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SUPERUNIFIED THEORIES BASED ON THE GEOMETRY OF LOCAL (SUPER-) GAUGE INVARIANCE†

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ABSTRACT

Based on the geometry of local (super-) gauge invariance, a general theoretical framework for constructing superunified theories is given. The main ingredients of this approach are (a) a general method of constructing invariants for superunified theories, and (b) the concept of "constrained" geometries for the description of gravity and supergravity as well as the choice of their gauge groups. It is argued that any unified theory must contain gravity, and then, to retain the invariances of pure gravity theory, such a theory must be a superunified one. The general formalism is then applied, respectively, to pure gravity, gravity coupled to Yang-Mills, simple supergravity, and SO(2)-extended supergravity. Important properties of these theories are discussed in detail.

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I. INTRODUCTION

Supersymmetries as invariance groups of particle physics may be considered both from the global and local points of view. Since their introduction in 4-dimensional space-time,[1-5] a number of attempts, suggestions, or conjectures about the relevance of these symmetries as global or local invariance groups have been made.[1-16] More recently, locally supersymmetric (supergravity) actions have been constructed by several authors.[17-19]

In this paper we propose to make a detailed study of superunified theories from the point of view of the geometry of local (super-) gauge invariance.[20] The merits of this approach have been argued and stressed several times before:[6,10,14,16,20] Supersymmetry groups are direct generalizations of Lie groups and contain fermi-bose symmetry. Given that ordinary local symmetries are gauge symmetries associated with Lie groups, it is natural to expect that local supersymmetries are (super-) gauge symmetries associated with supergroups. Superunified theories are thus the next step in the generalization of non-abelian gauge theories.

To motivate their relevance, it will be recalled[10,21] that exact local (Lie) gauge invariance is defined over Minkowski space and endows space-time with a richer structure than that implied by special relativity alone. The combined geometrical structure known as a fiber bundle provides a unified picture of theories based on local internal gauge symmetries. To proceed further, it is desirable to include gravity in such a scheme (see below for the underlying reasons). Despite the fact that a gauge theory of gravitation presents a number of novel features not found in gauge theories of internal symmetry, the local gauge principle is especially suited for the couplings of matter (gauge fields at least) to gravity which is already a local theory. Such a unification has been carried out elsewhere.[14] Here we want to refine the idea of (super-) unification still further by pointing out the intimate relation between the space-time of general relativity and exact super (-Lie) gauge invariance. As mentioned above, exact local (Lie) gauge invariance is ideally suited to the space-time of special relativity. This is because the only gauge invariance involved here is that associated with internal symmetries, and there can be no conflict between this and the global Lorentz transformations of special relativity. However, once one proceeds to the space-time of general relativity and views gravitation as a gauge theory, a conflict between the gauge invariance of the gravity gauge group and that of internal symmetries can in fact arise. It will be explained in section IV that a geometric description of gravitation involves not an ordinary fiber bundle but a "constrained" one. As a result some gauge fields transform according to a non-linear realization of the (super-) gauge group. In arbitrary couplings of matter fields to gravity, aside from the fact that such couplings are not purely geometrical, the invariance under non-linear transformations of the gravity gauge group is almost always destroyed. It is only when the matter and gravity fields together form a representation of a supersymmetry group that the invariance under the non-linear transformations can be regained. One is therefore led to conclude that just as local (Lie) gauge invariance is naturally suited to the space-time of special relativity, local super (-Lie) gauge invariance is naturally linked with the space-time of general relativity. Conversely, gauge theories based on supergroups which have homogeneous Lorentz group as a subgroup necessarily involve gravitation.

Our geometrical point of view provides other reasons in support of a superunified scheme. An important criterion for a unified theory is that all fundamental fields in the theory appear on the same footing. Such is not the case in arbitrary couplings of matter fields to gravity. On the other hand in superunified theories it is possible to interpret all fields as gauge fields transforming according to the adjoint representation of the supergauge group. This removes the ambiguity of assigning fundamental fields to various representations of the unbroken symmetry group. As a by-product of this, one finds that, e.g., in supergravity theory the mysterious transformation properties assigned[17-19] to various fields under local supersymmetry transformations are just local (super-) gauge transformations for the independent fields of the supergroup. The origin of the non-linear transformations are also understood to be due to the geometrical constraints of the theory.

From a more practical point of view, one can offer at least two reasons in favor of a geometrically superunified theory. Firstly, it is found that arbitrary couplings of matter fields to gravity destroy the renormalizability of the theory. Since in such theories some of the gauge symmetry of the pure gravity theory is also destroyed, one may argue that the non-renormalizability may be related to the loss of the symmetry of the original theory. The symmetry which could (hopefully!) render the theory renormalizable can be regained only within the context of a superunified theory. Moreover, in many unified theories of weak, electromagnetic, and strong interactions, to maintain Baryon number conservation, one encounters mass scales of the order of Plank mass $m_h \sim 10^{19}$ Gev. Since at such scales, gravitational effects can no longer be ignored, a unified theory of this kind must of necessity include gravitation. Then for reasons enumerated above, it must be a geometrically superunified theory.

This paper is organized as follows: In section II a number of useful mathematical concepts are reviewed for reader's convenience. A more detailed modern presentation has been given elsewhere.[16] Section III is devoted to methods of constructing invariants. Starting from the action integrals of reference 20, in which the dependence on gauge fields comes only through the components of curvature tensor, we proceed to a general method of constructing invariants in a fiber bundle. For pure gravity and simple supergravity they reduce, as they must, to those given in reference 20. Such general invariants appear to be indispensible when extended supergravity theories are considered.[22]

In section IV we introduce the concept of a "constrained" geometry for the purpose of matching the number of degrees of freedom allowed in a geometrical theory to those required by physics. The specific theories dealt with in this paper are those which involve gravity. But the concept is potentially useful in other contexts as well.[22] The constrained geometry point of view provides a more direct justification for the geometric description of gravity and supergravity presented in reference 20. It shows, in particular, how the invariants constructed in section III can be constrained to describe these theories.

In section V, the general developments of sections II-IV are applied successively to pure gravity, gravity coupled to Yang-Mills, simple supergravity, and SO(2)-extended supergravity. For gravity and simple supergravity it is shown that in this approach both the actions and the equations of motion depend fundamentally on the concept of connection (gauge potential). As a result, one gets a more general theory of gravity than that of Einstein's. It reduces to Einstein's theory when one assumes the existence of an inverse vierbein or, equivalently, of a metric. Section VI is devoted to a discussion of results and conclusions.

II. MATHEMATICAL PRELIMINARIES

In this section we discuss a number of mathematical topics which are necessary to describe our general formalism. Since the modern extension of various notions of differential geometry to superspaces with bose and fermi coordinates has been treated elsewhere,[16] we will not distinguish between Lie groups and supergroups or Lie algebras and superalgebras and describe them all in the unified notation of reference 16.

Consider a continuous group or supergroup G. Let L be the Lie (super)-algebra of G and $\{X_A\}$ a basis in L satisfying the generalized commutation relations

$$\begin{aligned} [X_A, X_B\} &\equiv X_A X_B - (-)^{\sigma_A \sigma_B} X_B X_A \\ &= f_{AB}^{\ \ C} X_C \end{aligned} \qquad (2.1)$$

where σ_A is the "grade" of the generator X_A. In this paper we will be mainly concerned with cases in which $\sigma_A = 1$ if A refers to a fermion and $\sigma_A = 0$ if A refers to a boson.

