other hand we can also consider open strings which end along some curve on the boundary of AdS, corresponding to Wilson loops [68, 69].

An important example of the former is the so-called 'giant magnon' of Hofman and Maldacena [9], which is dual to a single elementary excitation in the gauge theory picture. More general states containing arbitrary numbers of bound or scattering states of magnons correspond to more general classical string solutions [15, 16, 34]. These solutions can be constructed algebraically using the dressing method [34, 61], a well-known technique [28, 44] for generating solutions of classically integrable equations.

Here we turn our attention to the latter, demonstrating the applicability of the dressing method to the problem of constructing certain new Euclidean minimal area surfaces in anti-de Sitter space[1]. To apply the dressing method it is necessary to choose some solution of the classical equations of motion to use as the 'vacuum', which is then 'dressed' to build more general solutions. For the giant magnon system considered in [34, 61] it was natural to choose as vacuum the solution describing a pointlike string moving at the speed of light around the equator of the S^5, since this state corresponds to the natural vacuum in the spin chain picture.

For the present problem we choose as vacuum a particular solution, shown in Fig. (4.1), originally used by Kruczenski [70] to study the cusp anomalous dimension via AdS/CFT. It is the minimal area surface which meets the boundary of global AdS_3 along four intersecting light-like lines. This solution was generalized, and given a new interpretation, by Alday and Maldacena [17], who gave a prescription for computing planar gluon scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills at strong coupling using the AdS/CFT correspondence and found perfect agreement with the structure predicted on the basis of previously conjectured iteration relations for perturbative multiloop gluon amplitudes [71, 18, 72, 73, 74].

The Alday-Maldacena prescription is (classically) computationally equivalent to the problem of evaluating a Wilson loop composed of light-like segments. According to the AdS/CFT dictionary, such a Wilson loop is computed by evaluating the area of the surface in Fig. (4.1). The interpretation of this surface in terms of a gluon scattering process suggests calling this kind of solution a 'giant gluon.'

We dress the giant gluon to find new minimal area surfaces in AdS_3 and AdS_5 whose edges trace out more complicated, timelike curves on the boundary of AdS. It is not clear whether these new solutions have any interpretation as a scattering process of the type studied in [17], although they do have straightforward interpretations in terms of Wilson loops. However, when calculating a Wilson loop one usually first specifies a curve on the boundary of AdS and then finds the minimal area surface

[1] The dressing method has also been used to construct Minkowskian worldsheets in de Sitter space [54]

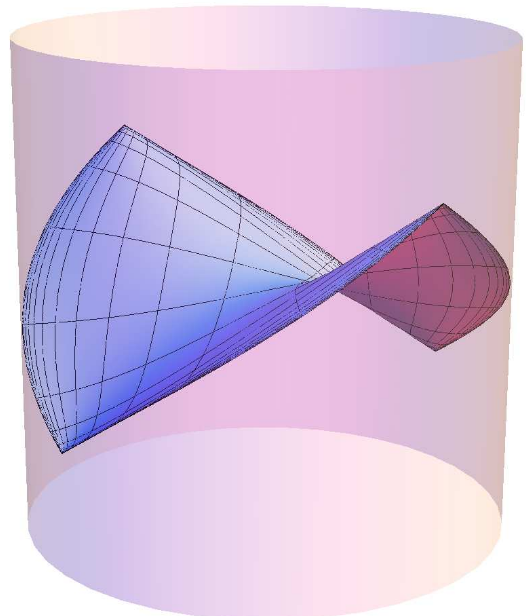

Figure 4.1: The 'giant gluon' solution (4.4.0.20) in AdS_3 global coordinates. The gluons follow the four light-like segments on the boundary of AdS_3 where the worldsheet ends.

bounding that curve. In contrast, the dressing method provides the minimal area surface without telling us the curve that it spans, i.e. without telling us which Wilson loop it is calculating. That information must be read off directly by analyzing the solution to see where it reaches the boundary of AdS, a procedure that we will see is rather nontrivial.

The outline of this work is as follows. In section 3 we demonstrate the applicability of the dressing method, focusing on the AdS_3 case which is simpler because there the problem can be mapped into the $SU(1,1)$ principal chiral model. In section 4 we discuss the dressed giant gluon in AdS_3, display explicit formulas for a special case of the solution, and analyze in detail the edge of the worldsheet on the boundary of AdS_3. In section 5 we turn to the more complicated construction for AdS_5 solutions using the $SU(2,2)/SO(4,1)$ coset model, and present some examples.

The main goal of this work is to demonstrate the applicability of the dressing method. Although we consider a few examples, they amount to only a small subset of the simplest possible solutions. It would be very interesting to more fully explore the parameter space of solutions that can be obtained. It would also be interesting to evaluate the (regulated) areas of these solutions, thereby calculating the corresponding Wilson loops in gauge theory. The giant gluon shown in Fig. (4.1) can actually be related [75], by analytic continuation and a conformal transformation, to a closed string energy eigenstate (a limit of the GKP spinning string [47]). It would be interesting to see whether it is possible to relate more general Euclidean worldsheets of the type we consider to various closed string states.

4.3 AdS Dressing Method

The dressing method [28, 29] is a general technique for constructing solutions of classically integrable equations. As we review shortly, at the heart of the method lies the ability to transform nonlinear equations of motion into a linear system for an auxiliary field. Here we apply this very general method to the specific problem of constructing minimal area Euclidean worldsheets in anti de-Sitter space. Initially we restrict our attention to AdS_3, where the problem relates to the $SU(1,1)$ principal chiral model, deferring the slightly more complicated AdS_5 case to section 4. Many of the equations in this section are similar to those appearing in [34, 61], which the reader may consult for further details. The two most significant differences compared to the $SU(2)$ principal chiral model considered in [34] are that we use complex coordinates $z, \bar{z}$ on the worldsheet, which is now Euclidean, and that the indefinite $SU(1,1)$ metric significantly changes the behavior of the solutions compared to $SU(2)$.

We parameterize AdS_d with $d+1$ embedding coordinates $\vec{Y}$ subject to the constraint

$$\vec{Y} \cdot \vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = -1. \tag{4.3.0.1}$$

Minimal area worldsheets are given by solutions to the conformal gauge equations of motion

$$\partial\bar{\partial}\vec{Y} - \vec{Y}\left(\partial\vec{Y} \cdot \bar{\partial}\vec{Y}\right) = 0 \tag{4.3.0.2}$$

subject to the Virasoro constraints

$$\partial\vec{Y} \cdot \partial\vec{Y} = \bar{\partial}\vec{Y} \cdot \bar{\partial}\vec{Y} = 0. \tag{4.3.0.3}$$

Here and throughout this work we use complex coordinates

$$z = \frac{1}{2}(u_1 + iu_2), \qquad \bar{z} = \frac{1}{2}(u_1 - iu_2), \tag{4.3.0.4}$$

with

$$\partial = \partial_1 - i\partial_2, \qquad \bar{\partial} = \partial_1 + i\partial_2. \tag{4.3.0.5}$$

Our first step is to recast the system (4.3.0.2), (4.3.0.3) into the form of a principal chiral model for a matrix-valued field g satisfying the equation of motion

$$\bar{\partial} A + \partial \bar{A} = 0 \tag{4.3.0.6}$$

in terms of the currents

$$A = i\partial g\, g^{-1}, \qquad \bar{A} = i\bar{\partial} g\, g^{-1}. \tag{4.3.0.7}$$

Note that the relation

$$\bar{\partial} A - \partial \bar{A} - i[A, \bar{A}] = 0 \tag{4.3.0.8}$$

follows automatically from (4.3.0.7).

To see how this is done let us consider for simplicity first the AdS_3 case. Here we use the coordinates $\vec{Y}$ to parameterize an element g of $SU(1,1)$ according to

$$g = \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & \bar{Z}_1 \end{pmatrix}, \qquad Z_1 = Y_{-1} + iY_0, \qquad Z_2 = Y_1 + iY_2, \tag{4.3.0.9}$$

which satisfies

$$g^\dagger M g = M, \qquad M = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.3.0.10}$$

and

$$\det g = -\vec{Y} \cdot \vec{Y} = +1. \tag{4.3.0.11}$$

It is easy to check that the systems (4.3.0.2), (4.3.0.3) and (4.3.0.6), (4.3.0.8) are equivalent to each other under this change of variables.

Next we transform the nonlinear second-order system (4.3.0.6), (4.3.0.7) for $g(z, \bar{z})$ into a linear, first-order system for an auxiliary field $\Psi(z, \bar{z}, \lambda)$ at the expense of introducing a new complex parameter λ called the spectral parameter. Specifically, the two equations (4.3.0.6), (4.3.0.7) are equivalent to

$$i\partial\Psi = \frac{A\Psi}{1 + i\lambda}, \qquad i\bar{\partial}\Psi = \frac{\bar{A}\Psi}{1 - i\lambda}. \tag{4.3.0.12}$$

For later convenience we have rescaled our definition of λ in this equation by a factor of i compared to the conventions of [34, 61].

To apply the dressing method we begin with any known solution g (which we refer to as the 'vacuum' for the dressing method, though we emphasize that any solution may be chosen as the vacuum) and then solve the linear system (4.3.0.12) to find $\Psi(\lambda)$ subject to the initial condition

$$\Psi(\lambda = 0) = g. \tag{4.3.0.13}$$

In addition we impose on $\Psi(\lambda)$ the $SU(1,1)$ conditions

$$\Psi^\dagger(\bar{\lambda})M\Psi(\lambda) = M, \qquad \det \Psi(0) = 1. \tag{4.3.0.14}$$

The purpose of the factor of i mentioned below (4.3.0.12) is to avoid the need to take $-\bar{\lambda}$ instead of $\bar{\lambda}$ in the first relation here.

Then we make a 'gauge transformation' of the form

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda). \tag{4.3.0.15}$$

If $\chi(\lambda)$ were independent of z and $\bar{z}$ this would be an uninteresting $SU(1,1)$ gauge transformation. Instead we want $\chi(\lambda)$ to depend on z and $\bar{z}$ but in such a way that $\Psi'(\lambda)$ continues to satisfy (4.3.0.12) and hence $\Psi'(0)$ provides a new solution to (4.3.0.6), (4.3.0.8). For AdS_3 it is not hard to show that this is accomplished by taking $\chi(\lambda)$ to have the form

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \tag{4.3.0.16}$$

where λ_1 is an arbitrary complex parameter and P is a projection operator onto any vector of the form $v_1 \equiv \Psi(\bar{\lambda}_1)v$ for any constant vector v. Concretely, P is therefore given by

$$P = \frac{v_1 v_1^\dagger M}{v_1^\dagger M v_1}. \tag{4.3.0.17}$$

As in [34] there is a minor remaining detail that (4.3.0.16) has

$$\det \chi(\lambda) = \bar{\lambda}_1/\lambda_1 \tag{4.3.0.18}$$

so in order for g' to lie in $SU(1,1)$ rather than $U(1,1)$ we should rescale g' by the constant phase factor $\sqrt{\lambda_1/\bar{\lambda}_1}$ to ensure that it has unit determinant. To summarize, the desired dressed solution is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1} P \right] \Psi(0). \tag{4.3.0.19}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution may then be read off from g' using the parameterization (4.3.0.9). The resulting solution is characterized by the complex parameter λ_1 and the choice of the constant vector v.

4.4 AdS_3 Solutions

In this section we obtain new solutions for worldsheets in AdS_3 via the dressing method, taking as 'vacuum' the giant gluon solution [70, 17]

$$\vec{Y} = \begin{pmatrix} Y_{-1} \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cosh u_1 \cosh u_2 \\ \sinh u_1 \sinh u_2 \\ \sinh u_1 \cosh u_2 \\ \cosh u_1 \sinh u_2 \end{pmatrix}. \tag{4.4.0.20}$$

Using the AdS_3 parameterization (4.3.0.9) we find from (4.3.0.7) that

$$\begin{aligned} A &= 2\begin{pmatrix} -\cosh u_2 \sinh u_2 & i\cosh^2 u_2 \\ i\sinh^2 u_2 & +\cosh u_2 \sinh u_2 \end{pmatrix}, \\ \bar{A} &= 2\begin{pmatrix} -\cosh u_2 \sinh u_2 & i\sinh^2 u_2 \\ i\cosh^2 u_2 & +\cosh u_2 \sinh u_2 \end{pmatrix}. \end{aligned} \tag{4.4.0.21}$$

Then a solution to the linear system (4.3.0.12) for $\Psi(\lambda)$ is[2]

$$\Psi(\lambda) = \begin{pmatrix} m_- \operatorname{ch} Z \operatorname{ch} u_2 + i m_+ \operatorname{sh} Z \operatorname{sh} u_2 & m_- \operatorname{sh} Z \operatorname{ch} u_2 + i m_+ \operatorname{ch} Z \operatorname{sh} u_2 \\ m_+ \operatorname{sh} Z \operatorname{ch} u_2 - i m_- \operatorname{ch} Z \operatorname{sh} u_2 & m_+ \operatorname{ch} Z \operatorname{ch} u_2 - i m_- \operatorname{sh} Z \operatorname{sh} u_2 \end{pmatrix} \tag{4.4.0.22}$$

where

$$m_+ = 1/m_- = \left(\frac{1+i\lambda}{1-i\lambda}\right)^{1/4}, \qquad Z = m_-^2 z + m_+^2 \bar{z}. \tag{4.4.0.23}$$

The solution (4.4.0.22) has been chosen to satisfy the desired constraints (4.3.0.14) as well as the initial condition

$$\Psi(0) = \begin{pmatrix} \cosh u_1 \cosh u_2 + i \sinh u_1 \sinh u_2 & \sinh u_1 \cosh u_2 + i\cosh u_1 \sinh u_2 \\ \sinh u_1 \cosh u_2 - i\cosh u_1 \sinh u_2 & \cosh u_1 \cosh u_2 - i \sinh u_1 \sinh u_2 \end{pmatrix}, \tag{4.4.0.24}$$

correctly reproducing the giant gluon solution (4.4.0.20) embedded into $SU(1,1)$ according to (4.3.0.9). The dressed solution g' is then given by (4.3.0.19).

