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Entangling quantum fields via a classical gravitational interaction

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Abstract. We consider the coupling of quantum fields to classical gravity in the formalism of ensembles on configuration space, a model that allows a consistent formulation of interacting classical and quantum systems. Explicit calculations show that there are solutions for which two initially independent quantum fields evolve into an entangled state, even though their interaction occurs solely via a common classical gravitational field. Thus in contrast to recent suggestions, an observed generation of entanglement would not provide a definitive test of the nonclassical nature of gravity.

1. Introduction

The publication of proposals for witnessing nonclassical features of gravity, by Bose *et al.* [1] and by Marletto and Vedral [2], has encouraged a discussion about the possibility of generating entanglement between quantum systems which only interact via a classical gravitational field, as well as new proposals for looking for evidence of quantum gravity in laboratory experiments [3, 4, 5, 6, 7, 8, 9]. Whether entanglement is possible under these circumstances depends on *which* hybrid model is used to describe the interaction of classical and quantum sectors [7]. While some hybrid models of classical-quantum interactions seem to exclude entanglement, other models, in particular the formalism of ensembles on configuration space [10], allow for it. Thus an observed generation of entanglement cannot provide a definitive test of the nonclassicality of gravity without additional assumptions concerning the nature of classical-quantum interactions.

The purpose of this paper is to provide fully relativistic calculations showing that (a) there are solutions for which two quantum fields are in an entangled state even though their interaction occurs solely via a common classical gravitational field, and (b) such entangled solutions can evolve from initially unentangled ones. These calculations are carried out using the formalism of ensembles on configuration space.

The paper is organized as follows. In section 2, we give a brief introduction to the basic aspects of the configuration ensemble approach. Section 3 is more general in nature, consisting of a brief overview on consistency requirements for models of quantum-classical interactions and a summary of previous results on entanglement production for some specific models. In section 4, we return to the configuration ensemble approach and discuss its application to quantum matter fields coupled to a classical gravitational field. We focus on the midisuperspace formulation of spherical gravity, in particular on the case where the quantum sector consists of two quantized

scalar fields. In section 5 we consider a perturbative approach and show that there are solutions for which the two quantum fields are in an entangled state, and in section 6 we discuss how entangled solutions can evolve from initially unentangled ones. Finally, in section 7, we provide some concluding remarks.

2. Ensembles on configuration space describing classical, quantum and mixed classical-quantum systems

The formalism of ensembles on configuration is a general framework that can describe classical, quantum and hybrid systems [10, 11, 12]. It forms a natural starting platform for several axiomatic approaches to reconstructing quantum theory [10, 13, 14, 15, 16, 17] and it can describe the coupling of ensembles of quantum fields to classical spacetimes [10, 11, 18, 19].

2.1. Classical, quantum, and mixed classical-quantum systems

Start from the assumption that the configuration of a physical system is an inherently statistical concept. The system will then be described by an ensemble of configurations, with probability density P, with $P \geq 0$ and $\int dx\, P(x,t) = 1$. To describe dynamics, introduce an *ensemble Hamiltonian* $H[P,S]$, where S is an auxiliary field that is canonically conjugate to P. The equations of motion take the form

$$\frac{\partial P}{\partial t} = \frac{\delta H}{\delta S}, \qquad \frac{\partial S}{\partial t} = -\frac{\delta H}{\delta P}, \tag{1}$$

where $\delta/\delta P$ ($\delta/\delta S$) denotes the functional derivative with respect to P (S).

The following ensemble Hamiltonians lead to equations that describe the evolution of quantum and classical non-relativistic particles of mass m:

$$H_C[P,S] = \int dx\, P\left[\frac{|\nabla S|^2}{2m} + V(x)\right], \qquad H_Q[P,S] = H_C[P,S] + \frac{\hbar^2}{4}\int dx\; P\frac{|\nabla \log P|^2}{2m}. \tag{2}$$

