

On conformal supergravity and projective superspace

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ABSTRACT: The projective superspace formulation for four-dimensional $\mathcal{N} = 2$ matter-coupled supergravity presented in arXiv:0805.4683 makes use of the variant superspace realization for the $\mathcal{N} = 2$ Weyl multiplet in which the structure group is $SL(2, \mathbb{C}) \times SU(2)$ and the super-Weyl transformations are generated by a covariantly chiral parameter. An extension to Howe's realization of $\mathcal{N} = 2$ conformal supergravity in which the tangent space group is $SL(2, \mathbb{C}) \times U(2)$ and the super-Weyl transformations are generated by a real unconstrained parameter was briefly sketched. Here we give the explicit details of the extension.

KEYWORDS: Extended Supersymmetry, Superspaces, Supergravity Models

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1 Introduction

Long ago, Howe [1] proposed superspace formulations for four-dimensional $\mathcal{N} \leq 4$ conformal supergravity theories [2–5] by explicitly gauging $SL(2, \mathbb{C}) \times U(\mathcal{N})$ and identifying appropriate constraints on the torsion of curved superspace. In the case $\mathcal{N} = 1$, which had been earlier elaborated in a somewhat different but equivalent setting in [6], the approach of [1] was utilized [7] to provide a unified description for the known off-shell realizations (i.e., the old minimal, new minimal and non-minimal formulations) for $\mathcal{N} = 1$ Poincaré supergravity and the corresponding matter couplings. In the $\mathcal{N} = 2$ case, few applications of Howe’s formulation have appeared – essentially only the demonstration in [1, 8] of how to obtain some off-shell formulations for pure $\mathcal{N} = 2$ Poincaré supergravity by coupling the Weyl multiplet to compensating multiplets, generalizing the $\mathcal{N} = 2$ superconformal tensor calculus [9]. No general discussion of matter couplings within the superspace setting of [1] has been given. Of course, there is a simple historical explanation for that. Even in rigid $\mathcal{N} = 2$ supersymmetry, the adequate approaches for generating off-shell supermultiplets and supersymmetric actions appeared only in 1984; they go under the names harmonic superspace [10, 11] and projective superspace [12–15].¹ The relation of the approach of [1] to the harmonic superspace formulation for $\mathcal{N} = 2$ supergravity and its matter couplings [11, 16] has not been elucidated in detail, except for a short and incomplete discussion in [17].

¹The relationship between the *rigid* harmonic and projective superspace formulations is spelled out in [18]. For a recent discussion, see also [19].

A year ago, we developed a projective superspace formulation for 4D $\mathcal{N} = 2$ supergravity and its matter couplings [20].² In that work, we used an alternative superspace formulation for $\mathcal{N} = 2$ conformal supergravity. It differs from that given in [1] in the following three points: (i) the structure group is identified with $SL(2, \mathbb{C}) \times SU(2)$; (ii) the geometry of curved superspace is subject to the constraints introduced by Grimm [21]; (iii) the super-Weyl transformations are generated by a covariantly chiral but otherwise unconstrained superfield. In [20], we also briefly sketched the correspondence between the two superspace formulations for conformal supergravity. In the present note, we explicitly extend the approach of [20] to the case of Howe’s formulation for conformal supergravity.

This paper is organized as follows. In section 2 we first review the formulation of [1] for $\mathcal{N} = 2$ conformal supergravity, and present the finite form for the corresponding super-Weyl transformations. Using the latter result, we demonstrate how the formulation used in [20] emerges from Howe’s formulation upon gauge fixing the super-Weyl and local $U(1)$ symmetries. In section 3 we introduce a family of covariant projective supermultiplets and propose a locally supersymmetric and super-Weyl invariant action principle.

2 Conformal supergravity

We start by reviewing the superspace formulation for $\mathcal{N} = 2$ conformal supergravity proposed in [1].

2.1 Superspace geometry of conformal supergravity

Consider a curved four-dimensional $\mathcal{N} = 2$ superspace $\mathcal{M}^{4|8}$ parametrized by local coordinates $z^M = (x^m, \theta_i^\mu, \bar{\theta}_{\dot{\mu}}^i)$, where $m = 0, 1, \dots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $i = \underline{1}, \underline{2}$. The Grassmann variables θ_i^μ and $\bar{\theta}_{\dot{\mu}}^i$ are related to each other by complex conjugation: $\overline{\theta_i^\mu} = \bar{\theta}^{\dot{\mu}i}$. Following [1], we choose the structure group to be $SL(2, \mathbb{C}) \times SU(2)_R \times U(1)_R$, and let $M_{ab} = -M_{ba}$, $J_{ij} = J_{ji}$ and \mathbb{J} be the corresponding Lorentz, $SU(2)_R$ and $U(1)_R$ generators. Along with gauge fields for the three subgroups of the structure group, which are necessary to describe the multiplet of conformal supergravity, it is also useful to introduce an Abelian vector multiplet associated with an internal group $U(1)_Z$ with generator \mathbb{Z} such that $[M_{ab}, \mathbb{Z}] = [J_{ij}, \mathbb{Z}] = [\mathbb{J}, \mathbb{Z}] = 0$. One can think of \mathbb{Z} as a central charge operator. The central charge vector multiplet contains the graviphoton. The covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}i}) \equiv (\mathcal{D}_a, \mathcal{D}_{\underline{\alpha}}, \bar{\mathcal{D}}^{\dot{\underline{\alpha}}})$ have the form

$$\begin{aligned} \mathcal{D}_A &= E_A + \frac{1}{2} \Omega_A^{bc} M_{bc} + \Phi_A^{kl} J_{kl} + i \Phi_A \mathbb{J} + V_A \mathbb{Z} \\ &= E_A + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} + \Phi_A^{kl} J_{kl} + i \Phi_A \mathbb{J} + V_A \mathbb{Z} . \end{aligned} \tag{2.1}$$

Here $E_A = E_A^M \partial_M$ is the supervielbein, with $\partial_M = \partial/\partial z^M$, Ω_A^{bc} is the Lorentz connection, Φ_A^{kl} and Φ_A are the $SU(2)_R$ and $U(1)_R$ connections, respectively. Finally, the vector multiplet is described by V_A .