4.4.1 A special case

Since the general solution is rather complicated, we present here an explicit formula for the dressed solution for the particular choice of initial vector $v = \begin{pmatrix} 1 & i \end{pmatrix}$, with λ_1

[2]We will occasionally use sh, ch instead of sinh, cosh to compactify otherwise lengthy formulas.

arbitrary. We find that the dressed $SU(1,1)$ principal chiral field takes the form

$$g' = \begin{pmatrix} Z_1' & Z_2' \\ \bar{Z}_2' & \bar{Z}_1' \end{pmatrix} \tag{4.4.1.25}$$

where

$$Z_1' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_1}{D}, \qquad Z_2' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_2}{D} \tag{4.4.1.26}$$

in terms of the numerator factors

$$\begin{aligned}
\vec{N}_1 &= \begin{pmatrix} -(\bar{\lambda}_1|m|^2-\lambda_1)\cosh(Z+\bar{Z}) + i(\bar{\lambda}_1|m|^2+\lambda_1)\sinh(Z-\bar{Z}) \\ -(\lambda_1|m|^2+\bar{\lambda}_1)\sinh(Z-\bar{Z}) - i(\lambda_1|m|^2-\bar{\lambda}_1)\cosh(Z+\bar{Z}) \\ (\lambda_1-\bar{\lambda}_1)\bar{m}(\sinh(Z+\bar{Z}) - i\cosh(Z-\bar{Z})) \\ (\lambda_1-\bar{\lambda}_1)m(\cosh(Z-\bar{Z}) - i\sinh(Z+\bar{Z})) \end{pmatrix}, \\
\vec{N}_2 &= \begin{pmatrix} -(\lambda_1-\bar{\lambda}_1)\bar{m}(\sinh(Z+\bar{Z}) - i\cosh(Z-\bar{Z})) \\ -(\lambda_1-\bar{\lambda}_1)m(\cosh(Z-\bar{Z}) - i\sinh(Z+\bar{Z})) \\ +(\bar{\lambda}_1|m|^2-\lambda_1)\cosh(Z+\bar{Z}) - i(\bar{\lambda}_1|m|^2+\lambda_1)\sinh(Z-\bar{Z}) \\ +(\lambda_1|m|^2+\bar{\lambda}_1)\sinh(Z-\bar{Z}) + i(\lambda_1|m|^2-\bar{\lambda}_1)\cosh(Z+\bar{Z}) \end{pmatrix},
\end{aligned} \tag{4.4.1.27}$$

$\vec{Y}$ given in (4.4.0.20), and the denominator

$$D = (|m|^2-1)\cosh(Z+\bar{Z}) - i(|m|^2+1)\sinh(Z-\bar{Z}). \tag{4.4.1.28}$$

In these expressions

$$m = \left(\frac{1+i\lambda_1}{1-i\lambda_1}\right)^{1/2}, \qquad \bar{m} = \left(\frac{1-i\bar{\lambda}_1}{1+i\bar{\lambda}_1}\right)^{1/2}, \tag{4.4.1.29}$$

and

$$Z = z/m + m\bar{z}, \qquad \bar{Z} = \bar{z}/\bar{m} + \bar{m}z. \tag{4.4.1.30}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution are easily read off from (4.4.1.25) using (4.3.0.9). In Fig. (4.2) we plot a representative example of the solution (4.4.1.26). However before one can make sense of the plot we must understand the behavior of (4.4.1.26) at the boundary of AdS, which we address in the next subsection.

4.4.2 In search of the Wilson loop

Minimal area worldsheets in AdS_5 are related to Wilson loops in the dual gauge theory [68, 69]. According to the AdS/CFT dictionary, in order to calculate the

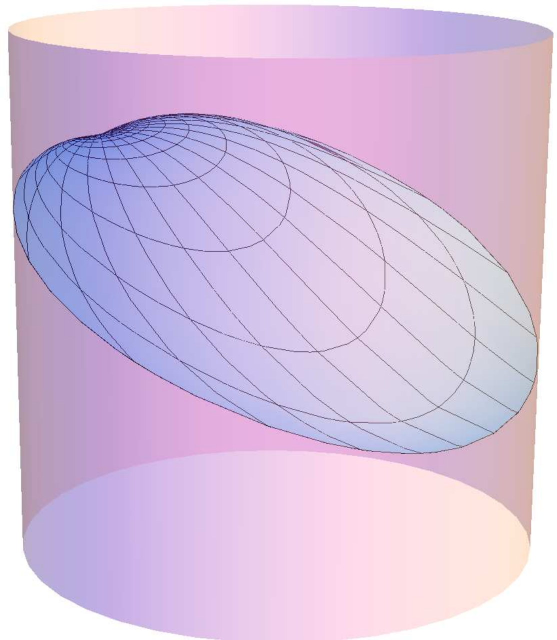

Figure 4.2: An example of a surface described by the solution (4.4.1.26) for the particular choice $\lambda_1 = 1/2 + i/3$.

expectation value of the Wilson loop for some closed path $\mathcal{C}$ on the boundary of AdS we should first find the minimal area surface (or surfaces) in AdS which spans that curve and then calculate e^{-A} where A is the (regulated) area of the minimal surface.

The solutions we have obtained by the dressing method turn this procedure on its head. In the previous subsection we displayed an explicit example of such a solution, which indeed describes a minimal area Euclidean worldsheet in AdS_3, but it is not immediately clear what the corresponding curve $\mathcal{C}$ is whose Wilson loop the solution computes. In order to answer this question we must look at (4.4.1.26) and find the locus $\mathcal{C}$ where the worldsheet reaches the boundary of AdS_3—this will tell us which Wilson loop we are computing.

In global AdS coordinates, the familiar radial coordinate ρ is related to the coordinates appearing in (4.3.0.9) according to

$$\cosh^2 \rho = |Z_1|^2, \qquad \sinh^2 \rho = |Z_2|^2. \tag{4.4.2.31}$$

Hence the boundary of AdS_3 lies at $Z_i = \infty$. Before proceeding with our complicated dressed solution let us pause to note that the giant gluon solution (4.4.0.20) reaches the boundary of AdS_3 precisely when $|u_1| \to \infty$ or $|u_2| \to \infty$. Moreover the four 'edges' of the worldsheet, at $u_1 \to +\infty$, $u_1 \to -\infty$, $u_2 \to +\infty$ and $u_2 \to -\infty$, sit on four separate null lines on the boundary of AdS_3 which intersect each other at four cusps [70, 17] to form the closed curve $\mathcal{C}$.

Looking at the dressed solution (4.4.1.26) we see a feature which makes it significantly more complicated to understand than the giant gluon. The presence of the nontrivial denominator factor

$$D = (|m|^2 - 1)\cosh(Z + \bar{Z}) - i(|m|^2 + 1)\sinh(Z - \bar{Z}) \tag{4.4.2.32}$$

in (4.4.1.26) means that the solution reaches the boundary of AdS_3 any time $D = 0$, which occurs at finite (rather than infinite) values of the worldsheet coordinates $z, \bar{z}$. In fact since D is periodic in Z (with period πi), the solution reaches the boundary of AdS_3 infinitely many times as we allow z (and hence Z) to vary across the complex plane. It is important to note that while D is periodic, the full solution is not.

If we define real variables U_i according to

$$Z = (U_1 + iU_2)/2, \qquad \bar{Z} = (U_1 - iU_2)/2 \tag{4.4.2.33}$$

then the locus $\tilde{C}$ of points on the worldsheet where the solution reaches the boundary of AdS_3 is

$$D = (|m|^2 - 1)\cosh U_1 + (|m|^2 + 1)\sin U_2 = 0. \tag{4.4.2.34}$$

This equation describes an infinite array of oval-shaped curves $\tilde{\mathcal{C}}_j$ periodically distributed along the U_2 axis and centered at $(U_1, U_2) = (0, 2\pi j + \pi/2)$. Note that the curves $\tilde{\mathcal{C}}_j$ in the worldsheet coordinates are not to be confused with their images $\mathcal{C}_j$ on the boundary of AdS_3 under the map (4.4.1.26). In particular the $\tilde{\mathcal{C}}_j$ are unphysical artifacts of the particular coordinate system we happen to be using on the worldsheet—only the curves $\mathcal{C}_j$ on the boundary are physically meaningful.

To summarize, we find that the solution (4.4.1.26) actually describes not one but infinitely many different minimal area surfaces in AdS_3, each spanning a different curve $\mathcal{C}_j$ on the boundary. In order to isolate any given worldsheet j we restrict the worldsheet coordinates U_1, U_2 to range over the interior of the curve $\tilde{\mathcal{C}}_j$. In particular, in order to find the area of the j-th worldsheet, and hence calculate the expectation value of the Wilson loop corresponding to the curve $\mathcal{C}_j$, one should integrate the induced volume element on the worldsheet only over the region $\tilde{\mathcal{C}}_j$. It would be interesting to pursue this calculation further, although we will not do so here.

4.4.3 A very special case

In the previous subsection we explained that the minimal area surfaces generated by the dressing method actually calculate infinitely many different Wilson loops. In general the solutions are sufficiently complicated that we find it necessary to analyze them numerically (one example is shown in Fig. (4.3)), but it is satisfying to analyze in detail one particularly simple example based on the solution (4.4.1.26) which itself is already a special case of the most general dressed solution.

Therefore we look now at the case $\lambda_1 = i$. Since the solution naively looks singular at this value we will carefully take the limit as $\lambda_1 \to i$ from inside the unit circle. To this end we consider

$$\lambda_1 = ia, \qquad m = \sqrt{\frac{1-a}{1+a}} \tag{4.4.3.35}$$

in the limit $a \to 1$. In this limit the equation for the boundary reduces to

$$\cosh U_1 = \sin U_2 \tag{4.4.3.36}$$

whose solutions are just points in the (U_1, U_2) plane.

In order to isolate what is going on near the point $(0, \pi/2)$ (for example) we should rescale the worldsheet coordinates by defining new coordinates x, y according to

$$U_1 = 2mx, \qquad U_2 = \frac{\pi}{2} + 2my \tag{4.4.3.37}$$

Then in the limit $a \to 1$ the equation becomes

$$0 = D = (1 - x^2 - y^2)(1-a) + \mathcal{O}(1-a)^2 \tag{4.4.3.38}$$

So now the edge of the worldsheet is the circle $x^2 + y^2 = 1$ in the (x, y) plane. Using (4.4.2.33) and (4.4.1.30) gives

$$u_1 = \frac{1}{2}x(1-a), \qquad u_2 = \frac{\pi}{4a}\sqrt{1-a^2} + \frac{1}{2a}(1-a)y. \tag{4.4.3.39}$$

Plugging these values and (4.4.3.35) into the solution (4.4.1.26) we can then safely take $a \to 1$, obtaining the surface

$$Z_1 = -i\frac{1+x^2+y^2}{1-x^2-y^2}, \qquad Z_2 = \frac{2ix-2y}{1-x^2-y^2}. \tag{4.4.3.40}$$

Switching now to Poincaré coordinates (R, T, X) according to the usual embedding

$$Z_1 = \frac{1}{2}\left(\frac{1}{R} + \frac{R^2 - T^2 + X^2}{R}\right) + i\frac{T}{R}, \qquad Z_2 = \frac{X}{R} + \frac{i}{2}\left(\frac{1}{R} - \frac{R^2 - T^2 + X^2}{R}\right) \tag{4.4.3.41}$$

we find

$$R = \frac{1 - x^2 - y^2}{x}, \qquad T = -\frac{1 + x^2 + y^2}{x}, \qquad X = -\frac{y}{x}. \tag{4.4.3.42}$$

Finally we note that this surface in AdS_3 satisfies

$$-T^2 + X^2 = -1 - \frac{(1 - x^2 - y^2)^2}{4x^2}. \tag{4.4.3.43}$$

At the edge of the worldsheet the second term on the right-hand side is zero, so we conclude that the solution (4.4.3.42) intersects the boundary of AdS_3 along the curve described by

$$-T^2 + X^2 = -1. \tag{4.4.3.44}$$

Interestingly this is a timelike curve whereas the giant gluon solution we started with traces out a path of lightlike curves on the boundary.

