Rapid Communications

Multiple-axion framework

Thomas C. Bachlechner,[1] Kate Eckerle,[1] Oliver Janssen,[2,3] and Matthew Kleban[2,3]
[1]*Department of Physics, Columbia University, New York 10027, New York, USA*
[2]*Center for Cosmology and Particle Physics, New York University, New York 10003, New York, USA*
[3]*New York University Abu Dhabi, P.O. Box 129188, Saadiyat Island, Abu Dhabi, UAE*

(Received 1 April 2017; published 5 September 2018)

We describe a systematic framework for periodic potentials of arbitrary dimension, such as those governing multiple axions. A novel type of alignment renders even complex theories analytically tractable. Theories with ~100 axions and random parameters have an exponential number of metastable vacua. Decay from a metastable minimum can occur via a thin-wall instanton and allows for a sufficient period of slow-roll inflation that ends in a vacuum containing axion dark matter and a cosmological constant, both consistent with current observations. Hence, this model can reproduce many macroscopic features of our Universe without tuned parameters.

DOI: 10.1103/PhysRevD.98.061301

I. INTRODUCTION

Axionic fields arise in a number of contexts [1–3]. Like quantized fluxes [4], theories of multiple axions [5] can naturally accommodate the observed cosmological constant. Furthermore, the unbroken discrete shift symmetry of axions provides a natural inflaton candidate [3,6–10], and the absence of direct detection and the issues with conventional dark matter models at subgalaxy length scales have led to a recent surge of interest in ultralight axions as dark matter [11–15]. In this paper we demonstrate that theories of multiple axions with a single energy scale near the fundamental scale and no small parameters can simultaneously fulfill all three of these roles. These theories also provide high energy metastable vacua that can serve as an initial or generic state for the Universe.

The potential governing N axions θ^i is protected by continuous shift symmetries $\theta^i \to \theta^i + c^i$ to all orders in perturbation theory, all of which are broken by $P \geq N$ leading nonperturbative effects at scales $\Lambda_I^4 \sim e^{-S_I}\Lambda_{\mathrm{UV}}^4$, where S_I denotes the action of the Ith instanton. The axions couple to each instanton through an integer charge matrix $\boldsymbol{\mathcal{Q}}^I$, resulting in a nonperturbative potential of the form

$$V_{\mathrm{np}} = \sum_{I=1}^{P} \Lambda_I^4[1 - \cos(\mathcal{Q}^I_j\theta^j + \delta^I)] + \cdots \tag{1}$$

where the phases of the instantons are denoted by δ^I and ellipses represent subleading terms [16].

A systematic analysis of (1) is made possible by a novel technique that renders theories of N axions computationally and analytically tractable. The technique is applicable well beyond axions, to a broad class of theories involving degrees of freedom with discrete shift symmetries. It allows us to efficiently identify both the exact and, crucially, the approximate symmetries of such theories. The result is a tool that can analyze many features of extremely complex energy landscapes with exponentially many local minima in polynomial time.

In this paper we present some key results, while further details will be published elsewhere [17–19].

II. ALIGNMENT

Consider a theory with N axions θ^i,

$$\mathcal{L} = \frac{1}{2}\partial\boldsymbol{\theta}^\top \boldsymbol{K}\partial\boldsymbol{\theta} - V_{\mathrm{np}}(\boldsymbol{\theta}) - V_0, \tag{2}$$

where $\boldsymbol{K}$ is the field space metric and V_0 denotes a background vacuum energy density.

The P leading terms in the axion potential (1) are invariant under the simple identifications $\mathcal{Q}^I_j\theta^j \cong \mathcal{Q}^I_j\theta^j + 2\pi$. To take advantage of this, we define the *lattice basis* by promoting the arguments of all P cosines to P independent fields ϕ^I, along with $P - N$ constraints, so that the ϕ^I are constrained to an N-surface Σ on which they parametrize the original potential (1),

$$\phi|_\Sigma = \boldsymbol{\mathcal{Q}}\boldsymbol{\theta} + \boldsymbol{\delta}. \tag{3}$$

Introducing $P - N$ Lagrange multiplier fields $\boldsymbol{\nu}$ that enforce this constraint, we see that the potential becomes

$$V_{\mathrm{np}} = \underbrace{\sum_{I=1}^{P} \Lambda_I^4[1 - \cos(\phi^I)]}_{\equiv V_{\mathrm{aux}}} + \boldsymbol{\nu}^\top \boldsymbol{\mathcal{R}}(\boldsymbol{\phi} - \boldsymbol{\delta}). \tag{4}$$

The $(P-N)\times P$ matrix $\mathcal{R}$ is not unique; one choice is any set of $P-N$ linearly independent rows of the projector $\mathcal{P}$ onto the orthogonal complement of Σ, $\mathcal{P} = \mathbb{1}_P - \mathcal{Q}(\mathcal{Q}^\top\mathcal{Q})^{-1}\mathcal{Q}^\top$.

