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Semiclassical Limit of The EPRL Spin Foam Model

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Abstract

One of the pressing issues of present-day theoretical physics is the need for the quantisation of gravity. Although Einstein's theory of general relativity (GR) has experienced tremendous success as a classical theory of gravity, it faces a number of problems, including the existence of singularities in high-curvature regimes, such as the centre of a black hole or the Big Bang. In such scenarios, quantum effects of the geometry of spacetime are posited to play an important role, thus begging the formulation of a theory of quantum gravity. Such an endeavour naturally leads to several different approaches, which generally take the main assumptions from either quantum field theory or general relativity and attempt to account for the other *a posteriori*.

In this dissertation, we opt for the use of general relativity as a starting point for quantising gravity and present the background independent non-perturbative canonical quantisation approach known as "loop quantum gravity". From this theory, it is possible to construct objects known as "spin foams", which allow us to explore its dynamics from a covariant perspective. More specifically, the spin foam quantization of general relativity is a path integral quantization based on the loop quantum gravity approach, where the gravitational field is described by "spin networks". These spin networks can be understood as Wilson loop variables for the Ashtekar formulation of GR. Within the Engle-Pereira-Rovelli-Livine (EPRL) spin foam model, the transition amplitudes are constructed by using the topological gravity theory based on the BF theory for the Lorentz group and the constraints which define GR are then imposed on them.

Consistency of any quantum gravity theory requires its correspondence to general relativity in the low-energy scale and this prompts us to analyse the semiclassical limit of the aforementioned EPRL model. The study of such a limit is performed through the use of the effective action approach from background field method of quantum field theory, which yields a generalisation of GR — in the sense that it allows non-metric geometries — as the semiclassical limit of the EPRL model. In other words, the leading term of the one-loop effective action of the EPRL model corresponds to the area-Regge action, which is based on the Regge action discretising gravity.

Finally, after presenting the review described above, we discuss a version of Regge calculus which takes triangle areas and 3d angles as variables, define a new convergent state sum from it and extend the theory in order to include a particular type of Lorentzian triangulation.

Keywords: spin foam, semiclassical limit, EPRL model, effective action, Regge calculus

Resumo

A teoria da relatividade geral de Einstein é a teoria gravitacional mais bem-sucedida. No entanto, é também uma teoria incompleta, uma vez que contém regimes em que diverge para o infinito e se torna incapaz de efectuar qualquer previsão. Tais regimes são referidos como “singularidades” e ocorrem em cenários de alta curvatura, tal como no centro de buracos negros e no “Big Bang”. Assim sendo, é necessário modificar esta teoria de forma a evitar singularidades. A principal solução proposta na literatura é a quantização da gravidade, uma vez que se supõe que no contexto referido as fluctuações quânticas da geometria desempenhem um papel importante na acção gravítica. Esta solução provém de uma longa história de quantizações de teorias clássicas que continham divergências ultravioletas. À semelhança de tais teorias — como por exemplo o modelo do átomo de hidrogénio —, espera-se que os problemas enfrentados pela relatividade geral sejam eliminados. Por outro lado, uma teoria de gravitação quântica aproximar-nos-ia de uma unificação das quatro forças fundamentais numa só “teoria de tudo”.

Relativamente à unificação da mecânica quântica com a relatividade geral, existem duas abordagens principais: a primeira adopta os pressupostos da mecânica quântica e tenta incluir a relatividade geral *a posteriori*, enquanto que a segunda faz o inverso. As teorias mais bem-sucedidas dentro de cada abordagem são, respectivamente, a teoria de cordas e a teoria de gravidade quântica em laços. Nesta dissertação defendemos a segunda, dada a sua natureza independente do fundo — isto é, independente da escolha de uma métrica de fundo — e não-perturbativa. De facto, é argumentado que, num regime de alta curvatura, o método perturbativo não é aconselhável, uma vez que existem várias possibilidades de escolha para o par fundo-perturbação da métrica, o que leva ao surgimento de múltiplos cones de luz, que por sua vez levam a relações de causalidade distintas e incompatíveis.

A teoria de gravidade quântica em laços é uma teoria quântica da geometria do espaço-tempo que consiste na quantização canónica (ou Hamiltoniana) da relatividade geral, originando uma teoria de gauge $SU(2)$, semelhante à de Yang-Mills. Contudo, embora a gravidade quântica em laços tenha tido muito sucesso — como por exemplo a concordância com a relatividade geral no limite clássico — e faça previsões únicas — como a discretização do espaço-tempo à escala de Planck e a possível resolução de singularidades, como por exemplo a substituição do “Big Bang” pelo “Big Bounce” —, a mesma sofre de alguns problemas, como a definição da dinâmica quântica no contexto referido.

A solução estudada para o problema da dinâmica envolve o uso de espumas de spin, que podem ser vistas como uma formulação de integral de caminho da gravidade quântica em laços. As espumas de spin são também uma “história quântica” das redes de spin, que por sua vez são as configurações permitidas para as excitações do campo. Em geral, os modelos de espumas de spin para a relatividade geral em quatro dimensões começam por discretizar a relatividade geral, depois quantizam a parte BF topológica da nova teoria discreta e finalmente impõem os constrangimentos simpliciais ao nível quântico. No caso do conhecido modelo Barrett-Crane, a quantização é efectuada na acção $SO(4)$ simplicial de Plebanski. Contudo, os constrangimentos de Plebanski — que reduzem a teoria de BF à relatividade geral — são impostos fortemente, o que leva a condições adicionais desnecessárias.

O modelo EPRL, por sua vez, é uma modificação do modelo Barrett-Crane, que discretiza a relatividade geral, quantiza a mesma e finalmente impõe e relaxa os constrangimentos de Plebanski. Este último passo diferencia este modelo do anterior e resolve muitos dos problemas enfrentados pelo mesmo. Em relação ao primeiro passo, a discretização é feita recorrendo ao uso de uma triangulação em 4d que é dual ao complexo representado pela espuma de spin e onde spins inteiros são atribuídos a faces e um elemento

da base do espaço de "intertwiners" a cada aresta. Consequentemente, obtemos o espaço de redes de spin da gravidade quântica em laços $SO(3)$ Hamiltoniana como espaço de estados da teoria, assim como uma amplitude de vértice covariante sob $SO(3)$ e $SO(4)$.

Uma vez que a consistência de qualquer modelo de gravidade quântica está intimamente ligada à convergência da mesma para a relatividade geral, no limite clássico, é importante certificarmo-nos de que o modelo EPRL se reduz à (ou converge para a) teoria de Einstein no limite de baixas energias. Para o efeito, usamos o método da acção efectiva, proveniente do método de campo de fundo da teoria quântica de campo. Este método consiste em aplicar a fórmula da acção efectiva aos modelos de espuma de spin e somar as fluctuações em volta de uma configuração clássica. A acção efectiva permite-nos obter o limite semiclássico de qualquer modelo de espuma de spin e pode ser aplicada a qualquer modelo de soma de estados de gravidade quântica. No caso do modelo de EPRL, é necessário modificar a amplitude do vértice de forma a obter o limite semiclássico correcto, ou seja, uma generalização da teoria da relatividade geral — no sentido em que permite geometrias não-métricas. A modificação necessária envolve a divisão da amplitude (mencionada acima) pelo produto de uma função homogénea de ordem 12 do spin e da soma das dimensões do spin elevadas a um número positivo e suficientemente grande. Desta forma, a soma de estados do modelo torna-se finita e converge para o limite supramencionado, ou seja, a acção área-Regge. Por outras palavras, o primeiro termo da accção efectiva de primeira ordem em $\hbar$ do modelo EPRL corresponde à acção área-Regge, baseada na acção Regge, que, por sua vez, discretiza a relatividade geral de Einstein. A acção área-Regge difere, contudo, da primeira, uma vez que usa as áreas das faces como variáveis, ao invés dos comprimentos das arestas, como na acção Regge.

Após o cálculo do limite semiclássico do modelo EPRL, apresentamos uma versão modificada do cálculo Regge, uma vez que este último é incapaz de definir uma medida invariante de gauge na integral de caminho. Esta teoria usa áreas de triângulos — seguindo os passos da teoria topológica BF —, assim como ângulos em 3d, que constrangem as áreas. O resultado é uma acção que impõe dois conjuntos de constrangimentos sobre a acção (comprimento-)Regge: o primeiro exige que a definição dos ângulos 2d em termos de ângulos 3d seja a mesma, independentemente do tetraedro escolhido (dos 2 possíveis para o primeiro ângulo), enquanto que o segundo inclui as áreas dos triângulos na definição da condição de fecho dos tetraedros. O efeito dos constrangimentos é a redução da acção área-ângulo Regge para a acção comprimento Regge.

Finalmente, é apresentada uma modificação da acção área-ângulo Regge que inclui triangulações Lorentzianas. Estas triangulações consistem na foliação do espaço tempo em camadas com 3 dimensões, representando cada uma um tempo discreto. Estas camadas possuem símplices espaciais de dimensão 3 e estão interligadas por arestas temporais, formando assim símplices temporais de dimensão 4. Os comprimentos das arestas espaciais são fixos e iguais para todas as arestas desse tipo e o mesmo acontece para as arestas temporais (com um comprimento diferente das anteriores). Obtendo a fórmula que nos dá os ângulos 2d em termos dos ângulos 3d para este tipo de símplices Lorentzianas permite-nos modificar os constrangimentos anteriores de forma a acomodar os casos Euclideano e Lorentziano.

A criação de um modelo de soma de estados para esta nova acção, assim como a modificação adequada desse modelo (por forma a garantir a sua convergência), a aplicação do método da acção efectiva ao mesmo e a generalização da nova acção para outros casos Lorentzianos são deixados para um trabalho futuro.

Palavras-Chave: espuma de spin, limite semiclássico, modelo EPRL, acção efectiva, cálculo de Regge

Contents

List of Figures v

List of Tables vii

List of Abbreviations ix

1 Introduction 1

1.1 Gravity . . . 1

1.2 Quantum Gravity . . . 2

2 Loop Quantum Gravity 5

2.1 Overview . . . 5

2.2 Spin Networks . . . 8

3 Spin Foams 11

3.1 Overview . . . 11

3.2 Spin Foam Model Prescription . . . 12

3.3 Spin Foam Models . . . 13

3.4 Quantising BF Theory . . . 15

4 The EPRL Model 17

4.1 Preliminary Model . . . 17

4.2 Extension of The Model . . . 21

5 The Semiclassical Limit of The EPRL Spin Foam Model 27

5.1 Overview . . . 27

5.2 Possible Methods . . . 29

5.3 The Effective Action Method . . . 31

5.4 Higher-Order Corrections . . . 35

6 Modified Regge Calculus 37

6.1 Area-angle Regge Calculus . . . 37

6.2 A Lorentzian Extension . . . 42

7 Open Problems and Conclusions 47

7.1 Open Problems . . . 47

7.2 Conclusions . . . 49

References 53

Appendices 57

Appendix A Lorentzian Volumes and Dihedral Angles 59

A.1 Lorentzian Volumes 59

A.2 Lorentzian Dihedral Angles 66

Appendix B Lorentzian Area-Angle Formulae 69

B.1 Derivation 69

List of Figures

1.2.1 Light cone derived from the choice of a background metric η_{ab} and (smaller) light cones resulting from the quantum perturbations added to said metric. It is an inconsistent representation of the gravitational field, since the field excitations are expected to obey the main light cone, which itself bears no meaning in the context of GR, a theory which does not admit any preferential reference frame or background [1]. 3

2.2.1 A spin network state where the vertices are connecting 3 edges (i.e. they are 3-valent). In this case, the contraction in the central vertex ψ is unique and given by a Clebsh-Gordan coefficient (or intertwiner), unlike for higher valence vertices, where a choice of a higher-order tensor from several coefficients is necessary [2]. 8

2.2.2 A 4-valent vertex can be defined as two 3-valent ones and other assignments of pairs of edges (e.g. 1-3, 2-4) are equally possible, provided we use the correct intertwiners in the wave function equation [2]. 9

3.1.1 A spin foam describing the transitions between 3 different spin networks, forming a 2-complex (left), and a transition vertex (right) within the spin foam. The transitions between spin networks occur from bottom to top and, as one can easily notice, with each transition, new links are created and the spins are reassigned at the vertices [3]. 12

6.2.1 Possible simplices in 2 (top), 3 (middle) and 4 (bottom) dimensions, (up to time reversal for 3 and 4 dimensions): $(2,1)$- and $(1,2)$-simplices (top left) and the resulting gluing (top right); $(3,1)$- (middle left) and $(2,2)$- (middle right) tetrahedra/simplices; $(4,1)$- (bottom left) and $(3,2)$- (bottom right) simplices. The 2-dimensional strip (top right) is eventually joined at both ends, forming a band with topology $S^1 \times [0,1]$ [4]. 43

List of Tables

B.1 Possible left- (LHS) and right-hand side (RHS) combinations (up to the commutation of the second and third terms in ϕ) for the equation 6.1.1, along with their geometric interpretation: the 4-simplex σ; the resulting tetrahedron $\sigma(k)$; the 2d angle α appearing on the left-hand side; and the 3d angle ϕ combinations. The notation used has been explained earlier. 71

List of Abbreviations

BC	Barrett-Crane
CERN	European Organisation for Nuclear Research
EPRL	Engle-Pereira-Rovelli-Livine
FK	Freidel-Krasnov
GR	General Relativity
LQG	Loop Quantum Gravity
QFT	Quantum Field Theory

Chapter 1

Introduction

1.1 Gravity

One of the staples of scientific theories is the fact that they are not immutable. In addition, the strength of a theory is highly dependent on the number of accurate predictions it makes. This means that any unaccounted phenomenon is enough to warrant a modification of the underlying theory, or even the development of a new one. This has often been the case throughout scientific history, including the history of gravitational physics.

Up until the last century, the reigning theory of gravity was Newton's law of universal gravitation, which treated it as a force between any two particles, which was proportional to the product of their masses and inversely proportional to the square of the distance between them. This theory boasted a number of important achievements, the most important of which was arguably the prediction of the existence of a planet beyond Uranus — later discovered and named Neptune —, which was influencing its orbit.

However, some conflicting observations were later made, including the discrepancy between the precession of the perihelion of Mercury's orbit and the respective Newtonian calculations. Initial attempts to explain this discrepancy by means of a new planet between Mercury and the Sun were unsuccessful and thus the road was paved for a new theory of gravity.

Such a theory came about in the beginning of the 20^{th} century, through Albert Einstein. This time, gravity was considered a geometric property of space and time — henceforth named spacetime — and a connection was established between the spacetime curvature and the energy and momentum of matter (and radiation) within it. In a nutshell, it can be summarised by the following quote, by John Archibald Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve". General relativity was a tremendous achievement: not only was it able to solve the problems plaguing Newton's theory — such as the aforementioned precession of Mercury — but it also introduced new important concepts and predictions, such as gravitational time dilation, lensing and redshift, as well as black holes and gravitational waves, all of which have been confirmed by numerous observations.

Einstein's theory of general relativity is currently the most complete, accurate and widely accepted description of gravity. However, despite the overwhelming evidence in its favour, there are a number of unsolved problems which render this theory incomplete, including the existence of spacetime singularities. These singularities occur in spacetime locations where the gravitational field of an object, its energy density and the spacetime curvature around it — which are related to one another via the Einstein field equations (EFE) — all become infinite. In such a scenario, as is the case for the centre of a black hole , as well as the Big Bang, the laws of physics break down and general relativity is unable to provide any prediction.

The problems faced by general relativity suggest that this theory of gravity is incomplete and thus should be modified, by analogy with the case of Newton's theory. More closely related to these issues, however, are the ultraviolet catastrophe of blackbody radiation and the classical model of the hydrogen atom. Both of these models are classical and entail singularities which are avoided by taking the appropriate quantum effects into account, at small (distance) scales [1].

These examples are taken as a sign that Einstein's theory should be modified so as to include quantum degrees of freedom. In fact, while general relativity has succeeded in describing the universe at cosmological scales, quantum mechanics has experienced the same for subatomic scales. It is therefore paramount to conciliate these two theories, since doing so would be a step further in formulating a "theory of everything", i.e. a theory encompassing all four fundamental forces under the same framework.

1.2 Quantum Gravity

As mentioned above, both general relativity and quantum mechanics benefit from a significant amount of experimental evidence. Nonetheless, the unification of both theories is not straightforward, meaning at least one of them requires some level of modification. Consequently, one must choose which aspects of each theory are assumed to be correct and will therefore be used as a starting point for the development of an all-encompassing framework [5].

At first glance, the most sensible approach might seem to be the quantum field theory (QFT) formalism, since it has been successfully used as a description of the other three fundamental forces. Nevertheless, even though QFT can be used as an effective field theory for gravity at low energies, this method is troublesome when applied to high energies, due to the fact that general relativity is nonrenormalisable in such scales.

Renormalisation techniques are essential in ridding quantum field theories of undesirable divergences, by introducing "counterterms" which end up cancelling them. However, when the required number of counterterms becomes infinite, so does the number of free parameters, thereby stripping the theory of any predictive power. In this kind of scenario, said theory is defined as "nonrenormalisable" and that is precisely the case with Einstein's theory [1].

One way of overcoming these difficulties is to substitute the concept of a point-like particle for a string-like — i.e. one-dimensional — one. This gives rise to "string theory", or "superstring theory", in its latest form. Despite being a mathematically elegant solution to the problem of unifying gravity and quantum mechanics and being a very prolific research area, superstring theory possesses significant shortcomings, the main one being the number of unverified assumptions that are made. In a daring challenge to Occam's razor, this approach relies on the existence of several extra dimensions of space, as well as new particles, all of which is yet to be observed at CERN, despite extensive experiments.

As a result, there is good reason to accept the main premisses of general relativity and build a new theory from there, in lieu of quantum mechanics, quantum field theory or particle physics. One of the fundamental aspects of general relativity is the fact that it satisfies the principle of general covariance, which implies that the laws encompassed by this theory are the same for any coordinate system or reference frame, thus being invariant under coordinate transformations and remaining unchanged for any observer. In addition, it is also a background independent theory, since the gravitational field is not attached to any fixed background structure.

Considering general relativity as the best description of gravity thus far [6], it is reasonable to borrow its aforementioned features — i.e. general covariance and background independence — for a quantum gravity theory [6]. In light of that, we can point further obstacles to using quantum field theory within this

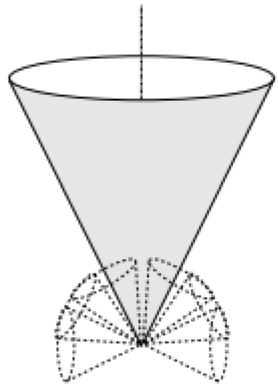


Figure 1.2.1: Light cone derived from the choice of a background metric η_{ab} and (smaller) light cones resulting from the quantum perturbations added to said metric. It is an inconsistent representation of the gravitational field, since the field excitations are expected to obey the main light cone, which itself bears no meaning in the context of GR, a theory which does not admit any preferential reference frame or background [1].

scenario. Firstly, QFTs are only defined for a fixed spacetime (background) geometry and are thus not generally covariant. Additionally, even though the ultraviolet divergences — i.e. high energy singularities — in these theories should in principle vanish upon accounting for quantum spacetime fluctuations, that can only be done by taking the dynamics of spacetime into account, which, by definition, forbids the use of fixed backgrounds [1].

An area closely related to background dependence is perturbation theory. Due to the relation between these two concepts, as well as the aforementioned problems with the former in the gravitational framework, a non-perturbative theory of gravity is generally considered to be the best approach when it comes to defining a quantum gravity theory. In order to understand the reasoning leading to this statement, we must first present the consequences of a perturbative approach to quantum gravity [1].

Perturbation theory is a method whereby a complex (quantum) system is described as the result of small perturbations to a simpler system, to which we know the solution. In the context of quantum gravity, this simpler system is regarded as a fixed background metric η_{ab}, on top of which we add metric fluctuations, h_{ab}. The resulting metric tensor is then

$$g_{ab} = \eta_{ab} + h_{ab}, \tag{1.2.1}$$

and this is appropriately applied to nearly-flat spacetime. However, in a high-curvature setting, such as the one where the study of quantum gravity is applied, the same spacetime metric can be equivalently represented by other background-perturbation pairs, meaning we can also have

$$g_{ab} = \eta'_{ab} + h'_{ab}, \tag{1.2.2}$$

where $\eta'_{ab} \neq \eta_{ab}$. This new background metric can represent a different light cone and thus lead to different causality relations from the original one. Since the background spacetime is what allows one to define causality relations in perturbation theory, it is contradictory to choose a background (and thus a causal structure or light cone) with no *a priori* physical meaning to which the gravitational field has to adhere, when in reality the quantum excitations of this field can generate different light cones. In other words, the mandatory background imposed by perturbation theory not only clashes with the covariant and background independent nature of general relativity, but is also devoid of any physical importance, when compared to other possible "backgrounds" (e.g. η'_{ab}) — as shown in Figure 1.2.1 [1].

The ambiguities presented above show that a standard perturbative approach to a covariant system is not a consistent one and thus the most logical path involves defining a non-perturbative theory of quantum gravity. Thus, to summarise the main ideas discussed in the last two chapters: it is crucial to formulate a theory of quantum gravity which is not only consistent with general relativity (by identifying it as its classical limit) [1] but also able to consistently describe the physics of singularities by taking quantum effects into account, at small scales. In order to accomplish the former goal, the resulting quantum gravity theory must obey the same principles as Einstein's, namely general covariance and background independence, as well as being non-perturbative. One such theory is named loop quantum gravity [6].

Chapter 2

Loop Quantum Gravity

2.1 Overview

Loop quantum gravity is an attempt to perform a mathematically sound background independent and non-perturbative quantisation of general relativity. Following the idea from general relativity that gravity is spacetime geometry, loop quantum gravity becomes a theory of quantum geometry that is diffeomorphism invariant, i.e. it is invariant under isomorphisms to other smooth manifolds [1].

The loop quantum gravity approach [7, 8] consists of using the canonical (i.e. Hamiltonian) quantisation of general relativity via connection variables. General relativity is then transformed into an $SU(2)$ gauge theory, similar to the Yang-Mills theory. However, even though this theory boasts many achievements — such as the consistency with general relativity in the classical limit — and makes unique predictions — such as the discretisation of space at the Planck scale —, it has also encountered obstacles, such as the problem of defining quantum dynamics within such a framework. Fortunately, this particular problem might be solved with the use of spin foams, which will be presented later in this chapter [1]. For now, we will describe how loop quantisation is performed [6].

As mentioned above, the Hamiltonian formulation of general relativity is the starting point for loop quantum gravity [6, 9]. In this formulation, we foliate the usual 4d spacetime into a series of spatial slices Σ evolving in time, with a 3-metric q_{ab} induced on them. In order to relate coordinates between different spatial slices, we use two Lagrange multipliers: the lapse function N and the shift vector N^a (where a is a spatial index). While the former measures the proper time, the latter measures changes in the spatial coordinates. These two multipliers impose the constraints which imbue general covariance in GR: the scalar/Hamiltonian constraint and the diffeomorphism/vector constraints. Thus, the information is embedded in the spatial metric q_{ab}, as well as in the extrinsic curvature K_{ab}.

In formulating GR as a gauge theory, LQG follows the Ashtekar-Barbero formalism [6, 10], which uses an $SU(2)$ gauge connection, called the Ashtekar-Barbero connection, A_a^i, and densitised triad, E_i^a (where both indices i are $SU(2)$) as canonical coordinates. In fact, one easily notices that the latter is the canonically conjugate momentum of the former. We define the aforementioned 3-metric using two other important elements, called the triad, e_i^a, and its inverse, the co-triad, e_a^i. The relations are then

$$q_{ab} = e_a^i e_b^j \delta_{ij}; \tag{2.1.1}$$

and

$$e_i^a e_b^j = \delta_i^j \delta_b^a, \tag{2.1.2}$$

while the densitised triad mentioned earlier is defined as

$$E_i^a = \sqrt{q} e_i^a. \tag{2.1.3}$$

To complete the canonical pair (A, E), we need only define the Ashtekar-Barbero connection:

$$A_a^i = \Gamma_a^i + \gamma K_a^i, \tag{2.1.4}$$

where $K_a^i = K_{ab} e_j^b \delta^{ij}$ is the extrinsic curvature and Γ_a^i is the spin connection consistent with E_i^a. Finally, we have the Poisson bracket

$$\left\{ A_i^a(x), E_b^j(y) \right\} = 8\pi G \gamma \delta_b^a \delta_i^j \delta(x-y), \tag{2.1.5}$$

where γ is the Immirzi parameter, which measures the size of the quantum of area in Planck units. Disregarding matter for now, we can express the constraints mentioned earlier, as well as a new one, in terms of the new variables (A, E):

- $G_i = \partial_a E_i^a + \epsilon_{ijk} \Gamma_a^j E^{ak} = 0$ is the new Gauss/gauge constraint, introduced due to the invariance of the Euclidean metric under $SU(2)$ rotations;
- $C_a = F_{ab}^i E_i^b = 0$ is the diffeomorphism/vector constraint;
- $C = \frac{1}{\sqrt{|det(E)|}} \epsilon_{ijk} \left[F_{ab}^i - (1+\gamma^2) \epsilon_{mn}^i K_a^m K_b^n \right] E^{aj} E^{bk} = 0$ is the scalar/Hamiltonian constraint,

where F_{ab}^i is the curvature tensor of the connection A.