More complicated cases must be studied numerically. In Fig. (4.3) we show the timelike curve on the boundary of AdS_3 that bounds the sample surface shown in Fig. (4.2).

4.5 AdS_5 Solutions

We now turn our attention to the dressing problem for worldsheets in AdS_5. This case is somewhat more complicated because it is not realized as a principal chiral model. Rather we use the $SU(2,2)/SO(4,1)$ coset model, parameterizing an element g of the coset in terms of the embedding coordinates $\vec{Y}$ according to [76]

$$g = \begin{pmatrix} 0 & +Z_1 & -Z_3 & +\bar{Z}_2 \\ -Z_1 & 0 & +Z_2 & +\bar{Z}_3 \\ +Z_3 & -Z_2 & 0 & -\bar{Z}_1 \\ -\bar{Z}_2 & -\bar{Z}_3 & +\bar{Z}_1 & 0 \end{pmatrix} \tag{4.5.0.45}$$

where

$$Z_1 = Y_{-1} + iY_0, \qquad Z_2 = Y_1 + iY_2, \qquad Z_3 = Y_3 + iY_4. \tag{4.5.0.46}$$

This parameterization satisfies

$$g^{\mathrm{T}} = -g, \qquad g^\dagger M g = M, \tag{4.5.0.47}$$

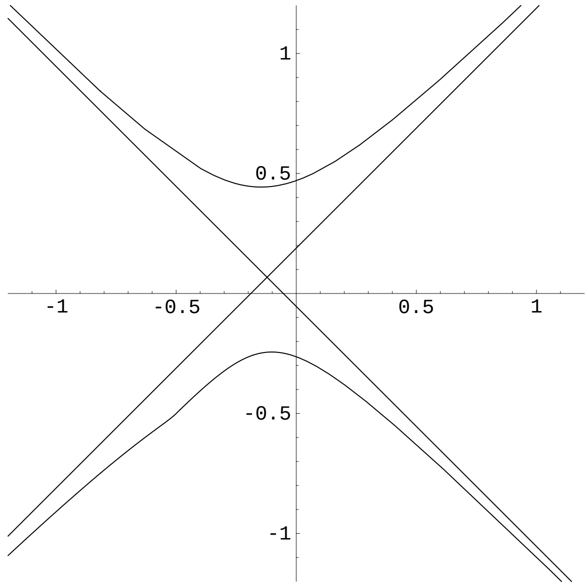

Figure 4.3: In this plot we consider, as an example, the solution (4.4.1.26) for the particular case $\lambda_1 = 1/2 + i/3$. As explained in the text, the solution actually corresponds to infinitely many Wilson loops on the boundary of AdS_3, one of which is the curve shown here in the (X, T) plane on the boundary of AdS_3 in Poincaré coordinates. The light-cone to which these timelike curves asymptote is also shown. The Wilson loop is of course a closed curve; the upper and lower branches shown here live on opposite sides of the The minimal area surface spanning this curve is shown in Fig. (4.2).

where

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \qquad (4.5.0.48)$$

and has determinant

$$\det g = -\vec{Y} \cdot \vec{Y} = 1. \qquad (4.5.0.49)$$

Taking again the giant gluon solution (4.4.0.20) (supplemented with $Y_3 = Y_4 = 0$) as the 'vacuum' we now find that the solution to the linear system (4.3.0.12) is

$$\Psi(\lambda) = \begin{pmatrix} 0 & +\operatorname{ch} u_1 \operatorname{ch} U_2 + im_- \operatorname{sh} u_1 \operatorname{sh} U_2 \\ -\operatorname{ch} U_1 \operatorname{ch} u_2 - im_+ \operatorname{sh} U_1 \operatorname{sh} u_2 & 0 \\ 0 & -\operatorname{sh} u_1 \operatorname{ch} U_2 - im_- \operatorname{ch} u_1 \operatorname{sh} U_2 \\ -m_+ \operatorname{sh} U_1 \operatorname{ch} u_2 + i \operatorname{ch} U_1 \operatorname{sh} u_2 & 0 \end{pmatrix}.$$
$$\begin{pmatrix} 0 & +m_- \operatorname{sh} u_1 \operatorname{ch} U_2 - i \operatorname{ch} u_1 \operatorname{sh} U_2 \\ +\operatorname{sh} U_1 \operatorname{ch} u_2 + im_+ \operatorname{ch} U_1 \operatorname{sh} u_2 & 0 \\ 0 & -m_- \operatorname{ch} u_1 \operatorname{ch} U_2 + i \operatorname{sh} u_1 \operatorname{sh} U_2 \\ +m_+ \operatorname{ch} U_1 \operatorname{ch} u_2 - i \operatorname{sh} U_1 \operatorname{sh} u_2 & 0 \end{pmatrix} \tag{4.5.0.50}$$

in terms of

$$U_1 = m_- z + m_+ \bar{z}, \qquad U_2 = (m_- z - m_+ \bar{z})/i, \qquad m_+ = 1/m_- = \left(\frac{1+i\lambda}{1-i\lambda}\right)^{1/2}. \tag{4.5.0.51}$$

The solution (4.5.0.50) has been chosen to satisfy the desired constraints

$$\Psi^\dagger(\bar{\lambda}) M \Psi(\lambda) = M, \qquad \det \Psi(0) = 1 \tag{4.5.0.52}$$

as well as the initial condition

$$\Psi(\lambda = 0) = g, \tag{4.5.0.53}$$

where g is the giant gluon solution (4.4.0.20) written in the embedding (4.5.0.45). Note that the symbols U_1, U_2 defined in (4.5.0.51) have been chosen because at $\lambda = 0$ they reduce to u_1, u_2.

4.5.1 Construction of the dressing factor

The dressing factor for this coset model takes the form

$$\chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P_1 + \frac{1/\lambda_1 - 1/\bar{\lambda}_1}{\lambda + 1/\bar{\lambda}_1} P_2. \tag{4.5.1.54}$$

In order to satisfy all the constraints on the dressed solution, we choose P_1 and P_2 as follows. First we choose P_1 to be the hermitian (with respect to the metric M) projection operator onto the vector $v_1 = \Psi(\bar{\lambda}_1) v$, where v is an arbitrary complex constant vector. Specifically, P_1 is then given as in (4.3.0.17) by

$$P_1 = \frac{v_1 v_1^\dagger M}{v_1^\dagger M v_1}, \tag{4.5.1.55}$$

which satisfies

$$P_1^2 = P_1, \qquad P_1^\dagger = M P_1 M \tag{4.5.1.56}$$

as desired. Next we choose

$$P_2 = \Psi(0) P_1^{\mathrm{T}} \Psi(0)^{-1}. \tag{4.5.1.57}$$

Because of (4.5.1.56) it is easy to check that P_2 also satisfies

$$P_2^2 = P_2, \qquad P_2^\dagger = M P_2 M, \tag{4.5.1.58}$$

so P_2 is also a hermitian projection operator; in fact it is easy to check that P_2 projects onto the vector

$$v_2 = \Psi(0) M \overline{v_1} \tag{4.5.1.59}$$

and hence can be written as

$$P_2 = \frac{v_2 v_2^\dagger M}{v_2^\dagger M v_2}. \tag{4.5.1.60}$$

Now let us explain the choice (4.5.1.57). Notice that

$$v_2^\dagger M v_1 = v_1^{\mathrm{T}} M \Psi(0)^\dagger M v_1 = v_1^{\mathrm{T}} \Psi(0)^{-1} v_1 \tag{4.5.1.61}$$

where we used $\Psi(0)^\dagger M \Psi(0) = M$. But since $\Psi(0)$ is antisymmetric, this is zero! So v_2 and v_1 are orthogonal, and hence

$$P_1 P_2 = P_2 P_1 = 0. \tag{4.5.1.62}$$

Using all of the above relations one can check that (4.5.1.54) satisfies the conditions

$$[\chi(\bar{\lambda})]^\dagger M \chi(\lambda) = M, \qquad \Psi^{\mathrm{T}}(0)\chi^{\mathrm{T}}(0) = -\chi(0)\Psi(0), \tag{4.5.1.63}$$

which guarantee that the dressed solution $\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda)$ continues to satisfy (4.5.0.47). As in the AdS_3 case we find that χ does not have unit determinant but rather

$$\det \chi(\lambda) = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \frac{\lambda - 1/\lambda_1}{\lambda - 1/\bar{\lambda}_1}. \tag{4.5.1.64}$$

We must therefore rescale the dressed solution $\Psi'(0) = \chi(0)\Psi(0)$ by a factor of $\sqrt{\lambda_1/\bar{\lambda}_1}$.

To summarize, the dressed solution g' is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left[1 + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1} P_1 + \frac{1/\lambda_1 - 1/\bar{\lambda}_1}{1/\bar{\lambda}_1} P_2 \right] \Psi(0) \tag{4.5.1.65}$$

in terms of (4.5.0.50) and the projection operators (4.5.1.55), (4.5.1.57). The solution is characterized by an arbitrary complex parameter λ_1 and the choice of a complex four-component vector v.

4.5.2 A special case

Since the general solution is again rather complicated we display only a special case, choosing the vector $v = \begin{pmatrix} 1 & i & 0 & 0 \end{pmatrix}$. We then find that the dressed solution g' has the form (4.5.0.45) with

$$Z_1' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_1}{D}, \qquad Z_2' = \frac{1}{|\lambda_1|}\frac{\vec{Y}\cdot\vec{N}_2}{D}, \qquad Z_3' = \frac{1}{|\lambda_1|}\frac{N_3}{D} \tag{4.5.2.66}$$

in terms of the numerator factors

$$\begin{aligned}
\vec{N}_1 &= \begin{pmatrix} -|m|^2\bar\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar U_1 + \operatorname{ch}U_2\operatorname{ch}\bar U_2) + \lambda_1\operatorname{sh}U_1\operatorname{sh}\bar U_1 + |m|^4\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar U_2 \\ -i|m|^2\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar U_1 + \operatorname{ch}U_2\operatorname{ch}\bar U_2) + i\bar\lambda_1\operatorname{sh}U_1\operatorname{sh}\bar U_1 + i|m|^4\bar\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar U_2 \\ -(\lambda_1-\bar\lambda_1)\bar m\operatorname{sh}U_1\operatorname{ch}\bar U_1 + i(\lambda_1-\bar\lambda_1)\bar m|m|^2\operatorname{ch}U_2\operatorname{sh}\bar U_2 \\ +i(\lambda_1-\bar\lambda_1)m\operatorname{ch}U_1\operatorname{sh}\bar U_1 - (\lambda_1-\bar\lambda_1)m|m|^2\operatorname{sh}U_2\operatorname{ch}\bar U_2 \end{pmatrix}, \\
\vec{N}_2 &= \begin{pmatrix} -(\lambda_1-\bar\lambda_1)\bar m\operatorname{sh}U_1\operatorname{ch}\bar U_1 + i(\lambda_1-\bar\lambda_1)\bar m|m|^2\operatorname{ch}U_2\operatorname{sh}\bar U_2 \\ +i(\lambda_1-\bar\lambda_1)m\operatorname{ch}U_1\operatorname{sh}\bar U_1 - (\lambda_1-\bar\lambda_1)m|m|^2\operatorname{sh}U_2\operatorname{ch}\bar U_2 \\ -|m|^2\bar\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar U_1 + \operatorname{ch}U_2\operatorname{ch}\bar U_2) + \lambda_1\operatorname{sh}U_1\operatorname{sh}\bar U_1 + |m|^4\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar U_2 \\ -i|m|^2\lambda_1(\operatorname{ch}U_1\operatorname{ch}\bar U_1 + \operatorname{ch}U_2\operatorname{ch}\bar U_2) + i\bar\lambda_1\operatorname{sh}U_1\operatorname{sh}\bar U_1 + i|m|^4\bar\lambda_1\operatorname{sh}U_2\operatorname{sh}\bar U_2 \end{pmatrix}, \\
N_3 &= \bar m(\lambda_1-\bar\lambda_1)(-i\operatorname{sh}U_1\operatorname{ch}\bar U_2 + |m|^2\operatorname{ch}U_1\operatorname{sh}\bar U_2),
\end{aligned} \tag{4.5.2.67}$$

$\vec{Y}$ again given in (4.4.0.20), and the denominator

$$D = -|m|^2(\operatorname{ch}U_1\operatorname{ch}\bar U_1 + \operatorname{ch}U_2\operatorname{ch}\bar U_2) + \operatorname{sh}U_1\operatorname{sh}\bar U_1 + |m|^4\operatorname{sh}U_2\operatorname{sh}\bar U_2. \tag{4.5.2.68}$$

In these expressions m and $\bar m$ are as in (4.4.1.29), with

$$U_1 = z/m + m\bar z, \qquad U_2 = (z/m - m\bar z)/i. \tag{4.5.2.69}$$

The real embedding coordinates $\vec{Y}'$ of the dressed solution may then be extracted from (4.5.0.46). It is straightforward, though somewhat tedious, to directly verify that the resulting $\vec{Y}'$ satisfies the equations of motion (4.3.0.2) and the Virasoro constraints (4.3.0.3), providing a check on our application of the dressing method.