The potential V_{np} is simply V_{aux} evaluated on the N-dimensional surface Σ defined by $\mathcal{R}(\boldsymbol{\phi}-\boldsymbol{\delta}) = \mathbf{0}$ (the equations of motion for $\boldsymbol{\nu}$). This constraint surface, illustrated in Fig. 1, slices through multiple distinct fundamental domains of the integer lattice $2\pi\mathbb{Z}^P$ on which V_{aux} is periodic. Each domain of this lattice is labeled by an integer vector $\boldsymbol{n}$ and defined by [20]

$$\|\boldsymbol{\phi} - 2\pi\boldsymbol{n}\|_\infty \le \pi. \tag{5}$$

Introducing the P fields ϕ represents a key part of our work because it allows us to easily identify both exact and approximate periodicities of the physical potential. To exhibit all symmetries of the physical potential, we define a basis of the integer lattice $2\pi\mathbb{Z}^P$ that is as aligned to the constraint surface Σ as possible. In particular, we can always choose N of the P integer basis vectors, $\boldsymbol{T}^{\parallel}$, to be parallel to the constraint surface (i.e., $\mathcal{P}\boldsymbol{T}^{\parallel} = \mathbf{0}$), while the remaining $P-N$ basis vectors, $\boldsymbol{T}^{\nparallel}$, are as parallel as possible (i.e., $\mathcal{P}\boldsymbol{T}^{\nparallel} \sim$ small). More precisely, $\mathcal{P}\boldsymbol{T}^{\nparallel}$ is a reduced basis of the rank $P-N$ lattice generated by $\mathcal{P}$, containing the shortest vectors under the ℓ_∞-norm [21]. Shifts by integer combinations of the vectors $\boldsymbol{T}_i^{\parallel}$ are exact symmetries of the potential (1), while shifts by integer combinations of the vectors $(\mathbb{1}_P - \mathcal{P})\boldsymbol{T}_a^{\nparallel}$ break the periodicity, but by the least amount possible (see Fig. 1).

We refer to theories in which the relative angles between the constraint surface and the aligned basis are small, $|\mathcal{P}\boldsymbol{T}_a^{\nparallel}|_\infty \ll 1$, as "well aligned." In addition to the N exact symmetries, well-aligned theories have $P-N$ approximate shift symmetries generated by the vectors $\boldsymbol{T}_a^{\nparallel}$.

When $N \gg 1$, the determinant of $\mathcal{Q}^\top\mathcal{Q}$ is large, so the denominators of the rational numbers appearing in the projector $\mathcal{P}$ are typically large. If these numbers were irrational, the angles would be arbitrarily small, and hence generically all the angles are small when $N \gg 1$ and P is not too large. In general, Minkowski's theorem provides an upper bound on the smallest angle [22]. We have verified numerically that, with large N, even theories with sparse charge matrices are generically very well aligned, so long as P is somewhat smaller than $2N$.

The phases $\boldsymbol{\delta}$ in (4) indicate the relative offset of the origin of the lattice basis from the constraint surface, and deserve some discussion. Clearly we can eliminate N of the P phases by continuous shifts of the axions. Furthermore, we can perform discrete shifts of ϕ to set the $P-N$ remaining phases to zero within some finite accuracy. This corresponds to choosing the lattice point closest to the constraint surface to be the origin of the lattice basis. Hence, in well-aligned theories, we can reduce the phases to zero within good accuracy. In fact, small phases are necessary for the lattice and kinetic alignment mechanisms of [23,24]; therefore, our new effect justifies these previously discussed varieties of axion alignment.

Let us apply this technology to find vacua. Wherever the constraint surface is close to the center of a fundamental domain, a quadratic expansion of the auxiliary potential yields the vacuum locations

$$\phi_{\mathrm{vac},\boldsymbol{m}} = 2\pi(\mathbb{1}_P - \boldsymbol{\Delta})\boldsymbol{T}^{\nparallel}\boldsymbol{m} + \mathcal{O}(\boldsymbol{\Delta}\boldsymbol{T}^{\nparallel}\boldsymbol{m})^2. \tag{6}$$

Here

$$\boldsymbol{\Delta} = \mathrm{diag}(\Lambda_I^{-1})\mathcal{R}^\top(\mathcal{R}\mathrm{diag}(\Lambda_I^{-1})\mathcal{R}^\top)^{-1}\mathcal{R},$$

$\boldsymbol{m}$ is an integer $(P-N)$-vector, and $\mathbb{1}_P - \boldsymbol{\Delta}$ is the (generally nonorthogonal) projector onto the constraint surface which yields approximate vacua of the on-shell potential. The quadratic approximation is valid as long as the vacuum is well inside a fundamental domain.