In the loop quantisation, the phase space is described by holonomies and fluxes [6], which are later transformed into quantum variables. The advantages of using these variables are threefold: they are diffeomorphism invariant, as required by the main goal of LQG; the holonomies behave well under gauge transformations; and the Poisson bracket of these variables is well-defined. Put simply, the holonomy of a connection is a mathematical concept used for determining how much a parallel transport around a closed loop fails to preserve the transported information. For a connection A, its holonomy along an edge e is

$$h_e(A) = P e^{\int_e dx^a A_a^i(x) \tau_i}. \tag{2.1.6}$$

In the equation above, P is the path ordering and τ_i are the $SU(2)$ generators. The conjugate momentum of the holonomy is the flux of the densitised triad — which in LQG plays the role of an electric field — over surfaces S, defined as

$$E(S, f) = \int_S f^i E_i^a \epsilon_{abc} dx^b dx^c, \tag{2.1.7}$$

where f^i is an $SU(2)$-valued function used to smear the flux [6]. Finally, the Poisson bracket of these two variables is given by

$$\{E(S, f), h_e(A)\} = 2\pi G \gamma \epsilon(e, S) f^i \tau_i h_e(A), \tag{2.1.8}$$

where $\epsilon(e, S) = 0$ if e and S do not intersect one another or if $e \subset S$. If e and S intersect at one point, we get $\epsilon(e, S) = \pm 1$, depending on their relative orientation.

The so-called holonmy-flux algebra presented here is associated to a kinematical Hilbert space, defined as the completion of the space of cylindrical functions, with respect to the Ashtekar-Lewandowski measure [6]. We define a generalised connection $\bar{A}_e$ as a map from an analytic path within a spatial slice to $SU(2)$, while a collection of such paths intersecting at most at their ends is called a graph. For a closed graph Γ with n edges, a cylindrical function with respect to said graph is written as

$$\psi_\Gamma(\bar{A}) = f_\Gamma(\bar{A}_{e_1}, ..., \bar{A}_{e_n}), \tag{2.1.9}$$

where $\bar{A}$ is an element of $SU(2)$, f_Γ is a function on $SU(2)^n$ and e is a path on a spatial slice. We also define

$$Cyl = \bigcup_\Gamma Cyl_\Gamma \tag{2.1.10}$$

as the space of all cylindrical functions with respect to any graph in a spatial sector. Furthermore, we can use the Ashtekar-Lewandowski measure to obtain the inner product on this space:

$$\left\langle \psi_\Gamma, \psi_\Gamma^{'} \right\rangle = \int_{SU(2)^n} \prod_{e \subset \Xi_{\Gamma\Gamma'}} dh^e \bar{\psi}_\Gamma \psi_\Gamma^{'}, \tag{2.1.11}$$

where Ξ is a graph such that $\Gamma, \Gamma^{'} \subset \Xi_{\Gamma\Gamma'}$, $\Gamma^{'}$ is the graph with respect to which $\psi_\Gamma^{'}$ is cylindrical and dh is the normalised Haar measure on $SU(2)$. We can now define the kinematical Hilbert space as the Cauchy completion of the space of all cylindrical functions in the Ashtekar-Lewandoswki norm.

The basis of this space is composed of so-called spin network states, which are cylindrical functions with respect to a graph where each edge is coloured using an irreducible (and non-trivial) representation of $SU(2)$. Naturally, by definition of basis, we know that any cylindrical function can be expanded in these states. We will delve into these spin network states later on.

We can now define the quantum operators for the flux and the holonomies. The former is given by

$$\hat{E}_\Gamma(S, f) = i2\pi G\hbar \sum_{e \subset \Gamma} \epsilon(e, S) Tr \left(f^i \tau_i \bar{A}_e \frac{\partial}{\partial \bar{A}_e} \right), \tag{2.1.12}$$

while the latter acts by multiplication [6]. In defining any general quantum operator, we must perform regularisation, which comprises the following steps:

1. The spatial manifold is triangulated into tetrahedra;
2. We use the Riemann sum over these cells, instead of the usual integral over the manifold;
3. We assign a regularised expression (in terms of the basic operators) to each cell, which must converge to the classical one when the cell vanishes;
4. If the expression is densely defined on the kinematical Hilbert space, it can be written as a quantum operator, independently of the regularisation.

As mentioned earlier, one of the most important consequences of this procedure is the discretisation of the spectra of such geometrical operators as area and volume, which thus give us an equally discrete manifold, at the Planck scale. This unique result is extremely useful when bypassing the singularities of general relativity.

The remaining step is to solve the quantum constraints, i.e. the Gauss, the diffeomorphism and the Hamiltonian constraints. Although the first two are solvable, finding the solution to the last one is a complicated task, as there are ambiguities when quantising the corresponding operator.

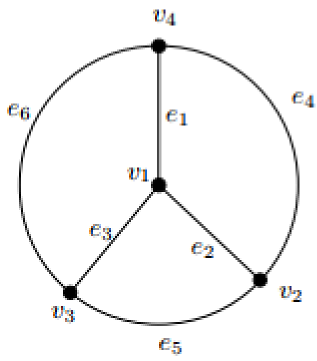


Figure 2.2.1: A spin network state where the vertices are connecting 3 edges (i.e. they are 3-valent). In this case, the contraction in the central vertex ψ is unique and given by a Clebsh-Gordan coefficient (or intertwiner), unlike for higher valence vertices, where a choice of a higher-order tensor from several coefficients is necessary [2].

After having defined the loop quantisation of gravity, we are left with the task of studying the dynamics of such a system and this is where the spin foam approach is used. However, before introducing the concept of spin foam, it is imperative to first discuss spin networks, as the former can be seen as the evolution (or quantum history) of the latter.

2.2 Spin Networks

In loop quantum gravity, the space of spin networks corresponds to the initial kinematical Hilbert space and the goal is to reduce it to a physical one, where all the constraints are satisfied. While the solution to the Gauss constraint lies within the former, the same does not hold for the diffeomorphism ones, which forces us to define the latter (larger) space [2].

LQG posits that the Planck-scale geometry is foam-like and that the field excitations only occur on specific configurations of edges connected by vertices, called spin networks [2]. These networks are graphs Γ which are discrete, finite and embedded in the (continuous) spatial manifold Σ. The edges $e_i \in \Gamma$ are connected at the vertices $v \in \Gamma$ and each one has a holonomy $h_e\left[A\right]$ of the gauge connection A. The Hilbert space of spin networks is given by the basis of wave functions over the configuration space [1] and these functions assign a complex number to each configuration of the gauge connection. Thus, one can represent the wave function on the spin network of the graph Γ as a function ψ of the holonomies of said graph, thereby obtaining a similar equation to the one presented earlier (in (2.1.9)), i.e.

$$\Psi_{\Gamma,\psi}\left[A\right] = \psi\left(h_{e1}\left[A\right], h_{e2}\left[A\right], ...\right). \tag{2.2.1}$$

In order to satisfy the Gauss constraint, the wave function needs to be $SU(2)$-invariant. consequently, since the wave function gives us a complex number, this number must be $SU(2)$-invariant as well and this is achieved by contracting the indices of the holonomies with invariant tensors placed at the vertices [2].

Due to the lack of uniqueness in the description of 4-valent or higher vertices, it is the case that most spin networks can be represented by several wave-functions of the same kind as the one in Figure 2.2.1 — where $\psi = \left(\rho_{j_1}(h_{e_1}[A])\right)_{\alpha_1\beta_1}\left(\rho_{j_2}(h_{e_2}[A])\right)_{\alpha_2\beta_2}\left(\rho_{j_3}(h_{e_3}[A])\right)_{\alpha_3\beta_3} C^{j_1j_2j_3}_{\beta_1\beta_2\beta_3}...$. For example, When writing the wave function of a spin network where 4 edges (e_1 to e_4) meet at one vertex, it is necessary

[1] The quantum configuration space is given by the space of holonomies [6].

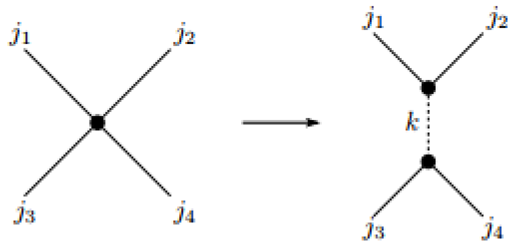


Figure 2.2.2: A 4-valent vertex can be defined as two 3-valent ones and other assignments of pairs of edges (e.g. 1-3, 2-4) are equally possible, provided we use the correct intertwiners in the wave function equation [2].

to group the edges into pairs. Say we choose (1,2) and (3,4). Then the wave function will be similar to the one in Figure 2.2.1 and differ only on the intertwiners, which become $C^{j_1 j_2 k}_{\beta_1 \beta_2 \beta} C^{j_3 j_4 k}_{\beta_3 \beta_4 \beta}$, where the spin k can be any chosen value between $|j_1 - j_2|$ and $j_1 + j_2$, as expected from quantum mechanics. Furthermore, we can switch to a different wave function representation by means of a transformation of the Clebsh-Gordan coefficients to the new ones representing the new coupling of the edges. Pictorially, this implies that a 4-valent vertex can be split into two virtual 3-valent vertices, with a virtual edge joining both and having a spin k attached to it, as shown in Figure 2.2.2.

The wave functions are cylindrical and act on the connection A only on a set of measure 0 — analogously to the Dirac delta function acting on other functions [2]. As a result, the Hilbert space of spin networks is given by the linear combinations of these functions over all the graphs, just as stated in (2.1.10). Additionally, the product of two cylindrical functions (even on different spin networks) is a cylindrical function and their scalar product, which is diffeomorphism invariant, vanishes for different graphs, $\Gamma \neq \Gamma'$, and is identical to (2.1.11) for coinciding graphs [2]. Lastly, we define the kinematical Hilbert space in the same manner as the one done earlier, meaning it is composed of all linear superpositions of spin network states Ψ_n with finite norm, i.e. $\Psi = \sum_{n=1}^{\infty} a_n \Psi_n$, with $||\Psi||^2 < \infty$. The main feature of this Hilbert space, as well as the one which differentiates the approach mentioned here from others, is its non-separability [2].

Spin networks allow us to describe the quantum geometry of space, as it is a description of a quantum state of the gravitational field on a spatial sector Σ of the spacetime manifold. However, for a description of spacetime, i.e. for a description of the evolution of spin networks, we need to ascend one dimension higher and follow the spin foam approach.

Chapter 3

Spin Foams

3.1 Overview

As discussed earlier, spin foams are the proposed solution to the problem of defining the dynamics of loop quantum gravity. They arise from the formal definition of the exponentiation of the scalar constraint and this approach can be seen as a path integral formulation of LQG [3]. When dealing with the scalar constraint, one can define the "projection operator" P from the kinematical Hilbert space to the kernel of the physical one [3]. The formal expression is given by

$$P = \prod_{x \in \Sigma} \delta(\hat{\mathcal{S}}(x)) = \int \mathcal{D}[N] e^{i\hat{\mathcal{S}}[N]}, \quad \hat{\mathcal{S}}[N] = \int dx^3 N(x) \hat{\mathcal{S}}(x), \tag{3.1.1}$$

where $N(x)$ is the lapse function and $\hat{S}$ is the scalar constraint. When P is applied to any state in the kinematical Hilbert space, the result is a solution of the set of constraints imposed on H_{kin} — the scalar, diffeomorphism and Gauss ones —, also called quantum Einstein equations (QEE) [3]. Moreover, apart from allowing us to obtain the space of solutions of the QEEs, this projection operator also defines the inner product for said space, imbuing it with the same structure as H_{phys} [3].

In the spin network basis (discussed earlier), the matrix elements of P can be thought of as the sum of transition amplitudes of evolving spin networks, in what might be called a quantum history. A graphical representation of this concept is given in Figure 3.1.1, where the spin network slices are clearly visible and similar to the ones in Figure 2.2.1.

More formally, we can designate the transition between two spin network states s and $s^{'}$ as a spin foam history and describe it by using a 2-complex bounded by these states, $F_{s \to s'}$ — such as the one in Figure 3.1.1 —, and a set of spin quantum numbers $\{j\}$ acting as labels for the edges e, faces f and vertices ν of the complex. As such, the so-called physical inner product is then given by

$$\begin{aligned} P_{ss'} &= \left\langle s^{'}, s \right\rangle_p \\ &= \langle Ps, s \rangle = \Sigma_{F_{s \to s'}} N(F_{s \to s'}) \Sigma_{\{j\}} \Pi_{f \subset F_{s \to s'}} A_f(j_f) \Pi_{e \subset F_{s \to s'}} A_e(j_e) \Pi_{\nu \subset F_{s \to s'}} A_\nu(j_\nu), \end{aligned} \tag{3.1.2}$$

i.e. it is a sum over the spin foam amplitudes, namely the 2-cell (e), 1-cell (f) and 0-cell (ν) ones, given respectively by the 3 different A_i in the equation above. The dependence on j_i, on the other hand, is referring to the quantum numbers defining the neighbourhood of some edge, face or vertex of F, which in the case of Figure 3.1.1 is specifically written as $A_\nu(j, k, l, m, n, s)$. Additionally, the remaining $N(F)$ is a normalisation factor.

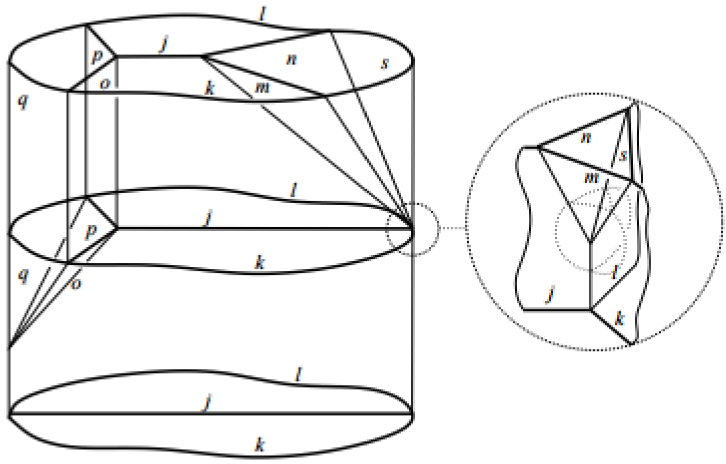


Figure 3.1.1: A spin foam describing the transitions between 3 different spin networks, forming a 2-complex (left), and a transition vertex (right) within the spin foam. The transitions between spin networks occur from bottom to top and, as one can easily notice, with each transition, new links are created and the spins are reassigned at the vertices [3].

The physical interpretation of a spin foam is that of a quantum history of spin network states: a set of transitions between quantum states of space or, equivalently, the evolution of the gravitational field. In other words, spin networks represent space, while spin foams represent spacetime. Furthermore, since the spin assignments give us the degrees of freedom of the gravitational field, they are responsible for representing the geometry of spacetime, not the shape of the spin foam itself. The obvious consequence is the expected background independence, as opposed to the lattices used in quantum field theory, for instance.

It is important to state that spin foams are an approach to the dynamical issue of LQG and thus are merely a tool, not a definitive theory. In fact, one can find numerous different models using the spin foam approach.

3.2 Spin Foam Model Prescription

In order to describe the spin foam approach to 4-dimensional general relativity, we shall follow the prescription for the 3-dimensional case [11], not only due to its increased simplicity, but also due to its chronological accuracy, regarding the origin of spin foams. In this 3d case, we have the following action for gravity (with cosmological constant Λ):

$$S_{3d} = \int \epsilon_{IJK} \left(e^I \wedge F^{JK}(\omega) + \frac{\Lambda}{6} e^I \wedge e^J \wedge e^K \right) \tag{3.2.1}$$

(where $e \equiv B$, as shown below).

The above theory is topological, meaning there are no local propagating gravitational degrees of freedom, which entails the fact that it can be discretised. In fact, a refined enough discretisation that takes into account all global degrees of freedom ensures that the new theory is discretisation-independent [11]. As a result, 3d gravity is closely related to the spin foam quantisation and we thus obtain a similar transition amplitude. Applying the above to Riemannian gravity, we obtain the Turaev-Viro model, for $\Lambda > 0$, and the Ponzano-Regge (state-sum) model (of 3-dimensional quantum gravity) for $\Lambda = 0$.

Next, we move on to 4D and present a theory which exists in any number of dimensions, called BF theory. BF is a topological field theory which naturally gives rise to a topological quantum field theory when quantised. Hence, we can use this theory as an intermediate one between 3- and 4-dimensional gravity, in which case we get the action

$$S_{BF} = \int B_{IJ} \wedge F^{IJ}(\omega), \tag{3.2.2}$$

where B^{IJ} is an $SO(\eta)$ Lie algebra valued (d-2)-form. One can easily check that $B_{IJ} = \epsilon_{IJK} e^K$ yields the previous case, i.e. 3-dimensional gravity. Additionally, BF theory is also a toplogical theory (in any dimension) and it can be quantised using the spin foam approach, which again provides a discretisation-independent theory.

Now if we use $B^{IJ} = * \left(e^I \wedge e^J\right)$ (where e^I, e^J are tetrads and $*$ is the Hodge star operator), we get a correspondence between the Hilbert-Palatini action and the 4d BF theory one, as well as a reduction from the latter to general relativity. The most important concretion of this method is the Plebanski action, which only differs from S_{BF} by one additional term:

$$S_{Pl} = \int \left(B_{IJ} \wedge F^{IJ}(\omega) + \frac{1}{2} \phi_{IJKL} B^{IJ} \wedge B^{KL} \right). \tag{3.2.3}$$

Here, ϕ is a field satisfying a number of symmetry constraints, including $\epsilon^{IJKL} \phi_{IJKL} = 0$. The extra term's purpose is to impose the simplicity constraints, which are

$$B^{IJ} \wedge B^{KL} = \sigma \mathcal{V} \epsilon^{IJKL} \iff \epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = \sigma \mathcal{V} \epsilon_{\mu\nu\rho\sigma}, \tag{3.2.4}$$

for the (non-degenerate) case $\mathcal{V} = \frac{1}{4!} \operatorname{Tr}(B \wedge B) \neq 0$. Here, σ denotes the sign which differentiates between Lorentzian and Riemannian cases. These constraints are aimed to reduce BF theory to general relativity, which is achieved because of the fact that the two sets of solutions for the simplicity constraints (for the same above $\mathcal{V}$), i.e. $B^{IJ} = * \left(e^I \wedge e^J\right)$ and $B^{IJ} = e^I \wedge e^J$, give the Hilbert-Palatini formulation and the extra term from the Holst action (respectively), when applied to the Plebanski action. These two give the entire Holst action, which is equivalent to the Palatini one, which in turn yields general relativity.

Spin foam models for 4D general relativity usually make use of the Plebanski action and quantise the correspondence between BF theory and GR, by first discretising the classical theory (i.e. inserting it into a simplicial complex), then quantising the topological BF part of the new (discretised) theory and finally imposing the simplicity constraints at the quantum level. In other words, we quantise before constraining.

Having presented the prescription under which most spin foam models come to be [11], we shall now review a number of them.

3.3 Spin Foam Models

In 1977, Plebanski equated the action of a constrained $SU(2)$ BF theory, below, to that of self-dual general relativity [12, 13, 14]:

$$S(B, A) = \int \operatorname{Tr}[B \wedge F(A)] - \psi_{ij} \left[B^i \wedge B^j - \frac{1}{3} \delta^{ij} B^k \wedge B_k \right]. \tag{3.3.1}$$

The correspondence between both theories is attained when we vary ψ_{ij}, which gives $\Omega^{ij} = 0$, where Ω^{ij} is the term being multiplied by ψ_{ij}, in the equation above. These are the constraints which reduce the action of BF theory to the one of self-dual Riemannian gravity.

From here, Reisenberger showed that the simplicial discretisation of the aforementioned action is well-defined and said action is recovered when the triangulation is refined [15]. Later on, he formulated a spin foam model from this discretisation, by promoting the B^i to operators, thereby imposing the constraints Ω^{ij} [16]. The resulting model is then

$$Z_{GR} = \int \prod_{e \in \mathcal{J}_\Delta} dg_e e^{-\frac{1}{2z^2}\hat{\Omega}^2} \sum_{\mathcal{C}:\{j\}\to\{f\}} \prod_{f\in\mathcal{J}_\Delta} \Delta_{j_f} \operatorname{Tr}\left[j_f\left(g_e^1 ... g_e^N\right)\right], \tag{3.3.2}$$

where $\hat{\Omega}^{ij} = \mathcal{J}^i \wedge \mathcal{J}^j - \frac{1}{3}\delta^{ij}\mathcal{J}^k \wedge \mathcal{J}_k$ and the constraints are imposed on a single 4-simplex amplitude, as a form of locality. The exponential function is used in order to impose the constraints sharply in the limit $z \to \infty$ [1], otherwise the algebra of the operators would not close, due to the way in which the constraints are implemented.

A generalisation of Reisenberger's model is one created by Freidel and Krasnov, in 1999 [14]. Their starting point was the BF action (first term) with an extra polynomial function of the B field (second term):

$$S(B, A) = \int \operatorname{Tr}\left[B \wedge F(A)\right] + \Phi(B). \tag{3.3.3}$$

Here, we can define the partition function $Z[J]$ as the functional integral over A and B of the exponential function of $i\int \operatorname{Tr}\left[B \wedge F(A)\right] + \operatorname{Tr}\left[B \wedge J\right]$, where J is an algebra-valued 2-form — we also discretise the manifold, as with most spin foam approaches. This model is useful for representing theories such as Yang-Mills, BF with cosmological terms or, if we define A and B as $SU(2)$-valued and Φ as the second term of equation (3.3.1), self-dual Riemannian gravity, as in Reisenberger's case. However, There are some differences between these two models, namely the fact that B^i is now represented by (commutative) functional derivatives, as opposed to (non-commutative) invariant vector fields, or the use of a potential term, instead of constraints.

In the case of Iwasaki [17], the spin foam model is again used for self-dual Riemannian gravity and is obtained by first performing a lattice discretisation of the Ashtekar formulation of general relativity. The consequent discrete action is obtained through a lattice path integral , using the Haar measure.

One of the main innovations in terms of spin foam models came with Barrett and Crane, in 1998. While the above models are closely related to loop quantum gravity, through the use of self-dual Riemannian gravity — where we get $SU(2)$ spin network states as boundaries —, they do not produce simple (nor closed) equations [1]. The Barret-Crane model [18, 19], on the other hand, is a much simpler one, based on the quantisation of the simplicial $SO(4)$ Plebanski action and on the concept of the quantum tetrahedron [20, 21]. The latter is essentially an assignment of an irreducible unitary representation of $SU(2)$ to each face of a tetrahedron, along with the requirement that this polyhedron's faces must close.

More recently, in 2007, Engle, Pereira, Rovelli and Livine published a modification of the Barrett-Crane model [22, 23, 24] (henceforth known as the "EPRL model"), which relaxes the Plebanski constraints — which reduce quantum BF theory to general relativity — imposed by the latter and thus is simpler and more closely related to LQG. The main problem with the previous approach was the non-commutativity of the quantum B field (and hence of the simplicity constraints as well), which entails a non-closed algebra, obtained from the commutation of Plebasnki constraints. Since these constraints imply additional classically non-existent (and possibly superfluous) conditions, one is able to relax them, thereby obtaining the simpler EPRL model from Barrett-Crane's.

Finally, a year later, Freidel and Krasnov developed a very similar model to EPRL [25]. In fact, it is identical to it for an Immirzi parameter $\gamma < 1$. This time, however, the approach involved the representation of the BF path integral of BF theory in terms of coherent states. As a result, it was possible to implement the Plebanski constraints semi-classically. The EPRL model benefitted from the FK model, by using the linear Plebanski constraints to reach a generalisation to arbitrary γ.

It is interesting to note that the last two models are widely regarded as the most advanced theories of 4-dimensional quantum gravity [1]. As a result, the EPRL model and its semiclassical limit will be discussed in the later sections.

3.4 Quantising BF Theory

Following Perez and Baez [1, 26], we learn that the action of (classical) BF theory is given by

$$S\left[B,\omega\right]=\int\limits_{\mathcal{M}}\left\langle B\wedge F(\omega)\right\rangle, \tag{3.4.1}$$

where G is a compact group with a Lie algebra $\mathfrak{g}$ having the inner product $\langle\rangle$, $\mathcal{M}$ is a d-dimensional manifold, B is a differential $(d-2)$-form, with values in $\mathfrak{g}$, F is the curvature 2-form and ω is a connection on a G-bundle over the manifold $\mathcal{M}$. Since all solutions to the equations of motion are locally related by gauge transformations, there are no local excitations within this theory [1]. As gauge symmetries of the action, we have the topological gauge transformation

$$\delta B=d_{\omega}\eta,\quad \delta\omega=0 \tag{3.4.2}$$

and the local G gauge transformations

$$\delta B=\left[B,\alpha\right],\quad \delta\omega=d_{\omega}\alpha, \tag{3.4.3}$$

where the d_ω are just the covariant exterior derivatives of the 0-forms (η and α) with values in $\mathfrak{g}$. These symmetries originate from the Bianchi identity ($d_\omega F(\omega)=0$) and the action above, respectively. Since the equation of motion solutions are pure gauge, it follows that the theory has only topological or global degrees of freedom.