Let H be a Lie subgroup of G. Then one can write

$$L = L_0 \oplus L_1$$

where L_0 is the Lie algebra of H and L_1 is the generator of elements homeomorphic to the quotient space G/H. This decomposition has the well known property that

$$[L_0,L_0] \subset L_0$$
$$[L_0,L_1] \subset L_1 \qquad (2.2)$$
$$[L_1,L_1\} \subset L_0$$

For semi-simple algebras, the most natural metric of L is the Killing metric[23]

$$g_{AB} = (-)^{\sigma_A \sigma_B} g_{BA} = \sum_{C,D} - (-)^{\sigma_C} f^{\ D}_{AC} f^{\ C}_{BD} \qquad (2.3)$$

It satisfies the identity

$$f^{\ B}_{DA} g^{CD} + f^{\ C}_{DA} g^{DB} = 0 \qquad (2.4)$$

For algebras such as the Poincaré algebra, which contain abelian invariant subalgebras, the metric (2.3) becomes degenerate, and one will have to make recourse to special methods such as Inönü-Wigner contraction[24] to implement the physical applications we have in mind.

Our geometrical approach is most conveniently realized in terms of a fiber bundle. Consider a fiber bundle $\underline{P}(G,M)$ with structure group G and a base manifold M. Let the set $\{y^I\} = \{x^\mu,\theta^\alpha\}, \mu = 0,\ldots,m; \alpha = 1,\ldots,n$ be a coordinate system in M, where x^μ's are bose-type and θ^α are fermi-type. To describe the geometry of $\underline{P}$ it is in most cases sufficient to consider the tangent space to a point of $\underline{P}$. Such a tangent space naturally breaks up into horizontal and vertical sectors. A connection in the bundle is introduced by specifying a gauge covariant basis

$$D_I = e_I + h^A_I X_A \qquad (2.5)$$

in the horizontal sector of the tangent space. The m + n + 1 quantities $D_I \equiv \{D_\mu,D_\alpha\}$ are generalizations of the conventional covariant derivatives D_μ, e_I are directional derivatives, and

$$h_I = h_I^A X_A \qquad (2.6)$$

is the connection in $\underline{P}$ with values in the (super-) Lie algebra of the group G. The quantities h_I^A are the connection coefficients or (when restricted to a cross-section of bundle) gauge fields which belong to the adjoint representation of G:

$$X_A h_I^B = -f^B_{AC} h_I^C \qquad (2.7)$$

To complete a basis in the tangent space, we add to D_I the set $\{\hat{X}_A\}$ which span the vertical sector and are isomorphic to the generators $\{X_A\}$ of G. By construction

$$[D_I, \hat{X}_A] = 0 \qquad (2.8)$$

When the base manifold is a real (m+1)-dimensional space spanned by $\{x^\mu\}$, the expression (2.5) takes the more familiar form

$$D_\mu = \partial_\mu + h_\mu^A X_A \qquad (2.9)$$

where ∂_μ's are ordinary partial derivatives.

A gauge (or supergauge) transformation is a local (super)-group transformation which relates gauge fields defined on one cross-section to those on any other. Thus, acting on the basis D_I, it gives

$$D_I \to D_I' = e_I + h_I'^A X_A = e^{-\varepsilon^A(y) X_A} D_I e^{+\varepsilon^B(y) X_B} \qquad (2.10)$$

For infinitesimal transformations

$$\delta_\varepsilon D_I = (D_I \varepsilon^A) X_A \qquad (2.11)$$

so that

$$\delta_\varepsilon h_I^A = D_I \varepsilon^A = \partial_I \varepsilon^A + f^A_{BC} \varepsilon^C h_I^B \qquad (2.12)$$

The curvature 2-form $\mathcal{R}$ of the fiber bundle is a horizontal 2-form with values in the Lie (super-) algebra of G. It can therefore be expanded in terms of the basis $\{X_A\}$:

$$\mathcal{R} = \mathcal{R}^A X_A \tag{2.13}$$

Given a complete set of basis 1-forms $\{\omega^I\} \equiv \{\omega^\mu, \omega^\alpha\}$, then in the notation of reference 16 the wedge products $\{\omega^I \diamond \omega^J\}$ form a complete set of basis two forms. Expanding $\mathcal{R}^A$ in such a basis, we get

$$\mathcal{R}^A = R^A_{IJ} \, \omega^I \diamond \omega^J \tag{2.14}$$

where R^A_{IJ} are the components of Riemann curvature tensor. Choosing the ω^μ to be the coordinate differentials dx^μ and specializing to a real base manifold, the expression (2.14) takes the more familiar form

$$\mathcal{R}^A = R^A_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{2.15}$$

Explicit expressions for the components R^A_{IJ} in terms of the connection coefficients can be calculated from the generalized bracket[16]

$$[D_I, D_J\} = -R^A_{IJ} X_A \tag{2.16}$$

Thus,

$$R^A_{IJ} = h^A_{I,J} - (-)^{\sigma_I \sigma_J} h^A_{J,I} + f^A_{BC} \, h^B_I \, h^C_J \tag{2.17}$$

the quantities R^A_{IJ} transform covariantly under the gauge transformations (2.12):

$$\delta_\varepsilon \, R^A_{IJ} = -f^A_{BC} \, \varepsilon^B \, R^C_{IJ} \tag{2.18}$$

In the following we will also make use of the "dual" of the 2-form $\mathcal{R}$. In general, given an n-dimensional manifold and its associated p-forms, the "duality" or "*" operation is one which maps a p-form onto an (n-p)-form. For example, in 4-dimensional space-time the dual of a

1-form is a 3-form, that of a 2-form is another 2-form, etc. In general, under duality mapping one gets from a p-form

$$\omega \xrightarrow{\text{duality}} (-)^{p-1} \, {}^*\omega \qquad (2.19)$$

In prăctice, one makes use of the Levi-Civita operator e, which is an nth rank tensor, to relate the components of a form and its dual. Given a basis g_I with a fixed orientation, define the numerical tensor ε as

$$\varepsilon^{12,\ldots,n} = \varepsilon(g_1,\ldots,g_n) = 1 \qquad (2.20)$$

where ε is antisymmetric under exchange of two indices except when they are both fermi-type in which case ε is symmetric. Then the components $e^{1,\ldots,n}$ of e are related to those of ε by an appropriate density. In 4-space-time

$$e^{\mu\nu\rho\lambda} = h^{-1} \, \varepsilon^{\mu\nu\rho\lambda} \qquad (2.21)$$

so that the components of 2-forms $\mathcal{R}$ and ${}^*\mathcal{R}$ are related by

$${}^*R^A_{\mu\nu} = e_{\mu\nu}{}^{\rho\lambda} \, R^A_{\rho\lambda} \qquad (2.22)$$

Finally, we recall the Jacobi identity for a super algebra:[25]

$$[X_A \, , [X_B \, , X_C\}\} + (-)^{\sigma_A\sigma_B} [X_C \, , [X_A \, , X_B\}\} + (-)^{\sigma_A\sigma_C} [X_B \, , [X_C \, , X_A\}\} = 0 \qquad (2.23)$$

The insertion of the bracket (2.16) into this super Jacobi identity results in the super Bianchi identities for the superspace. When restricted to a real base manifold, they become ordinary Bianchi identities

$$D_{(\lambda} \, R^A_{\mu\nu)} = 0 \qquad (2.24)$$

where $(\lambda\mu\nu)$ stands for the cyclic permutation of indices.

III. CONSTRUCTION OF INVARIANT ACTION INTEGRALS

In this section we start with a review of the actions proposed in reference 20 and then consider a succession of generalizations which seem to be indispensible when matter couplings to gravity and supergravity are considered. Except in subsection C, we shall confine ourselves to a real base manifold.