²The harmonic and projective superspace approaches to $\mathcal{N} = 2$ matter-coupled supergravity differ in (i) the structure of covariant off-shell supermultiplets used; and (ii) the locally supersymmetric action principle chosen.

The Lorentz generators with vector indices (M_{ab}) and spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$) are related to each other by the standard rule:

$$M_{ab} = (\sigma_{ab})^{\alpha\beta} M_{\alpha\beta} - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} M_{ab}, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} M_{ab}.$$

The generators of the structure group act on the spinor covariant derivatives as follows:³

$$\begin{aligned} [M_{\alpha\beta}, \mathcal{D}_\gamma^i] &= \varepsilon_{\gamma(\alpha} \mathcal{D}_{\beta)}^i, & [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{\mathcal{D}}_\gamma^i] &= \varepsilon_{\gamma(\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\beta})}^i, \\ [J_{kl}, \mathcal{D}_\alpha^i] &= -\delta_{(k}^i \mathcal{D}_{\alpha l)}, & [J_{kl}, \bar{\mathcal{D}}_i^{\dot{\alpha}}] &= -\varepsilon_{i(k} \bar{\mathcal{D}}_{l)}^{\dot{\alpha}}, \\ [\mathbb{J}, \mathcal{D}_\alpha^i] &= \mathcal{D}_\alpha^i, & [\mathbb{J}, \bar{\mathcal{D}}_i^{\dot{\alpha}}] &= -\bar{\mathcal{D}}_i^{\dot{\alpha}}, \end{aligned} \tag{2.2}$$

while $[\mathbb{Z}, \mathcal{D}_A] = 0$. Our notation and conventions coincide with those adopted in [20] and correspond to [22].

The entire gauge group is generated by local transformations of the form

$$\delta_{\mathcal{K}} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \quad \mathcal{K} = K^C \mathcal{D}_C + \frac{1}{2} K^{cd} M_{cd} + K^{kl} J_{kl} + iL \mathbb{J} + \tau \mathbb{Z}, \tag{2.3}$$

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary. Given a tensor superfield $\mathcal{U}(z)$, with its indices suppressed, it transforms as follows:

$$\delta_{\mathcal{K}} \mathcal{U} = \mathcal{K} \mathcal{U}. \tag{2.4}$$

The covariant derivatives obey the algebra

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} + iR_{AB} \mathbb{J} + F_{AB} \mathbb{Z}, \tag{2.5}$$

where $T_{AB}{}^C$ is the torsion, $R_{AB}{}^{kl}$, R_{AB} and $R_{AB}{}^{cd}$ are the curvatures and F_{AB} the vector multiplet field strength. To describe conformal supergravity, the torsion has to be subject to the following constraints [1]:

$$\begin{aligned} T_{\underline{\alpha}\underline{\beta}}{}^C &= T_{\underline{\alpha}\underline{\beta}}{}^{\dot{\gamma}} = 0, & T_{\alpha j}{}^{\dot{\beta}c} &= -2i\delta_j^i (\sigma^c)_{\alpha}{}^{\dot{\beta}}, \\ T_{\underline{\alpha}b}{}^c &= T_{ab}{}^c = 0, & T_{\alpha\dot{\alpha},\beta k}{}^{j\gamma} &= \frac{1}{2} \delta_\alpha^\gamma T_{\rho\dot{\alpha},\beta k}{}^{j\rho}. \end{aligned} \tag{2.6}$$

The gauge field V_A also has to obey covariant constraints to describe the vector multiplet. The vector multiplet constraints [23] are

$$F_{\alpha\beta}{}^{ij} = -2\varepsilon_{\alpha\beta} \varepsilon^{ij} \bar{W}, \quad F_i{}^{\dot{\alpha}\dot{\beta}} = 2\varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{ij} W, \quad F_{\alpha j}{}^{\dot{\beta}} = 0. \tag{2.7}$$

³The (anti)symmetrization of n indices is defined to include a factor of $(n!)^{-1}$.

The solution to the constraints is as follows:

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = 4S^{ij}M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}Y^{\gamma\delta}M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}^{\gamma\delta}\bar{M}_{\gamma\delta} \\ + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}S^{kl}J_{kl} + 4Y_{\alpha\beta}J^{ij} - 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{W}\mathbb{Z}, \quad (2.8a)$$

$$\{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_j^{\dot{\beta}}\} = -2i\delta_j^i(\sigma^c)_\alpha{}^{\dot{\beta}}\mathcal{D}_c + 4(\delta_j^i G^{\delta\dot{\beta}} + iG^{\delta\dot{\beta}i}{}_j)M_{\alpha\delta} + 4(\delta_j^i G_{\alpha\dot{\gamma}} + iG_{\alpha\dot{\gamma}}{}^i{}_j)\bar{M}^{\dot{\gamma}\dot{\beta}} \\ + 8G_\alpha{}^{\dot{\beta}}J_j{}^i - 4i\delta_j^i G_\alpha{}^{\dot{\beta}kl}J_{kl} - 2(\delta_j^i G_\alpha{}^{\dot{\beta}} + iG_\alpha{}^{\dot{\beta}i}{}_j)\mathbb{J}, \quad (2.8b)$$

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta^j] = -i(\tilde{\sigma}_\alpha)^{\dot{\alpha}\gamma}(\delta_k^j G_{\beta\dot{\alpha}} + iG_{\beta\dot{\alpha}}{}^j{}_k)\mathcal{D}_\gamma^k \\ + \frac{i}{2}\left((\sigma_a)_{\beta\dot{\gamma}}S^{jk} - \varepsilon^{jk}(\sigma_a)_\beta{}^{\dot{\delta}}\bar{W}_{\dot{\delta}\dot{\gamma}} - \varepsilon^{jk}(\sigma_a)^\alpha{}_\gamma Y_{\alpha\beta}\right)\bar{\mathcal{D}}_k^\gamma \\ + \frac{1}{2}R_{\alpha\beta}{}^{jcd}M_{cd} + R_{\alpha\beta}{}^{jkl}J_{kl} + iR_{\alpha\beta}{}^j\mathbb{J} + \frac{i}{2}(\sigma_a)_\beta{}^{\dot{\gamma}}\bar{\mathcal{D}}_j^{\dot{\gamma}}\bar{W}\mathbb{Z}. \quad (2.8c)$$