It is interesting to note two special cases of BF theory: Riemannian general relativity, for $G=SU(2)$ and $d=3$; and its spin foam 4-dimensional quantisation, for $G=Spin(4)$ and $d=4$. The latter can be related to general relativity by constraining the field $B^{IJ}_{ab}=\epsilon^{IJ}_{KL}e^K_a e^L_b$ (where e^I_a is the tetrad co-frame), in our BF theory action above. In doing so, we get the action of general relativity in four dimensions.

In order to avoid infinite volume factors (i.e. infrared divergences) within the transition amplitudes, we assume the compactness of G and deal with non-compact groups later on. Having said that, and adding the assumption of a compact and orientable manifold $\mathcal{M}$, we get the following partition function

$$\mathcal{Z}=\int\mathcal{D}[B]\mathcal{D}[\omega]\;e^{i\,S[B,\omega]}=\int\mathcal{D}[\omega]\;\delta(F(\omega)), \tag{3.4.4}$$

which is the volume of the space of flat connections on the manifold $\mathcal{M}$. The next step is to replace the manifold $\mathcal{M}$ with an arbitrary cellular decomposition Δ, associated to a dual 2-complex, Δ^*. The latter is comprised of a set of vertices, edges and faces — v, e, $f\in\Delta^*$, dual to d-, $(d-1)$-, $(d-2)$-cells $\in\Delta$, respectively. For simplicial decompositions Δ of $\mathcal{M}$ in 2, 3, and 4 dimensions, we get (dual) 2-complexes (Δ^*), whose faces are dual to 0-simplices (vertices), 1-simplices (edges) and 2-simplices (triangles) of the triangulations (Δ), respectively.

Considering a triangulation Δ, we obtain the link between the Lie algebra elements B_f, attached to faces of the dual 2-complex ($f\in\Delta^*$), and the B field:

$$B_f = \int_{(d-2)-cell} B, \tag{3.4.5}$$

i.e. B_f can be conceptualised as the integral of the (continuous) $(d-2)$-form B over the $(d-2)$-cell dual to the face $f \in \Delta^*$, or, alternatively, as the smearing of said form on the $(d-2)$-cells of Δ. Since we know that

$$\forall\ f \in \Delta^*, \exists!\ ((d-2) - cell) \in \Delta, \tag{3.4.6}$$

we can now label B_f, the discretisation of the B field.

Similarly, we discretise the connection ω by assigning group elements $g_e \in G$ to edges $e \in \Delta^*$, the former representing the holonomy of the connection along the latter, i.e.

$$g_e = Pe^{-\int_e \omega}, \tag{3.4.7}$$

where Pe is the (path-)ordered exponential. We can now write the partition function (or path integral) of the triangulation Δ of $\mathcal{M}$ as

$$z(\Delta) = \int \prod_{e\in\Delta^*} dg_e \prod_{f\in\Delta^*} dB_f e^{ib_f U_f} = \int \prod_{e\in\Delta^*} dg_e \prod_{f\in\Delta^*} \delta\left(g_{e_1} \dots g_{e_n}\right), \tag{3.4.8}$$

where we have used $U_f = g_{e_1} \dots g_{e_n}$ for the holonomy around faces f and we note that this equation is the discretised version of the previous path integral, presented in equation (3.4.4). Additionally, dg_e is the Haar measure, while dB_f is the Lebesgue one.

One can further modify this equation by using the left-translational invariance property of the Haar measure, as well, as the Peter-Weyl theorem, to obtain

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{\rho\}\to\{f\}} \int \prod_{e\in\Delta^*} dg_e \prod_{f\in\Delta^*} d_{\rho_f} \operatorname{Tr}\left[\rho_f(g_e^1 \dots g_e^N)\right], \tag{3.4.9}$$

where ρ are irreducible unitary representations of G and the Dirac delta distribution was modified according to

$$\delta(g) = \sum_\rho d_\rho \operatorname{Tr}\left[\rho(g)\right], \tag{3.4.10}$$

as implied by the latter theorem. The next simplification of this expression involves the integration over the connection, where we can use the fact that an edge e is always shared by d faces to conclude that the group elements g_e occur in d distinct traces, and thus write the projector

$$P^e_{inv}(\rho_1, \dots, \rho_d) := \int dg_e\ \rho_1(g_e) \otimes \dots \otimes \rho_d(g_e), \tag{3.4.11}$$

onto $Inv\left[\rho_1 \otimes \dots \otimes \rho_d\right]$.

Finally, one is able to obtain the spin foam amplitudes of $SO(4)$ BF theory:

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f:\{f\}\to\rho_f} \prod_{f\in\Delta^*} d\rho_f \prod_{e\in\Delta^*} P^e_{inv}(\rho_1, \dots, \rho_d). \tag{3.4.12}$$

Simply put, the BF amplitude of a 2-complex Δ^* is the sum of the natural contraction of the network of projectors P^e_{inv}, over all possible assignments of irreducible representations of G to faces f.

Chapter 4

The EPRL Model

4.1 Preliminary Model

Until the introduction of a new model by Engle, Pereira and Rovelli [22, 23] — and a subsequent extension by the same authors, in collaboration with Livine [24] —, the most studied one (in the four-dimensional Euclidean framework) was the so-called Barret-Crane (or BC) model [20, 21], which can be summarised as the straightforward result of a set of simplicity constraints being applied to a topological BF quantum field theory. Remarkably, these constraints correspond to the ones used to transform BF theory into general relativity, in their classical limit, and this theory has been shown to yield a number of correct n-point correlation functions, unlike other results from alternative perturbative approaches [22].

Nonetheless, some problems within the Barrett-Crane theory were later pointed out, including the failure of its boundary state space to match that of (non-perturbative) loop quantum gravity, the lack of a well-behaved volume operator and the erroneousness of the low-energy limit of some of the subsequent correlation functions. These problems were attributed to the gauge fixing used and the needlessly strong imposition of the (second-class) simplicity constraints — as mentioned before —, which fully constrains the intertwiner quantum numbers in a way that leads to the troublesome annihilation of physical degrees of freedom [22].

The solution proposed by the authors of the new model involves the discretisation of Euclidean general relativity through a triangulation, followed by its quantisation and weak imposition of constraints. As a result, we obtain the $SO(3)$ hamiltonian LQG spin network space as the state space of the theory, as well as a new, $SO(3)$- and $SO(4)$-covariant vertex amplitude, which may solve the problem of the incorrect BC correlation functions.

The derivation of the EPRL model starts with a fixed 4d triangulation Δ, which contains triangles f, tetrahedra e and 4-simplices v, associated to faces, edges and vertices of the dual 2-complex defining the spin foam, respectively — not unlike what we have dealed with in previous sections. We assign an integer spin j_f to each face and a basis element i_e of the intertwiner space to each edge. It is helpful to remember that: each edge is adjacent to 4 faces; whose representations are carried by 4 Hilbert spaces; whose tensor product contains, in turn, an $SO(3)$ invariant subspace; whose elements are the aforementioned intertwiners. By fixing the pairing of the 4 faces, we can obtain the basis yielded by the spin of the virtual link and thus get

$$\dim j = 2j + 1 \tag{4.1.1}$$

for the representation j. We also use the fact that an $SO(4)$ representation is tantamount to a pair of

$SU(2)$ ones, of the form (j^+, j^-), in order to obtain the equation

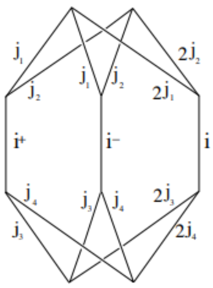
$$15j_{SO(4)}(j_f^+, j_f^-, i_e^+, i_e^-) = 15j(j_f^+, i_e^+)15j(j_f^-, i_e^-), \tag{4.1.2}$$

which tells us that the Wigner $15j$ symbol of the $SO(4)$ group — i.e. a function of 15 irreducible representations of $SO(4)$ — can be expressed as the product of 2 Wigner $SU(2)$ $15j$ symbols — equivalent to the Clebsch-Gordan coefficients.

Before unveiling the spin foam partition function, we must introduce a linear map f from the space of $SO(3)$ intertwiners between the $2j_1, \ldots, 2j_4$ representations to that of the $SO(4)$ ones, between the $(j_1, j_1), \ldots, (j_4, j_4)$ representations. We can thus expand f in terms of its linear coefficients in a given basis, i.e.

$$f\,|i\rangle = \sum_{i^+, i^-} f^i_{i^+ i^-} \left|i^+, i^-\right\rangle, \tag{4.1.3}$$

where said coefficients are written as

$$f^i_{i^+ i^-} = $$ 

, (4.1.4)

i.e. as the evaluation of the spin network above on the trivial connection.

Finally, the spin foam partition function of the EPRL model is

$$\begin{aligned} Z_{EPRL} &= \sum_{j_f i_e} \prod_f \left(\dim \frac{j_f}{2}\right)^2 \prod_v A\left(j_f, i_e\right) \\ &= \sum_{j_f i_e} \prod_f \left(\dim \frac{j_f}{2}\right)^2 \prod_v 15j_{SO(4)}\left(\left(\frac{j_f}{2}, \frac{j_f}{2}\right), f(i_e)\right) \\ &= \sum_{j_f i_e} \prod_f \left(\dim \frac{j_f}{2}\right)^2 \prod_v \sum_{i_e^+, i_e^-} 15j_{SO(4)}\left(\left(\frac{j_f}{2}, \frac{j_f}{2}\right), i_e^+, i_e^-\right) \prod_{e \in v} f^{i_e}_{i_e^+ i_e^-} \end{aligned} \tag{4.1.5}$$

Aside from having boundary states spanned by cubic graphs coloured with $SO(3)$ spins and intertwiners, we also note that this theory simply constitutes a modification of the Barrett-Crane one, whose (spin foam) partition function is given by the sum over half-integer spins j_f and the amplitude is an $SO(4)$ Wigner $15j$ symbol, i.e.

$$Z_{BC} = \sum_{j_f} \prod_f (\dim j_f)^2 \prod_v A_{BC}(j_f) \tag{4.1.6}$$

and

$$A_{BC} = 15j_{SO(4)}((j_f, j_f), i_{BC}). \tag{4.1.7}$$

It is clear that the intertwiner state space is the differentiating factor between the two models. While both theories draw their intertwiners from the $SO(4)$ intertwiner space between four simple representations,

$$H_e = \text{Inv}(H_{(j_1,j_1)} \otimes \ldots \otimes H_{(j_4,j_4)}), \tag{4.1.8}$$

the Barrett-Crane model only uses one intertwiner,

$$|i_{BC}\rangle = \sum_j (2j+1)\,|j,j\rangle \in H_e, \tag{4.1.9}$$

whereas the EPRL one uses the states given by the map $f\,|i\rangle$ above, which span a subspace $K_e \subset H_e$. In other words, while Barrett and Crane completely constrained the intertwiner degrees of freedom, the authors of the new model let them remain unconstrained and thus, instead of having a single intertwiner, we now have the space K_e. The reason for this change is the fact that the off-diagonal simplicity constraints need not be imposed strongly — as done by Barrett and Crane, where such imposition on H_e resulted in the single $|i_{BC}\rangle$ —, since they do not commute and are hence second-class. In fact, doing so has been shown to eliminate physical degrees of freedom in any given model.

To give a more formal perspective on this idea, we present the pseudoscalar $SO(4)$ Casimir operator for a given pair of faces $f \neq f'$ sharing an edge

$$C_{ff'} = \epsilon_{IJKL} B_f^{IJ} B_{f'}^{KL} \tag{4.1.10}$$

on the $\left(H_{(j_f,j_f)} \otimes H_{(j_{f'},j_{f'})}\right)$ representation, where ϵ is the totally antisymmetric tensor, the Bs are the generators of $SO(4)$ (with $I, J = 1, 2, 3, 4$) and the Einstein summation convention is implicit. The off-diagonal simplicity constraints are then

$$C_{ff'} = 0 \tag{4.1.11}$$

and follow from the fact that the external product of the bivectors of the faces of a tetrahedron (i.e. $B_{f_1} \ldots B_{f_4}$) vanishes. Nonetheless, due to the aforementioned problems regarding the strong imposition of such constraints, the authors of the (new) EPRL model opted to rewrite them, before imposing them weakly. Since the constraints simply mean that the tetrahedron must lie on a 3d subspace of 4d spacetime, we automatically get a normal to the tetrahedron, which can be defined as the timelike vector $n^I = (0, 0, 0, 1)$, without loss of generality. The result is then

$$\exists n : C = 2C_4 - C_3 = B_f^{IJ} B_f^{IJ} - B_f^{ij} B_f^{ij} = 0, \forall f \tag{4.1.12}$$

where i, j are spatial coordinates and C_4 and C_3 are the quadratic Casimir operators of $SO(4)$ and $SO(3) \leq SO(4)$, respectively. Naturally, imposing these constraints will generate a space from

$$\left(H_{(j_{f_1},j_{f_1})} \otimes \ldots \otimes H_{(j_{f_4},j_{f_4})}\right),$$

which yields K_e when projected onto the $SO(4)$ invariant-tensor space. Lastly, the coupling of the antisymmetric nature of the constraints and the symmetric nature of the $f\,|i\rangle$ states is what ensures that the former vanish weakly.

Having presented the EPRL model and compared it to the previous one, we shall establish the connection between the former and the quantisation of discretised general relativity, as well as the loop quantum gravity framework.

The first step towards this goal is to discretise general relativity on a Regge geometry, by using a simplicial decomposition Δ. Said geometry should be flat on each 4-simplex and thus allow for the curvature to be located on the "bones" (f). In fact, the ($SO(4)$) curvature associated to the triangle f within the tetrahedron t is encoded on the "link" of each bone and is represented by the rotation matrix $U_f(t)$, defined below. The remaining tools needed for the quantisation are:

- 4-simplices, tetrahedra t and triangles, respectively dual to vertices v, edges (e) and faces f, in the dual 2-complex;
- co-tetrad one-forms e, covering each t and v ;
- $B_f(t) = \int_f \star(e(t) \wedge e(t)) \in \mathfrak{g} = SO(4), \forall f \in t$, where $\star$ is the $\mathbb{R}^4$ Hodge star operator and $\mathfrak{g}$ is the algebra;
- Group elements $V_{vt} = V_{tv}^{-1} \in G = SO(4)$.

Furthermore, we also assume that, prior to varying the action, we have, for all f,

$$B_f(t)U_f(t,t') = U_f(t,t')B_f(t'), \quad U_f(t,t') = V_{tv_1}V_{v_1t_1}V_{t_1v_2}\dots V_{v_nt'}, \quad t,t' \in \text{Link}(f), \quad (4.1.13)$$

where the product of Vs is around the link of f, from t to t' (clockwise), as shown. The bulk action of e is then the sum over the faces of the trace of the product of B_f, V_{tv} and $V_{vt'}$. Additionally, the boundary terms are obtained by substituting the latter two terms by $U_f t, t'$.

We must now apply the constraints on the chosen independent variables B_f — instead of the tetrads —, $\forall f, f' \in t$: the aforementioned simplicity constraints and the closure constraint. While the former can now be written as

$$C_{ff'} = {}^*B_f(t).B_{f'}(t) = 0, \quad (4.1.14)$$

in both diagonal ($f = f'$) and off-diagonal ($f \neq f'$) cases, the latter deems the sum of B_f over the faces of a tetrahedron to vanish, i.e.

$$\forall t, \sum_{f \in t} B_f(t) = 0. \quad (4.1.15)$$

The dot represents the scalar product of the algebra and we also note the existence of an extra "dynamical simplicity constraint" — pertaining to triangles within the 4-simplices —, which is made obsolete when the remaining ones are satisfied.

In practice, While the closure constraint imbues the tetrahedra with gauge invariance and gives us the $SO(4)$ spin network states of the dual graph to the boundary triangulation as our state space, the simplicity ones — as discussed above — output a simple $SO(4)$ representation for each link, as well as K_e as the intertwiner space. Moreover, due to the correspondence between the boundary elements and those of the $SO(4)$ lattice gauge theory, we are able to the define the latter's Hilbert space as our (unconstrained) own.

The last steps towards obtaining the EPRL model's partition function involve the amplitude of a 4-simplex ($A[B_{tt'}]$) expressed in terms of the conjugate variables, i.e.

$$A[U_{tt'}] = \int dB_{tt'} e^{-i\sum \text{Tr}[B_{tt'}U_{tt'}]} A[B_{tt'}], \quad A[B_{tt'}] = \int dV_{vt} e^{i\sum \text{Tr}[B_{tt'}V_{tv}V_{vt'}]}, \quad (4.1.16)$$

which is then given by

$$A[j_{tt'}^{\pm}, i_t^{\pm}] = 15j_{SO(4)}(j_{tt'}^{+}, j_{tt'}^{-}, i_t^{+}, i_t^{-}), \tag{4.1.17}$$

when reverted to the spin network basis. Then, this amplitude, along with the constraints provided, will lead to the required partition function (4.1.5).

Finally, we note that the state space of the quantisation presented here is isomorphic (and physically equivalent) to the one of $SO(3)$ loop quantum gravity, since the quantum operators of both theories can be identified with the same classical counterparts.

4.2 Extension of The Model

In a subsequent publication [23], Engle, Pereira, Rovelli and Livine extended the original model to include a finite Immirzi parameter γ, while covering both the Euclidean and Lorentzian frameworks, once again reinforcing the link between canonical loop quantum gravity and the (4-dimensional) spin foam approach. The resulting theory possesses the same state space as the one of loop quantum gravity, for all values of the Immirzi parameter and within both the Euclidean and Lorentzian cases. The authors also conclude that the area spectrum of the spin foam theory is discrete, as in the LQG approach, and further add that the triangulation independence of the model can be recovered once the switch is made to the corresponding group field theory.

In order to obtain the extension to the previous model, one generally follows the same prescription as before, starting by the discretisation of general relativity, the assumption that (4.1.13) holds and the imposition of the closure and simplicity constraints (shown above). A number of modifications is nonetheless required for the new version to be obtained.

We begin with the observation that the simplicity constraints admit two sets of solutions, identified by $B = {}^*e \wedge e$ and $B = e \wedge e$. While both spawn (the Holst formulation of) general relativity, the former does so using the Newton constant G and the Immirzi parameter γ, whereas the latter assumes G_γ and $\frac{\pm 1}{\gamma}$ — (-1) for the Lorentzian theory — for the same quantities, respectively. The solution to this ambivalence is a reformulation of the off-diagonal constraints themselves, which are hence replaced by

$$\forall t, \exists n_I : \forall f \in t, C_f^J := n_I({}^*B_f(t))^{IJ} = 0, \tag{4.2.1}$$

where n_I is the vector depicting the normal one-form to t, which is present in the covariant LQG and spin foam formulations, as well as in the previous subsection. Consequently, only the former set of solutions is obtained, when this new stronger condition is satisfied.

Following, we obtain the discretised action

$$\begin{aligned} S =& \frac{-1}{2\kappa}\left(\sum_{f\in \text{int}\,\Delta} \text{Tr}\left[B_f(t)U_f(t,t) + \frac{1}{\gamma}{}^*B_f(t)U_f(t,t)\right]\right) \\ &- \frac{-1}{2\kappa}\left(\sum_{f\in \partial\Delta} \text{Tr}\left[B_f(t)U_f(t,t') + \frac{1}{\gamma}{}^*B_f(t)U_f(t,t')\right]\right), \end{aligned} \tag{4.2.2}$$

where $U_f(t,t)$ is the holonomy starting at t and going around the entire link, $\kappa = 8\pi G$, $B_f(t) \in \mathfrak{g}$, $U_f(t,t') \in G$ are the boundary variables and the sums are respectively over the interior and boundary of the simplicial decomposition Δ. When using $B = {}^*e \wedge e$, the continuous version of S becomes the Holst action, which is equivalent to the Palatini action and, in turn, to the Einstein-Hilbert action of general relativity. The final step in this discretisation of general relativity is to write the simplicity constraints in terms of the variable

$$J_f(t) = \frac{1}{\kappa}\left(B_f(t) + \frac{1}{\gamma}{}^*B_f(t)\right) \tag{4.2.3}$$

conjugate to $U_f(t,t')$. The result, for finite $\gamma \neq 0, 1$, is

$$C_{ff} = {}^*J_f.J_f\left(1 + s\frac{1}{\gamma^2}\right) - s\frac{2}{\gamma}J_f.J_f = 0 \tag{4.2.4}$$

and

$$C_f^J = n_I\left(({}^*J_f)^{IJ} - s\frac{1}{\gamma}J_f^{IJ}\right) = 0, \tag{4.2.5}$$

for the diagonal ((4.1.14), $f = f'$) and off-diagonal (4.2.1) simplicity constraints, respectively. By assuming $n_I = \delta_I^0$, we obtain exclusively spacelike tetrahedra in the Lorentzian case and we can then further alter the last equation to

$$C_f^i = \frac{1}{2}\epsilon_{kl}^{0j}J_f^{kl} - s\frac{1}{\gamma}J_f^{0j} = L_f^j - s\frac{1}{\gamma}K_f^j = 0, \tag{4.2.6}$$

where ϵ and L are the generators of the $SO(3)$ subgroup that leaves n_I unchanged and K are the generators of the associated boosts. We mention that the restriction we applied on n_I can be undone by gauge invariance, yielding the general case. Furthermore, the closure constraint remains the same for both conjugate variables and will be imposed naturally by the quantum dynamics.

As for the quantisation of the now discretised version of general relativity, we begin by presenting the boundary Hilbert space

$$\mathcal{H} = L^2(G^{\times L}), \tag{4.2.7}$$

with L being the number of links (or boundary faces f) in the boundary graph Γ of the 2-complex Δ^* dual to the simplicial decomposition Δ of spacetime. We also have $G = Spin(4)$ for the Euclidean case and $G = SL(2,\mathbb{C})$ for the Lorentzian, where G is now the universal covering of the original group ($SO(4)$ and $SO(3,1)$, respectively). Furthermore, for a single boundary face f, the quantised $\hat{B} := B_f(t)$ operator is given by the inverse of the relation (4.2.3) with $\hat{J} := J_f(t)$.

Moving on to the quantisation of the constraint (4.2.4), we can obtain the strong operator equation

$$C_2\left(1 + \frac{s}{\gamma^2}\right) - \frac{2s}{\gamma}C_1 = 0, \tag{4.2.8}$$

for the (pseudo-)Casimir operators of $\mathfrak{g}$,

$$C_1 = J.J = 2(L^2 + sK^2) \tag{4.2.9}$$

and

$$C_2 = {}^*J.J = 4sL.K, \tag{4.2.10}$$

and the Casimir operator L^2 of the $SU(2)$ subgroup leaving n_I invariant. Although the contraint (4.2.4) commutes with the remaining, the constraints (4.2.6) do not close (as a Poisson algebra), prompting us to replace the latter by the "master" constraint

$$M_f := \sum_i (C^i)^2 = \sum_i \left(L^i - \frac{s}{\gamma}K^i\right)^2 = 0 \Rightarrow L^2\left(1 - \frac{s}{\gamma}\right) + \frac{s}{2\gamma^2}C_1 - \frac{1}{2\gamma}C_2 = 0. \tag{4.2.11}$$

Finally, by joining the quantised versions of (4.2.4) and (4.2.6), we obtain

$$C_2 = 4\gamma L^2. \tag{4.2.12}$$

Having presented the discretisation and subsequent quantisation of general relativity needed to obtain the extended model, we must now reach the Euclidean and Lorentzian partition functions of said model. Starting with the Euclidean version (and assuming suitable orderings, normalisations and $\hbar$ corrections), we have $G = Spin(4)$, where the unitary representation is given by two half-integers (j^+, j^-). As such, the operators C_1 and C_2 become

$$C_1 = 4\left[j^+(j^+ + 1) + j^-(j^- + 1)\right], \tag{4.2.13}$$

and

$$C_2 = 4\left[j^+(j^+ + 1) - j^-(j^- + 1)\right] \tag{4.2.14}$$

and, plugging them into the diagonal simplicity constraint (4.2.8), we have the solution

$$(j^+)^2 = \left[\left(\frac{\gamma + 1}{\gamma - 1}\right)(j^-)\right]^2. \tag{4.2.15}$$

Next, we focus on the case $\gamma > 0 \Rightarrow j^+ > j^-$ and extract the quantisation condition on γ from this last equation, plugging it into (4.2.12) and getting

$$k^2 = \left(\frac{2j^-}{1-\gamma}\right)^2 = \left(\frac{2j^+}{1+\gamma}\right)^2, \tag{4.2.16}$$

where k is the quantum number related to the $SU(2)$ Casimir L^2. The solutions to this equation are

$$k = \begin{cases} j^+ + j^- & , \quad 0 < \gamma < 1 \\ j^+ - j^- & , \quad \gamma > 1 \end{cases} \tag{4.2.17}$$

As shown before, the (unconstrained) Hilbert space is $\mathcal{H} = L^2(Spin(4)^{\times L})$. For each face we have

$$L^2(Spin(4)) = \bigoplus_{j^+j^-} \bar{H}_{j^+j^-} \otimes H_{j^+j^-} \tag{4.2.18}$$

and by adding the new constraints (4.2.15) and (4.2.17), we get the constrained subspace given by $\mathcal{H}_f = L^2(SU(2))$. One defines a projection π from the last equation to $\mathcal{H}_f$ by writing

$$\pi\left(D^{(j^+,j^-)}_{q^+q^-,q'^+q'^-}(g)\right) = D^{(j^+,j^-)}_{q^+q^-,q'^+q'^-}(u)c^{q^+q^-}_{m}c^{q'^+q'^-}_{m'}, \tag{4.2.19}$$

where: $g \in Spin(4)$; $u \in SU(2)$; $j^\pm$ are the aforementioned $Spin(4)$ representations; $q^\pm$ identify the basis in the j representation; D are the matrix elements of the irreducible representations of each group; and c are the Clebsch-Gordan coefficients, which transform the $SU(2)$ irreducible representations into the (j^+, j^-) one.