Vacua outside the region of validity of the quadratic expansion can be found by numerically minimizing (4). In general this is very time consuming for exponentially large numbers of vacua, but the tools developed above allow us to overcome this difficulty. We can select a small but representative set of all the vacua by sampling fundamental domains that have nonvanishing overlap with the constraint surface, while ensuring that the domains in the sample are not related by shift symmetries (approximate or exact). Hence, our approach allows for a reliable statistical analysis of theories that may have vastly more vacua than could conceivably be sampled individually.

Although our qualitative conclusions hold more generally [17–19], let us illustrate the power of our approach with the simplest example of equal scales $\Lambda_I = \Lambda$ and $P = N+1$ nonperturbative terms. Assuming that the entries of

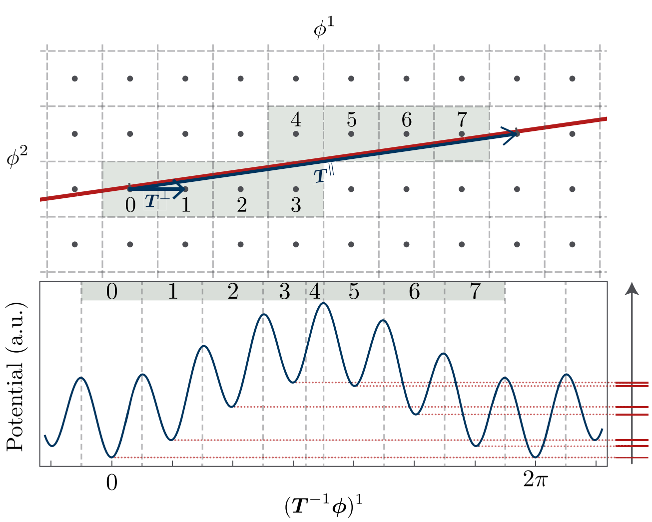

FIG. 1. Top: Constraint surface along with the lattice $2\pi\mathbb{Z}^P$ (gray dots). Arrows show the aligned basis vectors $\boldsymbol{T}^{\parallel}$ and $\boldsymbol{T}^{\nparallel}$. Bottom: Axion potential. Distinct fundamental domains are numbered and shaded.

$\mathcal{Q}$ are independent and identically distributed, we typically find $|\boldsymbol{\Delta T}^{\sharp}| \sim \det|\mathcal{Q}|^{-1}$, where $|\mathcal{Q}| = \sqrt{\mathcal{Q}^\top \mathcal{Q}}$. This allows us to determine the vacuum energies in the quadratic approximation

$$V_{\mathrm{vac},m} \approx \frac{1}{2}\Lambda^4 \left(\frac{2\pi m}{\det|\mathcal{Q}|}\right)^2 + V_0, \qquad m \in \mathbb{Z}. \tag{7}$$

The quadratic expansion is valid well within the fundamental domains (5), which gives an estimate of the number of vacua $m \lesssim \det|\mathcal{Q}|$. At large N the determinant of $|\mathcal{Q}|$ becomes extremely large, $\det|\mathcal{Q}| \approx \sigma_{\mathcal{Q}}^{P-1}\sqrt{P!}$, where $\sigma_{\mathcal{Q}}$ denotes the r.m.s. value of the entries of $\mathcal{Q}$ [25]. These estimates are valid in the universal regime, where at least a fraction $\gtrsim 3/N$ of the charge matrix entries are nonvanishing [26,27]. Therefore, even at moderately large N we find a vast number of minima whose locations and vacuum energy densities are given by (6) and (7). For example, with $N = 200$ and $\sigma_{\mathcal{Q}}^2 = 2/3$ we easily identified 10^{165} distinct vacua on a desktop computer. If $V_0 \sim -\Lambda^4$, this includes many minima with energies consistent with the observed vacuum energy of our Universe [4].

The highest vacuum in this simple example has an energy density $V_{\mathrm{vac,max}} \approx 0.14 \times P\Lambda^4$, well below the mean of the potential $P\Lambda^4$. Within the quadratic approximation, the vacua are distributed as $1/\sqrt{V}$, which yields a median vacuum energy density of roughly $V_{\mathrm{vac,max}}/4$. Neighboring vacua are easily identified in the lattice basis, typically lie at very different levels, and are separated by potential barriers of height $\gtrsim \Lambda$.

When the number of large nonperturbative effects grows at fixed $\sigma_{\mathcal{Q}}$, alignment eventually fails. However, in this regime we can switch to an approximate description in terms of an isotropic Gaussian random field whose correlation functions match that of the axion potential. Again, typical vacuum energies are well below the mean potential [17,28].

