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A relativistic wave equation with a local kinetic operator and an energy-dependent effective interaction for the study of hadronic systems

Research Article

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Abstract: We study a fully relativistic, two-body, quadratic wave equation for equal mass interacting particles. With this equation the difficulties related to the use of the square roots in the kinetic energy operators are avoided. An energy-dependent effective interaction, also containing quadratic potential operators, is introduced. For pedagogical reasons, it is previously shown that a similar procedure can be also applied to the well-known case of a one-particle Dirac equation. The relationships of our model with other relativistic approaches are briefly discussed. We show that it is possible to write our equation in a covariant form in any reference frame. A generalization is performed to the case of two particles with different mass. We consider some cases of potentials for which analytic solutions can be obtained. We also study a general numerical procedure for solving our equation taking into account the energy-dependent character of the effective interaction. Hadronic physics represents the most relevant field of application of the present model. For this reason we perform, as an example, specific calculations to study the charmonium spectrum. The results show that the adopted equation is able to reproduce with good accuracy the experimental data.

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1. Introduction

The study of three-dimensional relativistic wave equations for bound systems is a subject of relevant interest in atomic, nuclear and hadronic physics.

We anticipate that in the case of hadronic physics (that will be analyzed in detail in the present paper) the constituent quark models can give a good description of the hadronic spectra (at least of the low-lying states) if a relativistic dynamical model is adopted. But we recall that, also when studying the hyperfine corrections of the atomic energy levels, both the contributions related to the hadronic structure and the contributions given by the relativistic motion of the particles, must be carefully (and simultaneously) examined, as, for example, in the case of the so-called *proton radius puzzle* for the $e^- p$ and $\mu^- p$ bound systems [1, 2].

The simplest (and very commonly adopted) procedure to

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construct a relativistic dynamical model is to consider, in the Center of Mass (CM) of the system, a Hamiltonian operator [3, 4] containing the relativistic energy terms and the interaction operator in the form:

$$H = \sum_{i=1}^{N} \sqrt{\mathbf{p}_i^2 + m_i^2} + V, \qquad (1)$$

where N is the number of constituents, $\mathbf{p}_i$ and m_i respectively represent the momentum operator and the mass of the *i-th* constituent and, finally, V is the interaction operator. Throughout the present work we shall mainly deal with CM quantities, so that no special notation will be used for the corresponding operators. (When studying a generalized covariant form of our equation, a specific notation will be introduced).

In particular, the momentum operators $\mathbf{p}_i$ are always constrained by the CM condition, that is

$$\sum_{i=1}^{N} \mathbf{p}_i = 0 \ . \qquad (2)$$

In consequence, only $N-1$ *independent* relative momentum operators are actually used in the calculations.

In general, the interaction operator V depends on the relative coordinates of the constituents, on their spins and momenta. The form of the spin and momentum dependence is specifically determined by the relativistic and retardation effects associated to the interaction.

Considering the Hamiltonian of Eq. (1), the corresponding eigenvalue equation is:

$$H|\psi> = M|\psi> \qquad (3)$$

that is usually solved by means of a variational technique. This theoretical procedure for the study of relativistic bound systems presents, at least, the following formal and practical difficulties.

1. From a dynamical point of view, in Eq. (1) the negative-energy states (that, in the context of field theories, are transformed into free-antiparticle states) are excluded from the model at the very beginning of its formulation.

2. The insertion of the interaction V, as given in Eq. (1), is *exactly* consistent only for the zero component of a vector field. If a scalar (effective) field is considered, as it is usually done, in particular, for the study $c\bar{c}$ and $b\bar{b}$ spectra [5–8], one should add the corresponding scalar interaction operators to the constituent masses m_i by means of the substitution that will be discussed in Section 3. However, these scalar interaction operators would appear *in the square roots* of the relativistic energies, giving rise to very serious difficulties for the calculations, unless an approximate Taylor expansion of the square roots is performed.

3. Also considering the simplest two-body case and taking special forms for the interaction operator V, it is not easy to find analytic solutions to Eq. (3).

4. When using standard central potentials, the asymptotic behavior of the corresponding wave functions in the coordinate space cannot be determined analytically.

5. As for the variational (numerical) calculations, the matrix elements of the interacion operator V are conveniently calculated by means of a basis of trial wave functions in the coordinate space, given that the leading terms of the interaction are usually represented by central potentials. On the other hand, the matrix elements of the *nonlocal* relativistic kinetic terms must be calculated in the momentum space. To this aim one has to perform the Fourier transformation of the trial wave functions. We point out that such transformation can be done analytically only for some specific functions. The whole calculation requires, in any case, noticeable computational efforts.

   Moreover, as discussed in the previous points, the lack of analytic solutions and the difficulty of studying the asymptotic properties of the wave functions, do not allow for a check of the obtained numerical solutions.

These difficulties represent the main motivations to develop, in the present work, a different approach with the objective of constructing *fully relativistic* and *local* constituent quark models. In more detail, the kinetic (quadratic) terms of our equation will represent, without approximations, the relativistic motion of the quarks. The interaction terms will be introduced by means of *standard* substitutions. Their form, of phenomenological nature, will be chosen in order to reproduce the experimental spectra, taking into account (as far as possible) the general symmetries of the underlying field theory.

We recall that the difficulties related to many aspects of Quantum Chromo-Dynamics (QCD), in particular *confinement*, do not allow for direct analytic calculations. In consequence, we assume that the construction of QCD inspired quark *models*, (with different *phenomenological*

assumptions) represents, besides lattice investigations, a useful method for the study of hadronic spectroscopy.