As for the intertwiner spaces, we must first take the 4 faces of a boundary tetrahedron, assign them (or, equivalently, colour the 4 links of a node e in the graph Γ with) the representations (j_1^+, j_1^-) to (j_4^+, j_4^-) and impose

$$C_e = \sum_i M_{f_i} = 0 \tag{4.2.20}$$

on the tensor product of the representation states, so that we get the lowest $SU(2)$ irreducible representation for each face. Finally, we group average over $Spin(4)$ and obtain the physical intertwiner space for e. Projecting from the $Spin(4)$ to the $SU(2)$ intertwiner spaces

$$\text{Inv}_{Spin(4)}(\mathcal{H}_e), \quad \text{Inv}_{SU(2)}\left(\mathcal{H}_{j_1^+ \pm j_1^-} \otimes \ldots \otimes \mathcal{H}_{j_4^+ \pm j_4^-}\right),$$

respectively, is done by defining

$$\pi\left(C^{i_e^+, i_e^-}_{(q_1^+ q_1^-)\ldots(q_4^+ q_4^-)}\right) = C^{i_e^+, i_e^-}_{(q_1^+ q_1^-)\ldots(q_4^+ q_4^-)} \bigotimes_{i=1}^{4} c^{q_i^+ q_i^-}_{m_i}, \tag{4.2.21}$$

with C being the intertwiner given by a virtual link with the representation (i_e^+, i_e^-).

Finally, we define the vertex amplitude and the partition function of the Euclidean sector of the EPRL model. The former is written as

$$\begin{aligned} A(j_{ab}, i_a) &= \sum_{i_a^+ i_a^-} 15j\left(\frac{(1+\gamma)j_{ab}}{2}; i_a^+\right) 15j\left(\frac{|1-\gamma|\, j_{ab}}{2}; i_a^-\right) \bigotimes_a f^{i_a}_{i_a^+ i_a^-}(j_{ab}), \\ f^i_{i^+ i^-} &= i^{m_1 m_2 m_3 m_4} C^{i^+ i^-}_{(q_1^+ q_1^-)\ldots(q_4^+ q_4^-)} \bigotimes_{i=1}^{4} c^{q_i^+ q_i^-}_{m_i}, a, b = 1, \ldots, 5 \end{aligned} \tag{4.2.22}$$

where we have the familiar Wigner $SU(2)$ $15j$ symbols, as well as the 10 and 5 $SU(2)$ spins j and intertwiners i, respectively. The latter, on the other hand, is simply obtained through combining appropriate vertex, face and edge amplitudes:

$$Z_{EPRL_{Euclid}} = \sum_{j_f, i_e} \prod_f d_f \prod_v A(j_f, i_e), \quad d_f = (|1-\gamma| j_f + 1)(|1+\gamma| j_f + 1). \tag{4.2.23}$$

Analogously, in the Lorentzian case, the Casimir operators become

$$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4) \tag{4.2.24}$$

and

$$C_2 = n\rho, \tag{4.2.25}$$

for the unitary representation (n, ρ), with $n \in \mathbb{N}, \rho \in \mathbb{R}$. The diagonal simplicity constraints, in turn, become

$$n\rho\left(\gamma - \frac{1}{\gamma}\right) = \rho^2 - n^2 \Leftrightarrow \rho = \gamma n \vee \rho = -\frac{n}{\gamma}. \tag{4.2.26}$$

The first solution is selected once equation (4.2.12) is applied, with the additional condition $k = \frac{n}{2}$. The upshot of the constraints is the selection of the lowest $SU(2)$ irreducible representation in

$$\mathcal{H}_{(n,\rho)} = \bigoplus_{k \leq \frac{n}{2}} \mathcal{H}_k.$$

Once again, we derive a projection from the boundary Hilbert space to $SU(2)$, i.e. from $L^2(SL(2,\mathbb{C}))$ to $L^2(SU(2))$:

$$\pi\left(D^{n,\rho}_{jqj'q'}(g)\right) = D^{\frac{n}{2}}_{qq'}(u), \tag{4.2.27}$$

where we have the matrix D, just as before. We proceed with the definition of another projection, this time for the intertwiners. Following the same procedure as before, we have 4 links associated to the representations $(n_1, \rho_1) \ldots (n_4, \rho_4)$ and joined at a node $e \in \Gamma$. Again, we apply the constraint C_e strongly and then average over $SL(2,\mathbb{C})$, to get the physical intertwiner space. This time, the projection from $\mathrm{Inv}_{SL(2,\mathbb{C})}(\mathcal{H}_e)$ to $\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{\frac{n_1}{2}} \otimes \ldots \otimes \mathcal{H}_{\frac{n_4}{2}}\right)$ is

$$\pi\left(C^{(n_e,\rho_e)}_{(j_1,q_1)\ldots(j_4,q_4)}\right) = C^{(n_e,\rho_e)}_{(\frac{n_1}{2},q_1),\ldots(\frac{n_4}{2},q_4)}, \tag{4.2.28}$$

with the $SU(2)$ spin networks defining the boundary space. The resulting vertex amplitude is

$$\begin{aligned} A(j_{ab}, i_a) &= \sum_{n_a} \int d\rho_a (n_a^2 + \rho_a^2) \left(\bigotimes_a f^{i_a}_{n_a\rho_a}(j_{ab}) \right) 15j_{SL(2,\mathbb{C})}((2j_{ab}, 2j_{ab}\gamma); (n_a, \rho_a)), \\ f^i_{n\rho} &= i^{m_1m_2m_3m_4} \bar{C}^{n\rho}_{(j_1,m_1)\ldots(j_4,m_4)}, \end{aligned} \tag{4.2.29}$$

with the $SL(2,\mathbb{C})$ $15j$ wigner symbols and the representations j_1 to j_4 for the 4 links. Finally, the Lorentzian partition function is again composed of the vertex, edge and face amplitudes [27], which yield

$$Z_{EPRL_{Lorentz}} = \sum_{j_f, i_e} \prod_f (2j_f)^2(1+\gamma^2) \prod_v A(j_f, i_e). \tag{4.2.30}$$

As a final remark, we establish a connection between loop quantum gravity and the EPRL model, through the area spectrum of the latter. For a triangle dual to a face f, we have

$$A_4 = A_3 + \left(\frac{k\gamma^2}{\gamma^2 - s}\right)^2 sM_f, \quad A_4(f) = \frac{1}{2}({}^*B)^{IJ}({}^*B)_{IJ}, \quad A_3(f) = \frac{1}{2}({}^*B)^{ij}({}^*B)_{ij}, \tag{4.2.31}$$

where A_4 is the area operator for f and A_3 is its projection, as well as the operator in the canonical quantisation of general relativity. The relation above holds after the quantisation, as opposed to the classical (constrained) relation $A_3 = A_4$. By applying the constraints (4.2.8) and (4.2.12) to

$$A_3 = \left[\left(\frac{k\gamma^2}{\gamma^2 - s}\right)\left(\vec{K} - \frac{\vec{L}}{\gamma}\right)\right]^2, \tag{4.2.32}$$

we simply obtain (for both the Euclidean and Lorentzian cases)

$$A_3 = (k\gamma L)^2. \tag{4.2.33}$$

Using the Planck length l_P, where $l_P^2 = \hbar G$ (in natural units, i.e. $c = 1$), the resulting area spectrum is discrete and given by

$$A_3 = 8\pi l_P^2 \gamma \sqrt{k(k+1)}, \tag{4.2.34}$$

which matches the one from loop quantum gravity. Furthermore, it also corresponds to the result of applying the same simplicity constraints to the continuous covariant LQG spectrum. In other words, the continuous covariant LQG area spectrum becomes the discrete LQG one above, after being constrained by (4.2.8) and (4.2.12).

It is important to note that the main contribution — apart from the new vertex amplitude, which solves some of the problems faced by the Barrett-Crane model — is the connection made between the (4d) LQG and the spin foam approaches. In that regard, it was shown that the boundary space of the model is spanned by $SU(2)$ spin networks, similarly to the LQG case. Moreover, as mentioned in the previous paragraph, the area spectrum is the same for both theories. Both conclusions are sector-independent — where the sector is either Euclidean or Lorentzian.

Chapter 5

The Semiclassical Limit of The EPRL Spin Foam Model

5.1 Overview

Whether one is working within the framework of canonical non-perturbative quantum gravity (in terms of a connection formulation) — i.e. canonical loop quantum gravity — or a corresponding covariant (or path-integral) formulation of quantum gravity — i.e. spin foam models —, one issue of utmost importance remains: the semiclassical limit. This issue encapsulates the question of whether or not these quantum gravity theories converge to general relativity for $\hbar \to 0$, which is equivalent to small spacetime curvatures $R << 1/l_P^2$.

Such a question is both sensible and crucial, since any theory of quantum gravity (at high-energy scales) must concur with our current theory of classical (low-energy) gravity — also known as Einstein's theory of general relativity —, especially considering the overwhelming amount of experimental evidence and observations for the latter, contrasted with the current extreme difficulty involving observations and experiments at such high-energy scales as in the former. In other words, the current best method for evaluating the validity and consistency of a (non-perturbative) quantum gravity theory is to analyse its semiclassical limit and check whether the theory reduces to general relativity. The reason we call it the "semiclassical" limit, as opposed to the "classical" limit, is because, when matter is present, the semiclassical limit describes a situation where we have the classical gravitational field and quantum matter fields, i.e. QFT in curved spacetime.

Several methods have been used to study the semiclassical behaviour of spin foam models, including numerical simulations, symmetry reduction, the use of semiclassical states, the extraction of propagators from the models, etc. Nonetheless, the problem of finding the semiclassical limit (and its quantum corrections) remains one of the most difficult to overcome and systematically hinders the development of a realistic spin foam model of quantum gravity.

In this chapter, we focus on a particular approach to this issue, which employs the semiclassical expansion of the effective action of the spin foam model, and apply it to the EPRL theory. From past research, it hae been concluded that it is not possible to obtain enough information on the semiclassical limit of a spin foam model from its partition function alone, which is why the study of the large-distance asymptotics of the graviton propagator of models such as the Barrett-Crane was warranted. This particular model, however, was shown not to have a well-behaved propagator, due to the lack of intertwiners in the theory, causing its erroneous tensorial structure. In fact, intertwiners are now known to be crucial for the development of a complete Hilbert space of loop quantum gravity. Furthermore, progress was made regarding the type of (physical) boundary spin-network wavefunctions needed for a graviton propagator to present the correct large-distance asymptotics: such goal is achieved whenever the physical wavefunction has a specific Gaussian or Rovelli Gaussian form, in the large-spin limit.

For the case of the EPRL model, the large-spin asymptotics of the vertex amplitude can be written as

$$W(j, \vec{n}) \approx \frac{N_+ e^{i\alpha S_{\nu R}(j,\vec{n})} + N_- e^{-i\alpha S_{\nu R}(j,\vec{n})}}{V(j)}, \tag{5.1.1}$$

where

- $V(j)$ is the volume of the vertex tetrahedron; it is a homogeneous function of order 12;
- j are the spins assigned to the faces sharing the given vertex;
- $\vec{n}$ are the corresponding coherent state vectors;
- $S_{\nu R}$ is the corresponding 4-simplex Regge action;
- α is a constant; $\alpha = \gamma$ if the 4-simplex geometry is Lorentzian and $\alpha = 1$ if the 4-simplex geometry is Euclidean;
- $N_\pm \equiv N_\pm(j)$ are homogeneous functions of order 0 of j [28].

Note that, if all N_+ or all N_- were 0, one would obtain the intended asymptotics for the graviton propagator. In this case, the resulting state sum has a chance to be of the form

$$Z = \sum_j C(j) e^{i\gamma S_R(j)}, \tag{5.1.2}$$

where S_R is the area-Regge action [29] for the triangulation (see (5.2.12)). Then in the smooth spacetime limit, the state sum above could give the required Einstein-Hilbert action of classical general relativity. However, $N_\pm$ are different from zero in the general case, so that one will have a state sum

$$\tilde{Z} = \sum_j C(j) \left[e^{i\gamma S_R(j)} + \cdots + e^{-i\gamma S_R(j)} \right], \tag{5.1.3}$$

which was shown [30] not to give the correct semiclassical limit.

Let us focus on the state sum (5.1.2) and its classical limit, using the effective action approach [31]. From the definition of the one-loop effective action within the background field method, we have

$$e^{i\frac{\Gamma(j)}{\hbar}} = \sum_{\bar{j}} C(\bar{j}) e^{i\frac{S_R(j+\bar{j})}{\hbar}}. \tag{5.1.4}$$

It can be shown that this equation implies that the classical limit of Γ is $S_R(j)$. This is not the case for $\tilde{Z}$. However, if we modify the corresponding vertex amplitude such that the large-spin asymptotics is given by

$$\tilde{W}(j, \vec{n}) \approx \frac{e^{i\alpha S_{\nu R}(j)}}{V(j)}, \tag{5.1.5}$$

then the the corresponding effective action equation will be of the same form as (5.1.4).

The goal in the study of the EPRL model's semiclassical limit is then to show that there is a modification of the vertex amplitude yielding the asymptotics (5.1.5).

5.2 Possible Methods

We shall now describe several methods [31] which can be used to obtain the semiclassical limit of a spin foam model, whose partition function is generically given by

$$Z = \sum_{j,\iota} \prod_f W_2(j_f) \prod_l W_1(\iota_l) \prod_v W_0(j_{f(v)}, \iota_{l(v)}), \tag{5.2.1}$$

where, as we have seen before, j and ι are labels of the spin foam 2-complex σ dual to the simplicial complex resulting from a spacetime manifold triangulation. Furthermore, faces f and links l are respectively associated to the irreducible representations (spins) $j_f \in j$ and their intertwiners $\iota_l \in \iota$, both of the group $Spin(4)$, in 4 spacetime dimensions (or $Spin(3)$, in 3 dimensions). Since Z is just a complex number, we need to use the corresponding boundary wavefunctions or the effective action, in order to obtain the classical limit of the model. The former can be obtained by introducing a boundary for the spacetime manifold triangulation and the boundary spin network s as the boundary dual 1-complex γ with the labels j_b and ι_b, so that $s = (\gamma, j_b, \iota_b)$. For the boundary wavefunction $\Psi(s)$, in turn, we restrict the partition function above to include only spin foams bounded by the given spin network s. This amounts to performing the substitutions $W_i \to \tilde{W}_i$, $i \in \{0, 1, 2\}$, where the new amplitudes differ from the previous ones simply for the boundary faces and links, where specific gluing conditions need to be met. The wavefunction $\Psi_0(s)$ corresponds to a state from canonical loop quantum gravity

$$|\Psi_0\rangle = \sum_s \Psi_0(s) \, |s\rangle \,, \tag{5.2.2}$$

where $|s\rangle$ is the spin-network basis.

By using the loop transform

$$\Psi_0(A) = \sum_s \Psi_0(s) \langle A|s\rangle \,, \tag{5.2.3}$$

where $\langle A|s\rangle = W_s(A)$ are the spin-network wavefunctions, one can obtain the connection wavefunction, which can be written as

$$\Psi_0(A) = R(A) e^{\frac{iS(A)}{\hbar}} \,. \tag{5.2.4}$$

The remaining task is to prove that

$$S(A) = S_0(A) + \hbar S_1(A) + O(\hbar^2), \tag{5.2.5}$$

where the first term in the right-hand side obeys the Hamilton-Jacobi equation of canonical general relativity. Unfortunately, such a task is very difficult to perform, which paves the way for an alternative approach: making insertions in the (graviton) correlation functions of the single boundary state we started with. The result can be written as

$$G_n(x_1, \ldots, x_n) = \sum_{s,s'} \Psi_0^*(s) \Psi_0(s') \left\langle s \left| \hat{h}(x_1), \ldots, \hat{h}(x_n) \right| s' \right\rangle, \tag{5.2.6}$$

where $\hat{h}$ is the graviton operator. Even though the condition for a G_2 yielding the proper large-distance asymptotics has been found, it is not a sufficient one in order to obtain general relativity in the semiclassical limit. Instead, there must be a correspondence between each G_n and their Einstein-Hilbert (classical) counterpart. In other words, one must show that the Einstein-Hilbert action is the classical limit of the effective action defined by

$$\Gamma(h) = \sum_{n\geq 2} c_n \int dx_1 dt_1 \ldots \int dx_n dt_n \tilde{G}_n(x_1,t_1,\ldots,x_n,t_n) h(x_1,t_1)\ldots h(x_n,t_n), \tag{5.2.7}$$

where (x_i, t_i) are the usual spacetime coordinates and $\tilde{G}_n$ is the extension of G_n for the time coordinates. Despite the incremental ease relative to the previous wavefunction method, the correlation function method is still very difficult to use.

An alternative approach is the "background field method" (BFM). This method is commonly used in quantum field theory in order to obtain the one-loop effective action. The BFM method takes the effective action as a functional of an arbitrary spacetime metric g, as opposed to the previous method, where the deviation η from the flat metric $h = g - \eta$ was used. The one-loop effective action is calculated from

$$e^{i\frac{\Gamma(g)}{\hbar}} = \int \mathcal{D}h e^{i\frac{S(g+h)}{\hbar}}, \tag{5.2.8}$$

where $S(g)$ is the Einstein-Hilbert action, and h is now a deviation from an arbitrary background metric g.

The background field method can be applied to spin-foam models, but there are certain differences. For instance, in QFT, a renormalisation is required, in order to avoid the short-distance infinities, whereas in the spin foam case these infinities are avoided due to existence of the minimal area. Recall that the triangle areas are given by the positive half-integers — i.e. the $SU(2)$ spins j_f. These provide a Planck length (l_P) cut-off in the definition of the triangle areas, $A_f \propto l_P^2\sqrt{j_f(j_f+\frac{1}{2})}$. The gauge fixing procedure of QFT is also avoided, since the spins are diffeomorphism invariant, leaving only the problem of large-spin infinities. However, even that problem can be solved by using factors of the type j_f^{s-n}, $n \in \mathbb{N}$ for the spin foam vertex amplitude.

Applying the equation (5.2.8) to a spin foam model defined as in (5.2.1), we obtain

$$e^{i\Gamma(j,\iota)} = \sum_{j',\iota'} \prod_f W_2\left(j_f + j'_f\right) \prod_l W_1\left(\iota_l + \iota'_l\right) \prod_v A\left(j_{f(v)} + j'_{f(v)}, \iota_{l(v)} + \iota'_{l(v)}\right), \tag{5.2.9}$$

where we use a letter A for W_0 and the summation is over the fluctuations around the classical background, while the 2-complex is closed. We can further assume that the background spins are large and that the background labels form a stationary point of the area-Regge action $S_R(j)$, so that the calculation of Γ will become easier.

As mentioned earlier, we need the vertex amplitude A to have the asymptotics

$$A(j,\iota) \approx \frac{e^{i\alpha S_{vR}(j)}}{V(j)} = \frac{e^{i\alpha\sum_{f\supset v} j_f\theta_f}}{V(j)}, \tag{5.2.10}$$

as $j \to \infty$. Here the action is the vertex-Regge one, the θ_f are the dihedral angles and p denotes the (positive) order of the familiar homogeneous function V. Moreover, assuming the background spins are large, we can substitute the vertex amplitude in (5.2.9) by its asymptotic form and, by using

$$\sum_f j_f\delta_f = \sum_v S_{vR} + 2\pi\sum_f k_f j_f, \quad k_f \in \mathbb{Z}. \tag{5.2.11}$$

We obtain a multiple of a restricted area-Regge action [1]

$$S_R = \sum_f j_f \delta_f(j)\,. \tag{5.2.12}$$

and γ in the imaginary part of log $\prod_v A(j_f, \iota_l)$ provided that every simplex geometry is Lorentzian. δ_f is a Lorentzian deficit angle, which is given by

$$\delta_f = \begin{cases} 2\pi - \sum_{v\subset f} \theta'_{fv} & , \quad \text{spacelike} f \\ \sum_{v\subset f} \Theta_{fv} & , \quad \text{timelike} f \end{cases} , \tag{5.2.13}$$

where Θ is the boost parameter between the normal vectors of the 2 tetrahedra of the 4-simplex σ_v (dual to the vertex) sharing the spacelike dual triangle Δ_f and $\theta'_{fv} = \pi - \theta_{fv}$ is the interior dihedral angle of the same 4-simplex. Finally, we note that a spacelike f will yield a timelike Δ_f and vice versa.

5.3 The Effective Action Method

We now apply the prescription presented in the previous section — i.e. the effective action given in (5.2.9) — to the case of the EPRL spin foam model, as this model has a well-defined loop quantum gravity theory on a 3 dimensional boundary, which means it is an adequate theory to investigate. However, it is not clear whether the EPRL partition function is convergent, so that it should be regularised, if we want to obtain a well-defined effective action [32].

The EPRL model's partition function is of the form (5.2.1), since

$$Z_{EPRL} = \sum_{j,\iota} \prod_f \dim j_f \prod_v \sum_{n_1,\ldots,n_5 \geq 0} \prod_{a=1}^{5} \int_0^\infty d\rho_a \left(n_a^2 + \rho_a^2\right) f^{\iota_a}_{n_a\rho_a}(j)(15J)\left(2j_{bc}, 2\gamma j_{bc}; n_b, \rho_b\right), \tag{5.3.1}$$

where $W_2(j) = \dim j$, $W_1(j,\iota) = 1$ and W_0 is an integral and a sum of $15J$-symbols for the unitary representations (n,ρ) of the Lorentz group. It is useful to substitute W_0 by $W\left(j_{f(v)}, \vec{n}_{lf(v)}\right)$, where the $\vec{n}$ are four 3-dimensional unit vectors which are orthogonal to the triangle within the tetrahedron, respectively dual to the face f and link l.

Consequently, $W(j,\vec{n})$ will have the large-spin asymptotics of the form (5.1.1), where we have that $N_\pm \equiv N_\pm(\alpha) \in \mathbb{R}$ are order-1 homogeneous functions of the spins and $\alpha = \gamma$ or $\alpha = 1$, depending on whether the 4-simplex boundary geometry is Lorentzian or Euclidean, respectively. Moreover, $|W|$ is bounded and if $(j,\vec{n})$ at a given vertex do not form a Regge geometry, this implies that $W(\lambda j, \vec{n}), \lambda \to \infty$ will fall off faster than $\frac{1}{\lambda}^a, \forall a > 0$. Now, in order to obtain a vertex amplitude with asymptotics identical to (5.2.10) — instead of the type cos(S_R) —, we further change W to

$$\tilde{W} = \frac{VW + \sqrt{(VW)^2 - 4N_+N_-}}{2N_+}, \tag{5.3.2}$$

which gives the relation

$$W = \frac{N_+\tilde{W} + N_-\tilde{W}^{-1}}{V(j)} \tag{5.3.3}$$

and yields the desired asymptotics for large spins

[1] In the usual area-Regge action the dihedral angles are independent from the triangle areas, while in (5.2.12) the angles are specific functions of the areas.

$$\tilde{W}(j,\vec{n}) \approx e^{i\gamma S_{vR}(j)}\,. \tag{5.3.4}$$

Furthermore, in order to ensure that Z is convergent, we will divide $\tilde{W}$ by an appropriate power p of the product of the dimensions of the spins, $\prod_f (\dim j_f)^p$. We then define a new vertex amplitude

$$A(j,\vec{n}) = \frac{\tilde{W}(j,\vec{n})}{V(j)\prod_f (\dim j_f)^p}, \tag{5.3.5}$$

where the regularisation parameter p is triangulation independent and is large enough to deem the partition function Z_p (corresponding to (5.2.9)) convergent — one can take $p > 2$ for absolute convergence [32]. Such an A will have the large-spin asymptotics identical to the equation (5.2.10).

Finally, we wish to determine the form of the effective action for large background spins in (5.2.9), since then we will be able to write the asymptotic expression (5.2.10) in place of each vertex amplitude in (5.2.9). In the case of the EPRL model, using (5.2.9) for the one-loop effective action yields

$$e^{i\Gamma(j,\vec{n})} = \sum_{j',\vec{n}'}\prod_f \left(2(j_f + j'_f)+1\right)\prod_v A\left(j_{f(v)} + j'_{f(v)}, \vec{n}_{fl(v)} + \vec{n}'_{fl(v)}\right). \tag{5.3.6}$$

The large-spin limit of such an effective action then becomes

$$\begin{aligned} e^{i\Gamma(j,\vec{n})} &\approx N\sum_{j',\vec{n}'}\prod_f (j_f + j'_f)\prod_v \frac{e^{-i\gamma S_{vR}\left(j_{f(v)}+j'_{f(v)},\vec{n}_{fl(v)}+\vec{n}'_{fl(v)}\right)}}{V(j)\prod_f (j_f + j'_f)^p} \\ &\approx N\sum_{j',\vec{n}'}\prod_f (j_f + j'_f)^{1-p_f m_f} e^{iS_R(j+j',\vec{n}+\vec{n}')}, \end{aligned} \tag{5.3.7}$$

where $m_f \geq 2$ the number of vertices within a face f. Analogously to quantum field theory, we assume the background point $(j,\vec{n})$ is stationary for the area-Regge action $S_R(j)$ and that the 4-complexes are Lorentzian. This will simplify the procedure and will not affect the main behaviour of the effective action.