III. VACUUM TRANSITIONS

Vacuum transitions are important for at least two reasons. First, we should check whether they destabilize typical minima—that is, whether the decay rate is faster than the Hubble rate. This condition is most stringent for vacua with very low vacuum energy (such as those compatible with our Universe). Second, eternal inflation in a metastable minimum with relatively large vacuum energy is a compelling choice for the initial condition and/or generic state of our Universe (for instance [29,30]). Tunneling from such a minimum naturally sets up initial conditions for inflation [31]. If a sufficient number of e-folds of slow-roll inflation follows and the field trajectory ends in a minimum with very small vacuum energy—both of which are possible in these theories, and both of which are required by the criterion that galaxies are not exponentially rare [31–34]—this would account for much of the expansion history of our Universe.

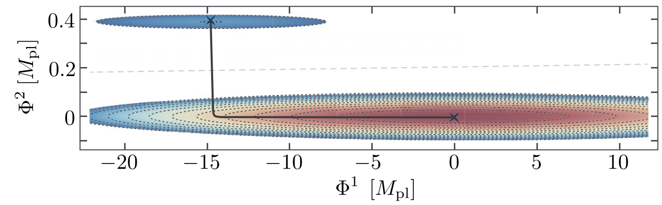

FIG. 2. Equal potential contours over a two-dimensional slice containing two vacua. The dashed line denotes the boundary of the fundamental domain. This allows for Coleman–de Luccia tunneling from a false vacuum to an inflationary slope.

The vacuum decay rate between vacua $\mathcal{A}$ and $\mathcal{B}$ in the thin-wall, no gravity approximation is given by $\Gamma_{\mathcal{BA}} \sim e^{-|B|}$, where

$$B = \frac{27\pi^2\sigma^4}{2(V_{\mathcal{B}} - V_{\mathcal{A}})^3} \tag{8}$$

and σ is the tension of the wall. A sufficient condition for stability on gigayear time scales is very roughly $\Gamma_{\mathrm{total}} < 10^{-10^3}$, where Γ_{total} includes a sum over $\lesssim 3^P$ neighboring vacua. To estimate the decay rate, we need to take the kinetic terms in (2) into account. The canonically normalized charge matrix is $\mathcal{Q}\boldsymbol{K}^{-1/2}$, and we denote its smallest singular value by q_{min}. The mass scale q_{min}^{-1} is the largest relevant scale of features in the axion potential. Due to isotropy, the tunneling path between generic neighbors is roughly $q_{\mathrm{min}}^{-1}/\sqrt{P}$, and so this provides a weak bound on the parameters of the theory: $\Lambda^4 \ll 10^3 \times q_{\mathrm{min}}^{-4}/P^3$. In the interesting parameter regime (e.g., $P^3 \sim 10^6$, $q_{\mathrm{min}}^{-1} \sim M_{\mathrm{pl}}$, $\Lambda \sim 10^{-2} M_{\mathrm{pl}}$) a vast number of vacua are extremely stable (see also [35]).

In general there is a tension between thin-wall vacuum decays that require $V'' \gg H^2$, and a sustained period of inflation that demands $V'' \ll H^2$ [36]. In the multiaxion models at hand, however, the hierarchies in the singular values of the canonically normalized charge matrix render the field space highly anisotropic, and these theories generically accommodate thin-wall vacuum transitions in bubbles that subsequently inflate in an open Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology. We give an example in Fig. 2 (note that the kinetic energy gained by the inflaton after tunneling does *not* cause the field to overshoot the inflationary plateau, for reasons explained in [31]).

IV. INFLATION

The implementation of inflation purely within effective field theories is famously tentative at best: physics above the cutoff can spoil an inflationary trajectory or destabilize the theory altogether. Despite this limitation it is still instructive to consider the cosmological observables that would arise in the absence of any such effects.

Shift symmetries provide for some of the most compelling, radiatively stable models of inflation [3] that can be embedded in string theory [37]. Most notable are variants of assisted inflation that exploit multiple axion shift symmetries to ensure the super-Planckian field space diameters required for large-field inflation, $\mathcal{D}_{\rm inf} \gtrsim M_{\rm pl}$ [6,7,38]. While the invariant diameters for single axions are sub-Planckian when the perturbative expansion is well controlled [39], axions are numerous in typical flux compactifications and generically allow for collective field space diameters that significantly exceed the ranges of the individual axions via kinetic and lattice alignment [27,40]. In lattice coordinates, the relevant boundaries of typical fundamental domains roughly correspond to a P-hypercube that has no special orientation with respect to the least massive axion, which thus is well aligned with one of the numerous diagonals. This observation generalizes the phenomenon of kinetic alignment [24,27] and generically yields an axion diameter as large as [17]

$$\mathcal{D}_{\rm inf} \sim 2\pi\sqrt{P} q_{\rm min}^{-1}, \tag{9}$$

which can exceed the field ranges of each individual axion by far if P is large (kinetic alignment) and/or the singular value of the charge matrix is small (lattice alignment).