The content of the present work will be developed as outlined in the following.

For pedagogical reasons, in order to introduce the main concepts (and techniques) of the present model, we shall first revise in Section 2 the well-known case of the one-body Dirac equation. For a special combination of a vector and a scalar potential, the original equation leads to a Schrödinger-like equation (for the upper components of the Dirac 4-spinor) with an energy-dependent effective potential. In more detail, in this equation the kinetic term is given by a *local* operator proportional to the square of the relative momentum. In this way, an energy-dependent effective interaction is introduced and the eigenvalue of the equation is given by a quadratic function of the physical energy of the system.

Obviously, this *one-body* Dirac equation cannot be applied as such to the study of constituent models for hadronic systems. For this reason, with a similar technique, we derive, in Section 3, our equation for the study of a two-body equal-mass system. Also in this case we obtain an energy dependent, Schrödinger-like equation in which the kinetic term is given by a *local* operator proportional to the square of the relative momentum. Our procedure essentially consists in writing the *free* equation for the relative momentum, total energy and constituent mass of the system, then inserting the scalar and vector interactions by means of standard minimal substitutions.

Our model is compared with more standard approaches based on three-dimensional reductions of the Bethe-Salpeter Equation (BSE) [9–14] and to a very complex procedure based on two-body Dirac equations of constraint dynamics [15].

Two generalizations are introduced in Section 4; in Subsection 4.1 a covariant version of the equation for the study of bound systems in any reference frame is proposed; in Subsection 4.2 the equation is also generalized to the case of two constituents with different mass.

In Section 5 we examine some specific cases in which the equation can be solved analytically.

Then, we consider a numerical technique to solve the equation in the case of an arbitrary interaction. As an example of physical relevance, in Section 6, we shall study, with a fit procedure, the general structure of the charmonium spectrum by using a definite form of the interaction. Furthermore, by using the same method of numerical solution, we shall also tentatively consider a possible *intrinsic* energy dependence of the interaction, showing that, in this way, an improvement of the fit of the spectrum can be obtained.

Finally, some conclusions will be drawn in Section 7.

Throughout the paper we set $\hbar = c = 1$.

2. The Dirac equation with a special combination of potentials

The content of the present Section has been deeply studied by many authors. See, for example, Refs. [19–22]. Here we only present the main results in order to introduce the reader to the case of two interacting particles that will be analyzed in the next Section 3.

We recall that the Dirac equation for a free particle is written, in the Hamiltonian form, as:

$$(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\psi = E_T\psi, \tag{4}$$

where E_T and $\mathbf{p}$ respectively represent the particle total energy and momentum, m is its mass and $\boldsymbol{\alpha}=\gamma^0\boldsymbol{\gamma}$, $\beta=\gamma^0$ are the Dirac matrices, for which, in the present work, the standard representation will used. In the Dirac 4-spinor

$$\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

the Pauli spinors φ and χ respectively represent the so-called *upper* and *lower* components.

We now insert in Eq. (4) the interaction with a scalar and a vector field. To this aim the following standard replacements are made:

$$m \to m + V_s \tag{5}$$

for the scalar field, and

$$E_T \to E_T - V_v \;;\; \mathbf{p} \to \mathbf{p} - \mathbf{V}_v \tag{6}$$

for the vector interaction. In the present work we only take the zero component V_v of the vector interaction disregarding the corresponding spatial (3-vector) part. More explicitly, we set $\mathbf{V}_v = 0$, considering that terms of this kind give rise only to higher order relativistic contributions. Furthermore, we take central interactions: $V_s = V_s(r)$ and $V_v = V_v(r)$.

With standard handling we can write the interacting Dirac equation directly in the form of two coupled equations for the upper and the lower components:

$$\boldsymbol{\sigma}\cdot\mathbf{p}\chi + V_v(r)\varphi + (m + V_s(r))\varphi = E_T\varphi \tag{7}$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}\varphi + V_v(r)\chi - (m + V_s(r))\chi = E_T\chi, \tag{8}$$

where the Pauli matrices $\boldsymbol{\sigma}$ have been introduced explicitly.
By means of Eq. (8) the lower components can be expressed in terms of upper ones as:

$$\chi = (E_T + m + V_s(r) - V_v(r))^{-1} \boldsymbol{\sigma} \cdot \mathbf{p}\varphi. \tag{9}$$

Incidentally, we recall that, for calculating the nonrelativistic limit, one makes the hypothesis that $E_T \simeq m$ and that $2m >> |V_s(r) - V_v(r)|$. In this way, the last equation gives the approximated expression

$$\chi \simeq \frac{1}{2m} \boldsymbol{\sigma} \cdot \mathbf{p}\varphi \tag{10}$$

that is replaced in Eq. (7), so that the standard Schrödinger equation for φ is finally obtained.
Here we follow an *exact* procedure considering the following special choice for the potential terms:

$$V_s(r) = V_v(r) = \frac{1}{2} U_D(r) \ . \tag{11}$$

Replacing in Eq. (7), with standard handling the following exact equation for φ is found:

$$\left[(E_T + m)^{-1} \mathbf{p}^2 + U_D(r) \right] \varphi = (E_T - m)\varphi, \tag{12}$$

where $E_T + m$ must be nonvanishing. This equation can be also written in an equivalent form that is more similar to a standard Schrödinger equation. To this aim, we first subtract from the total energy the particle rest mass, defining:

$$E_D = E_T - m. \tag{13}$$

Then, we multiply both sides of Eq. (12) by the quantity

$$\mathcal{F}(E_D) = 1 + \frac{E_D}{2m} \ . \tag{14}$$

Finally, introducing

$$\mathcal{E}_D(E_D) = E_D \cdot \mathcal{F}(E_D) \ , \tag{15}$$

Eq. (12) takes the form:

$$\left[\frac{\mathbf{p}^2}{2m} + U_D(r)\mathcal{F}(E_D) \right] \varphi = \mathcal{E}_D(E_D)\varphi \ . \tag{16}$$