By using

$$S_R(j+j') \approx S_R(j) + \frac{1}{2}\sum_{f,f'} S''_{Rff'}(j) j'_f j'_{f'}, \tag{5.3.8}$$

where S'' is the Hessian of S, we obtain

$$e^{i\Gamma(j)} \approx N e^{iS_R(j) - \sum_f m_f \ln j_f} \sum_{j'} e^{i\frac{\langle S''_R(j) j' j'\rangle}{2}} \prod_f \left(1 - m_f \frac{j'_f}{j_f} + \dots\right), \tag{5.3.9}$$

where the expression in parenthesis is the series expansion of

$$(j+j')^{-m} = j^{-m}\left(1 - \frac{j'}{j}\right)^{-m} = j^{-m}\left(1 - m\frac{j'}{j} + \dots\right) \tag{5.3.10}$$

in (5.3.7) and we also have $\langle S''_R(j) j' j'\rangle = \sum_{f,f'} S''_{Rff'}(j) j'_f j'_{f'}$.

In order to include A_f in the area-Regge action S_{AR}, we can use $j \to \frac{j.l_P^2}{l_P^2} \approx \frac{A_f}{l_P^2}$, yielding

$$S_{AR} = \sum_f j_f \delta(j) \longrightarrow S_{AR} = \frac{1}{l_p^2}\sum_f l_p^2 j_f \delta(j) \approx \frac{1}{l_p^2}\sum_f A_f \delta(j). \tag{5.3.11}$$

The next step is to simplify this expression by using an integral over a new variable $x_f = \frac{j'_f}{j_f}$, which amounts to using a sum of several integrals of the form

$$\int d^F x x_1^{n_1} \dots x_F^{n_F} e^{\frac{i}{2}\sum_{m,n} S''_{Rmn} x_m x_n}. \tag{5.3.12}$$

To solve these, in turn, we use

$$\begin{aligned} \left.\frac{dI(\mu)}{d\mu}\right|_{\mu=0}, \quad I(\mu) &= \int d^F x e^{\frac{i}{2}\sum_{m,n} S''_{Rmn} x_m x_n + \sum_m \mu_m x_m} \\ &= (2\pi i)^{\frac{F}{2}} \frac{e^{i\mu^T S_R^{''-1}\frac{\mu}{2}}}{\sqrt{\det\left(S''_R\right)}}, \quad \mu^T = (\mu_1 \dots \mu_F), \end{aligned} \tag{5.3.13}$$

where I is a generating function. Furthermore, we expand

$$(1+x)^{-m} = e^{-m\log(1+x)} = e^{-mx+\frac{mx^2}{2}+\dots}. \tag{5.3.14}$$

Finally, by putting these formulae together, we obtain

$$\begin{aligned} e^{i\Gamma} &\approx N e^{-\sum_f m_f \log(j_f) + iS_R(j)} \sum_{j'} e^{-\sum_f m_f \frac{j'_f}{j_f} + \frac{i}{2}\sum_{f,f'} \tilde{S}''_{Rff'} j'_f j'_{f'}} \\ &\approx N' e^{-\sum_f m_f \log(j_f) + iS_R(j)} \frac{e^{i\sum_{f,f'} m_f m_{f'} \frac{\tilde{G}_{ff'}(j)}{2 j_f j_{f'}}}}{\sqrt{\det\left(\tilde{S}''_R(j)\right)}} \\ &\approx N' e^{-\sum_f m_f \log(j_f) + iS_R(j) - \frac{1}{2}\operatorname{Tr}\left(\log\left(\tilde{S}''_R(j)\right)\right) + i\sum_{f,f'} m_f m_{f'} \frac{\tilde{G}_{ff'}(j)}{2 j_f j_{f'}}}, \\ \tilde{S}''_{Rff'} &= S''_{Rff'} - i\frac{m_f}{j_f^2}\delta_{ff'} \end{aligned} \tag{5.3.15}$$

where $\tilde{G}$ is the inverse matrix of $\tilde{S}''$.

We then apply the method above to the EPRL model, where we discover that

$$\begin{aligned} e^{i\Gamma(j,\vec{n})} &\approx N \sum_{j',\vec{n}'} e^{-\sum_f c_f\left(\log(j_f) + \frac{j_{f'}}{j_f}\right) + iS_R(j,\vec{n}) + \frac{1}{2}\left\langle S''_{Rjj} j' j' + 2S''_{Rjn} j' \vec{n}' + S''_{Rnn} \vec{n}' \vec{n}'\right\rangle}, \\ c_f &= p_f m_f - 1, \quad \tilde{S}''_{Rff'} = S''_{Rff'} - i\frac{c_f}{j_f^2}\delta_{ff'}, \end{aligned} \tag{5.3.16}$$

where $\tilde{S}''_{Rff'}$ are the elements of the matrix $\tilde{S}''_{jj'}$ and Gaussian integration yields

$$\begin{aligned} e^{i\Gamma(j,\vec{n})} &\approx N' e^{-\sum_f c_f \log(j_f) + iS_R(j,\vec{n}) - \frac{1}{2}\operatorname{Tr}\left(\log \tilde{S}''_R(j,\vec{n})\right) + \frac{1}{2}\sum_{f,f'} \tilde{c}_f \tilde{c}_{f'} G_{Rff'}(j,\vec{n})}, \\ \tilde{c}_f &= \frac{c_f}{j_f}, \quad \tilde{S}''_R, \quad = \begin{bmatrix} \tilde{S}''_{Rjj} S''_{Rjn} \\ S''_{Rjn} S''_{Rnn} \end{bmatrix}, \end{aligned} \tag{5.3.17}$$

where the elements of the matrix $\tilde{S}''_R$ are the matrices of the corresponding second-order partial derivatives of S_R, evaluated at $j = j(L)$ and $\vec{n} = \vec{n}(L)$, for a specific set of edge lengths L, whereas $G_{Rff'}$ is an element in the jj block of the matrix $S_R^{''-1}$. This means we have

$$\Gamma(j,\vec{n}) \approx S_R(j,\vec{n}) + \sum_f c_f \log(j_f) + \frac{1}{2}\,\mathrm{Tr}\left(\log\left(\tilde{S}''_R(j,\vec{n})\right)\right) + \sum_{f,f'} \frac{G_{Rff'}(j,\vec{n})}{2j_f j_{f'}}, \tag{5.3.18}$$

where we have followed the procedure of quantum field theory by performing $\Gamma \to \mathrm{Re}(\Gamma) + \mathrm{Im}(\Gamma)$. Moreover, the trace-log term is a discretisation as well as a regularisation of its QFT counterpart, since the latter diverges, whereas the former does not. We can write the equation above as

$$\Gamma(\lambda j,\vec{n}) \approx \lambda\tilde{\Gamma}_0(j,\vec{n}) + (\log\lambda)\tilde{\Gamma}_1(j,\vec{n}) + \tilde{\Gamma}_2(j,\vec{n}) + \lambda^{-1}\tilde{\Gamma}_3(j,\vec{n}) + \dots. \tag{5.3.19}$$

Finally, since the orders of the terms are respectively j, $\log j$ and $\frac{1}{j}$, we conclude that the leading term of the large-spin limit is the first one, and so the semiclassical limit of the effective action of the EPRL model is the area-Regge action

$$S_R(j,\vec{n}) = \frac{\gamma}{l_P^2}\sum_f j_f l_P^2 \delta(j_f) = \frac{1}{l_P^2}\sum_f A_f\delta(j_f) = \frac{S_{AR}}{l_P^2}. \tag{5.3.20}$$

From the first-order quantum corrections above, the second term comes from the face amplitude and the regularisation factor in the vertex amplitude and it does not have a counterpart in QFT (i.e. it is specific to spin foam models), whereas the third term is the aforementioned discretisation of the QFT trace-log term.

Once again, as done before, we can take the background spin-foam labels $(j,\vec{n})$ to correspond to the stationary points of each vertex amplitude. This implies that in each 4-simplex we could find an assigment of the edge lengths which then defines a local Regge geometry; however, the local edge-length assignments do not give a well-defined global assignment of the edge lengths unless we impose further constraints which would ensure that a common edge between two 4-simplices has the same length assignement.

Hence if we take the background spin-foam labels to correspond to a global Regge geomety, then

$$S_R(j,\vec{n}) = \gamma\sum_f j_f(L)\delta_f(L) = \frac{1}{8\pi l_P^2}\sum_f A_f(L)\delta_f(L)\,, \tag{5.3.21}$$

i.e. the area-Regge action becomes the usual Regge action.

Since the Regge action is the discretisation of the Einstein-Hilbert action, we note that an infinite refinement of the spacetime triangulation (the smooth spacetime limit) will yield the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi l_P^2}\int_M R(g)\sqrt{-g}\,d^4x\,, \tag{5.3.22}$$

where g is the smooth spacetime metric and $R(g)$ the corresponding scalar curvature.

The result (5.3.18) seems to imply that the classical limit of the EPRL model is the Regge action, or the EH action, in the smooth spacetime limit. However, this follows only if the background spin-foam labels correspond to a Regge geometry. In the general case, the background spin-foam labels $(j,\vec{n})$ should be arbitrary, and in this case one can only obtain the the area-Regge action. It is known that the area-Regge action does not give a metric geometry in the smooth spacetime limit, but what is called a twisted geometry [33].

5.4 Higher-Order Corrections

In the section above, we chose the "on-shell" background $(j, \vec{n}) = (j(L_0), \vec{n}(L_0))$ that formed a Regge geometry, where the set of edge lengths L_0 satisfied the corresponding equations of motion. We then generalised to the set of arbitrary edge lengths L. However, we would like to be more accurate regarding the quantum corrections beyond the first-order, so we use the exact formula for the QFT effective action [34]:

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = \int \mathcal{D}h e^{\frac{i}{\hbar}S(\phi+h) - \frac{i}{\hbar}\int dx \frac{\delta\Gamma}{\delta\phi(x)}h(x)}. \tag{5.4.1}$$

This equation can be solved perturbatively in $\hbar$ for an arbitrary background field ϕ, so that we obtain

$$\Gamma(\phi) = S(\phi) + \hbar\Gamma_1(\phi) + \hbar^2\Gamma_2(\phi) + \cdots . \tag{5.4.2}$$

The equivalent formula for the EPRL model is

$$\begin{gathered} e^{i\Gamma(j,\vec{n})} = \sum_{j'} \int d\vec{n}' \prod_f W_f(j+j') \prod_v W_v(j+j', \vec{n}+\vec{n}') e^{-i\left\langle \frac{\partial\Gamma}{\partial j}j' + \frac{\partial\Gamma}{\partial\vec{n}}\vec{n}' \right\rangle}, \\ \left\langle \frac{\partial\Gamma}{\partial j}j' + \frac{\partial\Gamma}{\partial\vec{n}}\vec{n}' \right\rangle = \sum_f \frac{\partial\Gamma}{\partial j_f}j'_f + \sum_{fl} \frac{\partial\Gamma}{\partial\vec{n}_{fl}}\vec{n}'_{fl}. \end{gathered} \tag{5.4.3}$$

As we can see, this formula differs from the one used previously by the spin foam higher-loop correction (in the exponential). Instead of an on-shell background, we use an off-shell one, with the set of edge lengths L. In the case where this background does not form a Regge geometry, the exponential $e^{i\Gamma}$ is exponentially suppressed.

Finally, the $\hbar$-expansion (5.4.2) becomes similar to the one in (5.3.19), i.e. the first 2 terms (S_R and Γ_1) are identical, whereas the remaining ones (i.e. $\Gamma_n = O\left(j^{-n+2}\right), \forall n \geq 2$) differ, due to the use of the new formula above.

Chapter 6

Modified Regge Calculus

6.1 Area-angle Regge Calculus

Despite tremendous strides within the area of spin foams and their convergence to the discretisation of general relativity known as Regge calculus [35], a few problems associated to the latter remain. For instance, much alike other non-perturbative approaches to the same topic, quantum Regge calculus is unable to provide a definition for a unique gauge-invariant measure in the path integral of quantum gravity [36]. Aiming to solve this problem, Rovelli suggested that area variables should replace the edge lengths as the fundamental variables in Regge calculus [37], similarly to the spin-foam-inspiring BF theory, where these area variables abound naturally. Hindrances to this approach include the excessive number of areas in a triangulation relative to the respective number of edges, which warrants the issuance of novel constraints. These constraints possess a non-local structure, which impedes their straightforward application.

The original method behind Regge calculus involves a simplicial triangulation of the spacetime manifold, followed by the the use of the ten edge lengths l_e of the given 4-simplex σ to convey the ten components of the metric tensor $g(\sigma)$ on it. In addition, gluing the 4-simplices is made easier, as the geometry of a tetrahedron shared by two 4-simplices is precisely characterised by the same edge lengths, implying that one need only perform said gluing on a common tetrahedron.

A naive implementation of the new area variables is nonetheless difficult. Singular (orthogonal) configurations and 2-to-1 ambiguities between the edge lengths and the areas complicate the expression of the metric tensor in terms of the new variables, while the insufficiency of these variables in the characterisation of any of the boundary tetrahedra implies that the entire 4-simplex must be used instead, leading to ambiguities in the geometries imposed by the 4-simplex pair on the shared tetrahedron and hence to metric discontinuities and non-local constraints in that pair.

The solution, therefore, is to include more variables — namely the tetrahedral (i.e. 3d) dihedral angles — and then constrain the entire set of variables (i.e. areas and angles) in such a way that we obtain the action of general relativity. This approach was debuted by Dittrich and Speziale [36], who presented a new discrete formulation of gravity solving the non-locality and complexity of the required constraints, with the added benefit provided by its connection to Plebanski's simplicty constraints.

Let us apply this new prescription. First, let us introduce the labels and the notation: we start with a 4-simplex σ and label its 5 vertices i, j, k, l, m. We then refer to lower-dimensional subsimplices by the vertex or vertices that need to be removed, in order to obtain them. For instance: the tetrahedron $\sigma(i)$ is the one obtained by removing the vertex i from σ; the triangle $\sigma(ij)$ is the one obtained by removing the vertices i, j from the same 4-simplex; the edge $\sigma(ijk)$ is obtained by removing i, j, k from σ; and $\sigma(ijkl)$ is the vertex m, obtained by removing the other 4 vertices from the 4-simplex. The respective volumes of each of the 4-simplex, tetrahedron, triangle, edge and vertex alluded to above are V, $V(i)$, $V(ij)$, $V(ijk)$ and $V(ijkl)$ — where $V(ijkl) = 0$. Similarly, we refer to the internal dihedral angles in the following manner: θ_{ij} is the 4d dihedral angle between the tetrahedra $\sigma(i)$ and $\sigma(j)$, joined by the triangle $\sigma(ij)$; $\phi_{ij,k}$ is the 3d dihedral angle between the triangles $\sigma(ik)$ and $\sigma(jk)$, joined by the edge $\sigma(ijk)$, all within the tetrahedron $\sigma(k)$; lastly, $\alpha_{ij,kl}$ is the 2d dihedral angle between the two edges $\sigma(ijk)$ and $\sigma(ijl)$, within the triangle $\sigma(ij)$ — shared, in turn, by the tetrahedra $\sigma(i)$ and $\sigma(j)$ within the 4-simplex σ.

In a closed 4-simplex, these quantities obey several important, relations. For example, we have

$$\cos(\alpha_{ij,kl}) = \frac{\cos(\phi_{ij,k}) + \cos(\phi_{il,k})\cos(\phi_{jl,k})}{\sin(\phi_{il,k}) sin(\phi_{jl,k})}. \tag{6.1.1}$$

In other words, it is possible to express the 2d dihedral angle α above by using 3 different 3d dihedral angles ϕ between the three triangles $\sigma(ik)$, $\sigma(jk)$ and $\sigma(lk)$ within the tetrahedron $\sigma(k)$. The method used to derive the equation above involves the affine metric associated to an n-simplex, which is used to obtain the scalar product of the normals to the subsimplices. The normals and their scalar product, in turn, are used to obtain the induced metric on a subsimplex and to compute the expression giving the 3d angle in terms of the affine metric and tetrahedra and triangle volumes. The generalised law of cosines and the upper-dimensional analogue of this expression are then employed to cancel said volume factors, resulting in the formula between 3d and 4d angles, which can finally be pushed one dimension down, yielding (6.1.1).

Since we assume σ to be closed, we infer that, within it, a triangle is shared by two tetrahedra, and so there are two possible choices of tetrahedra when defining the 2d angle α above, namely $\sigma(k)$ (as above) and $\sigma(l)$. The latter is thus written as

$$\cos(\alpha_{ij,lk}) = \frac{\cos(\phi_{ij,l}) + \cos(\phi_{ik,l})\cos(\phi_{jk,l})}{\sin(\phi_{ik,l})\sin(\phi_{jk,l})}. \tag{6.1.2}$$

The fact that the same angle can be given by two different expressions implies that those expressions are equivalent, and hence we have must have $\cos(\alpha_{ij,kl}) = \cos(\alpha_{ij,lk})$, i.e. $\cos(\alpha_{ij,kl}) - \cos(\alpha_{ij,lk}) = 0$. Therefore, we impose the constraint

$$\begin{aligned} \mathcal{C}_{kl,ij}(\phi) &\equiv \cos(\alpha_{ij,kl}) - \cos(\alpha_{ij,lk}) \\ &= \frac{\cos(\phi_{ij,k}) + \cos(\phi_{il,k})\cos(\phi_{jl,k})}{\sin(\phi_{il,k}) sin(\phi_{jl,k})} - \frac{\cos(\phi_{ij,l}) + \cos(\phi_{ik,l})\cos(\phi_{jk,l})}{\sin(\phi_{ik,l})\sin(\phi_{jk,l})} = 0. \end{aligned} \tag{6.1.3}$$

This yields 1 relation per 2d angle α of a given triangle, which implies there are 3 relations for each triangle, ergo 30 relations in total (i.e. for the 10 triangles of σ). By applying linearisation around non-degenerate configurations to the equation above, it is possible to conclude that only 20 of these 30 relations are independent.

Furthermore, a 4-simplex contains 10 4d angles θ_{ij} (between its 5 tetrahedra), which together form the 5×5 symmetric Gram matrix G, whose elements are defined by

$$G_{ij} = G_{ji} = \cos(\theta_{ij}), \quad \cos(\theta_{ii}) := -1, \quad \forall i, j \in \sigma. \tag{6.1.4}$$

For a closed and flat 4-simplex, we apply the condition $\det(G) = 0$ and obtain 9 independent quantities, which together with a scale factor form the metric variables parametrising the space of shapes of the 4-simplex. Similarly, we define the 4×4 symmetric Gram matrix for the tetrahedron $\sigma(k)$,

$$G^k(\phi) = \cos(\phi_{ij,k}), \quad \cos(\phi_{ii,k}) = -1, \quad \forall i,j,k \in \sigma, \tag{6.1.5}$$

where, as we can see, all the 6 3d angles ϕ belong to the tetrahedron $\sigma(k)$. The homologous constraint is then $\det(G^k(\phi)) = 0$.

We then join the constraint above to (6.1.3) and obtain the set

$$\det(G^k(\phi)) = 0, \quad \mathcal{C}_{kl,ij}(\phi) = 0, \tag{6.1.6}$$

which can be shown to have rank 21, by linearisation. We note that the former condition is local, while the second is not, since it involves two adjacent tetrahedra. Together, these are the relations defining the 3d angles of a 4-simplex. Furthermore, it suffices to apply the first set of constraints to the same 4-simplex, as the case for any pair of tetrahedra sharing the triangle yet not belonging to the same 4-simplex will automatically obey the constraints, through transitivity.

We are now ready to define general relativity on a discrete triangulated Riemannian manifold with no boundaries — for simplicity — through the use of areas A_t (of triangles t) and 3d angles ϕ_e^τ — where e is the edge shared by the triangles t, t' and τ is the tetrahedron containing them. In this notation, we write the constraints

$$\mathcal{C}^\sigma_{ee'}(\phi_e^\tau) \equiv \begin{cases} \mathcal{C}_{kl,ij} & , \quad e = \sigma(kli), e' = \sigma(klj) \\ 0 & , \quad \text{otherwise} \end{cases} = 0, \tag{6.1.7}$$

where $\mathcal{C}_{kl,ij}$ is given by (6.1.3) and the edges e and e' share the vertex m, all in the 4-simplex σ. In addition, we require a closure condition for the 5 tetrahedra within the 4-simplex. Such a condition is

$$N_\tau \equiv \sum_{t \in \tau} n_t = 0, \tag{6.1.8}$$

for a tetrahedron τ. We also have, by definition,

$$n_t . n_{t'} = -A_t A_{t'} \cos(\phi^\tau_{tt'}), \quad \cos(\phi^\tau_{tt}) := -1 \tag{6.1.9}$$

where n_t is the normal to the triangle t. We can also infer that, $\forall t \in \tau$,

$$\begin{aligned} N_\tau = 0 &\iff N_\tau . n_t = 0 \\ &\iff \left(\sum_{t' \in \tau} n'_t \right) n_t = 0 \\ &\iff n_t . n_t + \sum_{t' \in \tau \backslash \{t\}} n_{t'} . n_t = 0 \\ &\iff -A_t A_t \cos(\phi^\tau_{tt}) + \sum_{t' \in \tau \backslash \{t\}} (-A_t A_t \cos(\phi^\tau_{tt'})) = 0 \\ &\iff -A_t^2(-1) - A_t \sum_{t' \in \tau \backslash \{t\}} A_t \cos(\phi^\tau_{tt'}) = 0 \\ &\iff A_t \left(A_t - \sum_{t' \in \tau \backslash \{t\}} A_t \cos(\phi^\tau_{tt'}) \right) = 0 \\ &\iff A_t - \sum_{t' \in \tau \backslash \{t\}} A_t \cos(\phi^\tau_{tt'}) = 0 \end{aligned} \tag{6.1.10}$$

and so the constraints become

$$\mathcal{N}_t^\tau(A,\phi) \equiv A_t - \sum_{t'\in\tau\setminus\{t\}} A_{t'} \cos(\phi_{tt'}^\tau) = 0. \tag{6.1.11}$$

There are 4 constraints for each tetrahedron — 1 for each of the 4 triangles in τ — and thus 20 constraints for a 4-simplex — which contains 5 tetrahedra. Together, (6.1.7) and (6.1.11) constitute 50 constraints, forming a rank-30 system (obtained by linearisation). This implies that only 10 of the 40 variables (A_t, ϕ_e^τ) used are independent, which coincides with the number of kinematical degrees of freedom of discrete general relativity. The 40 variables obeying the 30 constraints ensure a well-behaved Regge triangulation and hence allow us to recover a unique set of edge lengths from the former, through the action

$$\begin{aligned} S\left(A_t, \phi_e^\tau, \mu_{ee'}^\sigma, \lambda_t^\tau\right) &= \sum_t A_t \epsilon_t(\phi) + \sum_\sigma \sum_{ee'\in\sigma} \mu_{ee'}^\sigma \mathcal{C}_{ee'}^\sigma(\phi) + \sum_\tau \sum_{t\in\tau} \lambda_t^\tau \mathcal{N}_t^\tau(A,\phi), \\ \epsilon_t &= 2\pi - \sum_{\sigma\ni t} \theta_t^\sigma \\ &= 2\pi - \sum_{\sigma\ni\sigma(ij)} \arccos\left(\frac{\cos(\phi_{ij,k}) - \cos(\phi_{ik,j})\cos(\phi_{jk,i})}{\sin(\phi_{ik,j})\sin(\phi_{jk,i})}\right) \end{aligned} \tag{6.1.12}$$

where ϵ is the deficit angle, μ and λ are Lagrange multipliers and the terms on the right-hand side represent the unconstrained area-angle Regge action and the local constraints (6.1.7) — gluing of adjacent tetrahedra — and (6.1.11) — closure of each tetrahedron —, in that order. In other words,

$$S\left(A_t, \phi_e^\tau, \mu_{ee'}^\sigma, \lambda_t^\tau\right) = S_R(l_e), \quad S_R(l_e) = \sum_t A_t(l_e)\epsilon_t(l_e), \tag{6.1.13}$$

where the right-hand side is the standard length-Regge action. The abridged method for reducing the area-angle variables to the length variables (as above) uses Heron's formula for the area of the triangles, to which we apply the 3d generalised law of cosines. The result is an expression for the volume of a tetrahedron in terms of the areas and angles, as well as the edge lengths in terms of said variables. The last step is to ensure that the latter expression gives the same result for any choice of tetrahedra, when constrained by (6.1.3). Thus, this method allows one to obtain the unique set of edge lengths from the given set of area and angle variables, after using the constraints (2.1.11) and (6.1.11).