4.6 Conventions

Here we summarize the standard conventions for global AdS_3 that we have used in preparing Fig. (4.1) and (4.2). We parametrize the $SU(1,1)$ group element (4.3.0.9) as

$$g = \begin{pmatrix} e^{+i\tau}\sec\theta & e^{+i\phi}\tan\theta \\ e^{-i\phi}\tan\theta & e^{-i\tau}\sec\theta \end{pmatrix}, \tag{4.6.0.70}$$

where τ is global time, ϕ is the azimuthal angle, and θ runs from 0 in the interior of the AdS_3 cylinder to $\pi/2$ at the boundary of AdS_3. In terms of these quantities the parametric plots in Figures 1 and 2 have Cartesian coordinates

$$(x, y, z) = (\theta\cos\phi, \theta\sin\phi, \tau) \tag{4.6.0.71}$$

and the boundary of AdS_3 is the cylinder $x^2 + y^2 = (\pi/2)^2$.

Chapter 5

Generating AdS String Solutions

5.1 Abstract

We use a Pohlmeyer type reduction to generate classical string solutions in AdS spacetime. In this framework we describe a correspondence between spikes in AdS_3 and soliton profiles of the sinh-Gordon equation. The null cusp string solution and its closed spinning string counterpart are related to the sinh-Gordon vacuum. We construct classical string solutions corresponding to sinh-Gordon solitons, antisolitons and breathers by the inverse scattering technique. The breather solutions can also be reproduced by the sigma model dressing method.

5.2 Introduction

Classical string solutions in $AdS_5 \times S^5$ have provided a lot of data in exploring various aspects of the AdS/CFT correspondence (see [47, 48, 49, 50] for review). Alday and Maldacena have given a prescription for computing gluon scattering amplitudes using AdS/CFT [17]. The prescription is equivalent to finding a classical string solution with boundary conditions determined by the gluon momenta. The value of the scattering amplitude is then related to the area of this solution. Using this prescription and the solution originally constructed in [70] they found agreement with the conjectured iteration relations for perturbative multiloop amplitudes for four gluons [71, 18, 72, 73, 74]. Several papers including [63, 77, 78, 75] have studied various aspects of the classical string solutions (see [79]–[95] for other developments). For the case of four and five gluons the results are fixed by dual conformal symmetry

[96, 90]. For a large number of gluons the amplitude at strong coupling was computed in [96] and it disagreed with the corresponding limit of the gauge theory guess [18]. In order to test the multiloop iterative structure of gauge theory amplitudes it would be very important to construct the string solution for six gluons and more.

Classical string theory on $R\times S^2$ (or $R\times S^3$) is equivalent to classical sine-Gordon theory (or complex sine-Gordon theory) via Pohlmeyer reduction [14]. De Vega and Sanchez showed that similarly string theory on AdS_2, AdS_3 and AdS_4 is equivalent to Liouville theory, sinh-Gordon theory and B_2 Toda theories respectively [54, 98, 99]. Moreover, a sine-Gordon-like action has been proposed for the full Green-Schwarz superstring in $AdS_5 \times S^5$ [56, 100]. Classical solitons in both theories should be in one to one correspondence. Indeed, giant magnon solutions on $R\times S^2$ and $R\times S^3$ map to one soliton solution in sine-Gordon and complex sine-Gordon respectively [9, 16].

Integrability of string theory on $AdS_5 \times S^5$ allows the use of algebraic methods to construct solutions of the nonlinear equations of motion. Given a vacuum solution of an integrable nonlinear equation, the dressing method provides a way to construct a new solution which also satisfies the equations of motion by using an associated linear system [28, 44]. In [34, 61] the dressing method was used to construct classical string solutions describing scattering and bound states of magnons on $R \times S^5$ and various subsectors, such as $R \times S^2$ and $R \times S^3$, by dressing the vacuum corresponding to a pointlike string moving around the equator of the sphere at the speed of light. In [62] it was used to construct solutions describing the scattering of spiky strings on a sphere [35] by starting with a different vacuum, a static string wrapped around the equator of the sphere.

In [63] the applicability of the dressing method to the problem of finding Euclidean minimal area worldsheets in AdS was demonstrated. We took as a vacuum the null cusp string solution constructed in [70] (which was later generalized and given a new interpretation in [17]). We dressed this vacuum and found new minimal area surfaces in AdS_3 and AdS_5. These solutions generically trace out timelike curves on the boundary, and might be relevant to studies of the propagation of massive particles in gauge theory. The vacuum solution [17, 70] can be related by analytic continuation and a conformal transformation to a closed string energy eigenstate (an infinite string limit of GKP string [47, 75]). In this work we outline the dressing method for Minkowskian worldsheets in AdS and construct new string solutions by starting with an infinite closed spinning string. We also show that the spikes of the long GKP string can be mapped to sinh-Gordon solitons at the boundary of AdS.

We use the inverse scattering method to construct string solutions corresponding to sinh-Gordon solitons, antisolitons, breathers and soliton scattering solutions. The sigma model solutions can be constructed in terms of wavefunctions of the Pohlmeyer reduced model [1] [102]. The advantage of this method is that it allows us to construct a string solution starting from any sinh-Gordon solution. All one has to do is to solve a linear system with coefficients depending on the chosen sinh-Gordon solution. Notice that in the dressing method one is also solving a linear system, but the difference is that in the dressing method the coefficients of the system depend on the chosen vacuum solution of the string equations, whereas in this method the coefficients depend only on the sinh-Gordon or reduced system solution. This is advantageous because any sinh-Gordon solution is generaly simpler than the corresponding sigma model solution.

This work is organized as follows. In section 3 we review the Pohlmeyer reduction and inverse scattering method for constructing string solutions from sinh-Gordon solutions. In section 4 various sinh-Gordon solutions are reviewed. In section 5 explicit string solutions are constructed and the physical meanings are discussed. It would be interesting to understand the physics of these new string solutions better. In section 6 we reproduce the breather solutions by the dressing method.

5.3 Pohlmeyer reduction for AdS strings

In this section we review the Pohlmeyer reduction for string theory in AdS_d space following [54]. We also review how to write down string solutions in terms of the wavefunctions of the sinh-Gordon inverse problem [102].

We parameterize AdS_d with $d+1$ embedding coordinates $\vec{Y}$ subject to the constraint

$$\vec{Y} \cdot \vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \cdots + Y_{d-1}^2 = -1. \tag{5.3.0.1}$$

The conformal gauge equation of motion for strings in AdS_d is

$$\partial\bar{\partial}\vec{Y} - (\partial\vec{Y} \cdot \bar{\partial}\vec{Y})\vec{Y} = 0 \tag{5.3.0.2}$$

[1] A different solution generating technique based on Pohlmeyer-type reduction was employed for string solutions on $AdS_3 \times S^1$ in [101].

subject to the Virasoro constraints

$$\partial\vec{Y}\cdot\partial\vec{Y} = \bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y} = 0. \tag{5.3.0.3}$$

Here we use coordinates z and $\bar{z}$ related to Minkowski worldsheet coordinates τ and σ by $z = \frac{1}{2}(\sigma - \tau),\ \bar{z} = \frac{1}{2}(\sigma + \tau)$ with $\partial = \partial_\sigma - \partial_\tau,\ \bar{\partial} = \partial_\sigma + \partial_\tau$.

Now let us show the equivalence of the string equations (5.3.0.2), (5.3.0.3) to the generalized sinh-Gordon model. To make the reduction we first choose a basis

$$e_i = (\vec{Y},\ \bar{\partial}\vec{Y},\ \partial\vec{Y},\ \vec{B}_4, \cdots,\ \vec{B}_{d+1}), \tag{5.3.0.4}$$

where $i = 1, 2 \cdots d+1$ and the vectors $\vec{B}_k$ with $k = 4, 5 \cdots d+1$ are orthonormal

$$\vec{B}_k\cdot\vec{B}_l = \delta_{kl}, \quad \vec{B}_k\cdot\vec{Y} = \vec{B}_k\cdot\partial\vec{Y} = \vec{B}_k\cdot\bar{\partial}\vec{Y} = 0. \tag{5.3.0.5}$$

Defining

$$\alpha \equiv \alpha(z,\bar{z}) = \ln(\partial\vec{Y}\cdot\bar{\partial}\vec{Y}), \tag{5.3.0.6}$$

$$u_k \equiv u_k(z,\bar{z}) = \vec{B}_k\cdot\bar{\partial}^2\vec{Y}, \tag{5.3.0.7}$$

$$v_k \equiv v_k(z,\bar{z}) = \vec{B}_k\cdot\partial^2\vec{Y}, \tag{5.3.0.8}$$

where $k = 4, 5 \cdots d+1$, the equation of motion for α becomes

$$\partial\bar{\partial}\alpha - e^{\alpha} - e^{-\alpha}\sum_{i=4}^{d+1} u_i v_i = 0. \tag{5.3.0.9}$$

This is called the generalized sinh-Gordon model. We can find the evolution of the vectors u_i and v_i by expressing the derivatives of the basis (5.3.0.4) in terms of the basis itself. In $d = 2$, $u = v = 0$ and the equation (5.3.0.9) becomes the Liouville equation. In $d = 3$ and $d = 4$ it can be reduced to sinh-Gordon and B_2 Toda models respectively [54].

Now let us discuss the $d = 3$ case in more detail. For the case of AdS_3, one can write an explicit formula for $\vec{B}_4$

$$B_{4a} \equiv e^{-\alpha}\epsilon_{abcd}\, Y_b\, \partial Y_c\, \bar{\partial} Y_d, \tag{5.3.0.10}$$

where $a, b, c, d = 1, 2, 3, 4$ and ϵ_{abcd} is the antisymmetric Levi-Civita tensor. The equations of motion can then be rewritten as

$$\bar{\partial}e_i = A_{ij}(z,\bar{z})e_j, \quad \partial e_i = B_{ij}(z,\bar{z})e_j, \tag{5.3.0.11}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \bar{\partial}\alpha & 0 & u \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & -ue^{-\alpha} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & \partial\alpha & v \\ 0 & -ve^{-\alpha} & 0 & 0 \end{pmatrix}. \tag{5.3.0.12}$$

The integrability condition $\partial A - \bar{\partial} B + [A, B] = 0$ implies $u = u(\bar{z})$, $v = v(z)$ and $\partial\bar{\partial}\alpha - e^{\alpha} - uve^{-\alpha} = 0$. We can make a change of variables

$$\alpha(z, \bar{z}) = \hat{\alpha}(z, \bar{z}) + \frac{1}{2}\ln(-u(\bar{z})v(z)) \tag{5.3.0.13}$$

to bring the equation (5.3.0.9) to a standard sinh-Gordon form

$$\partial\bar{\partial}\hat{\alpha} - 4\sinh\hat{\alpha} = 0. \tag{5.3.0.14}$$

5.3.1 Constructing string solutions from sinh-Gordon solutions

In this section we use the Pohlmeyer reduction to express solutions of the equations (5.3.0.2, 5.3.0.3) in terms of solutions of the sinh-Gordon equation (5.3.0.9) [102]. The idea is to first rewrite the matrices A_{ij} and B_{ij} which appear in (5.3.0.11) in a manifestly $SO(2,2)$ symmetric way. Then recalling that $SO(2,2)$ is isomorphic to $SU(1,1) \times SU(1,1)$ one can expand A_{ij} and B_{ij} in terms of $SU(1,1)$ generators. Defining

$$A_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) \\ -\frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) \end{pmatrix}, \tag{5.3.1.15}$$

$$A_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) & -\frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) \\ \frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) \end{pmatrix}, \tag{5.3.1.16}$$

$$B_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) \\ -\frac{i}{4}\bar{\partial}\alpha - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2} + e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2} - e^{\alpha/2}) \end{pmatrix}, \tag{5.3.1.17}$$

$$B_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) & -\frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) \\ \frac{i}{4}\partial\alpha + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2} - e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2} + e^{\alpha/2}) \end{pmatrix}, \tag{5.3.1.18}$$

we can rewrite equations (5.3.0.11) in terms of two unknown complex vectors $\phi = (\phi_1, \phi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ as

$$\bar{\partial}\phi = A_1\phi, \quad \partial\phi = A_2\phi, \tag{5.3.1.19}$$

$$\bar{\partial}\psi = B_1\psi, \quad \partial\psi = B_2\psi. \tag{5.3.1.20}$$

The vectors ϕ and ψ are normalized $\phi^\dagger\phi = \phi_1^*\phi_1 - \phi_2^*\phi_2 = \psi^\dagger\psi = \psi_1^*\psi_1 - \psi_2^*\psi_2 = 1$. In other words, given a solution $\alpha(z,\bar{z}), u(\bar{z})$ and $v(z)$ of the sinh-Gordon equation, we can find ϕ and ψ such that they solve the above linear system. Then the string solution is given by

$$\begin{aligned} Z_1 &\equiv Y_{-1} + iY_0 = \phi_1^*\psi_1 - \phi_2^*\psi_2, && (5.3.1.21) \\ Z_2 &\equiv Y_1 + iY_2 = \phi_2^*\psi_1^* - \phi_1^*\psi_2^*. && (5.3.1.22) \end{aligned}$$

This formula follows from the isomorphism between $SO(2,2)$ and the product of two copies of $SU(1,1)$ parametrized by the matrices $\begin{pmatrix} \phi_1 & \phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix}$ and $\begin{pmatrix} \psi_1 & \psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix}$.