We note that the previous equation contains, in the *l.h.s.* a standard, *local*, kinetic operator and an energy dependent effective interaction term. In the *r.h.s.* $\mathcal{E}_D$ does not represent the energy of the particle but is algebraically related to this (physically relevant) quantity by means of Eqs. (14) and (15). Furthermore, after solving Eq. (16), the lower components χ can be directly obtained from the solution φ by using Eqs. (9) and (11).
The relevant point for the present study is that the previous equation (obtained with the special condition of Eq. (11)), is *fully relativistic*, even though the kinetic term is given by a quadratic operator.
Finally, we also highlight that, due to the same condition of Eq. (11), no spin-orbit interaction is found. This property could be of some interest for the study of the nucleon spectrum.
In the following Section 3, we shall obtain a similar equation for the case of physical interest (in hadronic physics) of two equal-mass interacting particles.

3. The two-body equal-mass quadratic equation

The case of two-body *equal-mass* systems is very relevant in different areas of subatomic physics. We recall the case of Positronium in QED and of the Deuteron in nuclear physics. Moreover, in hadronic physics we mention the $q\bar{q}$ systems, in particular charmonium and Bottomonium, whose properties have been conveniently studied by means of a potential model with momentum dependent interaction terms.
In order to introduce our equation, as first step we write the quadratic equation for a two-body equal mass *free* system in the CM reference frame:

$$4\left[\mathbf{p}^2 + m^2\right] |\psi_{free}> = M^2 |\psi_{free}>, \tag{17}$$

where $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ is the CM relative momentum and M represents the invariant mass, *i.e.* the CM energy of the system. The previous equation admits positive and negative energy solutions.
With the aim of obtaining the *interacting* wave equation with the correct nonrelativistic limit, that will be explicitly given in Eq. (25), we introduce the interaction by means of the following *standard* substitutions:

$$m \to m + \frac{V_s(r)}{2} \tag{18}$$

and

$$M \to M - V_v(r) \ , \tag{19}$$

disregarding, as in the case of the Dirac Equation, the spatial part of the vector interaction.
In this way the following equation is obtained:

$$\left[4\mathbf{p}^2 \;+\; 4m^2 + V_s^2(r) + 4mV_s(r) \right.$$
$$\left. -\; V_v^2(r) + 2MV_v(r)\right]|\psi>= M^2|\psi> \qquad (20)$$

that represents the central result of the present work.
Note that, in this way, the scalar interaction $V_s(r)$ has been *exactly* introduced and does not require any approximated expansion, as, on the contrary, it is necessary in the standard treatment of Eqs. (1) and (3).
Formally the previous equation also holds for very small, or vanishing, values of the constituent mass m, opening the possibility of constructing light-quark constituent models. However, in this case, the form of the interaction operators V_v and V_s should be carefully studied considering their dynamical and relativistic properties.
In any case, when developing a realistic model, a dependence on the spin and momentum operators must be introduced.
In the present preliminary study we only focus our attention on a *simplified* model with the interaction represented by *central* potentials, *i.e.* with $V_v = V_v(r)$ and $V_s = V_s(r)$. Obviously, this choice does not allow to reproduce some relevant aspects of hadronic phenomenology but, on the other hand, it can be expected that the *leading* contributions of the interaction operators are, in any case, represented by central potentials of this kind.
We shall now perform some algebraic transformations in order to write the *same* equation in the form of an energy dependent, Schrödinger-like equation. The following transformations can be applied only when the constituent mass m is nonvanishing.
To this aim we first subtract from the total mass M the masses of the constituent particles, defining:

$$E = M - 2m \; . \qquad (21)$$

Then, dividing both sides by $4m$, with standard algebraic handling, the wave equation can be written as:

$$\left[\frac{\mathbf{p}^2}{m} \;+\; V_s(r) + V_v(r) + \frac{1}{4m}\left(V_s^2(r)\right.\right.$$
$$\left.\left. -\; V_v^2(r)\right) + \frac{E}{2m}V_v(r)\right]|\psi>= \mathcal{E}(E)|\psi>, \quad (22)$$

where the quantity $\mathcal{E}(E)$ is defined as:

$$\mathcal{E}(E) = E \cdot \left(1 + \frac{E}{4m}\right) \; . \qquad (23)$$

To avoid confusion, note that this definition has a different functional form with respect to $\mathcal{E}_D(E_D)$ given in Eq. (15); furthermore, in this case, the argument is E defined in Eq. (21).