Here the dimension-1 components of the torsion obey the symmetry properties

$$S^{ij} = S^{ji}, \quad Y_{\alpha\beta} = Y_{\beta\alpha}, \quad W_{\alpha\beta} = W_{\beta\alpha}, \quad G_{\alpha\dot{\alpha}}{}^{ij} = G_{\alpha\dot{\alpha}}{}^{ji} \quad (2.9)$$

and the reality conditions

$$\overline{S^{ij}} = \bar{S}_{ij}, \quad \overline{W_{\alpha\beta}} = \bar{W}_{\dot{\alpha}\dot{\beta}}, \quad \overline{Y_{\alpha\beta}} = \bar{Y}_{\dot{\alpha}\dot{\beta}}, \quad \overline{G_{\beta\dot{\alpha}}} = G_{\alpha\dot{\beta}}, \quad \overline{G_{\beta\dot{\alpha}}{}^{ij}} = G_{\alpha\dot{\beta}ij}. \quad (2.10)$$

The $U(1)_R$ charges of the complex fields are:

$$\mathbb{J}S^{ij} = 2S^{ij}, \quad \mathbb{J}Y_{\alpha\beta} = 2Y_{\alpha\beta}, \quad \mathbb{J}W_{\alpha\beta} = -2W_{\alpha\beta}, \quad \mathbb{J}W = -2W. \quad (2.11)$$

The dimension-3/2 components of the curvature appearing in (2.8c) have the following explicit form:

$$R_{\alpha\beta}{}^j{}_{cd} = -i(\sigma_d)_\beta{}^{\dot{\delta}}T_{ac\dot{\delta}}{}^j + i(\sigma_a)_\beta{}^{\dot{\delta}}T_{cd\dot{\delta}}{}^j - i(\sigma_c)_\beta{}^{\dot{\delta}}T_{da\dot{\delta}}{}^j, \quad (2.12a)$$

$$R_{\alpha\dot{\alpha}}{}^j{}_{kl} = -i\varepsilon^{j(k}\bar{\mathcal{D}}_{\dot{\alpha}}^{l)}Y_{\alpha\beta} - i\varepsilon_{\alpha\beta}\varepsilon^{j(k}\bar{\mathcal{D}}^{\dot{\delta}l)}\bar{W}_{\dot{\alpha}\dot{\delta}} - \frac{i}{3}\varepsilon_{\alpha\beta}\varepsilon^{j(k}\bar{\mathcal{D}}_{\dot{\alpha}q}S^{l)q} \\ + \frac{4}{3}\varepsilon^{j(k}\mathcal{D}_{(\alpha q}G_{\beta)\dot{\alpha}}{}^{l)q} + \frac{2}{3}\varepsilon_{\alpha\beta}\varepsilon^{j(k}\mathcal{D}_q^\delta G_{\delta\dot{\alpha}}{}^{l)q}, \quad (2.12b)$$

$$R_{\alpha\dot{\alpha}}{}^j{}_\beta = -\mathcal{D}_\beta^j G_{\alpha\dot{\alpha}} + \frac{i}{3}\mathcal{D}_{(\alpha k}G_{\beta)\dot{\alpha}}{}^{jk} + \frac{i}{2}\varepsilon_{\alpha\beta}\mathcal{D}_k^\gamma G_{\gamma\dot{\alpha}}{}^{jk}. \quad (2.12c)$$

The right-hand side of (2.12a) involves the dimension-3/2 components of the torsion which are expressed in terms of the dimension-1 tensors as follows:

$$T_{ab\dot{\gamma}}{}^k \equiv (\sigma_{ab})^{\alpha\beta}\mathcal{T}_{\alpha\beta\dot{\gamma}}{}^k - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\mathcal{T}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}^k, \quad (2.13a)$$

$$\mathcal{T}_{\alpha\beta\dot{\gamma}}{}^k = -\frac{1}{4}\bar{\mathcal{D}}_{\dot{\gamma}}^k Y_{\alpha\beta} + \frac{i}{3}\mathcal{D}_{(\alpha}G_{\beta)\dot{\gamma}}{}^k{}_l, \quad (2.13b)$$

$$\mathcal{T}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}{}^k = -\frac{1}{4}\bar{\mathcal{D}}_{\dot{\gamma}}^k \bar{W}_{\dot{\alpha}\dot{\beta}} - \frac{1}{6}\varepsilon_{\dot{\gamma}(\dot{\alpha}}\bar{\mathcal{D}}_{\dot{\beta})l}S^{kl} - \frac{i}{3}\varepsilon_{\dot{\gamma}(\dot{\alpha}}\mathcal{D}_q^\delta G_{\delta\dot{\beta})}{}^{kq}. \quad (2.13c)$$

The dimension-3/2 Bianchi identities are:

$$\mathcal{D}_\alpha^i S^{jk} = 0, \quad \bar{\mathcal{D}}_\alpha^i S^{jk} = i\mathcal{D}^{\beta(i} G_{\beta\dot{\alpha}}^{jk)}, \quad (2.14a)$$

$$\mathcal{D}_\alpha^i \bar{W}_{\dot{\beta}\dot{\gamma}} = 0, \quad (2.14b)$$

$$\mathcal{D}_{(\alpha}^i Y_{\beta\gamma)} = 0, \quad \mathcal{D}_\alpha^i S_{ij} + \mathcal{D}_j^\beta Y_{\beta\alpha} = 0, \quad (2.14c)$$

$$\mathcal{D}_{(\alpha}^i G_{\beta)\dot{\beta}}^{jk} = 0, \quad (2.14d)$$

$$\mathcal{D}_\alpha^i G_{\beta\dot{\beta}} = -\frac{1}{4}\bar{\mathcal{D}}_\beta^i Y_{\alpha\beta} + \frac{1}{12}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot{\beta}j} S^{ij} - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}^{\dot{\gamma}i}\bar{W}_{\dot{\gamma}\dot{\beta}} - \frac{i}{3}\varepsilon_{\alpha\beta}\mathcal{D}_j^\gamma G_{\gamma\dot{\beta}}^{ij}. \quad (2.14e)$$