We will now establish a connection between the area-angle Regge action presented above and the Plebanski action, beginning with the definition of Plebanski's constraints in terms of our variables:

$$B_{ij} = \pm e_{ijk} \wedge e_{ijl}, \quad k,l \neq i,j, \tag{6.1.14}$$

where B is the binormal to the triangle $\sigma(ij)$ and e are edge vectors $\sigma(ijk)$ and $\sigma(ijl)$, respectively. This is equivalent to writing

$$B^{ab} = e_{ijk}^a e_{ijl}^b \epsilon_{abcd} \quad k,l \neq i,j. \tag{6.1.15}$$

Now, applying the closure and simplicity constraints (for the geometric sector only) amounts to requiring

$$
\begin{aligned}
B_{ij}.B_{iv} &= \begin{cases} e_{ijk}^2 e_{ijl}^2 - (e_{ijk}e_{ijl})^2 & , \quad v=j \\ e_{ijk}^2(e_{ijl}.e_{ikl}) - (e_{ijk}e_{ijl})(e_{ijk}e_{ikl}) & , \quad v=k \end{cases} \\
&= \begin{cases} V_{ijk}^2 V_{ijl}^2 \sin(\alpha_{ij,kl}^2) & , \quad v=j \\ V_{ijk}^2 V_{ijl} V_{ikl}(\cos(\alpha_{il,jk}) - \cos(\alpha_{ij,kl})\cos(\alpha_{ik,jl})) & , \quad v=k \end{cases} \\
&= \begin{cases} V_{ij}^2 & , \quad v=j \\ V_{ij}V_{ik}V_{ij}V_{ik}\cos(\phi_{jk,i}) & , \quad v=k \end{cases} ,
\end{aligned}
\tag{6.1.16}
$$

which yields the inverse relation to (6.1.1), also satisfying (6.1.3). The entire Plebanski action is given by

$$
S = \int B \wedge F + \mu \mathcal{C}(B), \tag{6.1.17}
$$

where the constraints $\mathcal{C}$ reduce the topological BF theory (i.e. the first term) to general relativity. We can finally establish the following parallel: just as the Regge action is a discretisation of the Einstein-Hilbert action for general relativity ($\int \sqrt{g}R$), the action (6.1.12) is a discretisation of the Plebanski action for the same theory. In particular, the third term in (6.1.12) — i.e. the closure constraint — corresponds to the Gaussian one in BF theory, whereas the second therm gives the aforementioned simplicity constraints and the first and third terms combined discretise BF theory.

The theory introduced in this section manages to show how to correctly determine the geometry of all the elements of a triangulation, paving the way for a possible perturbation theory on a flat background. In the non-perturbative framework, on the other hand, we establish a connection to spin foam theory using the dynamical variables, which in the EPRL model can be given by the normals to the triangles. More explicitly, since scalar reduction of these variables generates the area-angle variables (A, ϕ), it is possible that the corresponding action introduced here acts as the semiclassical limit of the EPRL model, analogously to the connection between Regge calculus and the 3d case.

As we have previously seen, the usual path integral is given by

$$
Z = \int \mathcal{D}(\phi) e^{\frac{iS(\phi)}{\hbar}}, \tag{6.1.18}
$$

where the integration is performed over all paths. As a result, the state sum (written as the usual path integral) for the discrete action (6.1.12) becomes

$$
Z_{DS} = \sum_{A,\phi,\lambda,\mu} N(A) e^{\frac{iS_{DS}(A,\phi,\lambda,\mu)}{l_P^2}}, \tag{6.1.19}
$$

named after the authors of the aforementioned action. Moreover, as mentioned before, the spins j_f assigned to the faces f within the spin foam model framework essentially correspond to the areas of the triangles dual to the faces, since we have $A_f \propto l_p^2\sqrt{j_f(j_f+\frac{1}{2})} \propto j_f$ — the correspondence between intertwiners and the 3d dihedral angles ϕ can also be established. Hence, we can recast $N(A) \equiv N(j)$ in the equation above. The main goal is then to avoid large-distance (i.e. large-spin) infinities, by ensuring that Z_{DS} is finite and its vertex amplitude's large-spin asymptotics converges. In order to do so, one needs to introduce the appropriate negative powers of j_f in the spin foam vertex amplitude. Only then can we show that Z_{DS} defines the effective action which in the classical limit is equivalent to the Regge action. The solution involves the definition of N as

$$N(j) \sim \prod_f (\dim(j_f))^{-p}, \quad p > 0, \tag{6.1.20}$$

for a large enough value of p rendering the state sum Z_p finite.

6.2 A Lorentzian Extension

In this section, we present a particular Lorentzian discretisation of spacetime, along with the quantities it entails — such as volumes and dihedral angles —, in order to derive the Lorentzian counterpart of the (Euclidean) area-angle Regge calculus described above. We follow the approach by Ambjørn et al. [4] — and later provide general formulae [38], from which we derive their results (see appendix A) — and later modify the constraints of the area-angle Regge action [36] so that the new modified action includes this particular type of Lorentzian setting (see appendix B).

We start with a d-dimensional simplicial spacetime manifold, foliated into $(d-1)$-dimensional fixed topology spatial equilaterally triangulated simplicial manifolds (referred to as "slices"), which are connected via sets of d-dimensional simplices. we define proper time by the discrete labelling of the spatial slices and impose a fixed topology on these in order to maintain causality. Furthermore, we define the squared spacelike lengths to be a^2, whereas the timelike ones are $-\alpha a^2$, with positive α. Simplices are pieces of flat Minkowski space and the curvature of a simplicial manifold is thereby given by the specific gluing of said pieces. Without loss of generality, we deem the topology of the time-slices to be that of S^{d-1} and represent them each by a $(d-1)$-dimensional abstract triangulation, obtained by the gluing of $(d-1)$-simplices with spatial links of length a. In other words, we construct a $(d-1)$-dimensional piecewise linear geometry on S^{d-1} with Euclidean signature for each time-slice.

The connection between two consecutive S^{d-1} triangulations is as follows: the two triangulations $T_{d-1}(1)$ and $T_{d-1}(2)$ respectively associated with the time-slices having proper times 1 and 2 are joined such that the result is a d-dimensional piecewise linear geometry. The ensemble is made of d-simplices, has the two $(d-1)$-dimensional boundaries $T_{d-1}(1)$ and $T_{d-1}(2)$, and possesses the topology $[0,1] \times S^{d-1}$. As one may suspect, all spatial elements (such as links and subsimplices) are entirely contained within either slice (1 or 2) and the links connecting the two slices are timelike — with the respective lengths defined earlier. Additionally, any subsimplex containing at least 1 timelike length will be deemed "timelike", we consider the timelike distance between between the slices to be $\sqrt{\alpha}a$ and thus assign the proper length $-\alpha a^2$ to the simplical manifold with boundaries that is the ensemble.

The d-dimensional simplices glued together to form the ensemble are identified by the number of vertices on each slice. For instance, a $(d,1)$-simplex has d vertices (forming a $(d-1)$-simplex) in $T_{d_1}(1)$ and 1 in $T_{d-1}(2)$ and, as easily inferred, it has d timelike links joining the two sets of vertices. The remaining types of simplices are easy to visualise — e.g. Figure 6.2.1 — and continue from $(d-1,2)$ to $(1,d)$. Naturally, any two (i,j)- and (j,i)- simplices will be the time reversal of each other. In terms of gluing, any $(k+1,d-k)$-simplex can only be joined to a $(k+1,d-k)$-, a $(k,d+1-k)$- or a $(k+2,d-k-1)$-simplex.

We now define the volumes $V(k,d+1-k) \in \mathbb{R}^+$ and dihedral angles $\Theta_{(k,d+1-k)} \in \mathbb{C}$, $\mathrm{Re}(\Theta) \in [0,\pi]$ of the d-dimensional Minkowskian $(k,d+1-k)$-simplices, up to time reversal. we note that the range of Θ implies that the dihedral angles are uniquely defined by their sine and cosine.

For dimension $d=0$, (i.e. a point) we have, by convention,

$$V(1,0) = 1. \tag{6.2.1}$$

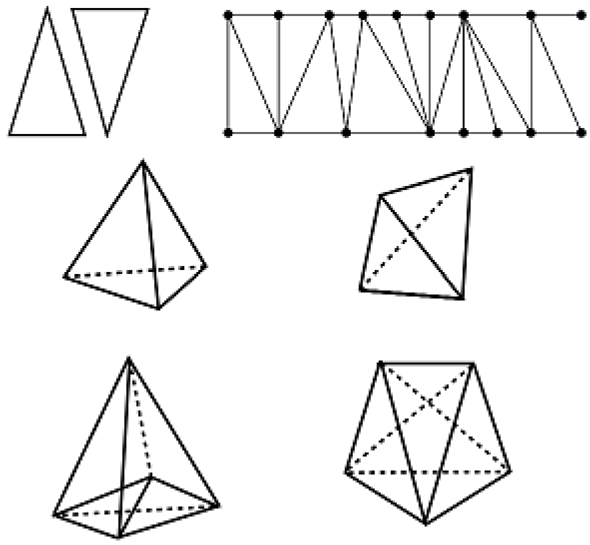


Figure 6.2.1: Possible simplices in 2 (top), 3 (middle) and 4 (bottom) dimensions, (up to time reversal for 3 and 4 dimensions): $(2,1)$- and $(1,2)$-simplices (top left) and the resulting gluing (top right); $(3,1)$- (middle left) and $(2,2)$- (middle right) tetrahedra/simplices; $(4,1)$- (bottom left) and $(3,2)$- (bottom right) simplices. The 2-dimensional strip (top right) is eventually joined at both ends, forming a band with topology $S^1 \times [0,1]$ [4].

For $d=1$, the volumes are as defined above, i.e.

$$V(2,0) = a \quad V(1,1) = \sqrt{\alpha}a, \tag{6.2.2}$$

for space- and timelike links, respectively. Analogously, for the triangles in $d=2$, we have

$$V(3,0) = \frac{\sqrt{3}}{4}a^2, \quad V(2,1) = \frac{1}{4}\sqrt{4\alpha+1}a^2, \tag{6.2.3}$$

depending on whether they are space- or timelike, respectively. The volumes in $d=3$ are as follows:

$$V(4,0) = \frac{1}{6\sqrt{2}}a^3, \quad V(3,1) = \frac{\sqrt{3\alpha+1}}{12}a^3 \quad V(2,2) = \frac{\sqrt{2\alpha+1}}{6\sqrt{2}}a^3, \tag{6.2.4}$$

where, naturally, the first simplex is spacelike and the rest are timelike. Finally, the $d=4$ volumes are given by

$$V(4,1) = \frac{1}{96}\sqrt{8\alpha+3}a^4 \quad V(3,2) = \frac{1}{96}\sqrt{12\alpha+7}a^4, \tag{6.2.5}$$

for the two possible timelike simplices.

The dihedral angles for $d=2$ are

$$\cos\Theta^{SL}_{(3,0)} = \frac{1}{2}, \quad \sin\Theta^{SL}_{(3,0)} = \frac{\sqrt{3}}{2} \tag{6.2.6}$$

$$\cos\Theta^{SL}_{(2,1)} = \frac{-i}{2\sqrt{\alpha}}, \quad \sin\Theta^{SL}_{(2,1)} = \frac{\sqrt{4\alpha+1}}{2\sqrt{\alpha}} \tag{6.2.7}$$

$$\cos\Theta^{TL}_{(2,1)} = \frac{2\alpha+1}{2\alpha}, \quad \sin\Theta^{TL}_{(2,1)} = \frac{-i\sqrt{4\alpha+1}}{2\alpha}, \tag{6.2.8}$$

where SL means "spacelike" and refers to the angles involving at least one spacelike edge, while TL means "timelike" and refers to angles involving 2 timelike edges. The case for $d=3$ is

$$\cos\Theta^{SL}_{(4,0)} = \frac{1}{3}, \quad \sin\Theta^{SL}_{(4,0)} = \frac{2\sqrt{2}}{3} \tag{6.2.9}$$

$$\cos\Theta^{SL}_{(3,1)} = \frac{-i}{\sqrt{3}\sqrt{4\alpha+1}}, \quad \sin\Theta^{SL}_{(3,1)} = \frac{2\sqrt{3\alpha+1}}{\sqrt{3}\sqrt{4\alpha+1}} \tag{6.2.10}$$

$$\cos\Theta^{TL}_{(3,1)} = \frac{2\alpha+1}{4\alpha+1}, \quad \sin\Theta^{TL}_{(3,1)} = \frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1} \tag{6.2.11}$$

$$\cos\Theta^{SL}_{(2,2)} = \frac{4\alpha+3}{4\alpha+1}, \quad \sin\Theta^{SL}_{(2,2)} = \frac{-i2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1} \tag{6.2.12}$$

$$\cos\Theta^{TL}_{(2,2)} = \frac{-1}{4\alpha+1}, \quad \sin\Theta^{TL}_{(2,2)} = \frac{2\sqrt{2}\sqrt{\alpha}\sqrt{2\alpha+1}}{4\alpha+1}, \tag{6.2.13}$$

where, again, SL and TL correspond to dihedral angles around space- and timelike links, respectively. Finally, the dihedral angles for $d = 4$ are

$$\cos\Theta^{SL}_{(4,1)} = \frac{-i}{2\sqrt{2}\sqrt{3\alpha+1}}, \quad \sin\Theta^{SL}_{(4,1)} = \frac{\sqrt{3}\sqrt{8\alpha+3}}{2\sqrt{2}\sqrt{3\alpha+1}} \tag{6.2.14}$$

$$\cos\Theta^{TL}_{(4,1)} = \frac{2\alpha+1}{2(3\alpha+1)}, \quad \sin\Theta^{TL}_{(4,1)} = \frac{\sqrt{4\alpha+1}\sqrt{8\alpha+3}}{2(3\alpha+1)} \tag{6.2.15}$$

$$\cos\Theta^{SL}_{(3,2)} = \frac{6\alpha+5}{2(3\alpha+1)}, \quad \sin\Theta^{SL}_{(3,2)} = \frac{-i\sqrt{3}\sqrt{12\alpha+7}}{2(3\alpha+1)} \tag{6.2.16}$$

$$\cos\Theta^{TL1}_{(3,2)} = \frac{4\alpha+3}{4(2\alpha+1)}, \quad \sin\Theta^{TL1}_{(3,2)} = \frac{\sqrt{4\alpha+1}\sqrt{12\alpha+7}}{4(2\alpha+1)} \tag{6.2.17}$$

$$\cos\Theta^{TL2}_{(3,2)} = \frac{-1}{2\sqrt{2}\sqrt{2\alpha+1}\sqrt{3\alpha+1}}, \quad \sin\Theta^{TL2}_{(3,2)} = \frac{\sqrt{4\alpha+1}\sqrt{12\alpha+7}}{2\sqrt{2}\sqrt{2\alpha+1}\sqrt{3\alpha+1}}, \tag{6.2.18}$$

where SL and TL refers to dihedral angles between two spacelike triangles and a spacelike and a timelike one, respectively. On the other hand, $TL1$ refers to dihedral angles between triangles belonging to $(2,2)$-tetrahedra, while $TL2$ refers to dihedral angles between a triangle within a $(3,1)$-tetrahedron and a $(2,2)$-one.

For completeness, we include the volume and spacelike dihedral angle for a Euclidean 4-simplex, respectively written as

$$V(5,0) = \frac{\sqrt{5}}{96}a^4 \tag{6.2.19}$$

and

$$\cos\Theta^{SL}_{(5,0)} = \frac{1}{4}, \quad \sin\Theta^{SL}_{(5,0)} = \frac{\sqrt{15}}{4}. \tag{6.2.20}$$

With these tools, we are able to derive the Lorentzian version of the area-angle Regge Calculus. We will use a left superscript "L" for the Lorentzian quantities, in order to distinguish them from the (previous) Euclidean ones, and we begin with the counterpart to equation (6.1.1), which is written as

$$\cos\left({}^{L}\alpha_{ij,kl}\right) = \frac{\cos\phi_{kl,i} + \cos\phi_{jl,i}\cos\phi_{jk,i}}{\sin\phi_{jl,i}\sin\phi_{jk,i}}. \tag{6.2.21}$$

On the other hand, (6.1.2) becomes

$$\cos\left({}^L\alpha_{ji,kl}\right) = \frac{\cos\phi_{kl,j} + \cos\phi_{il,j}\cos\phi_{ik,j}}{\sin\phi_{il,j}\sin\phi_{ik,j}}, \tag{6.2.22}$$

and thus we obtain the Lorentzian counterpart to (6.1.3):

$$\begin{aligned} {}^L\mathcal{C}_{ij,kl}(\phi) &\equiv \cos\left({}^L\alpha_{ij,kl}\right) - \cos\left({}^L\alpha_{ji,kl}\right) \\ &= \frac{\cos\phi_{kl,i} + \cos\phi_{jl,i}\cos\phi_{jk,i}}{\sin\phi_{jl,i}\sin\phi_{jk,i}} - \frac{\cos\phi_{kl,j} + \cos\phi_{il,j}\cos\phi_{ik,j}}{\sin\phi_{il,j}\sin\phi_{ik,j}} \\ &= 0. \end{aligned} \tag{6.2.23}$$

Furthermore, (6.1.6) becomes

$$\det({}^LG^k(\phi)) = 0, \quad {}^L\mathcal{C}_{ij,kl}(\phi) = 0, \tag{6.2.24}$$

for

$${}^LG^k_{ij}(\phi) = \cos\phi_{ij,k}, \quad \cos\phi_{ii,k} = -1. \tag{6.2.25}$$

In other words, the Lorentzian and Euclidean Gram matrices are the same. As for the Lorentzian constraints, the first one (analogous to (6.1.7)) is

$${}^L\mathcal{C}^\sigma_{ee'}(\phi^\tau_e) \equiv \begin{cases} {}^L\mathcal{C}_{ij,kl} & , \quad e = \sigma(ijk), e' = \sigma(ijl) \\ 0 & , \quad \text{otherwise} \end{cases} = 0, \tag{6.2.26}$$

while the second one (analogous to (6.1.1)) is

$${}^L\mathcal{N}^\tau_t(A,\phi) \equiv A_t - \left| \sum_{t'\in\tau\setminus\{t\}} A_{t'} \left[\mathrm{Re}(\cos(\phi^\tau_{tt'})) + \mathrm{Im}(\cos(\phi^\tau_{tt'})) \right] \right| = 0. \tag{6.2.27}$$

Finally, The Lorentzian version of the area-angle Regge action (6.1.12) — denoted by the subscript "L" in S — is

$$\begin{aligned} S_L\left(A_t, \phi^\tau_e, \mu^\sigma_{ee'}, \lambda^\tau_t\right) &= \sum_t A_t\epsilon_t(\phi) \sum_\sigma \sum_{ee'\in\sigma} \mu^\sigma_{ee'} \left({}^L\mathcal{C}^\sigma_{ee'}(\phi)\right) + \sum_\tau \sum_{t\in\tau} \lambda^\tau_t \left({}^L\mathcal{N}^\tau_t(A,\phi)\right), \\ \epsilon_t &= 2\pi - \sum_{\sigma\ni t} \theta^\sigma_t \\ &= 2\pi - \sum_{\sigma\ni\sigma(ij)} \arccos\left(\frac{\cos(\phi_{ij,k}) - \cos(\phi_{ik,j})\cos(\phi_{jk,i})}{\sin(\phi_{ik,j})\sin(\phi_{jk,i})}\right), \end{aligned} \tag{6.2.28}$$

where the formula for the deficit angle ϵ_t requires no modification for the Lorentzian version.

One could then use this new Lorentzian area-angle Regge action to define a state sum ${}^LZ_{DS}$, derive the appropriate modifications in order to ensure it is finite and, using the effective action formula (as shown above), study its classical limit and its relation to the Regge action.

Chapter 7

Open Problems and Conclusions

7.1 Open Problems

Despite tremendous progress within the area of loop quantum gravity and the spin foam formalism, some important issues remain unsolved [1].

The first problem presented here is that of extracting physics from the spin foam models, i.e. deriving physical and geometrical meaning from the spin foam configurations. This issue arises upon attempting to fulfil the task of attributing quantum spacetime configurations to spin foams. For instance, in some cases, the projector from the kinematical to the physical Hilbert space reveals that the latter is one-dimensional, thereby negating the existence of classical degrees of freedom and consequently the existence of quantum (non-trivial) Dirac observables as well. The sum over spin foams in this context is simply over pure gauge degrees of freedom and thus it becomes impossible to infer any physical interpretation from it. When attempting to formulate a spacetime geometric interpretation of the spin foam configurations, one must bear in mind that such a task is difficult, since it is not straightforward to say that a spin foam sum is a sum over geometries. Instead, it is necessary to average over the gauge orbits obtained through the quantum constraints. In addition, the physical degrees of freedom are obtained from the kinematical ones by performing the sum over gauge histories — defining the aforementioned projector — within the spin foam framework. Furthermore, the physical interpretation of quantum numbers present in the spin foams is complicated, as is the study of other quantities, when equating spin foams to simply quantum spacetime configurations. Nonetheless, the kinematical aspects and quantum geometry, though not directly entailing physical meaning, can still be useful for the formulation of Dirac observables [39, 40]. For instance, gauge invariant quantities might not play a physical role themselves, but indeed do so when combined. In short, obtaining physically meaningful quantities from spin foams is an obstacle yet to be completely surpassed.

The second problem involves the characteristics required from the path integral measure [41]. When interpreting spin foam models in a general covariant manner, we are led to the conclusion that path integrals output the physical Hilbert space, while the transition amplitudes yield the scalar product. However, in order to ensure general covariance, we need the path integral measure to be gauge invariant or, in other words, anomaly-free. Also, requiring a well-defined transition amplitude entails providing gauge fixing conditions. The case of compact gauge groups constitutes an exception to this "rule", as well-defined transition amplitudes, along with anomaly-free measures, can be obtained without resorting to gauge fixing conditions — this is the case for standard lattice gauge theory, for example [1]. For Lorentzian cases, however, the internal orbits are infinitely voluminous and hence beg gauge fixing, in order to avoid divergences in the path integral. Luckily, there are usually appropriate gauge fixing conditions for these cases.

Another gauge freedom, diffeomorphism invariance, is better understood by establishing a parallel between the spin network and the spin foam contexts. Within the boundary spin network states of a spin foam dual to a fixed discretisation, we consider equivalence classes of these states, in order to implement 3-diffeomorphism invariance — according to the canonical approach. The discretisation itself requires the boundary graphs to be located within the dual 1-skeleton of the boundary complex and the resulting states correspond to a 3-diffeomorphism equivalence class. Therefore, discretisation is tantamount to partial gauge fixing of boundary 3-diffeomorphisms, with a remaining discrete symmetry — given by the discrete symmetries of the spin networks — that needs to be factored out. In parallel, spin foams in the path integral are partially gauge fixed by the 2-complex containing them, with the finite group of discrete spin foam symmetries playing the role of the remaining symmetry to be factored out, when calculating the transition amplitudes. Also in this case, this task can be performed, due to the combinatorial and finite nature of equivalent spin foams. Moreover, spin foams are anomaly-free if the amplitudes are invariant under the discrete symmetry mentioned above. Once again, such a model is possible to obtain, provided the correct transition amplitudes are given. We note that gauge symmetries are expected to play an extra role, apart from the one mentioned here.

Further problems regarding spin foams include the following binary interpretation: either spin foams are diffeomorphism equivalence classes of geometries or they are projectors to the solution space of constraints. For the latter case, one would need gauge symmetries.

In addition, discreteness is paramount: the discretisation of the manifold acts as a regulator for the model, although the fundamental "unregulated" theory is still discrete — as are its excitations —, with a continuous diffeomorphism invariant low-energy counterpart. Hence, gauge symmetries are also required to be described within a discrete framework. In order to maintain diffeomorphism invariance and the same symmetry as the low-energy theory, we need to use "perfect actions" as regulators. Usually, such regulators exist. Nevertheless, the discretisation method breaks general covariance, although it is possible to use the "consistent discretisation" approach to attempt to solve this difficulty. Said approach allows to investigate the gauge symmetries and proves that diffeomorphism invariance is broken by the discretisation, due to the lack of a diffeomorphism generator.

As is well-known, breaking a gauge symmetry introduces unwanted degrees of freedom in a theory, which is why the breaking of the diffeomorphism gauge symmetry (mentioned above) is unwelcome, as it may lead to inconsistencies with the continuous low-energy limit, i.e. general relativity. Even for global symmetries, this occurs and poses a serious threat to the consistency of the model.

Finally, an ambiguity arises when considering anomaly-free measures, as they can yield other such measures solely by being multiplied by a gauge invariant function. This is an unwelcome ambiguity which unfortunately plagues most spin foam models.

We now discuss the main problem of discretisation dependence within the spin foam models. As mentioned earlier, manifold discretisation is intended to be an intermediate step in attaining the full theory. The use of discretisation allows for the reduction of the functional integral from infinite dimensions to a finite set of variables, which amounts to imposing in the path integral only spin foams that can be obtained by the colouring of the 2-complex. In turn, the boundary spin network states are also restricted (by the boundary graph), which also limits the possible fluctuations. These fluctuations reach the Planck scale, since the spin values providing the colouring, and thus defining the geometry, can take any values — in fact, two spins differing by a unit value will generate area eigenvalues differing by approximately one Planck length squared. The discretisation dependence should then be eliminated by a properly defined spin foam path integral.

An interesting case is the one of topological theories, which use a fixed cellular decomposition and contain no local excitations (i.e. gravitons), hence remaining decomposition independent. However, other approaches have not been as successful in establishing background independence. In order to recover general covariance, we present two possible methods: discretisation refinement and Feynman diagrams.

The former posits that increasing the number of simplices in the triangulation fixing the topology restores the previously truncated degrees of freedom. The idea is that each refinement generates a dual 2-complex which is also a refinement of the previous one, thereby enlarging the set of possible spin foams (yet still including the previous ones), as well as the space of possible boundary spin networks. Consequently, in the limit of infinite refinements (or, equivalently, 4-simplices), one recovers the full kinematical sector. However, it is unknown whether the amplitudes of 4d models converge in such a limit.

