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DIRAC EQUATION ON SCHWARZSCHILD SPACE-TIME IN EDDINGTON-FINKELSTEIN COORDINATES

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Dirac equation on Schwarzschild space-time is separated in Eddington-Finkelstein type coordinates in terms of spin-weighted spherical angular functions.

1. INTRODUCTION

It is well known that the pioneering works treating Dirac equation on curved space time was realized by Fock and Ivanenko [1]. Subsequently the Diracequation on Schwarzschild spacetime (see f.e. 2]) was treated by Sokolov and Ivanenko [3] and Brill and Wheeler [4]. The separation of variables was perfected in these works using the standard Schwarzschild coordinates.

Unruh [5] has obtained approximate analytical solutions to Dirac equation in the long wave length limit of Schwarzschild background in the same coordinates. This work was developed in the sequel for Kerr and Kerr-Newman spacetimes (see for references [6,7]. It is of interest to investigate the Dirac equation on the interior region of a Schwarzschild black hole, i.e. for $r < r_g = 2GM/c^2$, where M is the mass of a black hole. This paper deals with the examination of Dirac's equation in Eddington-Finkelstein type coordinates [8—10] which permit to make this.

2. SCHWARZSCHILD GEOMETRY. EDDINGTON-FINKELSTEIN TYPE COORDINATES

Schwarzschild geometry is an empty spacetime out of a massive spherical body or a black hole. This spacetime is described by a metric of the form:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 v d\varphi^2) \qquad (1)$$

where $r > r_g = 2GM/c^2$.

It is interesting to see what is when the metric is considered for all the values of the radial coordinate r. Then, the solution is singular for $r = 0$ and $r = r_g$. The singularity $r = r_g$ is an unphysical singularity, as one may see no scalar polynomial of the curvature tensor and of the metric diverge as $r \to r_g$. This suggests this singularity is one which is a result of a bad choice of coordinates. We can eliminate this by defining

$$r^* = \int \frac{dr}{1 - \frac{r_g}{r}} = r + r_g \ln\left(\frac{r}{r_g} - 1\right) \tag{2}$$

and [8.9]

$$t' = t + r^* \tag{3}$$

We shall try to separate the Dirac equation in modified Eddington-Finkelstein coordinate [10]

$$t' = t + r^* - r \tag{4}$$

Using coordinates (t', r, θ, φ) the metric takes the form g' given by

$$ds^2 = \left(-\frac{r_g}{r}\right) dt'^2 - \left(1 + \frac{r_g}{r}\right) \cdot dr^2 - 2\frac{r_g}{r} dt'\, dr - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{5}$$

The connection components for our metric are:

$$\Gamma^0_{00} = \frac{1}{2}\frac{r_g^2}{r^3}, \quad \Gamma^0_{01} = \frac{1}{2}\frac{r_g}{r^2}\left(1 + \frac{r_g}{r}\right),$$

$$\Gamma^0_{11} = \frac{1}{2}\frac{r_g}{r^2}\left(2 + \frac{r_g}{r}\right),$$

$$\Gamma^0_{22} = -r_g,$$

$$\Gamma^0_{33} = -r_g \sin^2\theta,$$

$$\Gamma^1_{00} = \frac{1}{2}\frac{r_g}{r^2}\left(1 - \frac{r_g}{r^2}\right), \quad \Gamma^1_{01} = -\frac{1}{2}\frac{r_g^2}{r^3}, \tag{6}$$

$$\Gamma^1_{11} = -\frac{1}{2}\frac{r_g}{r^2}\left(1 + \frac{r_g}{r}\right),$$

$$\Gamma^1_{22} = -r\left(1 - \frac{r_g}{r}\right),$$

$$\Gamma^1_{33} = -r\left(1 - \frac{r_g}{r}\right)\sin^2\theta,$$

$$\Gamma^2_{12} = \frac{1}{r},$$

$$\Gamma^{2}_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^{3}_{13} = \frac{1}{r},$$

$$\Gamma^{3}_{23} = \operatorname{ctg}\theta.$$

The resting connection coefficients are null.