We can comprehensively sample the inflationary dynamics by marginalizing over the charges $\mathcal{Q}$ and phases $\boldsymbol{\delta}$ of the instantons, the kinetic matrix $\boldsymbol{K}$, and the constant energy density V_0. Since we are only interested in vacua that satisfy the selection bias constraint of (almost) vanishing cosmological constant we can marginalize over V_0 by considering only those values that ensure a vanishing energy density at the vacuum reached at the end of inflation. The tools developed above provide us with a representative sample of all vacua, so by considering all initial conditions that can terminate in that vacuum we find a representative collection of all possible cosmological histories.

We study the classical dynamics by solving the equations of motion for the fields and the Friedmann equation for the scale factor of a homogeneous FLRW cosmology, discarding any solution inconsistent with our selection bias [41,42]. Whenever the single field, slow-roll approximation is valid throughout the evolution [43], the scale of inflation, tensor-to-scalar ratio, and spectral index, respectively, are given by

$$E_{\rm inf} \approx 0.01 \times r^{1/4} M_{\rm pl}, \qquad r \approx 16\epsilon, \qquad n_{\rm s} \approx 1 - 2\eta - 4\epsilon, \tag{10}$$

where $\epsilon = -\dot{H}/H^2$ is the first, and $\eta = \ddot{\boldsymbol{\Phi}} \cdot \boldsymbol{e}_\parallel / |\dot{\boldsymbol{\Phi}}| H$ is the second slow-roll parameter projected onto the tangent of the trajectory $\boldsymbol{e}_\parallel$. All quantities should be evaluated at horizon exit. The single field approximation is valid when the acceleration transverse to the field velocity and nonadiabatic particle and/or string interactions are negligible [44–48].

To study inflation in our model, we sample the trajectories leading into each vacuum, with V_0 chosen in each case so that the vacuum has zero energy. We discard any trajectories with less than 60 e-folds of inflation. We find two qualitatively different regimes. Whenever inflation proceeds over a super-Planckian distance within one single fundamental domain, as is the case in the aligned axion inflation scenario discussed in [23,24,27,49], we find a lower bound of $r \gtrsim 0.07$, and the single field approximation is valid. In generic theories (assuming roughly constant Λ_I), the vacua are very low compared to the mean of the axion potential and therefore downward vacuum transitions can only source this regime.

However, when inflation proceeds at typical scales of the potential, much more diverse features in the potential are encountered, such as hilltops and saddle points (see also [50–53]). Even in very simple theories, we observed a wide range of $n_{\rm s}$ and values of r as low as $r \sim 10^{-4}$, but we speculate that much lower values of r are possible in more complex models. The single field approximation breaks down for some trajectories and it is not clear whether nonadiabatic perturbations decay by the end of inflation to allow for a simple treatment. These results motivate a future study of the multifield dynamics and perturbations. The corresponding initial conditions can be sourced by upward transitions or other mechanisms (e.g., [54,55]). Note that while $n_{\rm s}$ and r are independent of the overall scale of the potential, the power spectrum depends on the scale, and so to match observation, trajectories with smaller r must occur in models with correspondingly smaller values for the Λ_I.

A single multiaxion theory can accommodate a very diverse set of inflationary trajectories with significantly different cosmological observables. This finding highlights the necessity of a detailed understanding of inflationary initial conditions to satisfy even the most basic prerequisites for definite predictions in multiaxion theories.

V. DARK MATTER

If $\mathcal{Q}$ is not full rank, the leading potential (1) leaves at least one of the fields ($\Phi_{\rm light}$) with an unbroken continuous shift symmetry. It is generally believed that theories of quantum gravity do not permit continuous global symmetries, so there should exist a subleading instanton with action S_G that breaks the continuous shift symmetry by a term $V_G \sim M_{\rm pl}^4 e^{-S_G} \cos(\Phi_{\rm light}/f_{\rm light})$. A typical axion decay constant of the leading nonperturbative term is $f_{\rm light} \approx \sqrt{\pi/8} f_N$, where f_N^2 denotes the largest eigenvalue of the field space metric $\boldsymbol{K}$ [56]. A natural guess for S_G is provided by the weak gravity conjecture [57], which asserts that no gauge interaction is weaker than gravity [58]. Extending this conjecture to axions provides an upper bound on the action, $S_G \lesssim M_{\rm pl}/f_N$ [59,60]. The bound is approximately saturated by Euclidean wormholes that couple to N axions, for which $S_G \approx \sqrt{3\pi N} M_{\rm pl}/2 f_N$ [56,61,62]. This allows us to estimate the mass of the lightest axion:

$$m_{\rm light} \approx \frac{M_{\rm pl}^2}{f_{\rm light}} e^{-S_G/2}.$$

A mass of roughly 10^{-22} eV has the virtue that it ameliorates the problems conventional CDM models have at sub-kpc scales by suppressing structure below the Compton wavelength m_{light}^{-1} [11–13]. Choosing $N \sim 100$ and $f_{\text{light}} \approx .04M_{\text{pl}}$ yields $m_{\text{light}} \approx 10^{-22}$ eV. Remarkably, with these numbers axion misalignment generates roughly the correct dark matter abundance:

$$\Omega_{\text{axion}} \sim 0.2 \times \left(\frac{f_{\text{light}}}{.04M_{\text{pl}}}\right)^2 \left(\frac{m_{\text{light}}}{10^{-22}\ \text{eV}}\right)^{1/2}. \tag{11}$$

The eigenvalue $f_N \sim f_{\text{light}}$ is typically related to the smallest singular value of the charge matrix by $q_{\min}^{-1} \lesssim N f_N$. The typical diameter of the fundamental domain is roughly $\mathcal{D}_{\text{inf}} \lesssim 2\pi P^{1/2} q_{\min}^{-1}$ (9) [17,27]. Hence, with $P \approx N \approx 100$ multiaxion models can accommodate both Planckian field space diameters $\mathcal{D}_{\text{inf}} \sim M_{\text{pl}}$ and light axions that reproduce the observed dark matter abundance $\Omega_{\text{axion}} \sim 0.1$. We provide a detailed discussion of reheating in [18,19].

It is something of a miracle that this analysis gives parameters with the correct range of values to both give the correct dark matter abundance and help solve this problem (see also [12,13]). Given this surprising observation, one may be led to speculate that dark matter could be related to nonperturbative gravitational physics.

The combination of this "fuzzy" dark matter and high-scale inflation can lead to an overproduction of isocurvature modes. Avoiding this probably requires $r \lesssim \text{few} \times 10^{-4}$ (see, for instance, [14,15]). Values of r this small appear for inflationary trajectories even in the simple $P = N + 1$ model discussed above, and we expect larger $P - N$ to provide even more diversity.

ACKNOWLEDGMENTS

We would like to thank Lam Hui, Austin Joyce, Nemanja Kaloper, David J. E. Marsh, Liam McAllister, and Alberto Tejedor for useful discussions. We thank Eva Silverstein for suggesting the title. The work of T. B. and K. E. was supported by the Department of Energy (DOE) under Grant No. DE-SC0011941. The work of M. K. is supported in part by the National Science Foundation (NSF) through Grant No. PHY-1214302, and he acknowledges membership at the New York University-East China Normal University (NYU-ECNU) Joint Physics Research Institute in Shanghai. O. J. is supported by the James Arthur Fellowship.