We now discuss some general aspects of our equation.
In the first place, we check the consistency of the model calculating the nonrelativistic expansion of Eq. (22) in (inverse) powers of c or, equivalently, of p/m. In this expansion the nonrelativistic terms are of the order c^0 and the relativistic corrections are of the order c^{-2}.
We also point out that, in Eq. (22), the terms $\frac{\mathbf{p}^2}{m}$ and, conventionally, $V_s(r)$, $V_v(r)$ are of the order c^0. Also, the factor $1/m$ that appears in the last two terms of the *l.h.s.* is of the order c^{-2}.
To obtain a Schrödinger equation *plus* relativistic corrections, let us focus our attention on the last term of the *l.h.s.*, that is $\frac{E}{2m}V_v(r)$, and on the quadratic energy term of the *r.h.s.*, that is $\frac{E^2}{4m}$. By replacing E , in these terms, with an approximated expression of the order c^0, we obtain a wave equation that is correct up to the order c^{-2}. Practically, we substitute E with the corresponding operator of the orden c^0, that is:

$$E \rightarrow \frac{\mathbf{p}^2}{m} + V_s(r) + V_v(r) \; . \qquad (24)$$

Then, with standard handling, one finds the Schrödinger equation *plus* relativistic corrections, in the form:

$$\left[\frac{\mathbf{p}^2}{m} \;+\; V_s(r) + V_v(r) - \frac{\mathbf{p}^4}{4m^3}\right.$$
$$\left. -\; \frac{1}{4m^2}\{\mathbf{p}^2, V_s(r)\}\right]|\psi>= E|\psi>, \qquad (25)$$

where now the eigenvalue E is approximated up to the order c^{-2}. In more detail, the first three terms of the *l.h.s.* represent the nonrelativistic contributions of the order c^0. The fourth term is the standard relativistic correction to the kinetic energy, of the order c^{-2}. The last term, also of the order c^{-2}, is a momentum dependent relativistic correction specific of the scalar interaction; in fact, it can be also straightforwardly obtained by means of the mass replacement of Eq. (18) in the nonrelativistic kinetic operator $\frac{\mathbf{p}^2}{m}$. Further terms of relativistic corrections would be given by the (peculiar) adopted dynamical model also including the spatial part of the vector potential.

In the second place, we discuss the relationship of our model with other relativistic approaches.
In general, relativistic wave equations for bound systems are derived performing a three-dimensional reduction of

the BSE or, equivalently, considering, in the scattering amplitude, the *box* and *crossed-box* fourth order Feynman graphs. In this way a great variety of equations has been constructed. Some examples are given in Refs. [9–14].

Obviously, for a given relativistic equation, the physical results strictly depend on the choice of the quasipotential that represents the interaction. In the present preliminary work, we shall not analyze in detail this point. We only try to reproduce, in Section 6, the general structure of the charmonium spectrum by using phenomenological scalar and vector potentials.

The most relevant aspect of our model, defined in Eq. (20) is the quadratic character of of the equation that, at least in the free limit, gives rise to positive and negative energy solutions.

The same property is found when a three-dimensional reduction of the BSE is performed. Let us consider, as an example, the analysis shown in a standard textbook [12] for the study of positronium. By using the *instantaneous approximation* for the interaction, the BSE is reduced to a three-dimensional equation in which the CM energy of the system (disregarding the interaction) is:

$$M = \pm 2\sqrt{\boldsymbol{p}^2 + m^2}\ . \tag{26}$$

(In our notation, M stands for E and $\sqrt{\boldsymbol{p}^2 + m^2}$ for ω of Ref. [12]).

Note that our quadratic model also gives the two signs of the energy that, as shown in [12], are related to the propagation of $(++)$ and $(--)$ fermionic states. In that equation, the $(+-)$ and $(-+)$ states are completely suppressed, while, in another (similar) relativistic model [11], they only appear as closed-channel, virtual states. In this way, our equation and the two models cited above do not admit spurious solutions with $M = 0$, avoiding the so-called *continuum dissolution* problem, discussed, for example, in Ref. [14].

Another interesting model with quadratic kinetic operators has been also proposed in the context of two-body Dirac equations of constraint dynamics [15]. The authors consider, for each particle, a Dirac-like equation. Then, these equations are coupled by the interaction terms. Furthermore, in that work a detailed theoretical discussion about the construction of relativistic equations is also given.

Concluding, we also mention that a specific aspect of our equation consists in the presence of the quadratic interaction terms $V_v^2(r)$ and $V_s^2(r)$. In the case of small coupling constants, their contributions would be negligible. On the other hand, for interaction operators with strong couplings, their effects should be carefully studied in the context of the adopted phenomenological model.

Finally, the very simple and transparent procedure that is used to introduce the interaction can be easily generalized in order to construct realistic constituent quark models with relativistic interactions.

4. Relevant generalizations of the relativistic equation

In this Section we consider two interesting generalizations of our equation: in Subsection 4.1 a covariant version that can be applied in any reference frame will be given; in Subsection 4.2 a version of the equation for two particles with different mass will be studied.

These generalizations are useful for further developments and applications of the model but are not necessary to understand the contents of the following parts of the paper. For this reason, the uninterested reader can skip the present Section.

4.1. The covariant form of the equation

This generalization is very relevant for the study of scattering processes of bound systems and, in turn, for determining their electroweak form factors. We recall that in these processes the bound system necessarily possesses different momenta in the initial and final state, so that a covariant formalism, that can be applied to a generic reference frame, is required.

For convenience, we start this generalization by writing Eq. (20) of our model in the following form:

$$\left[4(\boldsymbol{p}^2 + m^2) - M^2\right] |\psi_{CM}> = W(M, r)|\psi_{CM}>, \tag{27}$$

where $W(M, r)$ collectively represents all the terms containing the interaction:

$$W(M, r) = V_s^2(r) + 4mV_s(r) - V_v^2(r) + 2MV_v(r)\ . \tag{28}$$

To avoid confusion, we point out that the two-body CM reference frame, represents, in present context, the *rest frame* of the bound system.

We note that the relativistic transformation properties of the momentum can be more clearly analyzed than the transformation properties of the position operator. For this reason we shall write our equation in the momentum space.