The Bianchi identities for the vector multiplet are

$$\bar{\mathcal{D}}_i^\alpha W = 0, \quad (2.15a)$$

$$\left(\frac{1}{4}\mathcal{D}^\gamma(i\mathcal{D}_\gamma^j) + S^{ij}\right)W = \left(\frac{1}{4}\bar{\mathcal{D}}_{\dot{\gamma}}^i\bar{\mathcal{D}}^{\dot{\gamma}j} + \bar{S}^{ij}\right)\bar{W} \equiv \Sigma^{ij}, \quad \bar{\Sigma}^{ij} = \Sigma_{ij}. \quad (2.15b)$$

Using the anti-commutation relations (2.8a) and (2.8b), the Bianchi identities (2.14a) and (2.15a), one can check that eq. (2.15b) implies the following relations:

$$\mathcal{D}_\alpha^i \Sigma^{jk} = \bar{\mathcal{D}}_\alpha^i \Sigma^{jk} = 0. \quad (2.16)$$

It should be pointed out that the vector multiplet field strength, F_{ab} , is expressed in terms of the covariantly chiral scalar W and its conjugate as follows:

$$F_{ab} = -\frac{1}{8}(\sigma_{ab})_{\beta\gamma}\mathcal{D}^{\beta k}\mathcal{D}_k^\gamma W - \frac{1}{8}(\bar{\sigma}_{ab})_{\dot{\beta}\dot{\gamma}}\bar{\mathcal{D}}^{\dot{\beta}k}\bar{\mathcal{D}}_k^{\dot{\gamma}}\bar{W} - \frac{1}{4}\left((Y_{ab} + W_{ab})(W + \bar{W}) + \frac{i}{2}\varepsilon_{abcd}(Y^{cd} - W^{cd})(W - \bar{W})\right). \quad (2.17)$$

2.2 Super-Weyl transformations

The constraints (2.6) were shown in [1] to be invariant under infinitesimal super-Weyl transformations generated by a real unconstrained parameter $U = \bar{U}$. We find the finite form of such a transformation to be

$$\mathcal{D}'_\alpha{}^i = e^U \left(\mathcal{D}_\alpha^i + 4(\mathcal{D}^{\gamma i}U)M_{\gamma\alpha} - 4(\mathcal{D}_{\alpha k}U)J^{ki} - (\mathcal{D}_\alpha^i U)\mathbb{J} \right), \quad (2.18a)$$

$$\bar{\mathcal{D}}'_{\dot{\alpha}i} = e^U \left(\bar{\mathcal{D}}_{\dot{\alpha}i} + 4(\bar{\mathcal{D}}_{\dot{\alpha}}^{\dot{j}}U)\bar{M}_{\dot{\gamma}\dot{\alpha}} + 4(\bar{\mathcal{D}}_{\dot{\alpha}}^k U)J_{ki} + (\bar{\mathcal{D}}_{\dot{\alpha}i}U)\mathbb{J} \right), \quad (2.18b)$$

$$\begin{aligned} \mathcal{D}'_{\alpha\dot{\alpha}} &= e^{2U} \left(\mathcal{D}_{\alpha\dot{\alpha}} + 2i(\bar{\mathcal{D}}_{\dot{\alpha}k}U)\mathcal{D}_\alpha^k + 2i(\mathcal{D}_\alpha^k U)\bar{\mathcal{D}}_{\dot{\alpha}k} + 2(\mathcal{D}^\gamma{}_\alpha U)M_{\gamma\alpha} + 2(\mathcal{D}_\alpha{}^{\dot{\gamma}}U)\bar{M}_{\dot{\gamma}\dot{\alpha}} \right. \\ &\quad - 4i(\mathcal{D}^{\gamma k}U)(\bar{\mathcal{D}}_{\dot{\alpha}k}U)M_{\gamma\alpha} + 4i(\mathcal{D}_\alpha^k U)(\bar{\mathcal{D}}_k^{\dot{\gamma}}U)\bar{M}_{\dot{\gamma}\dot{\alpha}} \\ &\quad \left. + 8i(\mathcal{D}_\alpha^k U)(\bar{\mathcal{D}}_{\dot{\alpha}}^l U)J_{kl} + \frac{i}{2}(\mathcal{D}_\alpha^k U)(\bar{\mathcal{D}}_{\dot{\alpha}k}U)\mathbb{J} \right). \end{aligned} \quad (2.18c)$$

These relations imply that the dimension-1 components of the torsion transform as

$$W'_{\alpha\beta} = e^{2U} W_{\alpha\beta}, \quad (2.19a)$$

$$Y'_{\alpha\beta} = e^{2U} \left(Y_{\alpha\beta} - (\mathcal{D}_{(\alpha}^k \mathcal{D}_{\beta)k} U) - 4(\mathcal{D}_{(\alpha}^k U)(\mathcal{D}_{\beta)k} U) \right), \quad (2.19b)$$

$$S'_{ij} = e^{2U} \left(S_{ij} - (\mathcal{D}_{(i}^\gamma \mathcal{D}_{j)\gamma} U) + 4(\mathcal{D}_{(i}^\gamma U)(\mathcal{D}_{j)\gamma} U) \right), \quad (2.19c)$$

$$G'_{\alpha\dot{\alpha}} = e^{2U} \left(G_{\alpha\dot{\alpha}} - \frac{1}{4}[\mathcal{D}_\alpha^k, \bar{\mathcal{D}}_{\dot{\alpha}k}]U - 2(\mathcal{D}_\alpha^k U)(\bar{\mathcal{D}}_{\dot{\alpha}k} U) \right), \quad (2.19d)$$

$$G'_{\alpha\dot{\alpha}}{}^{ij} = e^{2U} \left(G_{\alpha\dot{\alpha}}{}^{ij} + \frac{i}{2}[\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}}^j]U \right). \quad (2.19e)$$

In the infinitesimal case, the above transformation laws reduce to those given in [1]. Of special importance for our consideration below is the fact that the right-hand side in (2.19e) contains no contribution quadratic in derivatives of U .