For example, the equations of motion derived from $H_Q[P,S]$ are given by

$$\frac{\partial P}{\partial t} + \nabla.\left(P\frac{\nabla S}{m}\right) = 0, \qquad \frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 P^{1/2}}{P^{1/2}} = 0 \tag{3}$$

while the equations of motion derived from $H_C[P,S]$ are the same as Eq. (3) but with $\hbar = 0$. The first equation in Eq. (3) is a continuity equation, the second equation is the classical Hamilton-Jacobi equation when $\hbar = 0$ and a modified Hamilton-Jacobi equation when $\hbar \neq 0$. Defining $\psi = \sqrt{P}\; e^{iS/\hbar}$, Eq. (3) takes the form

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi, \tag{4}$$

which is the usual form of the Schrödinger equation. It is straightforward to extend the formalism in a natural way to allow for mixed quantum-classical systems. A mixed quantum-classical ensemble Hamiltonian is given by [10, 11]

$$H_{QC}[P,S] = \int dq\, dx\, P\left[\frac{|\nabla_x S|^2}{2M} + \frac{|\nabla_q S|^2}{2m} + \frac{\hbar^2}{4}\frac{|\nabla_q \log P|^2}{2m} + V(q,x,t)\right]. \tag{5}$$

Here q denotes the configuration space coordinate of a quantum particle of mass m and x that of a classical particle of mass M, and $V(q,x,t)$ is a potential energy function describing the

quantum-classical interaction. The equations of motion for the joint probability density $P(q,x)$ and its conjugate $S(q,x)$ follow from H_{QC} as

$$\frac{\partial P}{\partial t} = -\nabla_q.\left(P\frac{\nabla_q S}{m}\right) - \nabla_x.\left(P\frac{\nabla_x S}{M}\right), \qquad \frac{\partial S}{\partial t} = -\frac{|\nabla_q S|^2}{2m} - \frac{|\nabla_x S|^2}{2M} - V + \frac{\hbar^2}{2m}\frac{\nabla_q^2 P^{1/2}}{P^{1/2}}. \quad (6)$$

These can be rewritten as a nonlinear Schrödinger equation for the 'hybrid' wave function $\psi = \sqrt{P}e^{iS/\hbar}$, with a similar nonlinear equation for the case of two classical particles. Such nonlinearity does not automatically lead to difficulties in either case, essentially because the form of classical observables is fundamentally different to that of quantum ones, see below.

2.2. Observables

The *state* of a system is determined by specifying P and S. *Observables* are defined as suitable functionals of P and S (arbitrary functionals $A[P,S]$ are not necessarily observables because these have to satisfy certain mild requirements, see [10] for a detailed discussion). Given observables $A[P,S]$ and $B[P,S]$, define their Poisson bracket

$$\{A,B\}_{PB} = \int dx \left(\frac{\delta A}{\delta P}\frac{\delta B}{\delta S} - \frac{\delta A}{\delta S}\frac{\delta B}{\delta P}\right). \quad (7)$$

This gives us an *algebra of obervables*. A critical physical distinction between classical and quantum systems (or classical and quantum components of a composite hybrid system) is that they have quite different sets of observables, and distinct algebras for these observables [10].

For a purely classical configuration space labelled by position x, the classical observable C_f corresponding to the phase space function $f(x,k)$ (where k is the momentum) is defined by the functional

$$C_f[P,S] := \int dx\, P\, f(x, \nabla_x S). \quad (8)$$

For a purely quantum configuration space labelled by the possible outcomes q of some complete basis set $\{|q\rangle\}$ of a Hilbert space $\mathcal{H}$ (i.e., $\int dq\, |q\rangle\langle q| = \hat{\mathbb{1}}$, with integration replaced by summation for discrete ranges of q), the quantum observable $Q_{\hat{M}}$ corresponding to the Hermitian operator $\hat{M}$ is defined by the functional

$$Q_{\hat{M}}[P,S] := \langle\psi|\hat{M}|\psi\rangle, \quad (9)$$

where $|\psi\rangle \in \hbar$ is the wave function defined via $\langle q|\psi\rangle = \sqrt{P(q)}e^{iS(q)/\hbar}$.