As first step, we introduce the rest frame, momentum space, wave function:

$$\psi_{CM}(\boldsymbol{p}) = <\boldsymbol{p}|\psi_{CM}> \tag{29}$$

and the momentum space interaction matrix-elements:

$$W(M, \boldsymbol{p}, \boldsymbol{p}') = < \boldsymbol{p}|W(M, r)|\boldsymbol{p}' > \ . \tag{30}$$

Recall that, in general, $W(M, \boldsymbol{p}, \boldsymbol{p}')$ depends on $\boldsymbol{p}$, $\boldsymbol{p}'$ only through $\boldsymbol{p}^2$, $\boldsymbol{p}'^2$ and the scalar product $\boldsymbol{p} \cdot \boldsymbol{p}'$; in particular, for the case local interactions, that we are presently considering, the dependence is on $(\boldsymbol{p} - \boldsymbol{p}')^2$.
With the help of the previous positions, the integral equation of our model in the momentum space takes the standard form:

$$\left[4(\boldsymbol{p}^2 + m^2) - M^2\right] \psi_{CM}(\boldsymbol{p}) = \int d^3p' W(M, \boldsymbol{p}, \boldsymbol{p}')\psi_{CM}(\boldsymbol{p}') \ . \tag{31}$$

We now pass to a generic reference frame, where the total 4-momentum of the system is $P^\mu = (E, \boldsymbol{P})$, with $E = \sqrt{\boldsymbol{P}^2 + M^2}$. Furthermore, the intrinsic 4-momentum in the generic reference frame is taken as an *on-shell* quantity: $q^\mu = (\epsilon(\boldsymbol{q}), \boldsymbol{q})$ with $\epsilon(\boldsymbol{q}) = \sqrt{\boldsymbol{q}^2 + m^2}$. Note that $\boldsymbol{q}_{CM} = \boldsymbol{p}$. We also point out that q^μ can identified with 4-momentum of one of the two particles of the bound system.
We now express the CM invariant quantities in terms of the 4-momenta of the generic frame:

$$\epsilon(\boldsymbol{p}) = P^\mu q_\mu \frac{1}{M} \tag{32}$$

and, consequently

$$\boldsymbol{p}^2 = \left(P^\mu q_\mu \frac{1}{M}\right)^2 - m^2 \ . \tag{33}$$

The *same* equations hold for the primed momenta, replacing $\boldsymbol{p}$ with $\boldsymbol{p}'$ and q_μ with q'_μ. Furthermore, we can write:

$$\boldsymbol{p} \cdot \boldsymbol{p}' = -q^\mu q'_\mu + P^\mu q_\mu P^\nu q'_\nu \frac{1}{M^2} \ . \tag{34}$$

In this way we can express the interaction matrix-elements as a function of covariant arguments:

$$W(M, \boldsymbol{p}, \boldsymbol{p}') = \bar{W}(P^\mu, q^\mu, q'^\mu) \ . \tag{35}$$

We now make use of the *covariant* integration over $\boldsymbol{q}$:

$$\int \frac{d^3q}{\epsilon(\boldsymbol{q})} \ldots = \int \frac{d^3p}{\epsilon(\boldsymbol{p})} \ldots \tag{36}$$

so that the *normalized* wave function in the generic reference frame is:

$$\psi(\boldsymbol{q}) = \left[\frac{\epsilon(\boldsymbol{p})}{\epsilon(\boldsymbol{q})}\right]^{1/2} \psi_{CM}(\boldsymbol{p}) \ , \tag{37}$$

with $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$ in the CM wave function of the *r.h.s.*. By using the previous results, with standard handling, Eq. (31) is finally written in a generic reference frame as:

$$\left[\frac{4}{M^2}(P^\mu q_\mu)^2 - M^2\right] \psi(\boldsymbol{q}) = \int d^3q' \left[\frac{\epsilon(\boldsymbol{p})}{\epsilon(\boldsymbol{q})}\right]^{1/2} \bar{W}(P^\mu, q^\mu, q'^\mu) \left[\frac{\epsilon(\boldsymbol{p}')}{\epsilon(\boldsymbol{q}')}\right]^{1/2} \psi(\boldsymbol{q}'), \tag{38}$$

where $\epsilon(\boldsymbol{p})$ and $\epsilon(\boldsymbol{p}')$ are given by Eq. (32).

Clearly, this procedure spoils the locality of the model in a generic reference frame. However, in general, one can determine the spectrum of the system and the wave functions in the CM (performing the Fourier transformation to momentum space), then with the help of Eq. (37) one can obtain the wave functions in a generic frame in order to calculate the form factors of the bound system.

4.2. The case of two particles with different mass

This second generalization is mainly relevant for the study of D, D_s, B and B_s mesons, that are bound states of two quarks with different mass. This specific aspect of the hadronic phenomenology was thoroughly studied in Ref. [7]. Here, we adapt that theoretical procedure to our relativistic equation. We recall that our generalized equation can be also used for the study of the baryon spectroscopy in the quark-diquark model [16, 17], avoiding the use of stan-

dard Hamiltonians with nonlocal kinetic terms. For this study, a similar technique was also used in the framework of other approaches [15, 18].
Here, we start by introducing for the two particles of mass m_i $(i = 1, 2)$, their total energies E_i, so that, working in the CM reference frame, the eigenvalue equation for noninteracting particles (that replaces Eq. (17)) takes the form:

$$\left[2\boldsymbol{p}^2 + m_1^2 + m_2^2\right] |\psi_{free}> = \left[E_1^2 + E_2^2\right] |\psi_{free}> , \quad (39)$$