A. The Actions Depending Only on Curvature

Consider the most general integral over a cross-section of the bundle, whose dependence on the gauge fields h^A_μ comes only through the components of the curvature tensor, and which is invariant under general coordinate transformations:

$$I(Q) = \int R^A \wedge R^B \, Q_{AB} \; d^4x \qquad (3.1)$$

$$= \int R^A_{\mu\nu} \, R^B_{\rho\lambda} \, Q_{AB} \, \varepsilon^{\mu\nu\rho\lambda} \; d^4x \qquad (3.2)$$

where $\varepsilon^{\mu\nu\rho\lambda}$ is the numerical Levi-Civita tensor (2.20), and Q_{AB} are constants, antisymmetric if G is a supergroup and the pair (A,B) belongs to L_1, but symmetric otherwise. This action does not transform irreducibly under G but can be decomposed into irreducible pieces; the corresponding Q_{AB} would then be the relevant Clebsh-Gordon coefficients. The variation of (3.2) under an arbitrary variation of the gauge fields h^A_μ is given by

$$\delta I(Q) = 4\int d^4x \; \delta h^A_\mu \, h^B_\nu \, R^C_{\lambda\sigma} \left(f^D_{BC} \, Q_{AD} - f^D_{AB} \, Q_{DC} \right) \varepsilon^{\mu\nu\lambda\sigma} \qquad (3.3)$$

The Bianchi identities (2.24) have been used in the derivation. Whenever the expression in the parentheses in the integrand vanishes, so does δI, and I becomes a topological invariant. This happens in particular when $Q_{AB} = g_{AB}$ given by (2.3).

When the variation of the gauge fields h_μ^A is restricted to the infinitesimal gauge transformations (2.12), then the variation of the action (3.2) is given by

$$\delta_\varepsilon I = -2\int \varepsilon^C \, R_{\mu\nu}^D \, R_{\lambda\sigma}^B \, f^A_{\ CD} \, Q_{AB} \, \varepsilon^{\mu\nu\lambda\sigma} \, d^4x \tag{3.4}$$

B. The Action Depending on Curvature and Its Dual

The action (3.1) has the correct form for describing pure gravity and simple supergravity. When couplings to matter are considered, the structure of (3.1) does not lead to correct equations of motion for extended gravity or supergravity. In particular, starting from (3.1) or (3.2) it appears impossible to obtain the correct Yang-Mills Lagrangian. One possibility is to consider actions of the form

$$J(Q) = \int d^4x \; {}^*R^A \wedge R^B \; Q_{AB} \tag{3.5}$$

$$= \int d^4V \; e_{\mu\nu}^{\ \ \rho\lambda} \, R_{\rho\lambda}^A \, R_{\sigma\delta}^B \, Q_{AB} \, e^{\mu\nu\sigma\delta} \tag{3.6}$$

where $e^{\mu\nu\rho\lambda}$ is the covariant Levi-Civita tensor (2.21). As far as pure gravity or gravity coupled to Yang-Mills is concerned, the action J(Q) has all the required properties of the Einstein-Yang-Mills Theory. In fact, it reproduces all the results of reference 14.[25] Its extension to supergravity, however, presents a number of problems. For example, the spin-3/2 equation obtained in this way does not have all the required constraints. It appears that at least one of the constraints would have to be imposed from outside. What one needs then is an action which in the absence of matter takes the form (3.2) but for matter couplings it gives the correct Yang-Mills action which is of the form (3.6). We will therefore consider generalizations of I(Q) and J(Q).

C. A More General Class of Invariants

The arguments in this subsection are equally applicable to cases in which the base manifold is real space-time or the superspace. Given the curvature 2-form $\mathcal{R}$ (2.13) of a bundle, we want to construct the most general scalar or scalar density of that bundle. Let e_J be a set of vector fields dual to the one forms ω^I:

$$\langle \omega^I , e_J \rangle = \delta^I_J \tag{3.7}$$

then one can construct a set of basis tensor-valued 0-forms

$$e_I \diamond e_J = \frac{1}{2}\left[e_I \oplus e_J - (-)^{\sigma_I \sigma_J} e_J \oplus e_I\right] \tag{3.8}$$

These are dual to the basis 2-forms $\omega^K \diamond \omega^L$:

$$\langle \omega^K \diamond \omega^L , e_I \diamond e_J \rangle = \delta^K_I \, \delta^L_J \tag{3.9}$$

Since $\mathcal{R}$ is a 2-form, we can construct a (super) Lie algebraic valued object $\mathcal{R}_{IJ}$ by the mapping (contraction)

$$\begin{aligned} \mathcal{R}_{IJ} &= \langle \mathcal{R} , e_I \diamond e_J \rangle \\ &= R^A_{KL} \, X_A \, \langle \omega^K \diamond \omega^L , e_I \diamond e_J \rangle \\ &= R^A_{IJ} \, X_A \end{aligned} \tag{3.10}$$

$\mathcal{R}_{IJ}$ is a tensor valued object with values in the (super) Lie algebra of the group G. To obtain a scalar, we contract it with the most general tensor valued 1-form Ω^{IJ} associated with the algebra $\{X_A\}$. Let $\{\omega^B\}$ be a set of basis 1-forms dual to $\{X_A\}$:

$$\langle \omega^B , X_A \rangle = \delta^B_A \tag{3.11}$$

Then we can expand Ω^{IJ} in this basis:

$$\Omega^{IJ} = \Omega^{IJ}_{A} \, \omega^{A} \tag{3.12}$$

Thus the scalar R that we seek is given by

$$\begin{aligned} R &= \langle \Omega^{IJ} , \mathcal{R}_{IJ} \rangle \\ &= \Omega^{IJ}_{A} \, R^{A}_{IJ} \end{aligned} \tag{3.13}$$

The quantities R^{A}_{IJ} are the covariant field strength tensors and are given by geometry in terms of the gauge fields as in (2.17). The object of the game is to obtain suitable Ω^{IJ}_{A}'s. These quantities are in general quite complex, and it is usually necessary to impose additional requirements to obtain an explicit form relevant to a particular application. We consider here a few of the special cases which are of interest in simple or extended supergravity. Before specializing, we note that the invariant action constructed from (3.13) has the form

$$I(\Omega) = \int d^{m+n+1} V \, \Omega^{IJ}_{A} (x,\theta) \; R^{A}_{IJ} (x,\theta) \tag{3.14}$$

where $d^{m+n+1}V$ is the invariant volume element in a space with m+1 bose and n fermi dimensions. The general method of handling such integrals, in particular in regard to the integration over the fermi coordinates will be discussed elsewhere.[22] Here we shall confine ourselves to a real 4-dimensional manifold in which case (3.14) reduces to

$$I(\Omega) = \int d^{4}V \, \Omega^{\mu\nu}_{A} (x) \, R^{A}_{\mu\nu} (x) \tag{3.15}$$

Consider now special cases of (3.15). Let

$$\begin{aligned} \Omega^{\mu\nu}_{A} &= e^{\mu\nu\rho\lambda} \; R^{B}_{\rho\lambda} \; Q_{AB} \\ &= h^{-1} \; \varepsilon^{\mu\nu\rho\lambda} \; R^{B}_{\rho\lambda} \; Q_{AB} \end{aligned} \tag{3.16}$$

where h at this point is the determinant of a suitable 4x4 matrix h^{i}_{μ}. Then it is trivial to see that with this Ansatz for $\Omega^{\mu\nu}_{A}$ (3.15) becomes identical with (3.2). It is also easy to see that with

$$\Omega^{\mu\nu}_{A} = e^{\mu\nu\rho\lambda}\, e_{\rho\lambda}{}^{\sigma\delta}\, R^{B}_{\sigma\delta}\, Q_{AB} \tag{3.17}$$

(3.15) reduces to (3.6). Encouraged by recovering some known results, we consider next a case applicable to Einstein-Yang-Mills and SO(2)-extended supergravity.

$$\Omega^{\mu\nu}_{A} = h^{-1}\, \Omega^{\mu\nu\rho\lambda}_{AB}(x)\, R^{B}_{\rho\lambda} \tag{3.18}$$

For this choice we get from (3.15)

$$I(\Omega) = \int d^4x\, \Omega^{\mu\nu\rho\lambda}_{AB}(x)\, R^{B}_{\rho\lambda}\, R^{A}_{\mu\nu} \tag{3.19}$$

Clearly, this contains (3.2) and (3.6) as special cases.