3. DIRAC EQUATION

One generalizes γ matrices from the flat space-time to function of position on space-time manifold and imposes the following condition:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I \tag{7}$$

The Dirac equation is

$$(i\gamma^{\mu}D_{\mu} - \mu)\psi = 0 \tag{8}$$

with D_{μ} covariant derivative:

$$D_{\mu} = \partial_{\mu} - \Gamma_{\mu} \tag{9}$$

where Γ_{μ} are the Fock-Ivanenko [2] coefficients:

$$\Gamma_{\mu} = -\frac{1}{4}\gamma^{\nu}(\gamma_{\nu,\mu} - \gamma_{\rho}\Gamma^{\rho}_{\nu\mu}) \tag{10}$$

The γ matrices are

$$\gamma^{t} = \left(1 - \frac{r_g}{r}\right)^{-1/2}\gamma^{(0)} + \frac{r_g}{r}(\sin\theta\cos\varphi\gamma^{(1)} + \sin\theta\sin\varphi\gamma^{(2)} + \cos\theta\gamma^{(3)})$$

$$\gamma^{r} = \left(1 - \frac{r_g}{r}\right)^{1/2}\cdot(\sin\theta\cos\varphi g^{(1)} + \sin\theta\sin\varphi\gamma^{(2)} + \cos\theta\,\gamma^{(3)})$$

$$\gamma^{\theta} = \frac{1}{r}\cdot(\cos\theta\cos\varphi\gamma^{(1)} + \cos\theta\sin\varphi\gamma^{(2)} - \sin\theta\gamma^{(3)})$$

$$\gamma^{\varphi} = \frac{1}{r\sin\theta}\cdot(-\sin\varphi\gamma^{(1)} + \cos\varphi\gamma^{(2)}) \tag{11}$$

$$\gamma_{t} = \left(1 - \frac{r_g}{r}\right)^{1/2}\cdot\gamma_{(0)}$$

$$\gamma_{r} = \left(1 - \frac{r_g}{r}\right)^{-1/2}\left(-\frac{r_g}{r}\gamma_{(0)} + \sin\theta\cos\varphi\gamma_{(1)} + \sin\theta\sin\varphi\gamma_{(2)} + \cos\theta\gamma_{(3)}\right)$$

$$\gamma_{\theta} = r(\cos\theta\cos\varphi\gamma_{(1)} + \cos\theta\sin\varphi\gamma_{(2)} - \sin\theta\gamma_{(3)})$$

$$\gamma_{\varphi} = r\sin\theta(-\sin\varphi\gamma_{(1)} + \cos\varphi\gamma_{(2)})$$

where $\gamma^{(a)}$ and are the well-known γ matrices for flat spacetime which satisfy

$$\gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)} = 2\eta^{(a)(b)} = \text{diag}\,(1, -1, -1, -1) \tag{12}$$

It is straightforward to calculate the Γ_μ coefficients, which are as follows

$$\Gamma_t = -\frac{r_g}{4r^2}(\sin\theta\,\text{cis}\,\varphi\gamma^{(0)}\gamma^{(1)} + \sin\theta\sin\varphi\gamma^{(0)}\gamma^{(2)} + \cos\theta\gamma^{(0)}\gamma^{(3)})$$

$$\Gamma_r = \frac{r_g^2}{4r^3}\left(1 - \frac{r_g}{r}\right)^{-1}(\sin\theta\cos\varphi\gamma^{(0)}\gamma^{(1)} + \sin\theta\sin\varphi\gamma^{(0)}\gamma^{(2)} + \cos\theta\gamma^{(0)}\gamma^{(3)})$$

$$\Gamma_\theta = -\frac{1}{2}\left(1 - \left(1 - \frac{r_g}{r}\right)^{1/2}\right)(\cos\varphi\gamma^{(1)}\gamma^{(3)} + \sin\varphi\gamma^{(2)}\gamma^{(3)}) \tag{13}$$

$$\Gamma_\varphi = \frac{1}{2}\left(1 - \left(1 - \frac{r_g}{r}\right)^{1/2}\right).$$

$$\cdot\,\sin\theta\,(\sin\theta\gamma^{(1)}\gamma^{(2)} + \cos\theta\,\sin\varphi\gamma^{(1)}\gamma^{(3)} - \cos\theta\cos\varphi\gamma^{(2)}\gamma^{(3)})$$