where, also in this case, $\boldsymbol{p}$ represents the CM relative momentum, with the standard assumption $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$. The interaction is introduced analogously to Eqs. (18) and (19) by means of the following substitutions:

$$m_i \to m_i + \frac{V_s(r)}{2} \quad (40)$$

and

$$E_i \to E_i + \frac{V_v(r)}{2} \ . \quad (41)$$

Then, the mass of the system, *i.e.* its total energy in the CM, is standardly introduced as:

$$M = E_1 + E_2 \ . \quad (42)$$

In order to construct a three-dimensional model, the energy difference of the two particles must be fixed. To this aim we use the following standard equation:

$$E_1^2 - E_2^2 = m_1^2 - m_2^2 = M \cdot (E_1 - E_2) \ . \quad (43)$$

Note that for two equal-mass particles, each particle has the same energy, that is $M/2$.

By replacing the previous positions in Eq. (39), one finally obtains:

$$\left[4\mathbf{p}^2 + 2(m_1^2 + m_2^2) + V_s^2(r) + 2(m_1 + m_2)V_s(r) - V_v^2(r) + 2MV_v(r)\right] |\psi> = F(M, m_1, m_2)|\psi> \ , \quad (44)$$

with

$$F(M, m_1, m_2) = \frac{(m_1^2 - m_2^2)^2}{M^2} + M^2 \ . \quad (45)$$

The last two equations represent the generalization of Eq. (20). Note that the *l.h.s.* of Eq. (44) is now given by the function defined in Eq. (45). The value of the mass M can be found by applying to this specific equation the numerical technique discussed in Section 6.

5. The study of some analytic cases

A very relevant point of our equation is the possibility of determining analytic solutions that can be used as starting point for the study of realistic models.
To this aim we consider, for two equal-mass constituents, three special choices of the potentials that can allow to solve, with relatively simple calculations, the *exact* equations of Section 3.

Choice 1. In this case, analogously to the one-body Dirac equation, we shall obtain an equation similar to Eq. (16) by taking the scalar and vector potentials satisfying the same relation of Eq. (11), that is:

$$V_s(r) = V_v(r) = \frac{1}{2}U(r) \ . \quad (46)$$

With this choice Eq. (22) takes the form

$$\left[\frac{\mathbf{p}^2}{m} + \left(1 + \frac{E}{4m}\right) U(r)\right] |\psi> = \mathcal{E}(E)|\psi> \quad (47)$$

that is analogous to Eq. (16).
By introducing the *effective energy-dependent mass* as

$$\mu(E) = \left(1 + \frac{E}{4m}\right) m \ , \quad (48)$$

Eq. (47) can be conveniently written in the form:

$$\left[\frac{\mathbf{p}^2}{\mu(E)} + U(r)\right] |\psi> = E|\psi> \ . \quad (49)$$

We have obtained a relativistic wave equation with the same structure of a standard two-body, equal-mass non-relativistic Schrödinger equation, but now the factor

$\mu(E)$ introduces the energy dependence of the relativistic model. In consequence, if one considers a potential $U(r)$ for which the corresponding wave equation can be solved analytically, then the values of E for Eq. (49) are obtained by replacing m with $\mu(E)$ in the expression of the nonrelativistic eigenvalues.
Let us discuss the two following examples: a Coulombic potential plus a constant term and a harmonic oscillator potential.
In the first case one has:

$$U(r) = -\frac{\beta_u}{r} + U_0 \ . \tag{50}$$

Recalling the standard eigenvalues of the nonrelativistic equation

$$E_n^{NR} = -\frac{m\beta_u^2}{4n^2} + U_0 \ , \tag{51}$$

by replacing m with $\mu(E_B)$ of Eq. (48), with standard algebra one finds:

$$E_n = \left(-\frac{m\beta_u^2}{4n^2} + U_0\right) \cdot \left(1 + \frac{\beta_u^2}{16n^2}\right)^{-1} \ , \tag{52}$$

being $n = 1, 2, 3.....$ The angular momentum quantum number is l, with the condition $l \leq n-1$.
For the case of a harmonic potential

$$U(r) = \frac{1}{2}kr^2 \ , \tag{53}$$

one has the nonrelativistic energies in the form:

$$E_N^{NR} = \left(N + \frac{3}{2}\right)\omega \ , \tag{54}$$

where $N = 0, 1, 2...$ and the angular momentum is $l = N, N-2, N-4, ... \geq 0$. Furthermore, ω is defined as:

$$\omega = \left(\frac{2k}{m}\right)^{1/2} \ . \tag{55}$$

Then, replacing the mass m with $\mu(E)$ of Eq. (48) one finds:

$$E_N \cdot \left(1 + \frac{E_N}{4m}\right)^{1/2} = \left(N + \frac{3}{2}\right)\omega \tag{56}$$

that can be easily solved numerically for E_N.
We point out that in the two examples discussed above the degeneracy of the energy levels remains the same as in the corresponding nonrelativistic cases.