IV. PHYSICAL DEGREES OF FREEDOM, CHOICE OF GAUGE GROUPS, AND "CONSTRAINED" GEOMETRIES

The prominent feature of a geometrical approach is that the physically interesting quantities are almost automatically supplied by geometry. However, of the infinite class of possible geometries, the choice of the physically relevant ones requires additional input. In this section we show how the necessity to maintain the correct number of degrees of freedom to describe gravitation imposes restrictions on the geometry of an unconstrained fiber bundle thus distinguishing some components of of the curvature tensor from the rest.[27] The reader not familiar with the content of reference 20 may wish to proceed to the next section and then return to this section for an understanding of the underlying logic.

Consider first pure gravity. From the point of view of a gauge theory, one would like to describe it by an appropriate number of gauge fields. As pointed out elsewhere,[14] for gravity the gauge group is, at least in part, tied down to the structure of space-time, so that it must contain the homogeneous Lorentz group as a subgroup. For this group the gauge fields h_μ^{ij} have 24 independent components. On the other hand it is well known that in Einstein's theory a symmetric metric tensor with ten independent components is sufficient to describe gravity, so that a theory based on independent fields h_μ^{ij} could not be Einstein's theory. A way out of this dilemma is to introduce additional gauge fields h_μ^i with the required number of degrees of freedom by enlarging the gauge group and then impose a constraint by means of which h_μ^{ij} could be solved for in terms of h_μ^i. For pure gravity the relevant group is the Poincaré or the de Sitter group. Since the Lorentz subgroup of these groups is the only part which is directly tied down to the structure of space-time, the remaining part of the enlarged gauge group must be realized nonlinearly. This is, of course, consistent with the existence of a constraint among the gauge fields.

Next consider the form of the constraint equations from the point of view of geometry. For obvious reasons it must (a) be a covariant tensor equation with respect to the homogeneous Lorentz group and (b) involve the geometrical quantities of the enlarged fiber bundle. For Poincaré as well as the de Sitter group the components of the curvature tensor are $R_{\mu\nu}^A = \left\{R_{\mu\nu}^{ij}, R_{\mu\nu}^i\right\}$, i, j=0,..,3. Of these $R_{\mu\nu}^{ij}$ include $R_{\mu\nu}^{oij}$, which is the curvature tensor of a bundle with SL(2,c) as gauge group and which enters Einstein's equations. Therefore, other than the Bianchi identities (2.24), one would not want to impose any conditions on them. This means that the constraint equation must involve $R^{i}_{\mu\nu}$. The general form of the constraint equations must then be

$$R^{i}_{\mu\nu} = T^{i}_{\mu\nu} \quad ; \; i = 0, \ldots, 3 \tag{4.1}$$

In a torsion-free theory of gravity $T^{i}_{\mu\nu} = 0$ and one has

$$R^{i}_{\mu\nu} = 0 \quad \text{(torsion-free)} \tag{4.2}$$

We refer to the enlarged geometries satisfying constraints of the form (4.1) as "constrained" geometries. The constraint equations (4.1) are equal in number to the dependent fields h^{ij}_{μ} which we want to eliminate.

It is to be emphasized that the arguments presented above are quite general and do not depend on a particular Lagrangian or action. They describe how one may geometrically realize the independent degrees of freedom of a physical theory. Of course, the action that one writes down must be compatible with (4.1) or (4.2) and must reproduce them under variation. We note, however, that constraints following from a variational principle hold only on extremal paths and surfaces whereas (4.1) holds everywhere. Even from a practical point of view the distinction may become important in a quantum theory where one sums over not just the extremal paths but all paths.

The conditions (4.1) define a 6-dimensional hypersurface in the 10-dimensional fiber space of the bundle based on O(3,2) or Poincaré groups and naturally divide the components $\left\{R^{A}_{\mu\nu}\right\} = \left\{R^{ij}_{\mu\nu}, R^{i}_{\mu\nu}\right\}$ of the curvature tensor into two parts: "canonical" or "torsion" components $R^{i}_{\mu\nu}$ whose vanishing supply the necessary constraint equations and the components $R^{ij}_{\mu\nu}$ which we refer to as "pure" curvature components. This distinction can be made sharper, especially for Poincaré or other affine groups, by going to a horizontal basis $D_i = h_i^\mu D_\mu$ in which $R^i_{\mu\nu}$ appears as coefficients of D_i. For affine groups the vanishing of $R^i_{\mu\nu}$ is a mathematical theorem. The components of the "pure" curvature are the same in number as those of a bundle based on the group O(3,1) and can be related to them. In fact from (2.17)

$$R^a_{\mu\nu} = R^{oa}_{\mu\nu} + f^a_{ij}\, h^i_\nu\, h^j_\mu \tag{4.3}$$

where $R^{oa}_{\mu\nu}$ is the curvature tensor of the O(3,1) bundle and transforms according to its adjoint representation. Therefore, equations of motion and dynamics are determined by the components of "pure" curvature and not all of $\left\{R^A_{\mu\nu}\right\}$. This means that the sums in the expressions of the form $R^A \wedge R^B\, Q_{AB}$ are restricted to $R^{ij} \wedge R^{k\ell}\, Q_{ijk\ell}$, where $Q_{ijk\ell}$'s are the Clebsh-Gordon coefficients for an invariant product of two adjoint representations of SL(2,c).

The above arguments can be repeated for the case of supergravity. To be able to define supersymmetry transformations, one must have a gauge group with an adjoint representation which can accommodate not only h^{ij}_μ but also spin-3/2 fields h^α_μ. The smallest such supergroup is OSp(1,2c).[24] But again such a gauge theory must describe gravity with a correct number of degrees of freedom, and as in the case of pure gravity, this cannot be done with h^{ij}_μ. So again we enlarge the supergroup to OSp(1,4)[24] or its contracted Salam-Strathdee[1,4] form, with gauge fields $\left\{h^{ij}_\mu, h^\alpha_\mu, h^i_\mu\right\}$. Then a suitable constraint would effectively express h^{ij}_μ in terms of h^α_μ and h^i_μ. Again only the Lorentz subgroup of these groups which is directly tied down to the structure of space-time will be realized linearly.