The super-Weyl transformation of the vector multiplet field strength is

$$W' = e^{2U} W . \tag{2.20}$$

Using this result, one can derive the super-Weyl transformation of the descendant Σ^{ij} introduced in (2.15b). It is

$$\Sigma'_{ij} = e^{4U} \Sigma_{ij} . \tag{2.21}$$

2.3 Partial gauge fixing I

The torsion $G_{\alpha\dot{\alpha}}{}^{ij}$ turns out to be a pure gauge degree of freedom with respect to the super-Weyl symmetry. This means that

$$G_{\alpha\dot{\alpha}}{}^{ij} = -\frac{i}{2} [\mathcal{D}_\alpha^{(i}, \bar{\mathcal{D}}_{\dot{\alpha}}^{j)}] U , \tag{2.22}$$

for some real scalar superfield U . The simplest way to see this is to follow Howe's procedure of introducing the minimal supergravity multiplet [1].

Suppose that the Abelian vector multiplet, which was introduced in subsection 2.1, is such that $W \neq 0$ at each point of the superspace. Under the super-Weyl and local $U(1)_R$ transformations, the field strength changes as

$$W \rightarrow e^{2(U-iL)} W . \tag{2.23}$$

Such a combined transformation acts on $G_{\alpha\dot{\alpha}}{}^{ij}$ according to eq. (2.19e), for $G_{\alpha\dot{\alpha}}{}^{ij}$ is neutral with respect to \mathbb{J} . Since the transformation parameters U and L are real and unconstrained, it is in our power to choose the gauge

$$W = 1 \tag{2.24}$$

which completely fixes the super-Weyl and local $U(1)_R$ symmetries. What are the implications of this gauge fixing? First of all, the condition that W is covariantly chiral implies that $0 = \bar{\mathcal{D}}_i^{\dot{\alpha}} W = -2i \Phi_i^{\dot{\alpha}} = 0$, and therefore

$$\Phi_\alpha^i = \Phi_i^{\dot{\alpha}} = 0 . \tag{2.25}$$

Since the spinor $U(1)_R$ connections vanish, the gauge condition (2.24) and the Bianchi identity (2.15b) lead to

$$S^{ij} = \bar{S}^{ij} . \tag{2.26}$$

Similar arguments give

$$\begin{aligned} 0 &= \mathcal{D}_\alpha^i \bar{\mathcal{D}}_{\dot{\beta}}^j \bar{W} = 2i\varepsilon^{ij} (\sigma^a)_{\alpha\dot{\beta}} \mathcal{D}_a \bar{W} + 4\varepsilon^{ij} G_{\alpha\dot{\beta}} \bar{W} - 4iG_{\alpha\dot{\beta}}{}^{ij} \bar{W} \\ &= -4\varepsilon^{ij} \Phi_{\alpha\dot{\beta}} + 4\varepsilon^{ij} G_{\alpha\dot{\beta}} - 4iG_{\alpha\dot{\beta}}{}^{ij} \end{aligned}$$

and therefore

$$G_{\alpha\dot{\beta}}{}^{ij} = 0, \quad \Phi_{\alpha\dot{\beta}} = G_{\alpha\dot{\beta}}. \quad (2.27)$$

The first equation here tells us that $G_{\alpha\dot{\beta}}{}^{ij}$ vanishes upon imposing the super-Weyl + local $U(1)_R$ gauge condition (2.24). Recalling the super-Weyl transformation law of $G_{\alpha\dot{\beta}}{}^{ij}$, eq. (2.19e), we conclude that the general form for $G_{\alpha\dot{\beta}}{}^{ij}$ is given by eq. (2.22).

2.4 Partial gauge fixing II

In the above consideration, the vector multiplet played the role of a useful technical tool that allowed us to prove eq. (2.22). Since eq. (2.22) has been justified, we can undo the gauge condition (2.24) and return to the general case. Due to (2.22) and the super-Weyl transformation (2.19e), we can use the super-Weyl gauge freedom to choose

$$G_{\alpha\dot{\beta}}{}^{ij} = 0. \quad (2.28)$$

In this gauge, let us introduce new covariant derivatives $\tilde{\mathcal{D}}_A$ defined by the rule:

$$\tilde{\mathcal{D}}_\alpha^i = \mathcal{D}_\alpha^i, \quad \tilde{\mathcal{D}}_a = \mathcal{D}_a - iG_a \mathbb{J}. \quad (2.29)$$

Then, making use of the (anti) commutation relations (2.8a), (2.8b) and (2.8c), one can readily check the covariant derivatives $\tilde{\mathcal{D}}_A$ have no \mathbb{J} -curvature, $\tilde{R}_{AB} = 0$, and therefore the corresponding connection $\tilde{\Phi}_A$ is flat. We can choose $\tilde{\Phi}_A = 0$ by applying an appropriate local $U(1)_R$ transformation. As a result, the superspace geometry proves to reduce to the one used in [20] for the description of general supergravity-matter systems. This geometry corresponds to Grimm's curved superspace setting [21].