5.4 Review of sinh-Gordon solutions

The sinh-Gordon equation (5.3.0.14) has a vacuum solution

$$\hat{\alpha}_0 = 0 \quad \text{or} \quad \alpha_0 = \ln 2. \tag{5.4.0.23}$$

The one-soliton solutions are

$$\alpha_{s,\bar{s}} = \ln 2 \pm \ln\big(\tanh^2\gamma(\sigma - v\tau)\big), \tag{5.4.0.24}$$

where v is the velocity of the solitons and $\gamma = 1/\sqrt{1-v^2}$.

We can also consider solutions periodic in σ

$$\alpha'_{s,\bar{s}} = \ln 2 \pm \ln\big(\tan^2\gamma(\sigma - v\tau)\big). \tag{5.4.0.25}$$

Multi-soliton solutions can be constructed via the Bäcklund transformation. If we call the plus solution of (5.4.0.24) soliton and the minus solution antisoliton, the two-(anti)soliton solution is given by

$$\alpha_{ss,\bar{s}\bar{s}} = \ln 2 \pm \ln\left[\frac{v\cosh X - \cosh T}{v\cosh X + \cosh T}\right]^2, \tag{5.4.0.26}$$

where $X = 2\gamma\sigma$, $T = 2v\gamma\tau$, and the soliton-antisoliton solution is given by

$$\alpha_{s\bar{s}} = \ln 2 \pm \ln\left[\frac{v\sinh X - \sinh T}{v\sinh X + \sinh T}\right]^2. \tag{5.4.0.27}$$

Here the solutions are in the center of mass frame with $v_1 = -v_2 = v$.

If we analytically continue the soliton-antisoliton solution and take v to be imaginary, $v = iw$, we get the breather solution of the sinh-Gordon system

$$\alpha_B = \ln 2 \pm \ln\Big[\frac{w \sinh X_B - \sin T_B}{w \sinh X_B + \sin T_B}\Big]^2, \tag{5.4.0.28}$$

where $X_B = 2\sigma/\sqrt{1+w^2}$ and $T_B = 2w\tau/\sqrt{1+w^2}$. In order to make the center mass move with velocity v_c, one can make a boost by replacing $\sigma \to \gamma_c(\sigma - v_c\tau)$ and $\tau \to \gamma_c(\tau - v_c\sigma)$, where $\gamma_c = 1/\sqrt{1-v_c^2}$.

5.5 String solutions

5.5.1 Vacuum

Now let us look at some examples. Starting with the sinh-Gordon vacuum $u = 2, v = -2, \alpha_0 = \ln 2$, the results of solving the linear system (5.3.1.19), (5.3.1.20) are

$$\phi_1 = e^{-i\tau} \qquad \phi_2 = 0 \qquad \psi_1 = \cosh\sigma \qquad \psi_2 = -\sinh\sigma. \tag{5.5.1.29}$$

Then the Minkowskian worldsheet solution is given by (see fig. 5.1)

$$Z_1 = e^{i\tau}\cosh\sigma, \tag{5.5.1.30}$$

$$Z_2 = e^{i\tau}\sinh\sigma. \tag{5.5.1.31}$$

This is the infinite string limit of spinning string [47].

The Euclidean worldsheet solution is obtained by making the change $\tau \to -i\tau$. Then Y_0 and Y_2 become imaginary, thus effectively exchanging places. The Euclidean vacuum solution reads

$$\vec{Y}_E = \begin{pmatrix} \cosh\sigma\cosh\tau \\ \sinh\sigma\sinh\tau \\ \sinh\sigma\cosh\tau \\ \cosh\sigma\sinh\tau \end{pmatrix}. \tag{5.5.1.32}$$

This is the solution found in [70] which was used by the authors of [17] to calculate the scattering amplitude for four gluons.

The energy and angular momentum can be calculated after we introduce the cutoff

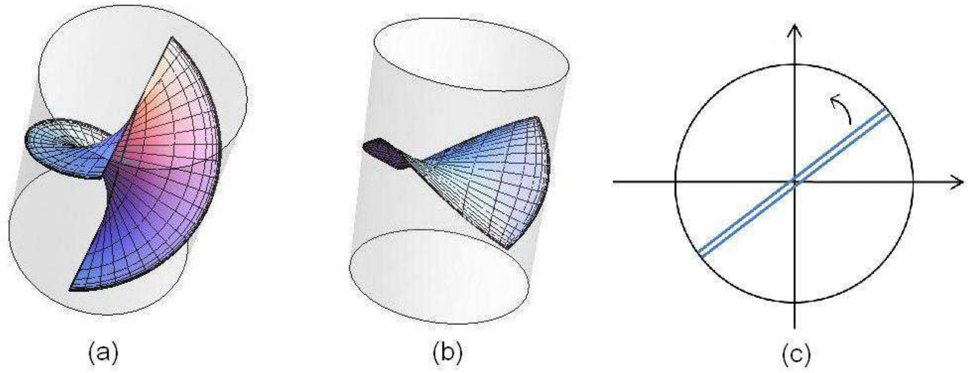

Figure 5.1: The vacuum solution in (a) Minkowskian and (b) Euclidean worldsheet plotted in AdS_3 coordinates. (c) Top view of Minkowskian vacuum solution. The boundary of the worldsheet touches the boundary of AdS space.

$\Lambda \gg 0$,

$$E = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \cosh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda}, \tag{5.5.1.33}$$

$$S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \sinh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda}, \tag{5.5.1.34}$$

$$E - S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{4\pi}{\sqrt{\lambda}}S, \tag{5.5.1.35}$$

which is exactly the result of [47].

5.5.2 Long strings in AdS_3 as sinh-Gordon solitons

Consider the GKP spinning string solution found in [47]

$$Z_1 = e^{i\tau}\cosh\rho(\sigma), \tag{5.5.2.36}$$

$$Z_2 = e^{i\omega\tau}\sinh\rho(\sigma), \tag{5.5.2.37}$$

where

$$\rho(\sigma) = am(i\sigma|1-\omega^2), \tag{5.5.2.38}$$

and am the Jacobi amplitude function. In the infinite string limit $\omega \to 1$ this solution reduces to (5.5.1.30), (5.5.1.31). The corresponding sinh-Gordon solution is given by

$$\alpha = \ln(2\rho'^2) = \ln\left(2dn^2(i\sigma|1-\omega^2)\right), \tag{5.5.2.39}$$

where dn is the Jacobi elliptic function.

Taking the (5.5.2.38) solution ${\rho'}^2 = \cosh^2\rho - \omega^2 \sinh^2\rho$ we can expand ρ near one of spikes (turning points of the string) and let $\omega = 1 + 2\eta$, where $\eta \ll 1$, to get

$$ {\rho'}^2 \sim e^{2\rho}(e^{-2\rho} - \eta). \tag{5.5.2.40} $$

Denoting $u = e^{-\rho}$ the above equation becomes

$$ {u'}^2 \sim u^2 - \eta. \tag{5.5.2.41} $$

If we choose the location of the spike to be at $\sigma = \sigma_0$, we find

$$ \rho(\sigma) = -\ln\big(\sqrt{\eta}\cosh(\sigma - \sigma_0)\big). \tag{5.5.2.42} $$

Now we can use the map (5.3.0.6) to find the sinh-Gordon solution corresponding to this spinning string

$$ \alpha = \ln(2{\rho'}^2) = \ln(2\tanh^2(\sigma - \sigma_0)) \tag{5.5.2.43} $$

This is exactly the one-soliton solution to the sinh-Gordon equation (5.3.0.9). Therefore, the long string limit of the spinning string solution [47] itself is a two-soliton configuration of the sinh-Gordon system and the solitons are located near the boundary of AdS.

5.5.3 One-soliton solutions

Let us describe the method of constructing string solutions corresponding to one-soliton sinh-Gordon solution in detail. Start with the sinh-Gordon solution

$$ \alpha_s = \ln 2 + \ln(\tanh^2\sigma). \tag{5.5.3.44} $$

The matrices entering into the linear system (5.3.1.19), (5.3.1.20) are given by

$$ A_1 = \begin{pmatrix} -i\coth 2\sigma & (i-1)\mathrm{csch}\, 2\sigma \\ -(i+1)\mathrm{csch}\, 2\sigma & i\coth 2\sigma \end{pmatrix}, \tag{5.5.3.45} $$

$$ A_2 = \begin{pmatrix} i\coth 2\sigma & -(i+1)\mathrm{csch}\, 2\sigma \\ (i-1)\mathrm{csch}\, 2\sigma & -i\coth 2\sigma \end{pmatrix}, \tag{5.5.3.46} $$

$$ B_1 = \begin{pmatrix} -i\mathrm{csch}\, 2\sigma & i\mathrm{csch}\, 2\sigma - \coth 2\sigma \\ -i\mathrm{csch}\, 2\sigma - \coth 2\sigma & i\mathrm{csch} 2\sigma \end{pmatrix}, \tag{5.5.3.47} $$

$$ B_2 = \begin{pmatrix} i\mathrm{csch}\, 2\sigma & -i\mathrm{csch}\, 2\sigma - \coth 2\sigma \\ i\mathrm{csch}\, 2\sigma - \coth 2\sigma & -i\mathrm{csch}\, 2\sigma \end{pmatrix}. \tag{5.5.3.48} $$

The spinors that solve the linear system are

$$\phi_1 = e^{-i\tau}\cosh(\frac{1}{2}\ln\tanh\sigma), \tag{5.5.3.49}$$

$$\phi_2 = -e^{-i\tau}\sinh(\frac{1}{2}\ln\tanh\sigma), \tag{5.5.3.50}$$

$$\psi_1 = (\tau+i)\cosh(\frac{1}{2}\ln\sinh 2\sigma) - \tau\sinh(\frac{1}{2}\ln\sinh 2\sigma), \tag{5.5.3.51}$$

$$\psi_2 = -(\tau+i)\sinh(\frac{1}{2}\ln\sinh 2\sigma) + \tau\cosh(\frac{1}{2}\ln\sinh 2\sigma). \tag{5.5.3.52}$$

Then we use (5.3.1.21, 5.3.1.22) to find the corresponding string solution (see fig. 5.2)

$$Z_1^s = \frac{e^{i\tau}}{2\sqrt{2}\cosh\sigma}\big(2\tau + i(\cosh 2\sigma + 2)\big), \tag{5.5.3.53}$$

$$Z_2^s = \frac{e^{i\tau}}{2\sqrt{2}\cosh\sigma}\big(-2\tau - i\cosh 2\sigma\big). \tag{5.5.3.54}$$

Because of the Lorentz invariance, we can always boost the solution as $\sigma \to \gamma(\sigma - v\tau), \tau \to \gamma(\tau - v\sigma)$. Notice this differs from the magnon case, where the boost symmetry of sine-Gordon translates into a non-obvious symmetry on the string side [9].

The Euclidean worldsheet solution is obtained by making the changes $\tau \to -i\tau$. Then Y_{-1} and Y_1 become imaginary, thus effectively exchanging places. The Euclidean one-soliton solution reads

$$\vec{Y}_E^s = \frac{1}{2\sqrt{2}\cosh\sigma}\begin{pmatrix} 2\tau\cosh\tau - \sinh\tau\cosh 2\sigma \\ -2\tau\sinh\tau + \cosh\tau(\cosh 2\sigma + 2) \\ -2\tau\cosh\tau + \sinh\tau(\cosh 2\sigma + 2) \\ 2\tau\sinh\tau - \cosh\tau\cosh 2\sigma \end{pmatrix}. \tag{5.5.3.55}$$

One can easily compute the energy and angular momentum

$$E = \int_{-\Lambda}^{\Lambda} d\sigma \frac{\sqrt{\lambda}}{16\pi\cosh^2\sigma}(1 + 8\tau^2 + 4\cosh 2\sigma + \cosh 4\sigma) \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2), \tag{5.5.3.56}$$

$$S = \int_{-\Lambda}^{\Lambda} d\sigma \frac{\sqrt{\lambda}}{16\pi\cosh^2\sigma}(1 + 8\tau^2 - 4\cosh 2\sigma + \cosh 4\sigma) \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2). \tag{5.5.3.57}$$

If we neglect the τ dependence since the exponential term is much larger than the square term, we have

$$E - S = \int_{-\Lambda}^{\Lambda} \frac{\sqrt{\lambda}}{2\pi}\cosh 2\sigma\,\mathrm{sech}^2\sigma\, d\sigma \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{8\pi}{\sqrt{\lambda}}S. \tag{5.5.3.58}$$

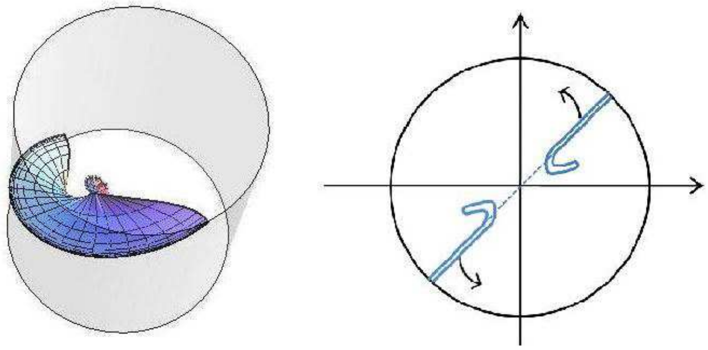

Figure 5.2: The one-soliton solution in (a) Minkowskian worldsheet plotted in AdS_3 coordinates. (b) Top view of the Minkowskian one-soliton solution. Please note the curvature of the string changes with the evolution of time.