The form of the constraint equations are as before dictated by considering the components $\left\{\hat{R}^A_{\mu\nu}\right\} = \left\{\hat{R}^a_{\mu\nu}, \hat{R}^\alpha_{\mu\nu}, \hat{R}^i_{\mu\nu}\right\}$ of the curvature tensor of the OSp(1,4) bundle. Writing (4.1) in the form

$$\hat{R}^i_{\mu\nu} \equiv R^i_{\mu\nu} - T^i_{\mu\nu} = 0 \tag{4.4}$$

and following the same arguments as in the case of pure gravity, it is clear that $\hat{R}^i_{\mu\nu} = 0$ are the correct constraint equations with $T^i_{\mu\nu}$ determined by spin-3/2 fields. Again the constraint equations define a 10-dimensional hypersurface in the 14-dimensional fiber space of the OSp (1,4) bundle and divide the components of $\left\{\hat{R}^A_{\mu\nu}\right\}$ into canonical components $\hat{R}^i_{\mu\nu} = 0$ and pure curvature components $\left\{\hat{R}^{ij}_{\mu\nu}, R^\alpha_{\mu\nu}\right\}$. It is the latter components which appear in expressions such as

$$R^A \wedge R^B\, Q_{AB} \to R^{ij} \wedge R^{k\ell}\, Q_{ijk\ell} + R^\alpha \vee R^\beta\, Q_{\alpha\beta} \tag{4.5}$$

The generalization of these concepts to include other matter fields is straightforward. Given supergroups such as OSp(N,4)[24] or SU(N|4) as gauge groups, the requirement that the gauge fields spanning their adjoint representations give a correct description of gravity naturally splits the components of their respective curvature tensors into two parts: "canonical" components $\hat{R}^i_{\mu\nu} = 0$ which provide the necessary constraint equations, and the "pure" curvature components $\left\{\hat{R}^A_{\mu\nu}, A \neq i\right\}$ which enter in the construction of invariants. We will make frequent use of this splitting in the next section. Note that these concepts can be extended to superspace in a straightforward manner.[22]

From the arguments presented above and elsewhere[14] it is quite clear that although gravitation can be formulated in terms of gauge fields as a local gauge theory, its geometry is different from that associated with the conventional treatment of nonabelian gauge theories. The former is constrained, the latter is not. It would be interesting to see the consequences of constraining the geometry of nonabelian gauge theories in a manner similar to gravity. For one thing, the constraints among the gauge fields would result in a smaller number of dynamically independent fields in the theory.[28] We hope to return to this topic in a separate publication.[22]

V. APPLICATIONS

The general developments of Section II-IV are now illustrated by applying them to special cases.

A. Pure Gravity[20]

As discussed in Section IV, to have the correct number of degrees of freedom to describe Einstein's theory, the gauge group cannot be SL(2,C). Nevertheless, let us consider a fiber bundle which has SL(2,C) as its structural group. Take the action to be of the form (3.1) in which the dependence on gauge fields h_μ^{ij} comes through the components of the curvature tensor $R_{\mu\nu}^{oij}$:

$$I^o(Q) = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda} \ Q_{AB} \ R_{\mu\nu}^{oA} \ R_{\rho\lambda}^{oB}$$

The requirements of invariance under Lorentz transformations and reflections completely determine Q_{AB}, so that one gets

$$I^o = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda} \ \varepsilon_{ijk\ell} \ R_{\mu\nu}^{oij} \ R_{\rho\lambda}^{ok\ell} \tag{5.1}$$

From (3.3) it follows that the total variation δI^o of I^o vanishes identically, so that I^o is a topological invariant or a total divergence. Therefore, it does not contribute to any equations of motion. With this in mind, let us now consider a fiber bundle with structural group Sp(4) which is the covering group of the de Sitter group O(3,2). The adjoint representation of this group is 10-dimensional with gauge potentials $\left\{h_\mu^A\right\} = \left\{h_\mu^{ij} = -h_\mu^{ji},\ h_\mu^i\right\}$. From (3.1) the action must have the general form

$$I(Q) = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda} \ Q_{AB} \ R_{\mu\nu}^A \ R_{\rho\lambda}^B$$

From the discussion of previous sections only the "pure" components $R^{ij}_{\mu\nu}$ of the curvature tensor appear in this integral, so that one has

$$I(Q) = \int d^4x \; \varepsilon^{\mu\nu\rho\lambda} \; Q_{ijk\ell} \; R^{ij}_{\mu\nu} \; R^{k\ell}_{\rho\lambda} \tag{5.2}$$

Then invariance under Lorentz transformations as well as reflections uniquely determines $Q_{ijk\ell}$:

$$I = \int d^4x \; \varepsilon^{\mu\nu\rho\lambda} \; \varepsilon_{ijk\ell} \, R^{ij}_{\mu\nu} \, R^{k\ell}_{\rho\lambda} \tag{5.3}$$

where from (2.19) or (4.3)

$$R^{ij}_{\mu\nu} = R^{oij}_{\mu\nu} + f^{ij}_{k\ell} \, h^k_\nu \, h^\ell_\mu \tag{5.4}$$

Substituting this into (5.3), one gets

$$I = I^o + \int d^4x \; \varepsilon^{\mu\nu\rho\lambda} \left\{ 2R^{oa}_{\mu\nu} \, h^j_\rho \, h^i_\lambda \, f^b_{ij} \, \varepsilon_{ab} + h^i_\mu \, h^j_\nu \, h^k_\rho \, h^\ell_\lambda \, f^a_{ij} \, f^b_{k\ell} \, \varepsilon_{ab} \right\} \tag{5.5}$$

where I^o is the total divergence given by (5.1). The first term in the integral is a gauge theory version of Einstein's Lagrangian, and the last part is the cosmological term. Dropping I^o and making an Inönü-Wigner contraction, one obtains the pure gravity action. We note that because in the process of contraction $Sp(4) \rightarrow I\,SL(2,c)$, the metric (2.3) becomes degenerate, one must retain the constrained Sp(4) until (5.5) is obtained and then let the contraction parameter (radius of de Sitter space) go to ∞ .

Variation of the action (5.5) with respect to the gauge potentials h^i_μ and h^{ij}_μ gives, respectively,

$$\varepsilon^{\mu\nu\rho\lambda} \; \varepsilon_{ab} \; f^b_{ij} \; h^i_\nu \; R^a_{\rho\lambda} = 0 \tag{5.6}$$

$$\varepsilon^{\mu\nu\rho\lambda} \; \varepsilon_{ab} \; f^b_{ij} \; h^i_\nu \; R^j_{\rho\lambda} = 0 \tag{5.7}$$

Notice that these equations as well as the actions (5.3) and (5.5) are based fundamentally on the concept of connection (gauge potentials) and are independent of whether the space-time manifold is metrizable or not. In this respect they are more general than Einstein's equations, and a quantum theory based on (5.6) and (5.7) may turn out to be different than the quantized version of Einstein's theory.

To obtain Einstein's equations from (5.6) and (5.7), one must further assume that the gauge potentials h^i_μ are invertible, i.e., that there exist objects h^ν_j such that

$$h^i_\mu \, h^\mu_j = \delta^i_j \tag{5.8}$$

Once this assumption is made, then it is possible[14] to identify the gauge potentials h^i_μ with the conventional "vierbeins." In this case the theory becomes, effectively a metric theory because then one can define a non-singular metric

$$g_{\mu\nu} = \eta_{ij} \, h^i_\mu \, h^j_\nu \tag{5.9}$$

Thus it is strictly speaking not correct to call the gauge potentials h^i_μ "vierbeins" except in the context of a metric theory.[14] With the assumption (5.8) equations (5.6) and (5.7) reduce to the usual equations of general relativity:[14,29]

$$R^i_{\mu\nu} = 0 \tag{5.10}$$

$$R^\lambda_\mu \equiv h^\nu_i \, h^\lambda_j \, R^{ij}_{\mu\nu} = 0 \tag{5.11}$$

Equations (5.10) are consistent, as they must be, with the constrained geometry discussed in the previous section and provides the constraints which cuts down the number of independent gauge fields to those necessary for describing gravity. In fact, it can be solved for h^{ij}_{μ} in terms of h^{i}_{μ}:[29]

$$h_{\nu ij} = \frac{1}{4}\Big[h^{\mu}_{j}\left(h_{i\mu,\nu} - h_{i\nu,\mu}\right) - h^{\mu}_{i}\left(h_{j\mu,\nu} - h_{j\nu,\mu}\right) + h^{\mu}_{j}\, h^{\lambda}_{i}\, h^{k}_{\nu}\left(h_{k\mu,\lambda} - h_{k\lambda,\mu}\right)\Big] \tag{5.12}$$

where

$$h_{i\mu} = \eta_{ij}\, h^{j}_{\mu} \;;\; h_{\mu ij} = \eta_{ik}\, \eta_{j\ell}\, h^{k\ell}_{\mu} \tag{5.13}$$

Insertion of h^{ij}_{μ} from (5.12) into (5.5) leads to the <u>second order form</u> of general relativity in which the dynamically independent variables are the gauge fields h^{i}_{μ}. The variations of these fields under local gauge transformations of the de Sitter group are given by (3.4).