Evaluating the Poisson bracket of any pair of classical observables C_f, C_g or quantum observables $Q_{\hat{M}}, Q_{\hat{N}}$ via Eq. (7) yields

$$\{C_f, C_g\} = C_{\{f,g\}}, \qquad \{Q_{\hat{M}}, Q_{\hat{N}}\} = Q_{[\hat{M},\hat{N}]/(i\hbar)} \quad (10)$$

where $\{f,g\} = \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial k_i} - \frac{\partial f}{\partial k_i}\frac{\partial g}{\partial x_i}\right)$ denotes the usual phase space bracket and $[\hat{M},\hat{N}]$ the usual commutator. Thus *Eq. (8) is an isomorphism between the algebra of observables C_f on configuration space and the algebra of observables f on classical phase space, and Eq. (9) is an isomorphism between the algebra of observables $Q_{\hat{M}}$ on configuration space and the algebra of quantum observables $\hat{M}$*. As the Poisson bracket properties (10) remain unchanged for mixed classical-quantum systems of the type described by the ensemble Hamiltonian of Eq. (5), this statement remains true even under interactions between the classical and quantum components. In particular, *the classical/quantum distinction is always maintained.*

2.3. Entanglement

Two ensembles with respective configuration spaces X and Y are defined to be *independent* if $P(x, y)$ and $S(x, y)$ satisfy $P(x, y) = P_X(x)P_Y(y)$ and $S(x, y) = S_X(x) + S_Y(y)$ (with the latter only required to hold up to some additive constant) [10, 11]. For quantum ensembles, note that independence is equivalent to a factorisable wave function $\psi = \sqrt{P}e^{iS/\hbar}$, and hence *any two quantum ensembles are either independent or entangled.*

The concept of entanglement remains meaningful in the general case [10]. However, it is important to note that the notion of 'entanglement' referred to here is not in the strong sense of Bell inequality violation, but in Schrödinger's original weaker sense that the properties of a joint ensemble cannot be decomposed into properties of the individual ensembles [20].

This can be understood by looking at a simple classical example. Consider a classical joint ensemble, corresponding to two classical particles described by respective configuration spaces X and Y, with probability density $P(x, y)$ and conjugate quantity $S(x, y)$. The product of two classical phase space functions $f(x, p_x)$ and $g(y, p_y)$ is itself a classical phase space function and the expectation value of this product corresponds to the classical observable [10]

$$C_{fg} = \langle fg \rangle = \int dxdy\, P(x, y)\, f(x, \partial_x S)\, g(y, \partial_y S). \tag{11}$$

Now, there is clearly a trivial hidden variable for any such observable. In particular, defining $\lambda := [x, y, S(x, y)]$, $P(\lambda) := P(x, y)$, $F(\lambda) := f(x, \partial_x S)$, and $G(\lambda) := g(y, \partial_y S)$, one has

$$\langle fg \rangle = \int d\lambda\, P(\lambda)\, F(\lambda)\, G(\lambda). \tag{12}$$

Hence, no Bell inequality can be violated via such observables [21].

Nevertheless, if the independence condition $S(x, y) = S_X(x) + S_Y(y)$ is *not* satisfied, then the 'hidden value' of the observable $f(x, p_x)$ for the *first* particle, i.e., $F(\lambda)$, will in general depend on the position of the *second* particle, via $p_x = \partial_x S(x, y)$. That is, while knowledge of the position and momentum of the first particle at a given time is sufficient to determine all observables for the particle at that time, it will not be sufficient to determine them at any later time: one needs to know the evolution of the joint quantity $\partial_x S(x, y)$. Moreover, if one locally perturbs the position of the second particle, from y to y', the corresponding perturbation of $S(x, y)$ to $S(x, y')$ will typically perturb the value of p_x in this model. Hence, *a kind of nonlocality, or inseparability, can be associated even with classical configuration space ensembles.* We will, by analogy with Schrödinger's original discussion [20], refer to this property as 'entanglement'. This leads to the following general definition which applies to *all* configuration space ensembles [10]:

> *A joint ensemble is entangled with respect to the joint configuration space $X \times Y$ if and only if $S(x, y) \not\equiv S_X(x) + S_Y(y)$ (up to some additive constant).*

For further details on entanglement for quantum, classical and hybrid systems, see Ref. [10].

3. On models of quantum-classical interactions, consistency requirements, and the generation of entanglement in some particular models

The task of finding a physically consistent approach for modelling interactions between classical and quantum systems is highly non-trivial. Thus there are many possible models of classical-quantum interactions in the literature, and these typically have mathematical or physical difficulties associated with them [7, 10, 11].