The energy is not conserved because there is momentum flow at the asymptotic end of the string and the string itself is not closed.

Similarly, the one-antisoliton string solution corresponding to $\alpha_{\bar{s}}$ is given by

$$Z_1^{\bar{s}} = \frac{e^{i\tau}}{2\sqrt{2}\sinh\sigma}\big(2\tau - i\cosh 2\sigma\big), \tag{5.5.3.59}$$

$$Z_2^{\bar{s}} = \frac{e^{i\tau}}{2\sqrt{2}\sinh\sigma}\big(-2\tau + i(\cosh 2\sigma - 2)\big), \tag{5.5.3.60}$$

whereas the periodic in σ string solutions mapping to α'_s and $\alpha'_{\bar{s}}$ are respectively

$$\vec{Y}'_s = \frac{1}{2\sqrt{2}\cos\sigma}\begin{pmatrix} 2\tau\cosh\tau - \sinh\tau\cos 2\sigma \\ 2\tau\sinh\tau - \cosh\tau(\cos 2\sigma + 2) \\ 2\tau\cosh\tau - \sinh\tau(\cos 2\sigma + 2) \\ -2\tau\sinh\tau + \cosh\tau\cos 2\sigma \end{pmatrix}, \tag{5.5.3.61}$$

$$\vec{Y}'_{\bar{s}} = \frac{1}{2\sqrt{2}\sin\sigma}\begin{pmatrix} 2\tau\cosh\tau + \sinh\tau\cos 2\sigma \\ 2\tau\sinh\tau + \cosh\tau(\cos 2\sigma - 2) \\ -2\tau\cosh\tau - \sinh\tau(\cos 2\sigma - 2) \\ 2\tau\sinh\tau + \cosh\tau\cos 2\sigma \end{pmatrix}. \tag{5.5.3.62}$$

Energy and angular momentum are singular for those solutions.

5.5.4 Two-soliton solutions

For the two-soliton solution α_{ss} in sinh-Gordon, the spinors are

$$\phi_1 = e^{i\tau}\frac{i\sqrt{1-v^2}\sinh T + iv\sinh T}{\sqrt{\cosh^2 T - v^2\cosh^2 X}}, \tag{5.5.4.63}$$

$$\phi_2 = e^{i\tau}\frac{v\sinh X}{\sqrt{\cosh^2 T - v^2\cosh^2 X}}, \tag{5.5.4.64}$$

$$\psi_1 = \frac{(\sqrt{1-v^2}\cosh X + i\sinh T)\cosh\sigma - \sinh X\sinh\sigma}{\sqrt{\cosh^2 T - v^2\cosh^2 X}}, \tag{5.5.4.65}$$

$$\psi_2 = \frac{(-\sqrt{1-v^2}\cosh X + i\sinh T)\sinh\sigma + \sinh X\cosh\sigma}{\sqrt{\cosh^2 T - v^2\cosh^2 X}}, \tag{5.5.4.66}$$

where $X = 2\gamma\sigma$, $T = 2v\gamma\tau$. The two-soliton string solution is [2]

$$Z_1^{ss} = e^{-i\tau}\frac{v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \sqrt{1-v^2}\mathrm{sh}X\mathrm{sh}\sigma + i\sqrt{1-v^2}\mathrm{sh}T\mathrm{ch}\sigma}{\mathrm{ch}T + v\mathrm{ch}X} \tag{5.5.4.67}$$

$$Z_2^{ss} = e^{-i\tau}\frac{v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \sqrt{1-v^2}\mathrm{sh}X\mathrm{ch}\sigma + i\sqrt{1-v^2}\mathrm{sh}T\mathrm{sh}\sigma}{\mathrm{ch}T + v\mathrm{ch}X} \tag{5.5.4.68}$$

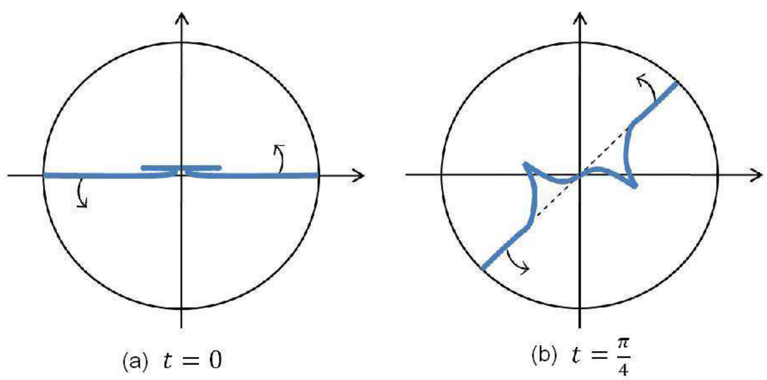

Figure 5.3: The Minkowskian two-soliton solution with $v = \frac{1}{\sqrt{5}}$ at different global time (a) $t = 0$, (b) $t = \pi/4$.

Fig. 5.3 shows the shape of the two-soliton string at two different global time instants. In fig. 5.3(a), the string is folded along the x axis, whereas in fig. 5.3(b), we find the usual bulk spikes.

[2]We occasionally use the notation sh and ch for sinh and cosh to simplify otherwise lengthy formulas.

The two-soliton solution can also be anallytically continued to the Euclidean worldsheet under the $\tau \to -i\tau$ change. Then Y_0 and Y_2 become imaginary and they effectively change place.

The two-antisoliton string solution can be constructed in the same way and it only differs from the two-soliton solution by three signs, the second and third terms in the numerator and the second term in the denominator.

For the soliton-antisoliton $\alpha_{s\bar{s}}$ solution, the result is

$$Z_1^{s\bar{s}} = e^{-i\tau}\frac{v\mathrm{sh}T\mathrm{ch}\sigma \pm \mathrm{sh}X\mathrm{ch}\sigma \mp \sqrt{1-v^2}\mathrm{ch}X\mathrm{sh}\sigma + i\sqrt{1-v^2}\mathrm{ch}T\mathrm{ch}\sigma}{\mathrm{sh}T \pm v\mathrm{sh}X} \tag{5.5.4.69}$$

$$Z_2^{s\bar{s}} = e^{-i\tau}\frac{v\mathrm{sh}T\mathrm{sh}\sigma \pm \mathrm{sh}X\mathrm{sh}\sigma \mp \sqrt{1-v^2}\mathrm{ch}X\mathrm{ch}\sigma + i\sqrt{1-v^2}\mathrm{ch}T\mathrm{sh}\sigma}{\mathrm{sh}T \pm v\mathrm{sh}X} \tag{5.5.4.70}$$

Finally, we take the breather solution of sinh-Gordon (5.4.0.28) and we solve the spinors from (5.3.1.19), (5.3.1.20) to find the string solution

$$Z_1^B = \frac{e^{-i\tau}}{\sin T_B \pm w\mathrm{sh}X_B}\big\{-w\sin T_B\mathrm{sh}\sigma \pm \mathrm{sh}X_B\mathrm{sh}\sigma \\ \mp \sqrt{1+w^2}\mathrm{ch}X_B\mathrm{ch}\sigma + i\sqrt{1+w^2}\cos T_B\mathrm{sh}\sigma\big\}, \tag{5.5.4.71}$$

$$Z_2^B = \frac{e^{-i\tau}}{\sin T_B \pm w\mathrm{sh}X_B}\big\{-w\sin T_B\mathrm{ch}\sigma \pm \mathrm{sh}X_B\mathrm{ch}\sigma \\ \mp \sqrt{1+w^2}\mathrm{ch}X_B\mathrm{sh}\sigma) + i\sqrt{1+w^2}\cos T_B\mathrm{ch}\sigma\big\}, \tag{5.5.4.72}$$

where $X_B = 2\sigma/\sqrt{1+w^2}$ and $T_B = 2w\tau/\sqrt{1+w^2}$.

5.6 *AdS* dressing method

The dressing method allows the construction of solutions to nonlinear classically integrable equations. Many of the equations here are similar to [63] and the reader may look there for further details. Here we use the dressing method to construct new string theory solutions on AdS_3 for a Minkowskian worldsheet.

We recast the system (5.3.0.2, 5.3.0.3) into the form of a principal $SU(1,1)$ chiral model for the matrix-valued field $g(z,\bar{z})$ that satisfies the equation of motion

$$\bar{\partial}A + \partial\bar{A} = 0, \tag{5.6.0.73}$$

where the currents A and $\bar{A}$ are given by

$$A = i\partial g g^{-1}, \tag{5.6.0.74}$$

$$\bar{A} = i\bar{\partial} g g^{-1}. \tag{5.6.0.75}$$

As an example we can consider the AdS_3 case and easily prove the equivalence of equations (5.6.0.73) to equations (5.3.0.2, 5.3.0.3) using the following $SU(1,1)$ parametrization

$$g = \begin{pmatrix} Y_{-1} + iY_0 & Y_1 + iY_2 \\ Y_1 - iY_2 & Y_{-1} - iY_0 \end{pmatrix} \tag{5.6.0.76}$$

that satisfies

$$g^\dagger M g = M, \qquad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \det\, g = 1. \tag{5.6.0.77}$$

The second order system (5.6.0.73) is equivalent to the first order system

$$i\partial\Psi = \frac{A\Psi}{1-\lambda}, \qquad i\bar{\partial}\Psi = \frac{\bar{A}\Psi}{1+\lambda} \tag{5.6.0.78}$$

for the auxiliary field $\Psi(z, \bar{z}, \lambda)$. The complex number λ is called the spectral parameter.

In order to apply the dressing method we start with any known solution that we call the vacuum and we solve (5.6.0.78) to find $\Psi(\lambda)$ subject to the condition

$$\Psi(\lambda = 0) = g. \tag{5.6.0.79}$$

Since we want $\Psi(\lambda)$ to be an $SU(1,1)$ element we further impose the unitarity constraint

$$\Psi^\dagger(\bar{\lambda}) M \Psi(\lambda) = M \tag{5.6.0.80}$$

and demand that

$$\det\, \Psi(0) = 1. \tag{5.6.0.81}$$

Furthermore we consider the transformation

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda) \tag{5.6.0.82}$$

and seek a $\chi(\lambda)$, the dressing factor, that depends on z and $\bar{z}$ in such a way that $\Psi'(\lambda)$ still satisfies (5.6.0.78). In that case $\Psi'(\lambda = 0)$ is a new solution to (5.6.0.73).

For the AdS_3 case we can take the dressing factor to be

$$\chi(\lambda) = I + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P, \tag{5.6.0.83}$$

where λ_1 is an arbitrary complex number and the projector P is given by

$$P = \frac{\upsilon_1 \upsilon_1^\dagger M}{\upsilon_1^\dagger M \upsilon_1}, \quad \upsilon_1 = \Psi(\bar{\lambda_1}) e, \tag{5.6.0.84}$$

where e is an arbitrary vector with constant complex entries called the polarization vector. The projector P does not depend on the length of the e vector.

The determinant of $\chi(\lambda)$ is $\bar{\lambda}_1/\lambda_1$ and if we want our solution to sit in $SU(1,1)$ we should rescale $\chi(\lambda)$ by the compensating factor $\sqrt{\frac{\lambda_1}{\bar{\lambda}_1}}$.