In the following, the Dirac equation is written as

$$i\left\{\left(1 + \frac{r_g}{r}(\mathrm{n}\alpha)\right)\partial_t + \frac{\left(1 - \frac{r_g}{r}\right)^{3/4}(\mathrm{n}\alpha)}{r}\left(\partial_r r\left(1 - \frac{r_g}{r}\right)^{1/4}\right) + \right.$$

$$\left. + \left(1 - \frac{r_g}{r}\right)^{1/2} i(\alpha\mathrm{n}) - \left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}}(\mathrm{n}\,\alpha)(\partial_r r) - \frac{1}{r}\left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}}(\mathrm{n}\alpha)\right\}\psi - \mu\psi = 0 \tag{14}$$

We have denoted:

$$\alpha^{(i)} = \gamma^{(0)}\gamma^{(i)},\ (\mathrm{n}\alpha) = \sin\theta\cos\varphi\alpha^{(1)} + \sin\theta\sin\varphi\alpha^{(2)} + \cos\theta\alpha^{(3)} \tag{15}$$

This equation will be separated if we'll consider the following solutions:

$$\psi_{\epsilon km} = \frac{e^{-i\epsilon t}}{r(1 - r_g/r)^{1/4}}\begin{pmatrix} G_k(r)\eta_{km}(\vartheta, \varphi) \\ -iF_k(r)(\mathrm{n}\,\sigma)\eta_{km}(\vartheta, \varphi)\end{pmatrix} \tag{16}$$

One obtains:

$$\rho\frac{dG_k(r)}{dr} + \left(\frac{k}{r} - \frac{i\epsilon}{\rho}\frac{r_g}{r}\right)G_\epsilon(r) = \left(\frac{\epsilon}{\rho} + \mu\right)F_k(r)$$

$$\rho\frac{dF_k(r)}{dr} - \left(\frac{k}{r} + \frac{i\epsilon}{\rho}\frac{r_g}{r}\right)F_k(r) = \left(-\frac{\epsilon}{\rho} + \mu\right)G_\epsilon(r) \tag{17}$$

where $\rho = (1 - r_g/r)^{1/2}$.

If one sets

$$G_k(r) = e^{i\epsilon r_g \ln\left(\frac{r}{r_g}-1\right)} \bar{G}_k(r) \tag{18}$$

and

$$F_k(r) = e^{i\epsilon r_g \ln\left(\frac{r}{r_g}-1\right)} \bar{F}_k(r) \tag{19}$$

we'll obtain the same equations as in [5]

$$\rho \frac{d\bar{G}_k(r)}{dr} + \frac{k}{r} \bar{G}_k(r) = \left(\frac{\epsilon}{r} + \mu\right) \bar{F}_k(r), \tag{20}$$

$$\rho \frac{d\bar{F}_k(r)}{dr} + \frac{k}{r} \bar{F}_k(r) = \left(-\frac{\epsilon}{r} + \mu\right) \bar{G}_k(r). \tag{21}$$

where k is a positive or negative nonzero integer, $l = \left|k - \frac{1}{2}\right| - \frac{1}{2}$, and

$$\gamma_{km} = \begin{cases} \begin{pmatrix} \left(\frac{l-m+1/2}{2l+1}\right)^{1/2} \Upsilon_{1,m-1/2}(\theta, \varphi) \\ \left(\frac{l+m+1/2}{2l+1}\right)^{1/2} \Upsilon_{1,m+1/2}(\theta, \varphi) \end{pmatrix}, & k > 0 \\ (-1)^k i \begin{pmatrix} \left(\frac{l+m+1/2}{2l+1}\right)^{1/2} \Upsilon_{1,m-1/2}(\theta, \varphi) \\ -\left(\frac{l-m+1/2}{2l+1)}\right)^{1/2} \Upsilon_{1,m+1/2}(\theta, \varphi) \end{pmatrix}, & k < 0 \end{cases} \tag{22}$$

with σ_i as the Pauli marices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

and

$$(n\sigma) = \sin\theta \cos\varphi\, \sigma_1 + \sin\theta \sin\varphi\, \sigma_2 + \cos\theta\, \sigma_3 \tag{24}$$

4. CONCLUSIONS

We have presented a procedure of separation of variables of Dirac equation in a Schwarzschild space time in Eddington-Finkelstein coordinates. The solutions obtained are valuable for analyzing the behavior of Dirac's wave function inside the events horizon of a black hole.

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