3.1. Consistency requirements

A number of consistency requirements have been proposed to evaluate whether a particular approach to classical-quantum interactions provides a viable model or not (see Ref. [10] for a discussion on various attempts at solving this problem). Almost all of the approaches which have been proposed run into difficulties with these requirements, but two types of models have been shown to satisfy them. Consistency of the model based on ensembles on configuration space is shown in Refs. [10, 12]; for models based on a mean-field approach [22, 23], it follows from the consistency of the model due to Elze, who showed that his model satisfies the requirements [24] and therefore that all equivalent mean-field models are consistent. We will not go over the consistency conditions here but instead refer the reader to these publications and to the references therein.

The particular approach considered by Bose *et al.* [1] and Marletto and Vedral [2] is a third type of model, based on formally embedding a classical system into a diagonal basis of some quantum system; i.e., equivalent to describing classical-quantum interactions via Koopman-type dynamics [25, 26, 27]. While this approach has some interesting properties, it has been shown that it already fails for simple examples, thus some of the basic consistency requirements proposed in the literature are not satisfied. For example, classical observables remain classical for a limited class of interactions only, which do not include some of the textbook examples of experimental setups like the standard Stern-Gerlach measurement interaction [28]. Peres and Terno have further shown that this approach does not reproduce the correct classical limit for classical-quantum oscillators, and indeed may result in a runaway increase of the classical oscillator amplitude [27, 29]. We see then that the approach already runs into serious difficulties with the simple problem of classical-quantum oscillators, and one would expect even greater difficulties if one were to fully work out the details of the interaction of quantum matter with a classical gravitational field using this model (which, to the best of our knowledge, has not been done yet). Further, predictions made by such models in the limit of perturbative Newtonian gravity appear to conflict with observational data [30].

3.2. Previous analyses of entanglement in mixed classical-quantum systems

We restrict our discussion of previous analyses of entanglement to the first two models discussed above as these are known to satisfy consistency requirements. The creation of entanglement for two quantum subsystems via a classical interaction has already been discussed in both cases.

For the case of mixed classical-quantum systems described by ensembles on configuration space, it has been shown that one can construct explicit examples in which entanglement between quantum subsystems is created via an interaction with a classical subsystem [7]. The results presented in sections 5 and 6 of this paper complement these earlier results with further examples involving two quantum fields which are entangled via a common classical gravitational field.

For the approach of Elze, detailed calculations are available for the entanglement dynamics of a system of two qubits and one classical oscillator [31]. In particular, it is shown that the concurrence remains constant if the two qubits have some initial entanglement and therefore that no additional entanglement is generated in this model. This result is consistent with the observation in Ref. [7] concerning the absence of entanglement creation in mean-field approaches, of which the Elze model is a particular formulation.

These results regarding entanglement creation reflect fundamental differences between the two models [10]. In particular, in a mean-field approach the classical particle follows a deterministic trajectory in phase space, rather than being described by an ensemble on configuration space. Thus, unlike the configuration ensemble approach, the mean-field approach cannot couple quantum fluctuations into correlated classical observables.

This reinforces a point made earlier, that the possibility of entanglement generation via a classical mediator depends on the particular approach used to model the interaction between

classical and quantum systems. While some hybrid models of classical-quantum interactions seem to exclude entanglement, other models, in particular the formalism of ensembles on configuration space, allow for it.

4. The coupling of scalar quantum fields to classical gravity in the spherically symmetric case

The analysis of section 2 for interacting particles can be extended to interacting fields, in particular to the coupling of scalar quantum fields to classical gravity. A detailed description of the formalism is given in [10]. Here we consider only the case of spherically symmetric spacetimes and the corresponding midisuperspace formulation of general relativity known as spherical gravity. For the case of spherical symmetry, the line element may be written in the form

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \Lambda^2\left(dr + N_r dt\right)^2 + R^2 d\Omega^2, \tag{13}$$

where the lapse function N and the shift function N_r are functions of the radial coordinate r and the time coordinate t. Thus the configuration space for the gravitational field consists of two fields, R and Λ. Spherically symmetric gravity is discussed in detail in a number of papers, mostly in reference to the canonical quantization of black hole spacetimes [32, 33, 34, 35, 36].