Putting everything together the new solution $g' = \Psi'(\lambda = 0)$ to the system (5.3.0.2, 5.3.0.3) is given by

$$g' = \sqrt{\frac{\lambda_1}{\bar{\lambda}_1}} \left(I + \frac{\lambda_1 - \bar{\lambda}_1}{-\lambda_1} P \right) g. \tag{5.6.0.85}$$

5.6.1 Breather solution

Here we apply the above dressing method to dress the vacuum in order to find new string theory solutions in AdS_3. As a vacuum we choose the solution (5.5.1.30), (5.5.1.31). Using the AdS_3 parametrization (5.6.0.76) we find that the currents A, $\bar{A}$ are given by

$$\begin{aligned} A &= \begin{pmatrix} 1 & ie^{2i\tau} \\ ie^{-2i\tau} & -1 \end{pmatrix}, &(5.6.1.86)\\ \bar{A} &= \begin{pmatrix} -1 & ie^{2i\tau} \\ ie^{-2i\tau} & 1 \end{pmatrix}. &(5.6.1.87)\end{aligned}$$

Then a solution to the system (5.6.0.78) subject to the unitarity constraints yields

$$\Psi(\lambda) = \begin{pmatrix} e^{i\tau}\left(\cosh Z - \frac{i\lambda \sinh Z}{\sqrt{1-\lambda^2}}\right) & \frac{e^{i\tau}\sinh Z}{\sqrt{1-\lambda^2}} \\ \frac{e^{-i\tau}\sinh Z}{\sqrt{1-\lambda^2}} & e^{-i\tau}\left(\cosh Z + \frac{i\lambda \sinh Z}{\sqrt{1-\lambda^2}}\right) \end{pmatrix}, \tag{5.6.1.88}$$

where

$$Z = z\left(\frac{1+\lambda}{1-\lambda}\right)^{1/2} + \bar{z}\left(\frac{1-\lambda}{1+\lambda}\right)^{1/2}. \tag{5.6.1.89}$$

The general solution, that can be read off from the components of the matrix field $g' = \chi g$ in terms of the polarization vector e is rather complicated, so we present here the full solution in the case of $e = (1 \;\; i)$. The dressed solution is

$$\begin{aligned} Y_{-1} + iY_0 &= e^{i\tau}\frac{N_1}{D}, &(5.6.1.90)\\ Y_1 + iY_2 &= e^{i\tau}\frac{N_2}{D}, &(5.6.1.91)\end{aligned}$$

where

$$\begin{aligned} N_1 = \sqrt{1-\lambda_1^2}\cosh Z_1((\bar\lambda_1-\lambda_1)\sqrt{1-\bar\lambda_1^2}\cosh\bar Z_1(\sinh\sigma - i\cosh\sigma) \\ -(\bar\lambda_1-1)\sinh\bar Z_1((\lambda_1+\bar\lambda_1)\cosh\sigma - i(\lambda_1-\bar\lambda_1)\sinh\sigma)) \\ +(\lambda_1-1)\sinh Z_1((\lambda_1-\bar\lambda_1)(\bar\lambda_1-1)\sinh\bar Z_1(i\cosh\sigma+\sinh\sigma) \\ +\sqrt{1-\bar\lambda_1^2}\cosh\bar Z_1((\lambda_1+\bar\lambda_1)\cosh\sigma + i(\lambda_1-\bar\lambda_1)\sinh\sigma)), \end{aligned} \tag{5.6.1.92}$$

$$\begin{aligned} N_2 = \sqrt{1-\lambda_1^2}\cosh Z_1((\bar\lambda_1-\lambda_1)\sqrt{1-\bar\lambda_1^2}\cosh\bar Z_1(\cosh\sigma - i\sinh\sigma) \\ +i(\bar\lambda_1-1)\sinh\bar Z_1((\lambda_1-\bar\lambda_1)\cosh\sigma + i(\lambda_1+\bar\lambda_1)\sinh\sigma)) \\ +(\lambda_1-1)\sinh Z_1((\lambda_1-\bar\lambda_1)(\bar\lambda_1-1)\sinh\bar Z_1(\cosh\sigma+i\sinh\sigma) \\ +\sqrt{1-\bar\lambda_1^2}\cosh\bar Z_1(i(\lambda_1-\bar\lambda_1)\cosh\sigma + (\lambda_1+\bar\lambda_1)\sinh\sigma)), \end{aligned} \tag{5.6.1.93}$$

$$D = 2|\lambda_1|\left((\lambda_1-1)\sqrt{1-\bar\lambda_1^2}\cosh\bar Z_1\sinh Z_1 - \sqrt{1-\lambda_1^2}(\bar\lambda_1-1)\cosh Z_1\sinh\bar Z_1\right), \tag{5.6.1.94}$$

where [3]

$$Z_1 = z\left(\frac{1+\lambda_1}{1-\lambda_1}\right)^{1/2} + \bar z\left(\frac{1-\lambda_1}{1+\lambda_1}\right)^{1/2}, \tag{5.6.1.95}$$

$$\bar Z_1 = z\left(\frac{1+\bar\lambda_1}{1-\bar\lambda_1}\right)^{1/2} + \bar z\left(\frac{1-\bar\lambda_1}{1+\bar\lambda_1}\right)^{1/2}, \tag{5.6.1.96}$$

This is precisely the same solution (5.5.4.71), (5.5.4.72) that we obtained in the previous section using the inverse scattering method as we can easily see by expressing the spectral parameter λ_1 in terms of center mass velocity v_1 and the frequency w_1 of the breather solution by

$$\lambda_1 = \frac{w_1 - iv_1}{w_1 v_1 - i}. \tag{5.6.1.97}$$

5.7 Conventions

Here we summarize the standard conventions for global AdS_3 that we have used in preparing the figures. We parameterize the $SU(1,1)$ group element as

$$\begin{aligned} Z_1 &= e^{it}\sec\theta, \\ Z_2 &= e^{i\phi}\tan\theta, \end{aligned}$$

[3] Z_1, $\bar Z_1$ should not to be confused with the embedding string coordinates in (5.3.1.21), (5.3.1.22).

where t is the global time, ϕ the azimuthal angle, and θ runs from 0 in the interior of the AdS_3 cylinder to $\pi/2$ at the boundary of AdS_3. In terms of these quantities the parametric plots in the figures have Cartesian coordinates

$$(x, y, z) = (\theta \cos \phi, \theta \sin \phi, t)$$

and the boundary of AdS_3 is the cylinder $x^2 + y^2 = (\pi/2)^2$.

Chapter 6

On Dyonic Giant Magnons on CP^3

6.1 Abstract

A new example of AdS/CFT duality relating IIA string theory on $AdS_4 \times CP^3$ to $\mathcal{N} = 6$ superconformal Chern-Simons theory has recently been provided by ABJM. By now a number of papers have considered particular giant magnon classical string solutions in the CP^3 background, corresponding to excitations in the spin chain picture of the dual field theory. In this work we apply the $CP^3 = SU(4)/S(U(3) \times U(1))$ dressing method to the problem of constructing general classical string solutions describing various configurations of giant magnons. As a particular application we present a new giant magnon solution on CP^3.

6.2 Introduction

Motivated by the work of Bagger, Lambert and Gustavsson [103] on maximally superconformal field theories in three dimensions, Aharony, Bergman, Jafferis, and Maldacena (ABJM) constructed [21] an $\mathcal{N} = 6$ superconformal Chern-Simons theory with $U(N) \times U(N)$ gauge symmetry at levels $(k, -k)$ that is believed to be dual to M-theory on $\mathrm{AdS}_4 \times S^7/Z_k$ (see also [104]). ABJM further considered the $N, k \to \infty$ limit keeping the 't Hooft coupling $\lambda = N/k$ fixed and conjectured that in this limit the $\mathcal{N} = 6$ field theory is dual to type IIA string theory on $\mathrm{AdS}_4 \times CP^3$.

Given the important role that integrability has played in exploring the structure of $\mathcal{N} = 4$ Yang-Mills theory and its dual, it is natural that this new example of AdS/CFT provides an arena for further studying aspects of integrability in gauge/string duality.

The worldsheet theory for IIA strings on $\mathrm{AdS}_4 \times CP^3$ has been constructed and its possible integrability explored in [105], while on the Chern-Simons side the anomalous dimensions of local operators are apparently encoded in in integrable spin chain Hamiltonian [106]. An exact magnon S-matrix for this spin chain has been proposed in [107], numerous tests of these proposals have been carried out in [108], and aspects of Wilson loops have been studied in [109].

Hofman and Maldacena [9] identified the string theory dual of an elementary magnon in the spin chain description of $\mathcal{N} = 4$ Yang-Mills theory as a a particular classical open string configuration on an $R \times S^2$ subset of $\mathrm{AdS}_5 \times S^5$, called the giant magnon. The study of giant magnons and their BPS bound states [15, 16] has provided a wealth of detailed information about AdS/CFT. Naturally therefore a number of papers [110, 112, 111, 113, 114] have explored in detail the properties of various giant magnon solutions relevant to the ABJM incarnation of $\mathrm{AdS}_4/\mathrm{CFT}_3$.

The dressing method of Zakharov and Mikhailov [28, 29] provides an algorithm to directly construct solutions of classically integrable equations. This method has proven useful for the construction of various giant magnon solutions, including magnons on spheres [34, 61, 62] and on anti-de Sitter space [25, 26, 27]. An explicit solution describing the scattering of N giant magnons on $R \times S^3$ was also presented in [115], and their dynamics on S^2 was studied in [116].

Since the equations of motion for a string on $R \times CP^3$ are also classically integrable, these techniques can be employed here as well. In this work we demonstrate the application of the dressing method for $SU(4)/S(U(3) \times U(1))$ coset model (due to Harnad et. al. [44]) to the problem of constructing CP^3 giant magnon solutions. An important feature of the dressing method, which has been exploited for example in [34, 61, 62, 115], is that repeated application can be used to generate explicit classical string solutions describing the scattering of any number of giant magnons (or, when applicable, bound states thereof). We present below an explicit solution (4.6) with the interesting feature that the solution depends explicitly on two parameters, and has a formula for the charge $\Delta - J$ which is identical to that of Dorey's two-charge dyonic giant magnon [16], yet (4.6) carries only a single SO(6) charge. We do not present any multi-magnon solutions here, but the algebra involved is no more complicated than for the solutions studied in [61, 115].

6.3 The CP^3 Model

The CP^3 model may be described by a complex four-component vector n with lagrangian density

$$\mathcal{L} = -\partial_\mu n^\dagger \cdot \partial^\mu n + (n^\dagger \cdot \partial_\mu n)(\partial^\mu n^\dagger \cdot n) - \Lambda(n^\dagger \cdot n - 1). \tag{6.3.0.1}$$

The Lagrange multiplier constrains the fields to lie on $S^7 \in C^4$, while the local $U(1)$ invariance of (6.3.0.1) allows us to identify $n \sim e^{i\Lambda(x)} n$, thereby reducing the configuration space to $S^7/U(1) = CP^3$. The action possesses an $SU(4)$ symmetry with Noether currents

$$J^a_\mu = 2\,\mathrm{Im}[(n^\dagger \cdot T^a \partial_\mu n) - (n^\dagger \cdot T^a n)(n^\dagger \cdot \partial_\mu n)], \tag{6.3.0.2}$$

where T^a are generators of $SU(4)$. The equations of motion (after eliminating the Lagrange multiplier) are

$$-\partial^2 n + (n^\dagger \cdot \partial^2 n)n + 2(n^\dagger \cdot \partial_\mu n)\partial^\mu n + 2(\partial_\mu n^\dagger \cdot n)(n^\dagger \cdot \partial^\mu n)n = 0. \tag{6.3.0.3}$$

To describe classical strings on $R \times CP^3$ (with a trivial time coordinate), the equations of motion must be supplemented with the Virasoro constraints

$$\begin{aligned} (\partial_+ n^\dagger \cdot \partial_+ n) - (n^\dagger \cdot \partial_+ n)(\partial_+ n^\dagger \cdot n) &= \frac{1}{4}, \\ (\partial_- n^\dagger \cdot \partial_- n) - (n^\dagger \cdot \partial_- n)(\partial_- n^\dagger \cdot n) &= \frac{1}{4}, \end{aligned} \tag{6.3.0.4}$$

where we have used light-cone coordinates $x_+ = \frac{1}{2}(x - t),\ x_- = \frac{1}{2}(x + t)$ and the derivatives are with respect to those coordinates, $\partial_+ = \partial_x - \partial_t,\ \partial_+ = \partial_x + \partial_t$.

Several classes of solutions to the equation of motion (6.3.0.3) and the Virasoro constraints (6.3.0.4) may be obtained by embedding known giant magnon solutions that live on S^2 or S^3 into CP^3 (an extensive discussion of these embeddings has been given in [111]). As a first example, let (X_1, X_2, X_3) be coordinates satisfying $X_1^2 + X_2^2 + X_3^2 = 1$. An isometric embedding $S^2 \to CP^3$ is given by

$$n^{\mathrm{T}} = \frac{1}{\sqrt{2(1 - X_3)}} \begin{pmatrix} X_1 + iX_2 & 1 - X_3 & 0 & 0 \end{pmatrix}. \tag{6.3.0.5}$$

In this manner any solution $X^i = (X_1, X_2, X_3)$ of string theory on $R \times S^2$

$$\begin{aligned} -\partial^2 X^i + (X \cdot \partial^2 X)X^i &= 0, \\ \partial_+ X \cdot \partial_+ X = \partial_- X \cdot \partial_- X &= 1, \end{aligned} \tag{6.3.0.6}$$

lifts to a solution of (6.3.0.3) and (6.3.0.4). In general it may be necessary to rescale the worldsheet coordinates x, t in order to satisfy (6.3.0.4) with the normalization shown. Such a rescaling does not affect the equations of motion (6.3.0.3). Similarly we can consider what has been called the "$S^2 \times S^2$" embedding in the literature. This is given by the map

$$n^{\mathrm{T}} = \frac{1}{2\sqrt{(1-X_3)}} \begin{pmatrix} X_1 + iX_2 & 1 - X_3 & X_1 + iX_2 & 1 - X_3 \end{pmatrix} \tag{6.3.0.7}$$

whose image inside CP^3 is actually [111] just a single S^2 partially rotated into two orthogonal directions compared to (6.3.0.5).