Choice 2. Another possible choice is to take only a scalar potential with a vanishing vector potential, setting $V_v(r) = 0$ in Eqs. (20) and (22). In this case the effective interaction is not energy-dependent. One can solve directly Eq. (20). Otherwise, Eq (22) can be solved as a standard Schrödinger equation where the potential is $V_s(r) + \frac{1}{4m}V_s^2(r)$ and the eigenvalue is $\mathcal{E}$. Then, the energy of the system is determined solving algebraically, with respect to E, the quadratic Eq. (23). Choosing the sign that gives the standard nonrelativistic limit, one has:

$$E = 2m\left[\left(1 + \frac{\mathcal{E}}{m}\right)^{1/2} - 1\right] \ . \tag{57}$$

An analytic solution can be obtained for a potential with a Coulombic behavior:

$$V_s(r) = -\frac{\beta_s}{r} \ . \tag{58}$$

In this case, by using a nonrelativistic result [23], it can be found that the eigenvalues have the form:

$$\mathcal{E}_{kl} = -m\beta_s^2\left[2k + 1 + \left[(2l+1)^2 + 4\beta_s^2\right]^{1/2}\right]^{-2} \ , \tag{59}$$

where the integer $k = 0, 1, 2, ...$ has been introduced and $l \leq k$ represents the angular momentum quantum number. Then E_{kl} is calculated by means of Eq. (57).

Choice 3. Finally, we consider an interaction with a vector potential and a vanishing scalar potential, taking $V_s(r) = 0$. In this case Eq. (20) can be conveniently written in the following form

$$\left[\mathbf{p}^2 + m^2\right]|\psi> = \left[\frac{M}{2} - \frac{V_v(r)}{2}\right]^2 |\psi> , \tag{60}$$

that is formally equivalent to a standard one-body Klein-Gordon equation in which the energy is replaced by $M/2$ and the zero component of the vector potential potential by $V_v(r)/2$.
An analytic solution can be found [24] for a Coulombic interaction:

$$V_v(r) = -\frac{\beta_v}{r} \ ; \tag{61}$$

it has the form:

$$M_{nl} = 2m\left[1 + \frac{\beta_v^2}{4\left[n - l - \frac{1}{2} + \left((l+\frac{1}{2})^2 - \frac{\beta_v^2}{4}\right)^{1/2}\right]^2}\right]^{-1/2} \ , \tag{62}$$

with the integer $n = 1, 2, 3, ...$ and the angular momentum quantum number $l \leq n-1$.

6. A numerical application to the charmonium spectrum

In order to show a possible application of our model to a real physical system for which the relativistic effects are usually considered relevant, we now study, as an example, the general structure of the charmonium spectrum. By using, for the interaction, only central potentials, we shall not take into account the effects related to the spin-spin, spin-orbit and tensor interaction. These *corrective contributions*, that are extremely relevant for a detailed study of charmonium spectroscopy [5–8], can be introduced perturbatively carefully considering the Lorentz transformation properties of the interaction operators.

The aim of the following analysis is only to demonstrate that our relativistic equation, with a *local* kinetic operator, can adequately reproduce the main structure of the charmonium spectrum for the low-lying resonances. No attempt is made here to study the *details* of this spectrum given by the *corrective contributions* nor to determine the quantum numbers of all the experimentally observed resonances.

More precisely, we shall reproduce a limited set of well-established resonances, neglecting the mass splittings due to the *corrective contributions* of the interaction. Following the standard quark model assignments for the resonance quantum numbers, we consider the S-wave states with radial excitation number $n = 1, 2, 3$. For the P-wave states we take $n = 1, 2$. The values of the masses (to be reproduced) for these resonances have been taken, for example, from Ref. [25]. For the D-wave states, we consider the resonances with $n = 1, 2$; in this case we directly try to reproduce the experimental values of the masses [26] of the ψ resonances at 3.773 GeV and 4.153 GeV.

We now describe the specific numerical method for studying the charmonium spectrum by means of the energy-dependent relativistic model introduced in this work. To be definite, we shall refer, in the following, to the general Eq. (22) but the same method can be straightforwardly adapted to Eq. (49) of the *Choice 1* and also, with minor changes, to Eq. (44) for the case of constituents with different mass.

Our numerical method schematically consists in the following three steps.

First, we recall that, in the CM, a two-body standard Schrödinger equation can be solved by means of a diagonalization-minimization procedure (DMP), that is by diagonalizing the Hamiltonian matrix in a basis of orthonormal trial wave functions. Furthermore, the *dimensional parameter*, on which the trial wave functions depend, is also varied and the minimization procedure is repeated, until the minimum value of the ground state energy is found.

The radial wave functions of the variational basis [27] used in the present work have the following form:

$$R_{nl}(r) = \frac{1}{\bar{r}^{3/2}} \left[\frac{n!}{\Gamma(2l+3+n)} \right]^{1/2} s^l L_n^{2l+2}(s) \exp(-s/2) \,, \tag{63}$$

where $\bar{r}$ is the variational *dimensional parameter* and $s = r/\bar{r}$ is introduced as an adimensional argument. Finally, the $L_n^{2l+2}(s)$ represent the standard Laguerre polynomials, being n the radial quantum number and l the angular momentum quantum number.

Second, for a given form of the potentials $V_s(r)$ and $V_v(r)$, we can find the approximate eigenvalues $\mathcal{E}$ associated to the *l.h.s.* operator of Eq. (22) by means of the previously described DMP. With respect to this point, we note that in Eq. (22), the *l.h.s.* operator *depends on* the energy E. Furthermore, the *r.h.s.* eigenvalue is a function of E, that is $\mathcal{E} = \mathcal{E}(E)$, as given by Eq. (23).

In consequence, varying the numerical value of E in the *l.h.s.*, we find the corresponding values of $\mathcal{E}$ solving the equation by means of the DMP, until the numerical value found for $\mathcal{E}$ is equal to the *r.h.s.* function $\mathcal{E}(E)$ of Eq. (23). The value of E for which this equality is found represents the numerical solution for the energy of the system. Its mass is obviously given by $M = E + 2m$.