4.1. The case of vacuum gravity

The most direct way of introducing a classical configuration space ensemble for gravity is to start from the Einstein-Hamilton-Jacobi (EHJ) equation [37, 38]. For the case of spherical gravity without matter fields, it takes the form $H_{\Lambda R} = 0$ with

$$H_{\Lambda R} = -\frac{1}{R}\frac{\delta S}{\delta R}\frac{\delta S}{\delta \Lambda} + \frac{\Lambda}{2R^2}\left(\frac{\delta S}{\delta \Lambda}\right)^2 + V, \qquad V = \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} \tag{14}$$

where we have set $c = G = \hbar = 1$. In spherical gravity, the momentum constraints of the full theory are replaced by a single (radial) diffeomorphism constraint, $(\delta S/\delta R)R' - \Lambda\left(\delta S/\delta\Lambda\right)' = 0$, where primes indicate derivatives with respect to r [38]. We will require that S be invariant under diffeomorphisms so that it automatically solves the momentum constraint.

An appropriate ensemble Hamiltonian for spherically symmetric gravity is given by

$$\mathcal{H} = \int dr\, N \int \mathrm{D}R\,\mathrm{D}\Lambda\, P H_{\Lambda R}. \tag{15}$$

where P is a probability density function (which, like S, is assumed to satisfy the diffeomorphism constraint) and $\mathrm{D}R\,\mathrm{D}\Lambda$ is an appropriate measure [10]. Assuming the constraints $\frac{\partial S}{\partial t} = \frac{\partial P}{\partial t} = 0$, the ensemble Hamiltonian of Eq. (15) leads to two equations for S and P, the EHJ equation of spherical gravity,

$$\int dr\, N\left[-\frac{1}{R}\frac{\delta S}{\delta R}\frac{\delta S}{\delta \Lambda} + \frac{\Lambda}{2R^2}\left(\frac{\delta S}{\delta \Lambda}\right)^2 + V\right] = 0, \tag{16}$$

and the continuity equation

$$\int dr\, N\left[\frac{\delta}{\delta R}\left(P\frac{1}{R}\frac{\delta S}{\delta \Lambda}\right) + \frac{\delta}{\delta \Lambda}\left(P\frac{1}{R}\frac{\delta S}{\delta R} - P\frac{\Lambda}{R^2}\frac{\delta S}{\delta \Lambda}\right)\right] = 0. \tag{17}$$

4.2. The addition of quantum scalar fields

The ensemble Hamiltonian of a hybrid system where matter is in the form of n minimally coupled quantized radially symmetric scalar fields ϕ_a of mass m is given by

$$\mathcal{H}_{\phi\Lambda R} = \int dr \int \mathrm{D}\phi\mathrm{D}\Lambda\mathrm{D}R\; P\, N\left[H^C_{\phi\Lambda R} + \frac{1}{8\Lambda R^2}\sum_{a=1}^{n}\left(\frac{\delta \log P}{\delta \phi_a}\right)^2\right], \tag{18}$$

where

$$H^C_{\phi\Lambda R} = H^C_{\Lambda R} + \sum_{a=1}^{n}\left[\frac{1}{2\Lambda R^2}\left(\frac{\delta S}{\delta\phi_a}\right)^2 + \frac{R^2}{2\Lambda}\phi_a'^2 + \frac{\Lambda R^2 m_a^2}{2}\phi_a^2\right], \tag{19}$$

is a purely classical term which now includes the coupling to scalar fields ϕ_a and the last term in Eq. (18) is an additional, non-classical term that must be included in the ensemble Hamiltonian when the scalar field is quantized (recall we have set $\hbar = 1$). Assuming again constraints $\frac{\partial S}{\partial t} = \frac{\partial P}{\partial t} = 0$, the corresponding equations are

$$\int dr\, N\left[H^C_{\phi\Lambda R} - \frac{1}{2\Lambda R^2}\sum_{a=1}^{n}\left(\frac{1}{\sqrt{P}}\frac{\delta^2\sqrt{P}}{\delta\phi_a^2}\right)\right] = 0, \tag{20}$$

and the continuity equation

$$\int dr\, N\left[\frac{\delta}{\delta R}\left(P\frac{1}{R}\frac{\delta S}{\delta\Lambda}\right) + \frac{\delta}{\delta\Lambda}\left(P\frac{1}{R}\frac{\delta S}{\delta R} - P\frac{\Lambda}{R^2}\frac{\delta S}{\delta\Lambda}\right) - \sum_{a=1}^{n}\frac{\delta}{\delta\phi_a}\left(P\frac{1}{\Lambda R^2}\frac{\delta S}{\delta\phi_a}\right)\right] = 0. \tag{21}$$