Next consider the infinitesimal transformations of the fields h^{ij}_{μ} in first order formalism. The existence of the constraint $R^{i}_{\mu\nu} = 0$ indicates that h^{ij}_{μ} must transform according to a non-linear representation of Sp(4) or ISL(2,c). To obtain the form of this transformation we note that the variation of (5.3) under the local infinitesimal transformation $\delta_{\varepsilon^i} h^{j}_{\mu}$ is given by

$$\delta^{(1)}_{\varepsilon^i} I = 2\int d^4x\, \varepsilon^{\mu\nu\rho\lambda}\, \varepsilon_{ab}\, f^{a}_{ij}\, R^{i}_{\mu\nu}\, R^{b}_{\rho\lambda}\, \varepsilon^{j} \tag{5.14}$$

The variation $\delta^{(2)}_{\varepsilon^i} I$ of (5.3) with respect to $\delta_{\varepsilon^i} h^{jk}_{\mu}$ must be such that

$$\delta_{\varepsilon^i} I = \delta^{(1)}_{\varepsilon^i} I + \delta^{(2)}_{\varepsilon^i} I = 0 \tag{5.15}$$

Looking at the variation of (5.3) with respect to a general variation of h_μ^{ij}, one finds a term similar in structure to (5.14):

$$\delta^{(2)}_{\varepsilon^i} = 4\int d^4x \; \varepsilon^{\mu\nu\rho\lambda} \; \varepsilon_{ac} \, f^c_{ij} \; \delta h^a_\mu \; h^i_\nu \; R^j_{\rho\lambda} \qquad (5.16)$$

solving for δh^a_μ from (5.14)-(5.16) we get

$$\delta h^a_\mu = \frac{1}{4} \, \varepsilon^{aij} \; \varepsilon_{b\ell k} \; h^\nu_i \; h^j_\lambda \left[h^k_\lambda \; R^b_{\mu\nu} - h^k_\nu \; R^b_{\mu\lambda} - h^k_\mu \; R^b_{\nu\lambda} \right] \varepsilon^\ell(x) \qquad (5.17)$$

It would be interesting to explore the consequences of this invariance in regard to the renormalizability of pure gravity. Since all the present day statements in regard to renormalizability of this theory are made in second order formalism,[30] it is not clear whether the quantized first and second order formalisms are equivalent.

B. Gravity Coupled to Yang-Mills

Without supersymmetry, the internal and space-time symmetries are quite distinct. Therefore, the gauge group for gravity coupled to Yang-Mills theory is of direct product form. From our point of view it is Sp(4) X SU(2) or its contracted form. Direct extension of the gravity action in terms of the curvature components to this case leads to a total divergence Pseudoscalar piece for the Yang-Mills part as can easily be verified. So we proceed with the general action (3.15) or its more restricted form (3.19):

$$I(\Omega) = \int d^4x \; \Omega^{\mu\nu\rho\lambda}_{AB} \; (x) \; R^A_{\mu\nu} \; R^B_{\rho\lambda} \qquad (5.18)$$

We require that (i) in the absence of Yang-Mills $I(\Omega)$ reduce to the gravity action (5.3) and (ii) it satisfy the same Lorentz, reflection, and general coordinate invariance properties as the gravity action (5.3). Thus

$$I(\Omega) = \int d^4x \left[\Omega^{\mu\nu\rho\lambda}_{ijk\ell} R^{ij}_{\mu\nu} R^{k\ell}_{\rho\lambda} + \Omega^{\mu\nu\rho\lambda}_{ab} F^a_{\mu\nu} F^b_{\rho\lambda}\right] \tag{5.19}$$

Requirement (i) implies that

$$\Omega^{\mu\nu\rho\lambda}_{ijk\ell} = \varepsilon^{\mu\nu\rho\lambda} \varepsilon_{ijk\ell} \tag{5.20}$$

Requirement (ii), in particular reflection invariance, implies that

$$\begin{aligned}\Omega^{\mu\nu\rho\lambda}_{ab} &= \frac{C}{2}\eta_{ab} \varepsilon^{\mu\nu\delta\sigma} e^{\rho\lambda}_{\delta\sigma} \\ &= C\eta_{ab}\, h\, g^{\mu\rho} g^{\nu\lambda} \quad ; \; C = \text{const.}\end{aligned} \tag{5.21}$$

We therefore have for Einstein-Yang-Mills theories

$$I = \int d^4x \left[\varepsilon^{\mu\nu\rho\lambda} \varepsilon_{ijk\ell} R^{ij}_{\mu\nu} R^{k\ell}_{\rho\lambda} + Ch\eta_{ab} g^{\mu\rho} g^{\nu\lambda} F^a_{\mu\nu} F^b_{\rho\lambda}\right] \tag{5.22}$$

Note that in this case the introduction of the metric tensor $g_{\mu\nu}$ or the inverse vierbeins h^{μ}_{i} is indispensible, so that the formulation of the theory in terms of connections alone appears to be not possible. As a special case of (5.22) one obtains the Einstein-Maxwell theory when internal symmetry group is U(1). Also note that the action (5.22) is not invariant under the non-linear transformation (5.17), and any attempt to make it so involves the introduction of additional fields such that the fields in the action form a supermultiplet [see, e.g., subsection D) below]. This may be cited as a reason for the nonrenormalizability of arbitrary couplings of matter fields to gravity. Only (supersymmetric) matter couplings which retain or enlarge invariances of pure gravity seem to be equally renormalizable.