The latter is based on the correspondence between spin foam amplitudes and Feynman diagram amplitudes of a group field theory [42]. The perturbative expansion of this field theory yields a combinatorial and thus background independent sum over discretisations and this fact constitutes an advantage over other approaches.

It is important to note that this perturbative approach usually leads to divergences, although physical information might still be retrieved through specific method. This convergence problem, along with the unknown physical role of the coupling constant, are the issues faced by the group field theory alternative, which nonetheless has the potential to solve diffeomorphism gauge symmetry and the discretisation dependence.

Finally, the problem of discretisation dependence is possibly linked to that of diffeomorphism symmetry, in the sense that implementing the latter might automatically entail the former [43].

The final open problem presented here is the Lorentz invariance in the low energy limit of the spin foam models [44], intimately linked to the problem of defining the dynamics of the physical states. More specifically, there have been proposed Lorentz invariance violating effects in quantum gravity, in the form of granularity of spacetime. However, the corresponding corrections to the Lagrangian have been found to be of small enough order to be in direct conflict with the observed Lorentz invariance of particle physics. Therefore, the aim of spin foam and loop quantum gravity theories ought to be showing that their low energy limit respects the observed low energy Lorentz invariance. Possible solutions include using an effective action without any Lorentz invariance violating effects, as well as including these effects only for dimension at least 5, due to some protecting mechanism.

7.2 Conclusions

Within the subject of quantising gravity, we have presented the loop quantum gravity framework, as well as the spin foam approach. We also showed 3d and 4d prescriptions for spin foams and further listed the most important models in the literature. Following the description of how to quantise BF theory, as done by many of the models, we reached our main topic: the EPRL model and its semiclassical limit. We described the model as it was introduced in the original paper and proceeded to show how its vertex amplitude can be modified in order for the state sum to yield the correct semiclassical limit — the Regge action — corresponding to a discretisation of general relativity, by using the effective action, closely related to the background field method of quantum field theory. However, the problems of non-locality, ambiguities and incompleteness surrounding area-Regge calculus prompted the discussion of a modification of this theory, which uses constrained areas and 3d angles to create an equivalent action to length-Regge and a better suited one than area-Regge. Then, we applied the aforementioned effective action method, as well as the same vertex modification to the state sum defined by the new area-angle

Regge action, in order to show it is finite and thus has Regge action as its classical limit. Finally, we shifted to the Lorentzian framework, by performing a Lorentzian discretisation of spacetime using a folliation into $(d-1)$-hypersurfaces, connected by timelike vertices of timelike 4-simplices. In other words, we assigned different time- and spacelike lengths to different 4-simplices, after which we wrote the corresponding volumes and dihedral angles for dimensions $d = 0, 1, 2, 3, 4$. Armed with these tools, we finally derived a Lorentzian version of the constraints present in the area-angle Regge action, thereby obtaining its Lorentzian version. This was followed by a few remarks on problems faced by the spin foam and loop quantum gravity approach, which are under investigation at the moment.

Another important aspect is the fact that the effective action approach provides a solution to the issue of obtaining the semiclassical limit of a spin foam model and can be applied to any state sum model of quantum gravity. In the case of the EPRL model one has to modify the vertex amplitude, as in (5.3.5), in order to obtain a correct semiclassical limit. The corresponding semiclassical limit is a theory which is a generalisation of general relativity, in the sense that this theory allows non-metric geometries.

This conclusion is an important one, since any quantum gravity theory should yield general relativity in its low-energy limit, especially, since this is the context in which we have the most amount of evidence for the consistency of the latter. In other words, any quantum gravity theory should still be consistent with the well-known and investigated classical counterpart, lest it be incomplete. Fortunately, progress has been made regarding the important subject of the semiclassical limit of spin foam models, as shown here. Specifically, equations (6.1.19) and (6.1.20) allow us to:

- Use the new Lorentzian area-angle action to define a state sum ${}^{L}Z_{DS}$;
- Derive the appropriate modifications in order to ensure its finiteness;
- Study its classical limit and its relation to the Regge action, through the use of the effective action formula.

Nevertheless, there is still work to be done in this and other related areas of quantum gravity and spin foams. Aside from the problems requiring further investigation presented earlier, there are other interesting avenues worth pursuing [1], such as:

- Strengthening the relation between canonical and covariant formulations of quantum gravity;
- Investigating spin foams through their correspondence to group field theory and Feynman diagrams;
- Deriving possible quantum gravity effects that might be observed at the current low-energy limit by present-day technology;
- Applying new discoveries from the field of loop quantum gravity and spin foams to that of loop quantum cosmology;
- Studying triangulations with timelike links and faces;
- Deriving the Regge action for such triangulations;
- Applying the effective action equation to such triangulations;
- Calculating the propagator in the context of these triangulations.

Regarding the last 4 issues and taking into account the Lorentzian triangulation used here, further investigation also includes, in particular:

- Studying Lorentzian triangulations with different space- and timelike length assignments.

In addition, we note that the smooth limit of the DS action is not yet known.

While an effort has been made here in order to establish preliminary concepts for tackling the latter tasks, a more thorough investigation of these has been left as future work.

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Appendices

Appendix A

Lorentzian Volumes and Dihedral Angles

Here, we derive the Lorentzian (and Euclidean) volumes and dihedral angles from Ambjørn et al. [4], for dimensions $d = 0, 1, 2, 3, 4$ explicitly, by using the general formulae from Tate and Visser [38], among other.

A.1 Lorentzian Volumes

Starting with the volumes, we provide them for distinct simplices up to time reversal. The cases for $d = 0, 1$ follow from convention and the length assignments, respectively, and are given by

$$V(1,0) = 1 \tag{A.1.1}$$

and

$$V(2,0) = a, \quad V(1,1) = \sqrt{\alpha}a, \tag{A.1.2}$$

for a point in $d = 0$ and space- and timelike links in $d = 1$, respectively. For triangles in $d = 2$, we use the well-known formula

$$A_t = \frac{1}{2}b.h \tag{A.1.3}$$

for the area A_t of a triangle t with base b and height h. For the case of a spacelike triangle, all sides have length a, meaning $b = a$. In addition, since the triangle is equilateral, its height bisects it into two right triangles, allowing us to use the pithagorean theorem, thereby obtaining

$$h = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \frac{\sqrt{3}}{2}a. \tag{A.1.4}$$

The area for a spacelike triangle is then

$$V(3,0) \equiv A_t^{SL} = \frac{1}{2}b.h = \frac{1}{2}a\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}a^2. \tag{A.1.5}$$

For the case of a timelike triangle, we have an isosceles triangle, with two sides of timelike length $\sqrt{\alpha}a$ and the remaining side, which we define to be the base, of spacelike length $b = a$. As a result, the height still bisects the triangle and so we can use the pithagorean theorem again, yielding,

$$h = \sqrt{(\sqrt{\alpha}a)^2 - \left(\frac{a}{2}\right)^2} = \sqrt{-\alpha a^2 - \frac{a^2}{4}} = \sqrt{-\frac{1}{4}(4\alpha + 1)a^2} = \frac{1}{2}\sqrt{4\alpha + 1}a, \tag{A.1.6}$$

where we have used $(\sqrt{\alpha}a)^2 = -\alpha a^2$, from our definition of squared timelike length. The resulting volume is then

$$V(2,1) \equiv A_t^{TL} = \frac{1}{2}b.h = \frac{1}{2}a\frac{1}{2}\sqrt{4\alpha+1}a = \frac{1}{4}\sqrt{4\alpha+1}^2. \tag{A.1.7}$$

For $d = 3$, we use the following formula for the volume V_τ of a tetrahedron τ:

$$V_\tau = \frac{1}{3}A_0.h, \tag{A.1.8}$$

for its height h and area of its base triangle $A_0 = A_t$. For a spacelike tetrahedron, $A_0 = A_t^{SL} = V(3,0)$, leaving only the task of deriving the height. By definition, the height forms a right angle with the plane containing the base triangle and similarly produces a right triangle with the hypotenuse corresponding to a tetrahedron side, implying its length is spacelike and thus a. With two sides of these triangles having respective lengths a and h, we simply need to derive the base side l, in order to obtain h through the pythagorean theorem. For that, we use the law of cosines,

$$a^2 = b^2 + c^2 - 2b.c\cos A, \tag{A.1.9}$$

for a triangle with sides a, b, c opposite the angles A, B, C. We choose the side a to be one of the sides of the base triangle of the tetrahedron, thus having length a. the remaining two sides meet at the intersection between the tetrahedron height and the plane containing the base triangle. Due to the properties of the height of a regular tetrahedron, we know that the triangle we just defined is isosceles, with the two remaining sides having length l. Furthermore, the line segment connecting the vertices of the base triangle to the previously described intersection point bisect the angles of said triangle, which are all equal to $\frac{\pi}{3}$, since it is equilateral. Consequently, $B, C = \frac{\pi}{6}$, and by the fact that the sum of internal angles of a flat Euclidean triangle corresponds to π, we can infer that the remaining angle is given by $A = \frac{2\pi}{3}$. We now have sufficient information to use the law of sines, which, for our triangle, becomes

$$a^2 = l^2 + l^2 - 2l.l\cos\left(\frac{2\pi}{3}\right) \implies l = \sqrt{\frac{a^2}{2\left(\left(1-\cos\left(\frac{2\pi}{3}\right)\right)\right.}} = \frac{a}{\sqrt{3}}. \tag{A.1.10}$$

Next, we use the pythagorean theorem to derive h in the triangle with sides of lengths h, l, a:

$$h = \sqrt{a^2 - l^2} = \sqrt{a^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \frac{2}{3}a. \tag{A.1.11}$$

Finally, we use the newly found height h in the formula for the volume of the tetrahedron, thus obtaining

$$V(4,0) \equiv V_\tau^{SL} = \frac{1}{3}V(3,0).h = \frac{1}{3}\frac{\sqrt{3}}{4}a^2\frac{2}{3}a = \frac{\sqrt{2}}{12}a^3 = \frac{1}{6\sqrt{2}}a^3. \tag{A.1.12}$$

For a timelike $(3,1)$-tetrahedron, the base triangle is still spacelike and so its area and l have the same values as in the spacelike case. The only change consists in the hypotenuse of the triangle with the remaining sides h and l, which is now a timelike link of length $\sqrt{\alpha}a$. We then get

$$h = \sqrt{(\sqrt{\alpha}a)^2 - l^2} = \sqrt{(\sqrt{\alpha}a)^2 - \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{\alpha + \frac{1}{3}}a. \tag{A.1.13}$$

The formula for the $(3,1)$-volume then becomes

$$V(3,1) \equiv A_{\tau}^{TL(3,1)} = \frac{1}{3}V(3,0).h = \frac{1}{3}\frac{\sqrt{3}}{4}a^2.\sqrt{\alpha + \frac{1}{3}}a = \frac{\sqrt{3\alpha+1}}{12}a^3. \tag{A.1.14}$$

Regarding the $(2,2)$-simplex, we use the Cayley-Menger determinant det$(\hat{B})$, where $\hat{B}$ is the $(d+2) \times (d+2)$ matrix — where d is the dimension of the simplex — given by a first row and column of the form $(0, 1 \ldots, 1)$ and the remaining elements B_{ij} are the squared lengths between the vertices i and j. Explicitly, for a tetrahedron with sides of lengths a, b, c, d, e, f in $d = 3$, $\hat{B}$ is written as

$$\hat{B} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & d^2 & e^2 \\ 1 & b^2 & d^2 & 0 & f^2 \\ 1 & c^2 & e^2 & f^2 & 0 \end{bmatrix}. \tag{A.1.15}$$

The volume of a simplex S in dimension d is then given by

$$V_d(S) = \frac{(-1)^{d+1}}{2^d (d!)^2} \det(\hat{B}) \tag{A.1.16}$$

and thus for a 3-simplex ($d = 3$), we have

$$V_3^2(S) = \frac{1}{288} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & d^2 & e^2 \\ 1 & b^2 & d^2 & 0 & f^2 \\ 1 & c^2 & e^2 & f^2 & 0 \end{vmatrix}. \tag{A.1.17}$$

within a $(2,2)$-simplex, we have 2 spacelike lenghts — joining each pair of points on each timeslice — and 4 timelike lengths — joining each point in the first timeslice to the other 2 in the second one — and so, remembering the squared space- and timelike length assignments (a^2 and $-\alpha a^2$, respectively), we can rewrite the equation above as

$$\begin{aligned} V_3^2(S) &= \frac{1}{288} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 \\ 1 & a^2 & 0 & -\alpha a^2 & -\alpha a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 & 0 & a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 & a^2 & 0 \end{vmatrix} \\ &= -\frac{1}{288} \begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & 0 & 0 \\ 1 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 & \alpha a^2 + a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 \end{vmatrix}, \end{aligned} \tag{A.1.18}$$

where we have swapped rows 1 and 2 — thus changing the sign of the determinant — and subtracted each row by row 1 — which does not change the determinant. Then, we swap rows 2 and 5 and cancel the leading coefficients in rows 3, 4 and 5 by adding the $\frac{1}{b}$, -1 and $\frac{1}{\alpha a^2}$ multiples of row 2, respectively. The result is then

$$V_3^2(S) = \frac{1}{288} \begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & \frac{-2\alpha a^2 - a^2}{\alpha} & \frac{-\alpha a^2 + a^2}{\alpha} & -\alpha a^2 \\ 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 0 & -\frac{1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \end{vmatrix}, \tag{A.1.19}$$

where the swapping of rows changed the sign of the determinant and the other operations left it invariant. The next step is to swap rows 3 and 5 and to cancel the leading coefficients in the new row 5 by subtracting the $-2\alpha a^2 + a^2$ multiple of the new row 3. The result is

$$V_3^2(S) = -\frac{1}{288} \begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & -\frac{1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \\ 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 0 & 0 & -a^2(4\alpha+3) & -4\alpha a^2 - a^2 \end{vmatrix}, \tag{A.1.20}$$

where, again, only the sign changed. Lastly, we swap rows 4 and 5 and and cancel the leading coefficients in row 5 by subtracting it by the $\frac{1}{4\alpha+3}$ multiple of row 4. The result is

$$V_3^2(S) = \frac{1}{288} \begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & -\frac{1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \\ 0 & 0 & 0 & -a^2(4\alpha+3) & -4\alpha a^2 - a^2 \\ 0 & 0 & 0 & 0 & \frac{8\alpha a^2 + 4a^2}{4\alpha+3} \end{vmatrix}, \tag{A.1.21}$$

where only the sign changed. Now that we have an upper triangular matrix, we know that its determinant is the product of its diagonal elements, yielding

$$\begin{aligned} V_3^2(S) &= \frac{1}{288} 1.\left(-\alpha a^2\right)\left(-\frac{1}{\alpha}\right)\left(-a^2(4\alpha+3)\right)\left(\frac{8\alpha a^2 + 4a^2}{4\alpha+3}\right) \\ &= \frac{1}{288}\left(-4a^6(2\alpha+1)\right) \\ \implies V(2,2) \equiv V_3(S) &= \sqrt{\frac{1}{288}\left(-4a^6(2\alpha+1)\right)} = \frac{\sqrt{2\alpha+1}}{6\sqrt{2}}a^3. \end{aligned} \tag{A.1.22}$$

We now move one dimension higher, to $d = 4$, where we need to find the volumes $V(5,0), V(4,1)$ and $V(3,2)$. For the first one, we have that all links and lengths are spacelike and so the formula used for the derivation of the previous volume becomes

$$V_4^2(S) = \frac{-1}{9216}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & a^2 & a^2 & a^2 \\ 1 & a^2 & 0 & a^2 & a^2 & a^2 \\ 1 & a^2 & a^2 & 0 & a^2 & a^2 \\ 1 & a^2 & a^2 & a^2 & 0 & a^2 \\ 1 & a^2 & a^2 & a^2 & a^2 & 0 \end{vmatrix}$$
$$= \frac{1}{9216}\begin{vmatrix} 1 & 0 & a^2 & a^2 & a^2 & a^2 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & -a^2 & 0 & 0 \\ 0 & a^2 & 0 & a^2 & -a^2 & 0 \\ 0 & a^2 & 0 & 0 & 0 & -a^2 \end{vmatrix}, \tag{A.1.23}$$

where rows 1 and 2 were swapped — thus changing the sign of the determinant — and each row from 3 to 6 was subtracted by the new row 1. Next, we swap rows 2 and 6 — changing the sign of the determinant again — and subtract rows 3 to 5 by the new row 2, which results in

$$V_4^2(S) = \frac{-1}{9216}\begin{vmatrix} 1 & 0 & a^2 & a^2 & a^2 & a^2 \\ 0 & a^2 & 0 & 0 & 0 & -a^2 \\ 0 & 0 & -a^2 & 0 & 0 & a^2 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \\ 0 & 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}. \tag{A.1.24}$$

Finally, we sequentially add the multiples $-\frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^2}\frac{1}{a^2}$ of rows 2 to 5 (respectively) to row 6, thereby obtaining

$$V_4^2(S) = \frac{-1}{9216}\begin{vmatrix} 1 & 0 & a^2 & a^2 & a^2 & a^2 \\ 0 & a^2 & 0 & 0 & 0 & -a^2 \\ 0 & 0 & -a^2 & 0 & 0 & a^2 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \\ 0 & 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{vmatrix}. \tag{A.1.25}$$

Again, the determinant is the product of the diagonal elements, i.e.

$$\begin{aligned} V_4^2(S) &= \frac{-1}{9216}.1.(a^2)(-a^2)^3.5 = \frac{-1}{9216}(-5a^8) = \frac{5}{9216}a^8 \\ \implies V(5,0) \equiv V_4(S) &= \sqrt{\frac{5}{9216}a^8} = \frac{\sqrt{5}}{96}a^4. \end{aligned} \tag{A.1.26}$$

For $V(4,1)$, the equation becomes

$$
\begin{aligned}
V_4^2(S) &= \frac{-1}{9216}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 1 & -\alpha a^2 & 0 & a^2 & a^2 & a^2 \\ 1 & -\alpha a^2 & a^2 & 0 & a^2 & a^2 \\ 1 & -\alpha a^2 & a^2 & a^2 & 0 & a^2 \\ 1 & -\alpha a^2 & a^2 & a^2 & a^2 & 0 \end{vmatrix} \\
&= \frac{1}{9216}\begin{vmatrix} 1 & 0 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -\alpha a^2 & \alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 \\ 0 & -\alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 \\ 0 & -\alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 & \alpha a^2 + a^2 \\ 0 & -\alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \end{vmatrix} \\
&= \frac{-1}{9216}\begin{vmatrix} 1 & 0 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & -\alpha a^2 & 0 & 0 & a^2 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \\ 0 & 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 0 & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \end{vmatrix} \\
&= \frac{1}{9216}\begin{vmatrix} 1 & 0 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \\ 0 & 0 & 0 & a^2 & a^2 & \frac{4\alpha a^2+a^2}{2\alpha+1} \\ 0 & 0 & 0 & 0 & a^2 & \frac{6\alpha a^2+2a^2}{2\alpha+1} \\ 0 & 0 & 0 & 0 & 0 & \frac{8\alpha a^2+3a^2}{2\alpha+1} \end{vmatrix},
\end{aligned} \tag{A.1.27}
$$

where the following changes were made, respectively from each line to the next:

- Rows 1 and 2 were swapped and rows 3 to 6 were subtracted by row 1;
- Rows 2 and 6 were swapped, rows 3 to 5 were subtracted by row 2 and row 6 was added by the multiple $\frac{1}{\alpha a^2}$ of row 2;
- Rows 3 and 6 were swapped, row 6 was added by the $\frac{\alpha a^2}{2\alpha+1}$ multiple of row 3, rows 4 and 6 were swapped, row 6 was added by row 4, row 5 and row 6 were swapped and row 6 was added by row 5 — all in that order.

We can now obtain the determinant by multiplying the diagonal elements, i.e.

$$
\begin{aligned}
V_4^2(S) &= \frac{1}{9216}.1.(-\alpha a^2)\left(\frac{2\alpha+1}{\alpha}\right)(a^2)^2\left(\frac{8\alpha a^2+3a^2}{2\alpha+1}\right) = \frac{1}{9216}(-a^8(8\alpha+3)) \\
\implies V(4,1) \equiv V_4(S) &= \sqrt{-\frac{1}{9216}a^8(8\alpha+3)} = \frac{1}{96}\sqrt{8\alpha+3}a^4.
\end{aligned} \tag{A.1.28}
$$

The remaining simplex $((3,2))$ has a volume given by

$$V_4^2(S) = \frac{-1}{9216}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 1 & a^2 & 0 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 & 0 & a^2 & a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 & a^2 & 0 & a^2 \\ 1 & -\alpha a^2 & -\alpha a^2 & a^2 & a^2 & 0 \end{vmatrix}$$

$$= \frac{1}{9216}\begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & 0 & 0 & 0 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 & \alpha a^2 + a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \end{vmatrix}$$

$$= \frac{-1}{9216}\begin{vmatrix} 1 & 0 & \alpha a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & \frac{-2\alpha a^2 - a^2}{\alpha} & \frac{\alpha a^2 + a^2}{\alpha} & \frac{\alpha a^2 + a^2}{\alpha} & a^2 \\ 0 & 0 & 0 & -a^2 & 0 & a^2 \\ 0 & 0 & 0 & 0 & -a^2 & a^2 \\ 0 & 0 & \frac{-1}{\alpha} & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \end{vmatrix} \tag{A.1.29}$$

$$= \frac{1}{9216}\begin{vmatrix} 1 & 0 & a^2 & -\alpha a^2 & -\alpha a^2 & -\alpha a^2 \\ 0 & -\alpha a^2 & -\alpha a^2 - a^2 & \alpha a^2 + a^2 & \alpha a^2 + a^2 & \alpha a^2 \\ 0 & 0 & -\frac{1}{\alpha} & \frac{2\alpha+1}{\alpha} & \frac{2\alpha+1}{\alpha} & 2 \\ 0 & 0 & 0 & -a^2(4\alpha+3) & -a^2(4\alpha+3) & -4\alpha a^2 - a^2 \\ 0 & 0 & 0 & 0 & a^2 & \frac{8\alpha a^2 + 4a^2}{4\alpha+3} \\ 0 & 0 & 0 & 0 & 0 & \frac{12\alpha a^2 + 7a^2}{4\alpha+3} \end{vmatrix},$$

where the following changes were made, respectively from each line to the next:

- Rows 1 and 2 were swapped and rows 3 to 6 were subtracted by row 1;
- Rows 2 and 6 were swapped, rows 4 and 5 were subtracted by row 2, row 3 was added by the multiple $\frac{1}{\alpha}$ of row 2 and row 6 was added by the multiple $\frac{1}{\alpha a^2}$ of row 2;
- Rows 3 and 6 were swapped, row 6 was subtracted by the $2\alpha a^2 + a^2$ multiple of row 3, rows 4 and 6 were swapped, row 6 was subtracted by the multiple $\frac{1}{4\alpha+3}$ of row 4, row 5 and row 6 were swapped and row 6 was added by row 5 — all in that order.

We can now finally obtain the determinant by multiplying the diagonal elements, i.e.

$$V_4^2(S) = \frac{1}{9216}.1.(-\alpha a^2)\left(-\frac{1}{\alpha}\right)(-(a^2(4\alpha+3)))(a^2)\left(\frac{12\alpha a^2 + 7a^2}{4\alpha+3}\right) = \frac{1}{9216}(-(a^8(12\alpha+7)))$$
$$\implies V(4,1) \equiv V_4(S) = \sqrt{\frac{1}{9216}(-(a^8(12\alpha+7)))} = \frac{1}{96}\sqrt{12\alpha+7}a^4. \tag{A.1.30}$$

We note that the simplex S in the formulae changes according to the context — as if we had named it S, S', S'',... for the different (i,j)-simplices.