5. Black hole with two scalar quantum fields in spherical gravity: entangled solutions

We now apply the formalism presented in the previous section to the case of two quantized scalar fields, ϕ_1 and ϕ_2, in the space-time of a classical black hole. We assume that the quantum fields act as a perturbation to the space-time; i.e., that the contribution to the gravitational field from the mass of the black hole is much larger than that of the quantum matter fields. Under these circumstances, it is appropriate to search for an approximate perturbative solution of Eqs. (20) and (21) based on an expansion in powers of ϕ_a [39]. The advantage of using such an approach is that it is possible to solve the equations iteratively, term by term, as is clear from the equations below. We will use the notation of [39] where $S^{(n)}$ stands for a functional of order $(\phi_a)^n$. While the term $S^{(0)}$ can be chosen freely, the higher order terms depend on the previous ones.

To carry out the calculation, it will be convenient to write the expression for $P[R,\Lambda,\phi_1,\phi_2]$ in the form

$$P = e^{-\left(F^{(0)}[R,\Lambda]\right)}\, e^{-\left(\sum_{n>0} F^{(n)}[R,\Lambda,\phi_1,\phi_2]\right)} =: P_A[R,\Lambda]\, P_B[R,\Lambda,\phi_1,\phi_2]. \tag{22}$$

P_A depends on the gravitational degrees of freedom only while P_B depends on both gravitational and scalar field degrees of freedom. Furthermore, we will require that, *up to order* $(\phi_a)^2$,

$$\frac{\delta P_B}{\delta\phi_a} = -\frac{\delta F^{(2)}}{\delta\phi_a}P_B, \qquad \frac{\delta P_B}{\delta h_{ij}} = -\frac{\delta F^{(2)}}{\delta h_{ij}}P_B, \qquad \frac{1}{\sqrt{P_B}}\frac{\delta^2\sqrt{P_B}}{\delta\phi_a^2} = -\frac{1}{2}\frac{\delta^2 F^{(2)}}{\delta\phi_a^2} + \frac{1}{4}\left(\frac{\delta F^{(2)}}{\delta\phi_a}\right)^2, \tag{23}$$

(note the terms on the right of the last equation are of order $(\phi_a)^0$ and $(\phi_a)^2$) respectively).

Our ansatz then is that the odd terms vanish, i.e., $F^{(1)} = F^{(3)} = 0$, so that P_B is to a first approximation a *Gaussian functional* with respect to the ϕ_a. While not an essential assumption, this choice seems physically reasonable as it implies a solution of the quantum sector that is in some respect close to the simplest solution for quantum field theory in curved space time (i.e., the ground state functional). The expression $\delta^2 F^{(2)}/\delta\phi_a^2$ needs to be regularized (such a term appears also in solutions of the Schrödinger functional equation). We will not consider the regularization problem here, we will simply assume that this term has been regularized and that it is finite. We will assume $\delta^2 F^{(2)}/\delta\phi_1^2 = \delta^2 F^{(2)}/\delta\phi_2^2$ and introduce the notation $C_F[R,\Lambda] := \frac{1}{2}\delta^2 F^{(2)}/\delta\phi_a^2$ for this term.