C. Simple Supergravity[20]

This theory can be developed in complete analogy with that of pure gravity. A different formulation will be given elsewhere.[22] The smallest supergroup which admits supersymmetry transformations and contains SL(2,C) as its Lie subgroup is OSp(1,2C). As discussed in section IV, this group does not give a correct description of gravity so that it will have to be enlarged. But since its topological invariants turn out to be relevant, consider a fiber bundle over real space-time with OSp(1,2C) as its structural group. The action (3.1) depending on the components of the curvature alone will then give

$$\hat{I}^o(Q) = \int d^4x \, \varepsilon^{\mu\nu\rho\lambda} \, Q_{AB} \, \hat{R}^{oA}_{\mu\nu} \, \hat{R}^{oB}_{\rho\lambda} \tag{5.23}$$

where now

$$\left\{\hat{R}^{oA}_{\mu\nu}\right\} = \left\{\hat{R}^{oij}_{\mu\nu} \, , \, \hat{R}^{o\alpha}_{\mu\nu}\right\} . \tag{5.24}$$

and

$$\left\{Q_{AB}\right\} = \left\{Q_{ijk\ell} \, , \, Q_{\alpha\beta}\right\} \tag{5.25}$$

The requirements of Lorentz and reflection invariance determine $\left\{Q_{AB}\right\}$ up to multiplicative constants. They will be completely fixed by the additional requirement that $\hat{I}^o$ be an OSp(1,2C) scalar. Thus one gets

$$\hat{I}^o = \int d^4x \, \varepsilon^{\mu\nu\rho\lambda} \left[\varepsilon_{ijk\ell} \, \hat{R}^{oij}_{\mu\nu} \, \hat{R}^{ok\ell}_{\rho\lambda} + \chi(C\gamma_5)_{\alpha\beta} \, \hat{R}^{o\alpha}_{\mu\nu} \, \hat{R}^{o\beta}_{\rho\lambda}\right] \tag{5.26}$$

where C = charge conjugation matrix, and χ is a normalizing parameter defined by

$$\varepsilon_{ab} \, f^{b}_{\alpha\beta} = \chi f^{\gamma}_{a\alpha} (C\gamma^5)_{\alpha\beta} \tag{5.27}$$

From (3.3) it is easy to verify that the general variation $\delta\hat{I}^o$ of $\hat{I}^o$ vanishes identically, so that it is a total divergence (topological invariant!).

For pure gravity the gauge group had to be enlarged from SL(2,C) to Sp(4), and the corresponding geometry constrained. Here by enlarging the SL(2C) subgroup of OSp(1,2C) to Sp(4) one arrives at OSp(1,4) for the supergravity gauge group. Consider then a fiber bundle with space-time manifold as base space and OSp(1,4) as structural group. The adjoint representation is 14-dimensional with gauge potentials $\left\{h^A_\mu\right\} = \left\{h^{ij}_\mu = -h^{ji}_\mu, h^\alpha_\mu, h^i_\mu\right\}$. The general form of the action is again given by (3.1):

$$\hat{I}(Q) = \int d^4x \, \varepsilon^{\mu\nu\rho\lambda} \, Q_{AB} \, \hat{R}^A_{\mu\nu} \, R^B_{\rho\lambda}$$

Proceeding to the constrained bundle of section IV, we note that of the components $\left\{\hat{R}^A_{\mu\nu}\right\} = \left\{\hat{R}^{ij}_{\mu\nu}, \hat{R}^\alpha_{\mu\nu}, \hat{R}^i_{\mu\nu}\right\}$ of the OSp(1,4) curvature tensor the "pure" components $\left\{\hat{R}^{ij}_{\mu\nu}, \hat{R}^\alpha_{\mu\nu}\right\}$ contribute to this action. Thus one gets

$$\hat{I}(Q) = \int d^4x \, \varepsilon^{\mu\nu\rho\lambda} \left[Q_{ijk\ell} \, \hat{R}^{ij}_{\mu\nu} \, \hat{R}^{k\ell}_{\rho\lambda} + Q_{\alpha\beta} \, \hat{R}^\alpha_{\mu\nu} \, R^\beta_{\rho\lambda}\right] \tag{5.28}$$

where

$$\begin{aligned} \hat{R}^{ij}_{\mu\nu} &= \hat{R}^{oij}_{\mu\nu} + f^{(ij)}_{k\ell} \, h^k_\nu \, h^\ell_\mu \\ &= R^{ij}_{\mu\nu} + f^{ij}_{\alpha\beta} \, h^\alpha_\nu \, h^\beta_\mu \end{aligned} \tag{5.29}$$

$$\hat{R}^\alpha_{\mu\nu} = \hat{R}^{o\alpha}_{\mu\nu} + f^\alpha_{i\beta} \left(h^i_\nu \, h^\beta_\mu - h^i_\mu \, h^\beta_\nu\right) \tag{5.30}$$

Since these curvature components involve those of OSp(1,2C) or of SL(2,C), and since the only quadratic terms in curvature components $R^{oij}_{\mu\nu}$ which we allow are those which are total divergences, then $Q_{ijk\ell}$ and $Q_{\alpha\beta}$ in (5.28) must have the same values as those in (5.26):

$$Q_{ijk\ell} = \varepsilon_{ijk\ell} \quad ; \qquad Q_{\alpha\beta} = \chi (C\gamma^5)_{\alpha\beta} \tag{5.31}$$

Thus

$$\hat{I} = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda} \left[\varepsilon_{ijk\ell} \ \hat{R}^{ij}_{\mu\nu} \ R^{k\ell}_{\rho\lambda} + \chi (C\gamma^5)_{\alpha\beta} \ \hat{R}^{\alpha}_{\mu\nu} \ \hat{R}^{\beta}_{\rho\lambda}\right] \tag{5.32}$$

Substituting (5.29) and (5.30) into (5.31) and using (5.27), one gets

$$\hat{I} = \hat{I}^o + I_S + I_C \tag{5.33}$$

where

$$I_S = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda}\left[2 \ \varepsilon_{ab} \ f^{\,b}_{\ ij} \ h^j_\rho \ h^i_\lambda \ R^{oa}_{\mu\nu} + 4\chi \ \hat{R}^{o\alpha}_{\mu\nu} \ h^\gamma_\rho \ h^i_\lambda \ f^{\,\beta}_{\ i\gamma} (C\gamma^5)_{\alpha\beta}\right] \tag{5.34}$$

$$I_C = \int d^4x \ \varepsilon^{\mu\nu\rho\lambda} \ \varepsilon_{ab} \ f^{\,a}_{\ ij}\left[-4 \ f^{\,b}_{\ \alpha\beta} \ h^\alpha_\mu \ h^i_\nu \ h^\beta_\rho \ h^j_\lambda + f^{\,b}_{\ k\ell} \ h^i_\mu \ h^j_\nu \ h^k_\rho \ h^\ell_\lambda\right] \tag{5.35}$$

In (5.33), $\hat{I}^o$, is a total divergence and can be dropped; I_S is an alternate form of supergravity action; and I_C contains cosmological terms which can be eliminated by an Inönü-Wigner contraction.

The variation of action (5.32) with respect to arbitrary variations of gauge fields gives the Euler-Lagrange equations

$$\varepsilon^{\mu\nu\rho\lambda} \ \varepsilon_{ab} \ f^{\,b}_{\ ij} \ h^i_\nu \ \hat{R}^j_{\rho\lambda} = 0 \tag{5.36}$$

$$\varepsilon^{\mu\nu\rho\lambda} \ f^{\,\alpha}_{\ i\beta} \ h^i_\nu \ \hat{R}^\beta_{\rho\lambda} = 0 \tag{5.37}$$

$$\varepsilon^{\mu\nu\rho\lambda}\left[2 \ \varepsilon_{ab} \ f^{\,b}_{\ ij} \ h^j_\nu \ \hat{R}^a_{\rho\lambda} + \chi (C\gamma^5)_{\gamma\beta} \ f^{\,\gamma}_{\ i\alpha} \ h^\alpha_\nu \ \hat{R}^\beta_{\rho\lambda}\right] = 0 \tag{5.38}$$

As in the case of pure gravity the action (5.32) and equations (5.36)-(5.38) are based entirely on the concept of connection (gauge fields) and are independent of the notion of a metric. To obtain the more familiar supergravity equations, one must make use of (5.8) or (5.9). Then writing

$$\left(\gamma_\mu\right)^\alpha_\beta = h^i_\mu f^\alpha_{i\beta}$$
$$\left(\gamma^\mu\right)^\alpha_\beta = \eta^{ij}\, h^\mu_i\, f^\alpha_{j\beta} \tag{5.39}$$