As anticipated, we incidentally point out that exactly the same procedure can be applied to solve Eq. (44) for two particles with different mass. In this case one has to apply the DMP to the *l.h.s.*, that in this case depends on M, until one finds the same numerical value for $F(M, m_1, m_2)$ in the *r.h.s.* of that equation.

Third, by using the solution procedure described above, a *fit* is performed on the free parameters of the model, minimizing the quantity:

$$\chi^2 = \sum_{nl} \left(M_{nl}^{theor} - M_{nl}^{exp} \right)^2 , \tag{64}$$

where equal weights have been assigned to all the charmonium resonances. The values of M_{nl}^{exp} have been taken according to the criteria discussed at the beginning of this Section. In this way the numerical values of the parameters that appears in $V_s(r)$ and $V_v(r)$ can be determined by the fit.

In detail, we point out that, after some trials, an interaction with equal vector and scalar contributions, as given by Eq. (46), was chosen to reproduce the charmonium spectrum. Furthermore, the potential $U(r)$ has been taken in the following form:

$$U(r) = U_0 \left[1 - \frac{1+\kappa}{e^{(\frac{r}{r_w})^p} + \kappa} \right] . \tag{65}$$

We note that the potential $U(r)$ of the previous expression is vanishing for $r = 0$ and tends to the finite value U_0 as $r \to \infty$. For this reason, that potential is not intended to reproduce confinement.

It depends on four free parameters: U_0, the adimensional parameters κ and p, and the length r_w, related to the width of the potential.

Moreover, in order to reproduce the short distance behavior of the strong $c\bar{c}$ interaction, we have also introduced a Coulombic vector potential, in the standard form:

$$V_v^C(r) = -\frac{4}{3}\frac{\alpha_s}{r} \ , \tag{66}$$

where α_s represents the strong effective adimensional coupling constant. The contributions of this interaction are calculated perturbatively.

We now show the numerical results of the calculation. Previously, we point out that two different strategies have been followed to assign the value of the charm quark mass m_c.

In *model a*, we fix the interval for m_c according to standard theoretical arguments for the *current* quark masses [26], that is $m_c = 1.275 \pm 0.025$ GeV.

In *model b*, assuming that the *constituent quark* masses can be different from the *current* ones, we leave m_c as a free parameter that is determined by the fit.

The results for M_{nl} of the two models are given in the columns *model a* and *model b* of Table 1. The obtained values for χ^2 are also shown in the last line of the table. As it could be expected, *model b* gives slightly better results than *model a*.

The values of the parameters of the two models, that are determined by the fit, are shown in the first two columns of Table 2.

Finally, as a test of applicability for our numerical solution technique, we also study a possible intrinsic energy-dependence of the interaction. Physically, effects of this kind may be produced by the suppression of the mesonic degrees of freedom of the interaction fields [28]. We introduce the *model c* replacing U_0 in Eq. (65) in the following way:

$$U_0 \to U_E = \bar{U}_0 + \lambda E^2 \ , \tag{67}$$

with $\bar{U}_0$ in GeV and λ in GeV^{-1}. In this model we fix the variation range of m_c as in *model a*. The other parameters have the same meaning of the corresponding quantities of *model a* and *model b*. The results are shown in the columns labelled as *model c* of Tables 1 and 2. A slight improvement of χ^2 is obtained.

Table 1. Values of M_{nl} (GeV) given by *model a*, *model b* and *model c* and the corresponding experimental values.

n	l	a	b	c	exp
1	0	3.07	3.06	3.07	3.07
2	0	3.68	3.67	3.67	3.67
3	0	4.06	4.07	4.05	4.04
1	1	3.50	3.50	3.51	3.53
2	1	3.94	3.94	3.93	3.93
1	2	3.77	3.77	3.77	3.77
2	2	4.13	4.14	4.13	4.15
χ^2		0.0019	0.0018	0.00085	

Table 2. Values of the parameters of *model a*, *model b* and *model c* determined by the fit.

	a	b	c
m_c(GeV)	1.255	1.219	1.264
U_0(GeV)	2.028	2.226	-
$\bar{U}_0$(GeV)	-	-	2.027
λ(GeV^{-1})	-	-	-0.0139
κ	-0.1048	-0.00235	-0.1061
$r_w(fm)$	1.211	1.210	1.209
p	1.481	1.329	1.481
α_s	0.0930	0.0836	0.1976

7. Conclusions and perspectives

In this work we have studied a relativistic model for composite systems with two equal-mass constituents, also generalizing the same model to the case of two particles with different mass. The model contains a *local* kinetic operator. More precisely, this operator is *quadratic* in the relative momentum so that a Schrödinger-like wave equation can be constructed. This equation is considerably easier to solve than other nonlocal relativistic equations.

As specific aspects of the present model, we note that quadratic potential terms are introduced; furthermore, for vector interactions, the effective potential contains an energy-dependent term and, also, the eigenvalue of our Schrödinger-like wave equation is algebraically related to the physical energy of the system.

For some central potentials of special form, the analytic solutions have been easily obtained. Otherwise, the locality of the kinetic operator allows to find the numerical solutions without performing the Fourier transformation of the wave functions to the momentum space.

A numerical method for solving our energy-dependent, local, wave equation has been developed.

As an example, a successful general description of the charmonium spectrum is obtained.

The model can be formally used to construct quark models with low-mass constituents, carefully considering the dynamical properties determined by the interaction terms. To this aim, spin-spin, spin-orbit and tensor interactions must be first consistently introduced.

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