Let us suppose that we have chosen the super-Weyl gauge condition (2.28) and also fixed the local $U(1)_R$ symmetry by setting $\Phi_\alpha^i = 0$. Eq. (2.28) does not completely fix the super-Weyl symmetry. In accordance with (2.19e), the residual gauge freedom is described by a parameter U constrained as

$$[\mathcal{D}_\alpha^{(i}, \tilde{\mathcal{D}}_{\dot{\alpha}}^{j)}]U = 0. \quad (2.30)$$

As pointed out in [20], the general solution of this equation is

$$U = \frac{1}{4}(\sigma + \bar{\sigma}), \quad \tilde{\mathcal{D}}_{\dot{i}}^{\dot{\alpha}}\sigma = 0, \quad \mathbb{J}\sigma = 0. \quad (2.31)$$

Here the parameter σ is covariantly chiral but otherwise arbitrary. As follows from (2.18a) and (2.18b), such a super-Weyl transformation must be accompanied by the following compensating $U(1)_R$ -transformation

$$\mathcal{D}'_A = e^{iL\mathbb{J}} \mathcal{D}_A e^{-iL\mathbb{J}}, \quad L = \frac{i}{4}(\sigma - \bar{\sigma}) \quad (2.32)$$

to preserve the gauge condition $\Phi_\alpha^i = 0$. The resulting transformation is

$$\mathcal{D}'_\alpha{}^i = e^{\frac{1}{2}\bar{\sigma}} \left(\mathcal{D}_\alpha^i + (\mathcal{D}^{\gamma i}\sigma)M_{\gamma\alpha} - (\mathcal{D}_{\alpha k}\sigma)J^{ki} \right), \quad (2.33a)$$

$$\tilde{\mathcal{D}}'_{\dot{\alpha}i} = e^{\frac{1}{2}\sigma} \left(\tilde{\mathcal{D}}_{\dot{\alpha}i} + (\tilde{\mathcal{D}}_{\dot{i}}^{\dot{\gamma}}\bar{\sigma})\bar{M}_{\dot{\gamma}\dot{\alpha}} + (\tilde{\mathcal{D}}_{\dot{\alpha}}^{\dot{k}}\bar{\sigma})J_{ki} \right). \quad (2.33b)$$

In the infinitesimal case, this super-Weyl transformation reduces to that given in [20]. The finite super-Weyl transformations of the covariant derivatives, eqs. (2.33a) and (2.33b), and of various components of the torsion were given in [24].

It is interesting to point out analogies between the 4D $\mathcal{N} = 2$ superspace formulation considered with that for 5D $\mathcal{N} = 1$ conformal supergravity⁴ [27]. In the five-dimensional case, the super-Weyl transformations are also generated by a real unconstrained parameter [27]. Moreover, the corresponding superspace torsion includes a vector-isovector component $C_{\hat{a}}^{ij} = C_{\hat{a}}^{ji}$, with the lower index being 5D vector, which can be gauged away by the super-Weyl transformations. This superfield is the 5D analogue of $G_{\alpha\dot{\alpha}}^{ij}$. In the gauge $C_{\hat{a}}^{ij} = 0$, the super-Weyl parameter obeys a constraint which is similar to (2.30).

3 Curved projective superspace

Matter couplings in supergravity are described in [20] in terms of covariant projective supermultiplets. In this section, we first generalize the concept of covariant projective supermultiplets to the case of Howe’s formulation for conformal supergravity, and then we present a locally supersymmetric and super-Weyl invariant action.

3.1 Covariant $O(n)$ supermultiplets

Consider a completely symmetric isotensor superfield $F^{i_1 \dots i_n} = F^{(i_1 \dots i_n)}$. For simplicity, we assume it to be neutral with respect to the central charge generator \mathbb{Z} in (2.1), $\mathbb{Z} F^{i_1 \dots i_n} = 0$, although this condition is not necessary for the discussion below. We require $F^{i_1 \dots i_n}$ to obey the constraints⁵

$$\mathcal{D}_{\alpha}^{(j} F^{i_1 \dots i_n)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(j} F^{i_1 \dots i_n)} = 0 . \tag{3.1}$$

Using the anti-commutation relations (2.8a) and (2.8b), one can check that these constraints are consistent provided the following conditions hold:

- (i) $F^{i_1 \dots i_n}$ is neutral with respect to \mathbb{J} ,

$$\mathbb{J} F^{i_1 \dots i_n} = 0 ; \tag{3.2}$$

- (ii) $F^{i_1 \dots i_n}$ is scalar with respect to the Lorentz group,

$$M_{ab} F^{i_1 \dots i_n} = 0 . \tag{3.3}$$

⁴The superconformal tensor calculus in five dimensions was developed in [25, 26].

⁵Constraints of the form (3.1) have a long history in rigid $\mathcal{N} = 2$ supersymmetry. For $n = 1$ they define an on-shell hypermultiplet [28]; the supermultiplet becomes off-shell if one allows for a non-vanishing intrinsic central charge, $\mathbb{Z} F^i \neq 0$. The case $n = 2$ was considered in [12, 29, 30] and corresponds to the off-shell $\mathcal{N} = 2$ tensor multiplet [31] provided F^{ij} is real. The case $n = 4$ was briefly discussed in [30] in the context of superactions, and it also played a key role in the relaxed hypermultiplet construction [32]. The constraints for arbitrary $n > 2$ first appeared in [33]. These constraints were shown in [14, 34] to provide alternative off-shell formulations for the hypermultiplet if $n = 2m$, with $m = 2, 3, \dots$, and $F^{i_1 \dots i_{2m}}$ is chosen to be real.

Thus, the transformation law (2.4) in the case of $F^{i_1 \dots i_n}$ becomes

$$\delta_{\mathcal{K}} F^{i_1 \dots i_n} = (K^C \mathcal{D}_C + K^{kl} J_{kl}) F^{i_1 \dots i_n}, \quad K^{kl} J_{kl} F^{i_1 \dots i_n} = \sum_{l=1}^n K^{i_l j} F^{j i_1 \dots \widehat{i_l} \dots i_n}. \quad (3.4)$$

One can associate with $F^{i_1 \dots i_n}$ a holomorphic tensor field on $\mathbb{C}P^1$, $F^{(n)}(u^+)$, defined as

$$F^{(n)}(u^+) = u_{i_1}^+ \dots u_{i_n}^+ F^{i_1 \dots i_n}, \quad F^{(n)}(c u^+) = c^n F^{(n)}(u^+), \quad c \in \mathbb{C} \setminus \{0\}, \quad (3.5)$$

with $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$ homogeneous coordinates for $\mathbb{C}P^1$.