[1] R. D. Peccei and H. R. Quinn, *CP* Conservation in the Presence of Instantons, Phys. Rev. Lett. **38**, 1440 (1977).
[2] P. Svrcek and E. Witten, Axions in string theory, J. High Energy Phys. 06 (2006) 051.
[3] K. Freese, J. A. Frieman, and A. V. Olinto, Natural Inflation with Pseudo-Nambu-Goldstone Bosons, Phys. Rev. Lett. **65**, 3233 (1990).
[4] R. Bousso and J. Polchinski, Quantization of four form fluxes and dynamical neutralization of the cosmological constant, J. High Energy Phys. 06 (2000) 006.
[5] T. C. Bachlechner, Axionic band structure of the cosmological constant, Phys. Rev. D **93**, 023522 (2016).
[6] S. Dimopoulos, S. Kachru, J. McGreevy, and J. G. Wacker, N-flation, J. Cosmol. Astropart. Phys. 08 (2008) 003.
[7] J. E. Kim, H. P. Nilles, and M. Peloso, Completing natural inflation, J. Cosmol. Astropart. Phys. 01 (2005) 005.
[8] L. McAllister, E. Silverstein, and A. Westphal, Gravity waves and linear inflation from axion monodromy, Phys. Rev. D **82**, 046003 (2010).
[9] N. Kaloper and L. Sorbo, A Natural Framework for Chaotic Inflation, Phys. Rev. Lett. **102**, 121301 (2009).
[10] N. Kaloper, A. Lawrence, and L. Sorbo, An ignoble approach to large field inflation, J. Cosmol. Astropart. Phys. 03 (2011) 023.
[11] W. Hu, R. Barkana, and A. Gruzinov, Cold and Fuzzy Dark Matter, Phys. Rev. Lett. **85**, 1158 (2000).
[12] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, String axiverse, Phys. Rev. D **81**, 123530 (2010).
[13] L. Hui, J. P. Ostriker, S. Tremaine, and E. Witten, On the hypothesis that cosmological dark matter is composed of ultra-light bosons, Phys. Rev. D **95**, 043541 (2017).
[14] D. J. E. Marsh, Axion cosmology, Phys. Rep. **643**, 1 (2016).
[15] A. Diez-Tejedor and D. J. E. Marsh, Cosmological production of ultralight dark matter axions, arXiv:1702.02116.
[16] Bold font denotes field space vectors and matrices; e.g., $\boldsymbol{\theta}$ is a vector with components θ^i, and we set $\hbar = 1$.
[17] T. C. Bachlechner, K. Eckerle, O. Janssen, and M. Kleban, Systematics of aligned axions, J. High Energy Phys. 11 (2017) 036.
[18] T. C. Bachlechner, K. Eckerle, O. Janssen, and M. Kleban, The axidental universe (to be published).
[19] T. C. Bachlechner, K. Eckerle, O. Janssen, and M. Kleban (to be published).
[20] The ℓ_∞-norm is defined as $\|\boldsymbol{v}\|_\infty \equiv \max_i\{|v^i|\}$.
[21] In practice, the basis $\boldsymbol{T}$ may be hard to find, but an approximation is easily obtained via the extended LLL algorithm [63], which finds a basis reduced under the ℓ_2-norm; see, for example, [64].
[22] H. Minkowski, *Gesammelte Abhandlungen von Hermann Minkowski* (B.G. Teubner, Leipzig and Berlin, 1911), Vol. 2.
[23] J. E. Kim, H. P. Nilles, and M. Peloso, Completing natural inflation, J. Cosmol. Astropart. Phys. 01 (2005) 005.
[24] T. C. Bachlechner, M. Dias, J. Frazer, and L. McAllister, Chaotic inflation with kinetic alignment of axion fields, Phys. Rev. D **91**, 023520 (2015).