We now give explicit solutions for $S^{(n)}$ for the first two terms in the expansion. This is already sufficient to demonstrate the existence of entanglement. The first equation in the expansion is

$$\left[-\frac{1}{R}\frac{\delta S^{(0)}}{\delta R}\frac{\delta S^{(0)}}{\delta \Lambda}+\frac{\Lambda}{2R^2}\left(\frac{\delta S^{(0)}}{\delta \Lambda}\right)^2+V\right]+\sum_{a=1}^{2}\left\{\frac{1}{2\Lambda R^2}\left(\frac{\delta S^{(1)}}{\delta \phi_a}\right)^2-\frac{C_F}{8\Lambda R^2}\right\}=0. \tag{24}$$

We choose the $S^{(0)}$ that solves the classical EHJ equation for a black hole; i.e, that makes the terms in square brackets equal to zero. The solution is well known [32, 36]. Thus, to zeroth order, we are dealing with a black hole space-time. With this choice of $S^{(0)}$, it is straightforward to find a solution for $S^{(1)}$,

$$S^{(1)}=\int dr\,\frac{\phi_1+\phi_2}{2}\sqrt{\frac{C_F}{2}}\;+\;C^{(1)}[R,\Lambda], \tag{25}$$

where $C^{(1)}$ is an arbitrary functional of the gravitational degrees of freedom. The next equation in the expansion is

$$-\frac{1}{R}\left(\frac{\delta S^{(0)}}{\delta R}\frac{\delta S^{(1)}}{\delta \Lambda}+\frac{\delta S^{(0)}}{\delta \Lambda}\frac{\delta S^{(1)}}{\delta R}\right)+\frac{\Lambda}{R^2}\frac{\delta S^{(0)}}{\delta \Lambda}\frac{\delta S^{(1)}}{\delta \Lambda}+\sum_{a=1}^{2}\left\{\frac{1}{2\Lambda R^2}\frac{\delta S^{(1)}}{\delta \phi_a}\frac{\delta S^{(2)}}{\delta \phi_a}\right\}=0. \tag{26}$$

The equation is linear in $S^{(2)}$. Except for $\delta S^{(2)}/\delta\phi_a$, all terms are known and they depend on $\phi_1+\phi_2$ only, so it is straightforward to solve for $S^{(2)}$. It is given by

$$\begin{aligned} S^{(2)} \;=\; & \frac{1}{2}\int dr\,(\phi_1+\phi_2)\left[-\frac{1}{R}\left(\frac{\delta S^{(0)}}{\delta R}\frac{\delta S^{(1)}}{\delta \Lambda}+\frac{\delta S^{(0)}}{\delta \Lambda}\frac{\delta S^{(1)}}{\delta R}\right)+\frac{\Lambda}{R^2}\frac{\delta S^{(0)}}{\delta \Lambda}\frac{\delta S^{(1)}}{\delta \Lambda}\right]\frac{2\Lambda R^2}{\sqrt{C_F/2}} \\ & +C^{(2)}[R,\Lambda,\phi_1-\phi_2] \end{aligned} \tag{27}$$

where $C^{(2)}=[R,\Lambda,\phi_1-\phi_2]$ is a quadratic but otherwise arbitrary functional of $\phi_1-\phi_2$. *Notice that in general* $S^{(2)}\neq S_1^{(2)}[R,\Lambda,\phi_1]+S_2^{(2)}[R,\Lambda,\phi_2]$*, which implies the entanglement of* ϕ_1 *and* ϕ_2, as per the discussion in section 2.3. This is the main result of this section.

The next term in the expansion is discussed in Ref. [40]. A perturbative solution requires also solving the continuity equation, Eq. (20), to the same order (the relevant equations are given in Ref. [40]). We do not carry out this step here, as the main purpose of the exercise, which is to show the existence of entangled states, is already accomplished with the solution of the EHJ equation.

6. The emergence of time and entanglement

Time did not play a role in the formalism used in the previous section. However, one would like to see whether entanglement can arise when such a hybrid system evolves. This *is* possible: since the gravitational field is treated classically, one may introduce a well defined *gravitational time* and derive a time-dependent equation for the quantum fields. Although a solution $S[\Lambda,R]$ of Eq. (16) is a functional of Λ and R, the two-dimensional space-time of spherical gravity can always be reconstructed by means of the rate equations [10, 33]

$$\dot{R}=-\frac{N}{R}\frac{\delta S}{\delta \Lambda}+N_r R',\qquad \dot{\Lambda}=-\frac{N}{R}\frac{\delta S}{\delta R}+\frac{\Lambda}{R^2}\frac{\delta S}{\delta \Lambda}+(\Lambda N_r)', \tag{28}$$

where N is the lapse function and N_r is the shift function in Eq. (13). The introduction of a gravitational time in this way has been discussed in the context of the semi-classical approximation to quantum geometrodynamics [38, 41, 42]).