It is useful to take the auxiliary variables u_i^+ to be inert⁶ under the local $SU(2)_R$ group, that is $[J_{kl}, u_i^+] = 0$, for their sole role is to describe $F^{i_1 \dots i_n}$ in terms of the index-free object $F^{(n)}(u^+)$. Then, the transformation law (3.4) can be rewritten as

$$\begin{aligned} \delta_{\mathcal{K}} F^{(n)} &= (K^C \mathcal{D}_C + K^{kl} J_{kl}) F^{(n)}, \\ K^{kl} J_{kl} F^{(n)} &= -\frac{1}{(u^+ u^-)} (K^{++} D^{(-1,1)} - n K^{+-}) F^{(n)}, \quad K^{\pm\pm} = K^{ij} u_i^{\pm} u_j^{\pm}, \end{aligned} \quad (3.6)$$

where

$$D^{(-1,1)} := u^{-i} \frac{\partial}{\partial u^{+i}}. \quad (3.7)$$

Eq. (3.6) involves an additional complex two-vector, u_i^- , which has to be linearly independent of u_i^+ , that is $(u^+ u^-) := u^{+i} u_i^- \neq 0$, and is otherwise completely arbitrary. It is important to note that since the u_i^+ are fixed and constant, $F^{(n)}(u^+)$ is not isoscalar. In this approach, the u_i^+ serve merely to totally symmetrize all $SU(2)_R$ indices.

Without imposing the constraints (3.1) and their corollaries (3.2) and (3.3), the above consideration can be naturally generalized. Namely, one can allow $F^{i_1 \dots i_n} = F^{(i_1 \dots i_n)}$ to carry any number of Lorentz indices and have a non-vanishing \mathbb{J} -charge. Let $F^{(n)}(u^+)$ be the homogeneous polynomial of degree n associated with $F^{i_1 \dots i_n}$. An operation of multiplication is naturally defined in the space of such polynomials, for given two homogeneous polynomials $F^{(n)}(u^+)$ and $F^{(m)}(u^+)$, their product $F^{(n+m)}(u^+) := F^{(n)}(u^+) F^{(m)}(u^+)$ is a homogeneous polynomials of degree $(n+m)$. If one introduces the differential operators $\mathcal{D}_{\alpha}^+ := u_i^+ \mathcal{D}_{\alpha}^i$ and $\bar{\mathcal{D}}_{\alpha}^+ := u_i^+ \bar{\mathcal{D}}_{\alpha}^i$, then

$$\mathcal{D}_{\alpha}^+ F^{(n)}(u^+) = u_j^+ u_{i_1}^+ \dots u_{i_n}^+ \mathcal{D}_{\alpha}^{(j} F^{i_1 \dots i_n)}, \quad \bar{\mathcal{D}}_{\alpha}^+ F^{(n)}(u^+) = u_j^+ u_{i_1}^+ \dots u_{i_n}^+ \bar{\mathcal{D}}_{\alpha}^{(j} F^{i_1 \dots i_n)}$$

are homogeneous polynomials of degree $(n+1)$. Here we have used the fact that the auxiliary variables u_i^+ are inert under the local $SU(2)_R$ group, $[J_{kl}, u_i^+] = 0$.

The example of $F^{(n)}$'s considered can naturally be extended to define more general *isotwistor* superfields. They are introduced similarly to the consideration given in the appendix in [20]. The only difference from [20] is that now an isotwistor superfield may have a non-vanishing \mathbb{J} -charge.

⁶This is similar to the approach often used in the context of higher spin field theories, see e.g. [35].

Let us now return to the constraints (3.1). They are equivalent to

$$\mathcal{D}_\alpha^+ F^{(n)} = \bar{\mathcal{D}}_\alpha^+ F^{(n)} = 0 . \tag{3.8}$$

When acting on isotwistor superfields, the differential operators \mathcal{D}_α^+ and $\bar{\mathcal{D}}_\alpha^+$ obey the following anti-commutation relations:

$$\{\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+\} = 4S^{++} M_{\alpha\beta} + 4Y_{\alpha\beta} J^{++} , \tag{3.9a}$$

$$\{\mathcal{D}_\alpha^+, \bar{\mathcal{D}}_\beta^+\} = 4iG_{\dot{\beta}}^{\gamma++} M_{\alpha\gamma} - 4iG_{\alpha}^{\dot{\gamma}++} \bar{M}_{\dot{\beta}\dot{\gamma}} + 8G_{\alpha\dot{\beta}} J^{++} - 2iG_{\alpha\dot{\beta}}^{++} \mathbb{J} , \tag{3.9b}$$

where we have defined

$$J^{++} := u_i^+ u_j^+ J^{ij} , \quad S^{++} := u_i^+ u_j^+ S^{ij} , \tag{3.10}$$

and similarly for $G_{\alpha\dot{\beta}}^{++}$. The constraints (3.8) are consistent because the integrability condition $J^{++} F^{(n)} = 0$ holds identically. The other integrability conditions for the constraints (3.8) are: $\mathbb{J} F^{(n)} = 0$ and $M_{ab} F^{(n)} = 0$. Following [20], the superfield $F^{(n)}$ will be called a covariant $O(n)$ supermultiplet.

As an example of $O(n)$ supermultiplets, we can consider the $O(2)$ multiplet

$$\Sigma^{++} = u_i^+ u_j^+ \Sigma^{ij} , \tag{3.11}$$

with Σ^{ij} defined in (2.15b).

Using O -type supermultiplets, $F^{(n)}$ and $H^{(m)}$, one can construct covariant rational supermultiplets of the form

$$R^{(n-m)}(u^+) = \frac{F^{(n)}(u^+)}{H^{(m)}(u^+)} , \tag{3.12}$$

which correspond to meromorphic tensor fields on $\mathbb{C}P^1$. The $R^{(p)}(u^+)$ possesses properties which are completely similar to (3.6) and (3.8). In the rigid supersymmetric case, rational supermultiplets were introduced in [14]. The above superfields are examples of covariant projective supermultiplets we will now introduce.