[25] N. R. Goodman, The distribution of the determinant of a complex Wishart distributed matrix, Ann. Math. Stat. **34**, 178 (1963).
[26] P. M. Wood, Universality and the circular law for sparse random matrices, Ann. Appl. Probab. **22**, 1266 (2012).
[27] T. C. Bachlechner, C. Long, and L. McAllister, Planckian axions in string theory, J. High Energy Phys. 12 (2015) 042.
[28] T. C. Bachlechner, On Gaussian random supergravity, J. High Energy Phys. 04 (2014) 054.
[29] A. D. Linde, Quantum creation of the inflationary universe, Lett. Nuovo Cimento **39**, 401 (1984).
[30] A. Vilenkin, Quantum cosmology and the initial state of the Universe, Phys. Rev. D **37**, 888 (1988).
[31] B. Freivogel, M. Kleban, M. Rodriguez Martinez, and L. Susskind, Observational consequences of a landscape, J. High Energy Phys. 03 (2006) 039.
[32] P. C. W. Davies and S. D. Unwin, Why is the cosmological constant so small?, Proc. R. Soc. A **377**, 147 (1981).
[33] S. Weinberg, Anthropic Bound on the Cosmological Constant, Phys. Rev. Lett. **59**, 2607 (1987).
[34] T. C. Bachlechner, Inflation expels runaways, J. High Energy Phys. 12 (2016) 155.
[35] A. Masoumi and A. Vilenkin, Vacuum statistics and stability in axionic landscapes, J. Cosmol. Astropart. Phys. 03 (2016) 054.
[36] A. Linde, Toy model for open inflation, Phys. Rev. D **59**, 023503 (1998).
[37] D. Baumann and L. McAllister, *Inflation and String Theory* (Cambridge University Press, Cambridge, England, 2015).
[38] A. R. Liddle, A. Mazumdar, and F. E. Schunck, Assisted inflation, Phys. Rev. D **58**, 061301 (1998).
[39] T. Banks, M. Dine, and E. Gorbatov, Is there a string theory landscape?, J. High Energy Phys. 08 (2004) 058.
[40] C. Long, L. McAllister, and J. Stout, Systematics of axion inflation in Calabi-Yau hypersurfaces, J. High Energy Phys. 02 (2017) 014.
[41] S. Groot Nibbelink and B. J. W. van Tent, Scalar perturbations during multiple field slow-roll inflation, Classical Quantum Gravity **19**, 613 (2002).
[42] L. C. Price, J. Frazer, J. Xu, H. V. Peiris, and R. Easther, MultiModeCode: An efficient numerical solver for multifield inflation, J. Cosmol. Astropart. Phys. 03 (2015) 005.
[43] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, A new approach to the evolution of cosmological perturbations on large scales, Phys. Rev. D **62**, 043527 (2000).
[44] X. Chen and Y. Wang, Quasi-single field inflation and non-gaussianities, J. Cosmol. Astropart. Phys. 04 (2010) 027.
[45] D. Green, B. Horn, L. Senatore, and E. Silverstein, Trapped inflation, Phys. Rev. D **80**, 063533 (2009).
[46] G. D'Amico, R. Gobbetti, M. Kleban, and M. Schillo, Unwinding inflation, J. Cosmol. Astropart. Phys. 03 (2013) 004.
[47] V. Assassi, D. Baumann, D. Green, and L. McAllister, Planck-suppressed operators, J. Cosmol. Astropart. Phys. 01 (2014) 033.
[48] R. Flauger, M. Mirbabayi, L. Senatore, and E. Silverstein, Productive interactions: Heavy particles and non-Gaussianity, J. Cosmol. Astropart. Phys. 10 (2017) 058.
[49] S. Dimopoulos, S. Kachru, J. McGreevy, and J. G. Wacker, N-flation, J. Cosmol. Astropart. Phys. 08 (2008) 003.
[50] T. Higaki and F. Takahashi, Natural and multi-natural inflation in axion landscape, J. High Energy Phys. 07 (2014) 074.
[51] T. Higaki and F. Takahashi, Axion landscape and natural inflation, Phys. Lett. B **744**, 153 (2015).
[52] M. Czerny, T. Higaki, and F. Takahashi, Multi-natural inflation in supergravity, J. High Energy Phys. 05 (2014) 144.
[53] M. Czerny, T. Higaki, and F. Takahashi, Multi-natural inflation in supergravity and BICEP2, Phys. Lett. B **734**, 167 (2014).
[54] K. R. Dienes, J. Kost, and B. Thomas, A tale of two timescales: Mixing, mass generation, and phase transitions in the early Universe, Phys. Rev. D **93**, 043540 (2016).
[55] K. R. Dienes, J. Kost, and B. Thomas, Kaluza-Klein towers in the early Universe: Phase transitions, relic abundances, and applications to axion cosmology, Phys. Rev. D **95**, 123539 (2017).
[56] T. C. Bachlechner, C. Long, and L. McAllister, Planckian axions and the weak gravity conjecture, J. High Energy Phys. 01 (2016) 091.
[57] The weak gravity conjecture in general is not inconsistent with large field inflation; see, e.g., [59,60,62,65–71].
[58] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, The string landscape, black holes and gravity as the weakest force, J. High Energy Phys. 06 (2007) 060.
[59] C. Cheung and G. N. Remmen, Naturalness and the Weak Gravity Conjecture, Phys. Rev. Lett. **113**, 051601 (2014).
[60] T. Rudelius, Constraints on axion inflation from the weak gravity conjecture, J. Cosmol. Astropart. Phys. 09 (2015) 020.
[61] N. Arkani-Hamed, J. Orgera, and J. Polchinski, Euclidean wormholes in string theory, J. High Energy Phys. 12 (2007) 018.
[62] M. Montero, A. M. Uranga, and I. Valenzuela, Transplanckian axions!?, J. High Energy Phys. 08 (2015) 032.
[63] A. Lenstra, H. Lenstra, and L. Lovász, Factoring polynomials with rational coefficients, Math. Ann. **261**, 515 (1982).
[64] W. van der Kallen, Implementations of extended LLL, 1998, http://www.staff.science.uu.nl/kalle101/lllimplementations.html.
[65] J. Brown, W. Cottrell, G. Shiu, and P. Soler, Fencing in the swampland: Quantum gravity constraints on large field inflation, J. High Energy Phys. 10 (2015) 023.
[66] J. Brown, W. Cottrell, G. Shiu, and P. Soler, On axionic field ranges, loopholes and the weak gravity conjecture, J. High Energy Phys. 04 (2016) 017.
[67] B. Heidenreich, M. Reece, and T. Rudelius, Weak gravity strongly constrains large-field axion inflation, J. High Energy Phys. 12 (2015) 108.
[68] B. Heidenreich, M. Reece, and T. Rudelius, Sharpening the weak gravity conjecture with dimensional reduction, J. High Energy Phys. 02 (2016) 140.
[69] P. Saraswat, Weak gravity conjecture and effective field theory, Phys. Rev. D **95**, 025013 (2017).
[70] A. Hebecker, P. Henkenjohann, and L. T. Witkowski, What is the magnetic weak gravity conjecture for axions?, Fortschr. Phys. **65**, 1700011 (2017).
[71] L. McAllister, P. Schwaller, G. Servant, J. Stout, and A. Westphal, Runaway relaxion monodromy, J. High Energy Phys. 02 (2018) 124.