In the case of magnons living on S^3 we can parameterize the unit 3-sphere with embedding coordinates $X^i = (X_1, X_2, X_3, X_4)$. Given any such solution X^i describing a classical string on $R \times S^3$ there are two possible natural embeddings into a solution of the CP^3 equations, given alternately by

$$n^{\mathrm{T}} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix} \tag{6.3.0.8}$$

or

$$n^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 & X_1 - iX_2 & X_3 - iX_4 \end{pmatrix}, \tag{6.3.0.9}$$

whose images are both $RP^3 \subset CP^3$ [111].

To provide a concrete example we remind the reader of the solution describing Dorey's dyonic magnon [16] on S^3,

$$\begin{aligned} X^1 + iX^2 &= e^{it/2}(\cos\tfrac{p}{2} + i\sin\tfrac{p}{2}\tanh\tfrac{u}{2}), \\ X^3 + iX^4 &= e^{iv/2}\sin\tfrac{p}{2}\,\mathrm{sech}\,\tfrac{u}{2}, \end{aligned} \tag{6.3.0.10}$$

where

$$u = i(Z(\lambda_1) - Z(\bar{\lambda}_1)), \qquad v = Z(\lambda_1) + Z(\bar{\lambda}_1) - t \tag{6.3.0.11}$$

in terms of

$$Z(\lambda) = \frac{x_+}{\lambda - 1} + \frac{x_-}{\lambda + 1}. \tag{6.3.0.12}$$

The resulting CP^3 solution involves putting two conjugate (and hence, oppositely charged) dyonic giant magnons together. The scaling of the world sheet coordinates (x, t) by $1/2$ compared to [16] leaves the equation of motion (6.3.0.3) intact but is necessary in order to preserve the normalization of the Virasoro constraints (6.3.0.4). In the above we have used the parameterization $\lambda_1 = re^{ip/2}$ of the spectral parameter, where p is the momentum of the magnon and r is related to its charge. In the limit $r \to 1$ we recover the "$S^2 \times S^2$" solution presented in [112].

More generally we can take any known giant magnon solution on S^3 (such as the general N-magnon solution found in [115]) and embed it into the CP^3 model, thus obtaining a new classing string solution moving on $R \times CP^3$ (again, it may be necessary to also scale the worldsheet coordinates to preserve the normalization given in (6.3.0.4)). Besides these 'trivial' solutions reviewed here, the CP^3 model admits more general solutions that can be obtained via the dressing method, to which we now turn our attention.

6.4 The Dressing Method for the CP^3 Coset

In order to apply the dressing method as outlined in [44] we first embed the CP^3 vector field n into an $SU(4)$ principal chiral field g. This may be done by noting that

$$CP^3 = \{g \in SU(4) : g\Omega g\Omega = 1\}, \qquad \text{with } \Omega = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (6.4.0.13)$$

In order to understand this embedding, first observe that if g satisfies $g^\dagger g = 1$ and $g\Omega g\Omega = 1$ then the matrix

$$P = \frac{1 + \Omega g}{2} \quad (6.4.0.14)$$

is a hermitian projection operator. Since $\det g = 1$ it follows that $\det(2P - 1) = -1$ so the rank of P must be either 1 or 3. In fact we can without loss of generality take P to have rank 1 since otherwise we could just replace $P \to 1 - P$ throughout this analysis. Then we identify the vector n as the (unit-normalized) image of P.

Conversely, given a unit vector n we take

$$g = \Omega(2P - 1) \qquad \text{with } P = nn^\dagger, \quad (6.4.0.15)$$

which is easily seen to satisfy $g\Omega g\Omega = 1$ and $g \in SU(4)$.

Under this embedding, the lagrangian (6.3.0.1) becomes proportional to

$$\mathcal{L} = \mathrm{Tr}[(g^{-1}\partial_\mu g)^2], \quad (6.4.0.16)$$

the equation of motion (6.3.0.3) becomes equivalent to the principal chiral model equation

$$\partial_+\partial_- g - \frac{1}{2}(\partial_+ g g^{-1}\partial_- g + \partial_- g g^{-1}\partial_+ g) = 0, \tag{6.4.0.17}$$

while the Virasoro constraints (6.3.0.4) map into

$$\mathrm{Tr}[(g^{-1}\partial_+ g)^2] = -2, \qquad \mathrm{Tr}[(g^{-1}\partial_- g)^2] = -2. \tag{6.4.0.18}$$

Next we recall Theorem 4.2 of [44]. Given any solution g of the $SU(4)$ principal chiral model which satisfies $g\Omega g\Omega = 1$, we first solve the auxiliary system

$$\partial_+\Psi = \frac{\partial_+ g g^{-1}\Psi}{1-\lambda}, \qquad \partial_-\Psi = \frac{\partial_- g g^{-1}\Psi}{1+\lambda} \tag{6.4.0.19}$$

to find $\Psi(\lambda)$ as a function of the auxiliary complex parameter λ, subject to the initial condition

$$\Psi(0) = g, \tag{6.4.0.20}$$

the $SU(4)$ constraints

$$\det\Psi(0) = 1, \qquad [\Psi(\bar\lambda)]^\dagger\Psi(\lambda) = 1, \tag{6.4.0.21}$$

as well as the coset constraint

$$\Psi(\lambda) = \Psi(0)\Omega\Psi(1/\lambda)\Omega. \tag{6.4.0.22}$$

With $\Psi(\lambda)$ in hand a new dressed solution to the coset model may be constructed algebraically. The input to specify a new solution is an arbitrary complex parameter λ_1 and an arbitrary complex four-vector e. In terms of this data the dressed solution is $g' = \Psi'(0)$ where

$$\Psi'(\lambda) = \left[1 + \frac{Q_1}{\lambda-\lambda_1} + \frac{Q_2}{\lambda - 1/\lambda_1}\right]\Psi(\lambda) \tag{6.4.0.23}$$

in terms of two matrices $Q_i = X_i F_i^\dagger$ specified by

$$F_1 = \Psi(\bar\lambda_1)e, \qquad F_2 = \Psi(0)\Omega\Psi(\bar\lambda_1)e \tag{6.4.0.24}$$

and the X_i are the solutions to

$$\begin{aligned} X_1\frac{F_1^\dagger F_1}{\lambda_1-\bar\lambda_1} + X_2\frac{F_2^\dagger F_1}{1/\lambda_1-\bar\lambda_1} &= F_1, \\ X_1\frac{F_1^\dagger F_2}{\lambda_1-1/\bar\lambda_1} + X_2\frac{F_2^\dagger F_2}{1/\lambda_1-1/\bar\lambda_1} &= F_2. \end{aligned} \tag{6.4.0.25}$$

6.5 Giant Magnon Solutions on CP^3

In order to obtain new giant magnon solutions on CP^3 via the dressing method described in the previous paragraph we first choose as the vacuum

$$n^{\mathrm{T}} = \begin{pmatrix} \cos(t/2) & \sin(t/2) & 0 & 0 \end{pmatrix}. \tag{6.5.0.26}$$

(a perhaps more obvious, but less useful, choice will be considered below). The scaling of the worldsheet time coordinate by 2 is necessary if we want our vacuum to satisfy the Virasoro constraints (6.3.0.4). Then the solution to the auxiliary system (6.4.0.19) that satisfies the initial condition and constraints is

$$\Psi(\lambda) = \begin{pmatrix} \cos Z(\lambda) & \sin Z(\lambda) & 0 & 0 \\ -\sin Z(\lambda) & \cos Z(\lambda) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.5.0.27}$$

where

$$Z(\lambda) = \frac{x_+}{\lambda - 1} + \frac{x_-}{\lambda + 1}. \tag{6.5.0.28}$$

Choosing (arbitrarily) the polarization vector

$$e^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & i & 0 \end{pmatrix}. \tag{6.5.0.29}$$

we find the solution

$$n^{\mathrm{T}} = \frac{1}{\sqrt{r}} \begin{pmatrix} n^1 & n^2 & n^3 & n^4 \end{pmatrix}, \tag{6.5.0.30}$$

specified by

$$\begin{aligned} n^1 &= +2(1-\lambda_1^2)\bar{\lambda}_1 \cos(t/2) + (1-|\lambda_1|^2)(\lambda_1 \cos(t/2 - iu) + \bar{\lambda}_1 \cos(t/2 + iu)) \\ &\qquad + (\lambda_1 - \bar{\lambda}_1)(\cos(t/2 - v) + |\lambda_1|^2 \cos(t/2 + v)), \\ n^2 &= -2(1-\lambda_1^2)\bar{\lambda}_1 \sin(t/2) - (1-|\lambda_1|^2)(\bar{\lambda}_1 \sin(t/2 + iu) + \lambda_1 \sin(t/2 - iu)) \\ &\qquad - (\lambda_1 - \bar{\lambda}_1)(\sin(t/2 - v) + |\lambda_1|^2 \sin(t/2 + v)), \\ n^3 &= -2i(\lambda_1 - \bar{\lambda}_1)(1-|\lambda_1|^2)\cosh(u/2 + iv/2), \\ n^4 &= 0. \end{aligned} \tag{6.5.0.31}$$

In the above the normalization factor r is given by

$$r = \sum_{i=1}^{4} \bar{n}^i n^i \tag{6.5.0.32}$$

and

$$u = i(Z(\lambda_1) - Z(\bar{\lambda}_1)), \qquad v = Z(\lambda_1) + Z(\bar{\lambda}_1) - t. \tag{6.5.0.33}$$

The solution (6.5.0.31) is identical to the one presented recently in [113] (including, coincidentally, an almost identical choice of polarization vector (6.5.0.29)).

Parameterizing $\lambda_1 = re^{ip/2}$ we find that the solution (6.5.0.30) carries a single nonzero $SU(4)$ charge J with dispersion relation of the form

$$\Delta - J = \frac{1+r^2}{2r}\left|\sin\frac{p}{2}\right|. \tag{6.5.0.34}$$

As usual for giant magnons, the charge J is itself infinite but the excitation energy $\Delta - J$ of the magnon above the ground state (a pointlike string moving at the speed of light) is finite. Remarkably the formula (6.5.0.34) is identical to the corresponding one for Dorey's dyonic giant magnon [16], but the solution (6.5.0.31) carries only a single macroscopic SU(4) charge and is hence not "dyonic" at all. The solution does reduce to the original Hofman-Maldacena magnon [9] with momentum p when $r \to 1$.

A second possible choice of vacuum is

$$n^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{it} & 1 & 0 & 0 \end{pmatrix}, \tag{6.5.0.35}$$

which differs from (6.5.0.26) by an $SU(4)$ rotation which, importantly, does not commute with Ω. In this case the solution to the linear system (6.4.0.19) is

$$\Psi(\lambda) = \frac{1}{\sqrt{1+\lambda}}\begin{pmatrix} \sqrt{\lambda}\,e^{+iZ(\lambda)} & +e^{+iZ(\lambda)} & 0 & 0 \\ -e^{-iZ(\lambda)} & \sqrt{\lambda}\,e^{-iZ(\lambda)} & 0 & 0 \\ 0 & 0 & \sqrt{1+\lambda} & 0 \\ 0 & 0 & 0 & \sqrt{1+\lambda} \end{pmatrix}. \tag{6.5.0.36}$$

If we choose as polarization vector

$$e^{\mathrm{T}} = \begin{pmatrix} 1 & i & 0 & 0 \end{pmatrix} \tag{6.5.0.37}$$

we find the solution

$$\begin{aligned} n^1 &= e^{+it/2}\left((\lambda_1 - \bar{\lambda}_1)|\lambda_1|^2 e^{iv} - (\lambda_1 - \bar{\lambda}_1)e^{-iv} - i(1-|\lambda_1|^2)\lambda_1 e^{u} - i(1-|\lambda_1|^2)\bar{\lambda}_1 e^{-u}\right), \\ n^2 &= e^{-it/2}\left((\lambda_1 - \bar{\lambda}_1)|\lambda_1|^2 e^{-iv} - (\lambda_1 - \bar{\lambda}_1)e^{iv} + i(1-|\lambda_1|^2)\lambda_1 e^{-u} + i(1-|\lambda_1|^2)\bar{\lambda}_1 e^{u}\right), \\ n^3 &= n^4 = 0, \end{aligned} \tag{6.5.0.38}$$

in terms u and v as before in (6.5.0.33). (This solution must of course be normalized to unit length as in (6.5.0.30).) This solution is a bound state of two HM giant magnons on $R \times S^2$ found in (5.14) of [34] under the identification $\lambda = e^{-q/2}e^{ip/2}$.

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