3.2 Covariant projective supermultiplets

By definition, a covariant projective supermultiplet of weight n , $Q^{(n)}(z, u^+)$, is a *scalar* superfield that lives on $\mathcal{M}^{4|8}$, is holomorphic on an open domain of $\mathbb{C}^2 \setminus \{0\}$ with respect to the homogeneous coordinates u_i^+ for $\mathbb{C}P^1$, and is characterized by the conditions:

- (i) it obeys the covariant constraints

$$\mathcal{D}_\alpha^+ Q^{(n)} = \bar{\mathcal{D}}_\alpha^+ Q^{(n)} = 0 ; \tag{3.13}$$

- (ii) it is a homogeneous function of u^+ of degree n , that is,

$$Q^{(n)}(z, c u^+) = c^n Q^{(n)}(z, u^+) , \quad c \in \mathbb{C} \setminus \{0\} ; \tag{3.14}$$

(iii) it is neutral with respect to \mathbb{J} :

$$\mathbb{J} Q^{(n)}(z, u^+) = 0 \quad (3.15)$$

(iv) the supergravity gauge transformations act on $Q^{(n)}$ as follows:

$$\begin{aligned} \delta_{\mathcal{K}} Q^{(n)} &= \left(K^C \mathcal{D}_C + K^{kl} J_{kl} \right) Q^{(n)}, \\ K^{kl} J_{kl} Q^{(n)} &= -\frac{1}{(u^+ u^-)} \left(K^{++} D^{(-1,1)} - n K^{+-} \right) Q^{(n)}. \end{aligned} \quad (3.16)$$

Using eqs. (3.9a) and (3.9b) one can see that these definitions are consistent. The integrability condition for the constraints (3.13) is $J^{++} Q^{(n)} = 0$, and clearly it holds identically.

What are admissible super-Weyl transformations of projective supermultiplets? Assuming that $Q^{(n)}$ transforms homogeneously under the super-Weyl transformations, the constraints (3.13) uniquely fix its transformation law:

$$\delta_U Q^{(n)} = 2nU Q^{(n)}. \quad (3.17)$$

On the space of covariant projective supermultiplets, one can introduce a generalized (smile) conjugation $Q^{(n)}(u^+) \rightarrow \tilde{Q}^{(n)}(u^+)$, with $\tilde{Q}^{(n)}$ also being a covariant projective supermultiplet. The smile-conjugation is defined in [20]. If n is even, one can consistently define real supermultiplets.

If one partially fixes the super-Weyl symmetry as in (2.28) as well as imposes the $U(1)_R$ gauge condition (2.25), the above definitions and properties reduce to those given in [20].

3.3 Action principle

Within the curved superspace setting under consideration, the construction of supersymmetric action principle is practically identical to that given in [20]. Let \mathcal{L}^{++} be a real projective multiplet of weight two, with the super-Weyl transformation law

$$\delta_U \mathcal{L}^{++} = 4U \mathcal{L}^{++}. \quad (3.18)$$

Associated with \mathcal{L}^{++} is the following functional:

$$S = \frac{1}{2\pi} \oint (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} E \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2}, \quad E^{-1} = \text{Ber}(E_A{}^M). \quad (3.19)$$

By construction, this functional is invariant under re-scalings $u_i^+(t) \rightarrow c(t) u_i^+(t)$, for an arbitrary function $c(t) \in \mathbb{C} \setminus \{0\}$, where t denotes the evolution parameter along the closed integration contour. Since $\mathbb{J}E = 0$ and $\mathbb{J}(W\bar{W}) = 0$, S is invariant under the local $U(1)$ transformations. Using this observation, the above functional can be shown to be invariant under arbitrary supergravity gauge transformations, eqs. (2.3) and (2.4), in complete analogy with [20]. Since E is invariant under the super-Weyl transformations,

$$\delta_U E = 0, \quad (3.20)$$

the transformation laws (2.20), (2.21) and (3.18) tell us that S is super-Weyl invariant.

In the super-Weyl and local $U(1)_R$ gauge defined by eqs. (2.28) and (2.25), the action (3.19) reduces to that proposed in [20].

The locally supersymmetric and super-Weyl invariant action (3.19) is suitable to describe the dynamics of general $\mathcal{N} = 2$ supergravity-matter system including the formulations of Poincaré supergravity introduced in [20, 37]. In particular this is true for chiral actions of the form

$$S_c = \int d^4x d^4\theta \mathcal{E} \mathcal{L}_c + \text{c.c.}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{L}_c = 0, \quad \mathbb{J} \mathcal{L}_c = -4\mathcal{L}_c, \quad \delta_U \mathcal{L}_c = 4U \mathcal{L}_c, \quad (3.21)$$

with \mathcal{E} the chiral density [8, 36]. The latter follows from the fact that S_c admits the following representation [37]:

$$\begin{aligned} S_c &= \frac{1}{2\pi} \oint (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} E \frac{W\bar{W} \mathcal{L}_c^{++}}{(\Sigma^{++})^2}, \\ \mathcal{L}_c^{++} &= -\frac{1}{4} V \left\{ \left((\mathcal{D}^+)^2 + 4S^{++} \right) \frac{\mathcal{L}_c}{W} + \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) \frac{\bar{\mathcal{L}}_c}{\bar{W}} \right\}, \end{aligned} \quad (3.22)$$

with $V(u^+)$ the tropical prepotential for the vector multiplet with field strength W , see [20] for the definition of $V(u^+)$.

4 Conclusion

For many years, Howe's superspace formulation for $\mathcal{N} = 2$ conformal supergravity [1] has remained a nice theoretical construction of purely academic interest. In the present paper, we demonstrated that the curved superspace setting of [1] is ideally suited for the construction of various matter couplings as well as a superspace action. For practical calculations, however, it is useful to work in the super-Weyl and local $U(1)_R$ gauge (2.28) and (2.25), in which the general supergravity-matter systems reduce to those presented in [20].

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