We once more set $S = S_A[R,\Lambda] + S_B[R,\Lambda,\phi_a]$, $P = P_A[R,\Lambda]\,P_B[R,\Lambda,\phi_a]$, but instead of a perturbative expansion, we introduce a suitable physical approximation. We choose S_A and P_A to be black hole solutions of the EHJ [32, 36] and continuity equations. Then, defining $\Psi[R,\Lambda,\phi_a;t) := \sqrt{P_B}\exp(iS_B)$ and assuming a weak fields limit (i.e., $\delta S_A/\delta R >> \delta S_B/\delta R$ and $\delta S_A/\delta\Lambda >> \delta S_B/\delta\Lambda$, see [40]), we derive the non-linear Schrödinger functional equation

$$i\hbar\dot{\Psi} = \hat{H}_f\Psi = \int dr\left[\sum_{a=1}^{2}\left\{\frac{1}{2\Lambda R^2}\frac{\delta^2}{\delta\phi_a^2} + \frac{R^2}{2\Lambda}(\phi_a')^2 + \frac{\Lambda R^2 m^2}{2}\phi_a^2\right\} + \Delta\right]\Psi, \tag{29}$$

where the non-linear correction term Δ is given by

$$\Delta = -\frac{1}{R}\frac{\delta S_B}{\delta R}\frac{\delta S_B}{\delta\Lambda} + \frac{\Lambda}{2R^2}\left(\frac{\delta S_B}{\delta\Lambda}\right)^2, \tag{30}$$

which may also be written in terms of Ψ and $\bar{\Psi}$. The term Δ is a new "correction" term that distinguishes the time evolution as evaluated by quantum field theory in curved space-time (where this term is absent) from the time evolution in the theory of ensembles on configuration space. We can now consider the following question: suppose that initially the two quantum fields ϕ_1 and ϕ_2 are *not* entangled. Can this non-linear time-dependent functional Schrödinger equation lead to their entanglement? The crucial point here is that there is no reason to believe that the term Δ will preserve non-entanglement of states, as it is quadratic in the functional derivatives of $S_B[R,\Lambda,\phi_1,\phi_2]$ with respect to R and Λ. One can argue as follows. Consider calculating the time evolution of the wavefunctional Ψ after an infinitesimally small time interval δt. If the initial state is not entangled so that

$$S_B[R,\Lambda,\phi_1,\phi_2;t=0) = S_B^1[R,\Lambda,\phi_1;t=0) + S_B^2[R,\Lambda,\phi_2;t=0), \tag{31}$$

the initial Δ will have in general mixed terms in ϕ_1 and ϕ_2 which will generically lead to entanglement, so that one would expect at time δt that

$$S_B[R,\Lambda,\phi_1,\phi_2;t=\delta t) \neq S_B^1[R,\Lambda,\phi_1;t=\delta t) + S_B^2[R,\Lambda,\phi_2;t=\delta t). \tag{32}$$

This suggests then that the interaction of ϕ_1 and ϕ_2 via a common gravitational field will in general create entanglement between them.

7. Discussion

Our main result is that entanglement between quantum fields may be generated via a classical gravitational interaction (section 6). This result is based on the configuration ensemble formalism (which is able to describe the coupling of quantum and classical systems more generally), with explicit calculations made for the case of black-hole spacetimes in spherical gravity under a weak-field approximation. The effective evolution equation for the quantum fields, Eq.(29), is defined with respect to a gravitational time.

The above result strongly supports the arguments made in [7], that observation of entanglement *per se*, in the experiments proposed by Bose *et al.* and by Marletto and Vedral [1, 2], does not necessarily imply that gravity is nonclassical in nature. Such an observation can only rule out some classical models of gravity, such as Koopmanian and mean-field models [7], but not all. In particular, entanglement appears to be ubiquitous in the configuration-ensemble model, as exemplified by the spherical gravity solutions in section 5 and the approximate evolution equation in section 6.

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