A.2 Lorentzian Dihedral Angles

Having presented the volumes for the Lorentzian simplices in $d = 0, 1, 2, 3, 4$, we shall now do the same for the dihedral angles in $d = 2, 3, 4$. The spacelike $(3, 0)$ triangles, are equilateral with spacelink link lengths a, implying that their 3 angles are also equal to each other and, since their sum must equal 2π, we conclude that they must measure $\frac{2\pi}{3}$ radians. As a result, their cosine and sine are, respectively,

$$\cos\Theta^{SL}_{(3,0)} = \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}, \quad \sin\Theta^{SL}_{(3,0)} = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}. \tag{A.2.1}$$

For the case of a timelike triangle, we have an isosceles triangle with 2 timelike (length $\sqrt{\alpha}a$) sides and 1 spacelike (length a) side. The cosine and sine formulas for the spacelike angle — i.e. the angle between a space- and a timelike link — are

$$\cos\Theta^{SL}_{(2,1)} = \frac{1}{2i}\left(\frac{S^2 - T_1^2 + T_2^2}{ST_1}\right), \quad \sin\Theta^{SL}_{(2,1)} = \frac{2}{ST_1}V(2,1), \tag{A.2.2}$$

where S, T_1 and T_2 are the spacelike and timelike edge lengths, respectively. However, since the timelike lengths are equal, the numerator in the first formula loses the two T terms and so, substituting by the assigned lengths, we obtain

$$\cos\Theta^{SL}_{(2,1)} = \frac{1}{2i}\left(\frac{a^2}{\sqrt{\alpha}a^2}\right) = \frac{-i}{2\sqrt{\alpha}}, \quad \sin\Theta^{SL}_{(2,1)} = \frac{2}{\sqrt{\alpha}a^2}\frac{1}{4}\sqrt{4\alpha+1}a^2 = \frac{\sqrt{4\alpha+1}}{.}2\sqrt{\alpha} \tag{A.2.3}$$

The case for a timelike angle — i.e. an angle between two timelike links begs the formula

$$\cos\Theta^{TL}_{(2,1)} = \frac{1}{2}\left(\frac{S^2 + T_1^2 + T_2^2}{T_1T_2}\right), \quad \sin\Theta^{TL}_{(2,1)} = \frac{2}{iT_1T_2}V(2,1). \tag{A.2.4}$$

Again, including the values in the formula yields

$$\begin{aligned} \cos\Theta^{TL}_{(2,1)} &= \frac{1}{2}\left(\frac{a^2 - 2\alpha a^2}{-\alpha a^2}\right) = \frac{2\alpha+1}{2\alpha}, \\ \sin\Theta^{TL}_{(2,1)} &= \frac{2}{-i\alpha a^2}\frac{1}{4}\sqrt{4\alpha+1}a^2 = \frac{-i\sqrt{4\alpha+1}}{2\alpha}. \end{aligned} \tag{A.2.5}$$

One dimension higher ($d = 3$), we start with the Euclidean $(4, 0)$-tetrahedron, using the formula which in its original form is written as

$$\cos\Theta^{SL}_{(4,0)} = \frac{-a^4 - (c^2 - e^2)(b^2 - f^2) + a^2(b^2 + c^2 - 2d^2 + e^2 + f^2)}{16A(a,c,e)A(a,b,f)}, \tag{A.2.6}$$

for the edge lengths a opposite to d, b opposite to e and c opposite to f, and for A as the area of the triangle with the specific edge lengths mentioned. Our case is however simple, since all 6 links are spacelike and so have length a. The result is

$$\begin{aligned} \cos\Theta^{SL}_{(4,0)} &= \frac{-a^4 - (a^2 - a^2)(a^2 - a^2) + a^2(a^2 + a^2 - 2a^2 + a^2 + a^2)}{16V(3,0)^2} \\ &= \frac{a^4}{16\left(\frac{\sqrt{3}}{4}a^2\right)^2} = \frac{1}{3}, \end{aligned} \tag{A.2.7}$$

whereas the sine can be obtained using the formula

$$\sin(x)^2 + \cos(x)^2 = 1, \forall x \implies \sin(x) = \sqrt{1-\cos(x)^2}, 0 \leq x \leq \pi, \tag{A.2.8}$$

giving us

$$\sin\Theta^{SL}_{(4,0)} = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}. \tag{A.2.9}$$

For the $(3,1)$ case, the formula for the cosine follows from the one above, while the one for the sine follows from the Lorentzian analog of the Euclidean sine law for dimension d, which is

$$\sin\Theta_E = \frac{dV_n V_{n-2}}{V_{n-1}V_{n-1}}, \tag{A.2.10}$$

where V_n is the n-simplex volume, V_{n-2} is the volume of the hinge where the dihedral angle is located and finally V_{n-1} and V'_{n-1} are the volumes of the two faces sharing said hinge. The resulting formulae are then

$$\begin{aligned}\cos\Theta^{SL}_{(3,1)} &= \frac{-S_1^4-(T_2^2-T_3^2)(S_2^2-S_3^2)+S_1^2(S_2^2+S_3^2+2T_1^2-T_2^2-T_3^2)}{16iV(2,1)V(3,0)},\\ \sin\Theta^{SL}_{(3,1)} &= \frac{3S_1V(3,1)}{2V(2,1)V(3,0)},\end{aligned} \tag{A.2.11}$$

which, for our length assignments, give

$$\begin{aligned}\cos\Theta^{SL}_{(3,1)} &= \frac{-a^4-(-\alpha a^2-(-\alpha a^2))(a^2-a^2)+a^2(a^2+a^2-2\alpha a^2-(-\alpha a^2)-(-\alpha a^2))}{16i\frac{1}{4}\sqrt{4\alpha+1}a^2\frac{\sqrt{3}}{4}a^2}\\ &= \frac{-i}{\sqrt{3}\sqrt{4\alpha+1}},\\ \sin\Theta^{SL}_{(3,1)} &= \frac{3a\frac{\sqrt{3\alpha+1}}{12}a^3}{2\frac{1}{4}\sqrt{4\alpha+1}a^2\frac{\sqrt{3}}{4}a^2} = \frac{2\sqrt{3\alpha+1}}{\sqrt{3}\sqrt{4\alpha+1}}.\end{aligned} \tag{A.2.12}$$

The timelike counterparts of these formulae are

$$\begin{aligned}\cos\Theta^{TL}_{(3,1)} &= \frac{T_1^4+(S_2^2+T_2^2)(S_3^2+T_3^2)-T_1^2(T_2^2+T_3^2+2S_1^2-S_2^2-S_3^2)}{16\left(\frac{1}{4}\sqrt{4\alpha+1}a^2\right)^2},\\ \sin\Theta^{TL}_{(3,1)} &= \frac{3\sqrt{\alpha}a\frac{\sqrt{3\alpha+1}}{12}a^3}{2\left(\frac{1}{4}\sqrt{4\alpha+1}a^2\right)^2}\end{aligned} \tag{A.2.13}$$

and, as such, we obtain

$$\begin{aligned}\cos\Theta^{TL}_{(3,1)} &= \frac{\alpha^2a^4+(a^2-\alpha a^2)(a^2-\alpha a^2)+\alpha a^2(-\alpha a^2-\alpha a^2+2a^2-a^2-a^2)}{16i\frac{1}{4}\sqrt{4\alpha+1}a^2\frac{\sqrt{3}}{4}a^2}\\ &= \frac{2\alpha+1}{4\alpha+1},\\ \sin\Theta^{TL}_{(3,1)} &= \frac{3a\frac{\sqrt{3\alpha+1}}{12}a^3}{2\frac{1}{4}\sqrt{4\alpha+1}a^2\frac{\sqrt{3}}{4}a^2} = \frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1}.\end{aligned} \tag{A.2.14}$$

For the $(2,2)$-tetrahedron, the formulae are, similarly,

$$\begin{aligned}\cos\Theta^{SL}_{(2,2)} &= \frac{S_1^4-(T_1^2-T_2^2)(T_3^2-T_4^2)+S_1^2(T_1^2+T_2^2+T_3^2+T_3^2+2S_2^2)}{16V(2,1)^2},\\ \sin\Theta^{SL}_{(2,2)} &= \frac{-i3S_1V(2,2)}{2V(2,1)^2},\end{aligned} \tag{A.2.15}$$

for the spacelike case and

$$\begin{aligned}\cos\Theta^{SL}_{(2,2)} &= \frac{T_1^4-(S_1^2+T_2^2)(S_2^2+T_4^2)+T_1^2(S_1^2+S_2^2+2T_3^2-T_2^2-T_4^2)}{16V(2,1)^2},\\ \sin\Theta^{SL}_{(2,2)} &= \frac{3T_1V(2,2)}{2V(2,1)^2},\end{aligned} \tag{A.2.16}$$

for the timelike one. The results are then

$$\begin{aligned}\cos\Theta^{SL}_{(2,2)} &= \frac{a^4-(-\alpha a^2-(-\alpha a^2))(-\alpha a^2-(-\alpha a^2))+a^2(-\alpha a^2-\alpha a^2-\alpha a^2-\alpha a^2+2a^2)}{16\left(\frac{1}{4}\sqrt{4\alpha+1}a^2\right)^2}\\ &= \frac{4\alpha+3}{4\alpha+1},\\ \sin\Theta^{SL}_{(2,2)} &= \frac{-i3a\frac{\sqrt{2\alpha+1}}{6\sqrt{2}}a^3}{2\left(\frac{1}{4}\sqrt{4\alpha+1}a^2\right)^2} = \frac{-i2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1}\end{aligned} \tag{A.2.17}$$

and

$$\begin{aligned}\cos\Theta^{SL}_{(2,2)} &= \frac{\alpha^2a^4-(a^2-\alpha a^2)(a^2-\alpha a^2)-\alpha a^2(a^2+a^2-2\alpha a^2+\alpha a^2+\alpha a^2)}{16V(2,1)^2}\\ &= \frac{-1}{4\alpha+1},\\ \sin\Theta^{SL}_{(2,2)} &= \frac{3\alpha a\frac{\sqrt{2\alpha+1}}{6\sqrt{2}}a^3}{2\left(\frac{1}{4}\sqrt{4\alpha+1}a^2\right)^2} = \frac{2\sqrt{2}\sqrt{\alpha}\sqrt{2\alpha+1}}{4\alpha+1},\end{aligned} \tag{A.2.18}$$

respectively. The formulae for the 4-simplices are simply cumbersome analogues of the ones used above for $d=3$.

Appendix B

Lorentzian Area-Angle Formulae

B.1 Derivation

In order to find the Lorentzian analogue of (6.1.1), we first attempt to apply The Euclidean version directly. This works for a Euclidean $(5,0)$-simplex, as expected, i.e.

$$\begin{aligned}\cos(\alpha_{ij,kl}) &= \frac{\cos(\phi_{ij,k}) + \cos(\phi_{il,k})\cos(\phi_{jl,k})}{\sin(\phi_{il,k})sin(\phi_{jl,k})} \\ \iff \cos\alpha^{SL}_{(3,0)} &= \frac{\cos(\phi^{SL}_{(4,0)}) + \cos(\phi^{SL}_{(4,0)})\cos(\phi^{SL}_{(4,0)})}{\sin(\phi^{SL}_{(4,0)})\sin(\phi^{SL}_{(4,0)})} \\ \iff \frac{1}{2} &= \frac{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}}{\frac{2\sqrt{2}}{3}\cdot\frac{2\sqrt{2}}{3}} \iff 1 = 1 \quad \Box,\end{aligned} \tag{B.1.1}$$

where we have used the spacelike 3d dihedral angles derived earlier. For the case of a $(4,1)$-simplex, several types of 2d angles are possible:

- $\alpha^{SL}_{(3,0)}$, the spacelike angle within the spacelike triangle;
- $\alpha^{SL}_{(2,1);(4,0)}$, the spacelike angle within a timelike triangle expressed in terms of 3d angles within the spacelike $(4,0)$-tetrahedron $\sigma(k)$ — i.e. k is the only vertex in the second timeslice;
- $\alpha^{SL}_{(2,1);(3,1)}$, the spacelike angle within a timelike triangle expressed in terms of 3d angles within the timelike $(3,1)$-tetrahedron $\sigma(k)$ — i.e. k is one of the 4 vertices in the first timeslice;
- $\alpha^{TL}_{(2,1)}$, the timelike angle within a timelike triangle.

The distinction between these angles is made because they are represented by different 3d angles, according to the equation above, namely the right-hand side of the original Euclidean equation is, respectively, SST, SSS, TSS, TTT, where this notation denotes the order in which space- S or timelike T 3d angles appear, e.g.

$$SST \equiv \frac{\cos\phi^{SL} + \cos\phi^{SL}\cos\phi^{TL}}{\sin\phi^{SL}\sin\phi^{TL}}, \tag{B.1.2}$$

where the type of 3d angle (i.e. $(3,1)$ or $(2,2)$) is unambiguously given by the 2d angle in question. In particular, we note that an exhaustive listing of the types of 2d angles and their respective 3d angles defining them is easily made by a listing of different assignments of the vetices i, j, k, l, m to the particular type of 4-simplex being studied. We also note that the vertice k will define the type of tetrahedron from where the 3d angles are withdrawn, since they all belong to the tetrahedra $\sigma(k)$, which amounts to removing the vertex k from the 4-simplex — e.g. removing the vertex k from the second timeslice in the $(4, 1)$-simplex will yield the second type of 3d angle listed (i.e. $\alpha^{SL}_{(2,1);(4,0)}$, for the appropriate choice of 2d angle, naturally), while removing k from the first timeslice and the same $(4, 1)$ configuration will yield the third type of 3d angle (i.e. $\alpha^{SL}_{(2,1);(3,1)}$). This must be taken into account, in order to test the Euclidean equation against the different combinations 2d and 3d angles. Finally, we note that, in our notation, the second and third letters commute, i.e. $SST \equiv STS$, since both the cosines and sines involves are multiplying each other, hence commuting. This saves us time when computing the different allowed combinatorics in this notation — i.e. $SSS, SST \equiv STS, TSS, STT, TST \equiv TTS, TTT$.

We provide an example where the Euclidean equation works, when applied to the $(4, 1)$-simplex, and one where it does not, respectively:

$$\begin{aligned} \cos\alpha^{SL}_{(3,0)} &= \frac{\cos(\phi^{SL}_{(3,1)}) + \cos(\phi^{SL}_{(3,1)})\cos(\phi^{SL}_{(3,1)})}{\sin(\phi^{SL}_{(3,1)})\sin(\phi^{TL}_{(3,1)})} \\ \iff \frac{1}{2} &= \frac{\frac{-i}{\sqrt{3}\sqrt{4\alpha+1}} + \frac{-i}{\sqrt{3}\sqrt{4\alpha+1}}\frac{2\alpha+1}{4\alpha+1}}{\frac{2\sqrt{3\alpha+1}}{\sqrt{3}\sqrt{4\alpha+1}}\frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1}} \\ \iff \frac{1}{2} &= \frac{-i}{2\sqrt{\alpha}} \quad \Rightarrow\Leftarrow \end{aligned} \tag{B.1.3}$$

and

$$\begin{aligned} \cos\alpha^{TL}_{(2,1)} &= \frac{\cos(\phi^{TL}_{(3,1)}) + \cos(\phi^{TL}_{(3,1)})\cos(\phi^{TL}_{(3,1)})}{\sin(\phi^{TL}_{(3,1)})\sin(\phi^{TL}_{(3,1)})} \\ \iff \frac{2\alpha+1}{2\alpha} &= \frac{\frac{2\alpha+1}{4\alpha+1} + \frac{2\alpha+1}{4\alpha+1}\cdot\frac{2\alpha+1}{4\alpha+1}}{\frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1}\cdot\frac{2\sqrt{\alpha}\sqrt{3\alpha+1}}{4\alpha+1}} \\ \iff 1 &= 1 \quad \square. \end{aligned} \tag{B.1.4}$$

The full results for the left- and right-hand side of equation (6.1.1) for the different configurations of 2d and 3d angles are given below.

Although many of the LHS and the RHS values are not equal, it can be seen that there are only three types of results: $\frac{1}{2}$, $\frac{-i}{2\sqrt{\alpha}}$ and $\frac{2\alpha+1}{2\alpha}$. As a result, the solution is simply a matter of matching the mismatched values.

The method for obtaining the new equation for the angle $\alpha_{ij,kl}$ is the following: begin by eliminating one of the remaining 2 vertices not involved in the angle (i.e. i or j); then, reference the 3d angle on the edge between m and the remaining vertex (i.e. j or i, respectively) first. For instance, if we choose to eliminate i, then we must refer to the 3d angle on the edge mj first in the new equation, followed by mk and ml, in any order. In other words, all the angles involved are in the tetrahedron $\sigma(k)$, the first angle is hinged at the edge $\sigma(ikl)$ and the remaining ones are hinged at $\sigma(ijl)$ and $\sigma(ijk)$, i.e. the modified equation is

$$\cos\alpha^{L}_{ij,kl} = \frac{\cos\phi_{kl,i} + \cos\phi_{jl,i}\cos\phi_{jk,i}}{\sin\phi_{jl,i}\sin\phi_{jk,i}}, \tag{B.1.5}$$

where the "first 3d angle referenced" is intended to mean the isolated angle in the left-hand side of the sum in the numerator — in this case, $\cos\phi_{kl,i}$, as intended. Furthermore, the label L identifies the fact that

Table B.1: Possible left- (LHS) and right-hand side (RHS) combinations (up to the commutation of the second and third terms in ϕ) for the equation 6.1.1, along with their geometric interpretation: the 4-simplex σ; the resulting tetrahedron $\sigma(k)$; the 2d angle α appearing on the left-hand side; and the 3d angle ϕ combinations. The notation used has been explained earlier.

σ	$\sigma(k)$	α	ϕ	LHS	RHS
$(5,0)$	$(4,0)$	$\alpha^{SL}_{(3,0)}$	SSS	$\frac{1}{2}$	$\frac{1}{2}$
$(4,1)$	$(4,0)$	$\alpha^{SL}_{(2,1)}$	SSS	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{1}{2}$
$(4,1)$	$(3,1)$	$\alpha^{SL}_{(3,0)}$	SST	$\frac{1}{2}$	$\frac{-i}{2\sqrt{\alpha}}$
$(4,1)$	$(3,1)$	$\alpha^{SL}_{(2,1)}$	TSS	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{1}{2}$
$(4,1)$	$(3,1)$	$\alpha^{TL}_{(2,1)}$	TTT	$\frac{2\alpha+1}{2\alpha}$	$\frac{2\alpha+1}{2\alpha}$
$(3,2)$	$(3,1)$	$\alpha^{SL}_{(2,1)}$	SST	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{-i}{2\sqrt{\alpha}}$
$(3,2)$	$(3,1)$	$\alpha^{SL}_{(2,1)}$	TTT	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{2\alpha+1}{2\alpha}$
$(3,2)$	$(3,1)$	$\alpha^{TL}_{(2,1)}$	TSS	$\frac{2\alpha+1}{2\alpha}$	$\frac{1}{2}$
$(3,2)$	$(2,2)$	$\alpha^{SL}_{(3,0)}$	STT	$\frac{1}{2}$	$\frac{2\alpha+1}{2\alpha}$
$(3,2)$	$(2,2)$	$\alpha^{SL}_{(2,1)}$	STT	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{2\alpha+1}{2\alpha}$
$(3,2)$	$(2,2)$	$\alpha^{SL}_{(2,1)}$	TTS	$\frac{-i}{2\sqrt{\alpha}}$	$\frac{-i}{2\sqrt{\alpha}}$
$(3,2)$	$(2,2)$	$\alpha^{TL}_{(2,1)}$	TTS	$\frac{2\alpha+1}{2\alpha}$	$\frac{-i}{2\sqrt{\alpha}}$

this equation is intended for the Lorentzian case. The fact that this new equation corrects the mismatch between the results in the table below can be easily checked. For example, the angle $\alpha^{SL}_{(3,0)}$ in the $(3,2)$-simplex will now yield the combination TSS, yielding the equality $\frac{1}{2} = \frac{1}{2}$ $\square$.

The analogue to (6.1.2) corresponds to swapping the labels $j \leftrightarrow i$ in the equation above, or instead swapping $k \leftrightarrow l$, as in the Euclidean equation. The result is

$$\cos\alpha^{L}_{ji,kl} = \frac{\cos\phi_{kl,j} + \cos\phi_{il,j}\cos\phi_{ik,j}}{\sin\phi_{il,j}\sin\phi_{ik,j}} = \cos\alpha^{L}_{ij,lk}. \tag{B.1.6}$$

We further obtain

$$\begin{aligned} \mathcal{C}^{L}_{ij,kl}(\phi) &\equiv \cos\alpha^{L}_{ij,kl} - \cos\alpha^{L}_{ji,kl} \\ &= \frac{\cos\phi_{kl,i} + \cos\phi_{jl,i}\cos\phi_{jk,i}}{\sin\phi_{jl,i}\sin\phi_{jk,i}} - \frac{\cos\phi_{kl,j} + \cos\phi_{il,j}\cos\phi_{ik,j}}{\sin\phi_{il,j}\sin\phi_{ik,j}} \\ &= 0, \end{aligned} \tag{B.1.7}$$

and hence the first constraint for the Lorentzian case is

$${}^{L}\mathcal{C}^{\sigma}_{ee'}(\phi^{\tau}_{e}) \equiv \begin{cases} {}^{L}\mathcal{C}_{ij,kl} & , \quad e = \sigma(ijk), e' = \sigma(ijl) \\ 0 & , \quad \text{otherwise} \end{cases} = 0. \tag{B.1.8}$$

For the second constraint, we apply the Euclidean version to the all the possible Lorentzian tetrahedra, i.e. $(4,0)$, $(3,1)$ and $(2,2)$. The results are, respectively,

$$\begin{aligned} \mathcal{N}^{(4,0)}_{(3,0)}(A,\phi) &\equiv A_t - \sum_{t' \in \tau \setminus \{t\}} A_{t'} \cos(\phi^{\tau}_{tt'}) = A_{(3,0)} - \sum_{i=1}^{3} A_{(3,0)_i} \cos\phi^{SL}_{(4,0)_i} \\ &= \frac{\sqrt{3}}{4}a^2 - 3\frac{\sqrt{3}}{4}a^2\frac{1}{3} = 0 \quad \square, \end{aligned} \tag{B.1.9}$$

$$
\begin{aligned}
\mathcal{N}^{(3,1)}_{(3,0)}(A,\phi) &\equiv A_t - \sum_{t'\in\tau\setminus\{t\}} A_{t'}\cos(\phi^{\tau}_{tt'}) = A_{(3,0)} - \sum_{i=1}^{3} A_{(2,1)_i}\cos\phi^{SL}_{(3,1)_i} \\
&= \frac{\sqrt{3}}{4}a^2 - 3\frac{1}{4}\sqrt{4\alpha+1}a^2\left(\frac{-i}{\sqrt{3}\sqrt{4\alpha+1}}\right) = 1+i \quad \Rightarrow\Leftarrow,
\end{aligned}
\tag{B.1.10}
$$

$$
\begin{aligned}
\mathcal{N}^{(3,1)}_{(2,1)}(A,\phi) &\equiv A_t - \sum_{t'\in\tau\setminus\{t\}} A_{t'}\cos(\phi^{\tau}_{tt'}) = A_{(2,1)} - 2A_{(2,1)}\cos\phi^{TL}_{(3,1)} - A_{(3,0)}\cos\phi^{SL}_{(3,1)} \\
&= \frac{1}{4}\sqrt{4\alpha+1}a^2 - 2\frac{1}{4}\sqrt{4\alpha+1}a^2\frac{2\alpha+1}{4\alpha+1} - \frac{\sqrt{3}}{4}a^2\frac{i}{\sqrt{3}\sqrt{4\alpha+1}} = -1+i \\
&\Rightarrow\Leftarrow,
\end{aligned}
\tag{B.1.11}
$$

$$
\begin{aligned}
\mathcal{N}^{(2,2)}_{(2,1)}(A,\phi) &\equiv A_t - \sum_{t'\in\tau\setminus\{t\}} A_{t'}\cos(\phi^{\tau}_{tt'}) = A_{(2,1)}\left(1 - 2\cos\phi^{TL}_{(2,2)} - \cos\phi^{SL}_{(2,2)}\right) \\
&= \frac{1}{4}\sqrt{4\alpha+1}a^2\left(1 - 2\frac{(-1)}{4\alpha+1} - \frac{4\alpha+3}{4\alpha+1}\right) = 0 \quad \square.
\end{aligned}
\tag{B.1.12}
$$

Although the equation does not match for two cases, a simple change will solve that problem: $\pm i \to 1$. This is equivalent to taking the real part of the cosine and the absolute value of the entire sum. This change gives us the Lorentzian version of the second constraint:

$$
{}^{L}\mathcal{N}^{\tau}_{t}(A,\phi) \equiv A_t - \left|\sum_{t'\in\tau\setminus\{t\}} A_{t'}\left[\mathrm{Re}(\cos(\phi^{\tau}_{tt'})) + \mathrm{Im}(\cos(\phi^{\tau}_{tt'}))\right]\right| = 0. \tag{B.1.13}
$$

In order to obtain the full Lorentzian version of the area-angle Regge action, the only task left is to check whether the Euclidean formula giving the 4d angles in terms of the 3d ones requires modification. The Euclidean version is

$$
\cos\Theta_{ij} = \frac{\cos\phi_{ij,k} - \cos\phi_{ik,j}\cos\phi_{jk,i}}{\sin\phi_{ik,j}\sin\phi_{jk,i}} \tag{B.1.14}
$$

and one can easily check that this equation applies to the Lorentzian case, without warranting any modifications. Consequently, we can now write the Lorentzian version for the area-angle Regge action:

$$
\begin{aligned}
S_L\left(A_t,\phi^{\tau}_{e},\mu^{\sigma}_{ee'},\lambda^{\tau}_{t}\right) &= \sum_t A_t\epsilon_t(\phi)\sum_{\sigma}\sum_{ee'\in\sigma}\mu^{\sigma}_{ee'}\left({}^{L}\mathcal{C}^{\sigma}_{ee'}(\phi)\right) + \sum_{\tau}\sum_{t\in\tau}\lambda^{\tau}_{t}\left({}^{L}\mathcal{N}^{\tau}_{t}(A,\phi)\right), \\
\epsilon_t &= 2\pi - \sum_{\sigma\ni t}\theta^{\sigma}_{t} \\
&= 2\pi - \sum_{\sigma\ni\sigma(ij)}\arccos\left(\frac{\cos(\phi_{ij,k}) - \cos(\phi_{ik,j})\cos(\phi_{jk,i})}{\sin(\phi_{ik,j})\sin(\phi_{jk,i})}\right),
\end{aligned}
\tag{B.1.15}
$$

where the formula for the deficit angle ϵ_t requires no modification for the Lorentzian version, as explained above. As a final remark, it is also easy to check that the Euclidean version of

$$
\det G^{k}(\phi) = 0 \tag{B.1.16}
$$

remains the same for the Lorentzian version.