We can write

$$\hat{R}^j_{\mu\nu} = 0 \tag{5.40}$$

$$\varepsilon^{\mu\nu\rho\lambda}\left(\gamma_\nu\right)^\alpha_\beta\, \hat{R}^\beta_{\rho\lambda} = 0 \tag{5.41}$$

$$\hat{R}^\lambda_\mu = \frac{1}{2}\chi\; e^{\lambda\nu\sigma\delta}\, h^\alpha_\nu (C\gamma^5\gamma\mu)_{\alpha\beta}\, \hat{R}^\beta_{\sigma\delta} \tag{5.42}$$

Equation (5.40) plays the same role in simple supergravity as (5.10) does in pure gravity. It is consistent with the constrained geometry of section IV and gives a constraint among the gauge fields of OSp(1,4). It can be solved for $\hat{h}^{ij}_\mu$ in terms of h^i_μ and h^α_μ :

$$\hat{h}_{\nu ij} = h_{\nu ij} + \frac{1}{4}\Big[h^\mu_j\, f_{i\alpha\beta}\, h^\alpha_\mu\, h^\beta_\nu - h^\mu_i\, f_{j\alpha\beta}\, h^\alpha_\mu\, h^\beta_\nu + h^\mu_j\, h^\lambda_i\, h^k_\nu\, f_{k\alpha\beta}\, h^\alpha_\mu\, h^\beta_\lambda\Big] \tag{5.43}$$

where $h_{\nu ij}$ is given by (5.12) and

$$f_{i\alpha\beta} = \eta_{ij}\, f^j_{\alpha\beta} \tag{5.44}$$

Insertion of $\hat{h}_{\nu ij}$ from (5.43) into the action (5.33) will result in the second order form of the supergravity theory in which the independent fields are h^i_μ's and h^α_μ's . Under local gauge transformations these field transform according to (3.4).

Next consider the infinitesimal supersymmetry transformations in first order formalism. The algebra of these transformations has been studied in detail by MacDowell.[31] Here we confine ourselves to local supersymmetry transformation. Following the analogy with the case of pure gravity, it is clear from the constraints $\hat{R}^i_{\mu\nu} = 0$ that the gauge potentials $\hat{h}^{ij}_{\mu\nu}$ must transform according to a nonlinear realization of OSp(1,4) or its contracted version. Then the arguments which led to the transformation (5.17) for pure gravity give in this case

$$\delta\hat{h}^{ij}_\mu = -\varepsilon^\alpha f^{\;ij}_{\alpha\beta}\, h^\beta_\mu - \frac{1}{2}\, \chi^{-1}\, \varepsilon^{ijk\ell}\, h^\nu_k\, h^\lambda_\ell\, \varepsilon^\alpha f^{\;\beta}_{m\alpha} (C\gamma^5)_{\beta\gamma} \cdot \left(2h^m_\lambda\, \hat{R}^\gamma_{\mu\nu} + h^m_\mu\, \hat{R}^\gamma_{\lambda\nu}\right) \tag{5.45}$$

After group contraction this expression becomes equivalent to the usual local supersymmetry transformation of $\hat{h}^{ij}_\mu$ [18]. Thus the combined transformations $\left\{\delta h^i_\mu\, ,\ \delta h^\alpha_\mu\, ,\ \delta\hat{h}^{ij}_\mu\right\}$ leave the action (5.32) invariant, and we have an example of a supersymmetric coupling of matter to gravity which has retained its invariance under nonlinear transformations of the type (5.17).

D. SO(2)-Extended Supergravity

The relation of this theory to simple supergravity is the same as Einstein-Maxwell or Einstein Yang-Mills theory is to pure gravity.

The main difference is that here the gauge group is not of direct product type. On replacing SO(1) in the OSp(1,4) of supergravity gauge group by SO(2) one arrives at OSp(2,4) for the gauge group of SO(2)-extended supergravity theory. As in subsection B an action based on components of curvature alone will not result in the usual action of spin-1 field, so that we must proceed with the general action (3.15). It is sufficient to start with the more restricted form (3.19):

$$I(\Omega) = \int d^4x \; \Omega^{\mu\nu\rho\lambda}_{AB} (x) \; R^A_{\mu\nu} \; R^B_{\rho\lambda}$$

Recalling the conditions of constrained geometry and imposing the same requirements (i) and (ii) as in subsection B, we have

$$I_2(\Omega) = \int d^4x \left[\Omega^{\mu\nu\rho\lambda}_{ijk\ell} \; \hat{R}^{ij}_{\mu\nu} \; R^{k\ell}_{\rho\lambda} + \Omega^{\mu\nu\rho\lambda}_{\alpha\beta} \; \hat{R}^{\alpha z}_{\mu\nu} \; \hat{R}^{\beta z}_{\rho\lambda} + \Omega^{\mu\nu\rho\lambda} \; \hat{F}_{\mu\nu} \; \hat{F}_{\rho\lambda} \right] ; \qquad (5.46)$$

where z = 1,2 to account for the fact that in the adjoint representation of OSp(2,4) there are two spin-3/2 gauge fields. From requirements (i) and (ii)

$$\begin{aligned} \Omega^{\mu\nu\rho\lambda}_{ijk\ell} &= \varepsilon^{\mu\nu\rho\lambda} \; \varepsilon_{ijk\ell} \\ \Omega^{\mu\nu\rho\lambda}_{\alpha\beta} &= \chi\varepsilon^{\mu\nu\rho\lambda} \; (C\gamma^5)_{\alpha\beta} \\ \Omega^{\mu\nu\rho\lambda} &= \frac{1}{8}\varepsilon^{\mu\nu\sigma\delta} \; e_{\sigma\delta}{}^{\rho\lambda} \\ &= \frac{1}{4}h \; g^{\mu\rho} \; g^{\nu\lambda} \end{aligned} \qquad (5.47)$$

Therefore,[32]

$$I_2 = \int d^4x \; \varepsilon^{\mu\nu\rho\lambda} \left[\varepsilon_{ijk\ell} \; \hat{R}^{ij}_{\mu\nu} \; R^{k\ell}_{\rho\lambda} + \chi (C\gamma^5)_{\alpha\beta} \; R^{\alpha z}_{\mu\nu} \; R^{\alpha z}_{\rho\lambda} + \frac{1}{8} e_{\rho\lambda}{}^{\sigma\delta} \; \hat{F}_{\mu\nu} \hat{F}_{\sigma\delta} \right] \qquad (5.48)$$

Further developments in extended supergravity will be given elsewhere.[22] Here again we have an extended theory of gravity in which the invariances of the type (5.17) are maintained.

VI. CONCLUDING REMARKS

We have presented a general theoretical framework for constructing superunified theories based on the geometry of (super-) gauge invariance: Just as local nonabelian gauge theories find a natural setting in Minkowski space-time, we have argued that any unification involving gravity must of necessity be a superunified theory. Otherwise, some of the invariances of pure gravity action are likely to be destroyed. We have also pointed out that this may be the reason for the nonrenormalizability of theories involving the arbitrary couplings of matter fields to gravity.

A general method of constructing invariants for superunified theories is given. Since they are based on the geometrical characteristics of a fiber bundle, it would be surprising if the actions for superunified theories turn out to be something other than special cases of those discussed in section III. The concept of "constrained" geomteries is introduced to match the degrees of freedom required in superunified theories to those of constrained fiber bundles. In this way one is led in an unambiguous way to the construction of the actions discussed in section V. It is hoped that the theoretical basis provided in this work will also solve the physically more interesting theories based on supergroups OSp(N,4) and SU(N|4).

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