REGGE POLES IN FIELD THEORY

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The investigation of the properties of field theoretic scattering amplitudes in the complex angular momentum plane probably can be approached at present only within the framework of an approximation scheme. It is doubtful whether general considerations can provide much information on the nature and location of the singularities in the ℓ-plane, though the assumption of a Mandelstam representation provides a certain domain of analyticity. The present paper is concerned with the properties in the ℓ-plane of the Bethe-Salpeter scattering amplitude in the ladder approximation, and is based on work by B.W. Lee and the author. This amplitude, though closely related to a potential scattering amplitude, has several features of field theory which are not contained in potential scattering:

(a) It is fully relativistic;
(b) It satisfies a unitary relation to which states of more than two particles contribute, i.e., it takes account of possibility of particle production;
(c) The crossed ladder graphs represent a set of Feynman diagrams for another scattering process, so that one may hope to interpret the Regge limit, $t \to \infty$, as the high-energy limit of a genuine scattering amplitude.

In terms of graphs, our aim is to discuss the behaviour in the ℓ-plane of the sum shown in Fig. 1

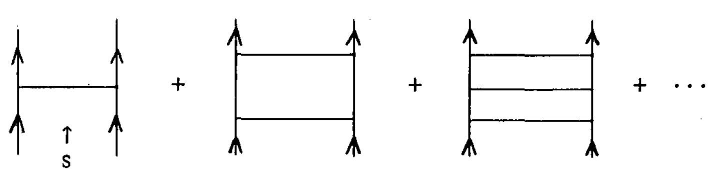

Fig. 1

and to investigate the connection with the high-energy limit of the graphs shown in Fig. 2

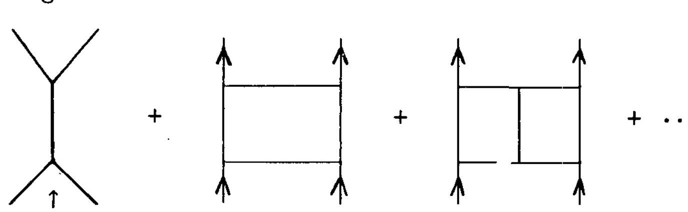

Fig. 2

For simplicity we consider the scattering of two J = 0 bosons of mass m through exchanges of a scalar boson of mass μ. We write the relevant Bethe-Salpeter equation for the scattering amplitude in momentum space as:

$$\langle \vec{p},\omega|T(s)|\vec{p}',\omega'\rangle = \langle \vec{p},\omega|B|\vec{p}',\omega'\rangle + \int \langle \vec{p},\omega|T(s)|\vec{p}'',\omega''\rangle \langle \vec{p}'',\omega''|K(s)|\vec{p}',\omega'\rangle d^3p''d\omega'' \qquad (1)$$

where

$$\langle \vec{p},\omega|B|\vec{p}',\omega'\rangle = [g^2/(2\pi)^4]\,[(\vec{p}-\vec{p}')^2+\mu^2 - i\epsilon - (\omega-\omega')^2]^{-1},$$

$$\langle \vec{p},\omega|K(s)|\vec{p}',\omega'\rangle = -iF^{-1}(\vec{p},\omega,s)\langle \vec{p},\omega,|B|\vec{p}',\omega'\rangle,$$

$$F(\vec{p},\omega,s) = [\vec{p}^2+m^2-(\sqrt{s}/2+\omega)^2]\,[\vec{p}^2+m^2-(\sqrt{s}/2-\omega)^2]\,.$$

These equations define a T matrix off the mass shell. s is the square of the total energy in the centre-of-mass system. The solution to Eq. (1) for physical scattering is to be evaluated at

$$\omega = \omega' = 0,$$

$$\vec{p} = \vec{u}_f\sqrt{(s/4)-m^2},$$

$$\vec{p} = \vec{u}_i\sqrt{(s/4)-m^2}.$$

The partial wave projection of Eq. (1) is of the form

$$T_\ell(s) = B_\ell + T_\ell(s)K_\ell(s) \qquad (2)$$

where

$$\langle p,\omega|B_\ell|p',\omega'\rangle = [g^2/(2\pi)^3]\,Q_\ell\{[p^2+p'^2+\mu^2-(\omega-\omega')^2]/2pp'\},$$

$$\langle p,\omega|K_\ell(s)|p',\omega'\rangle = -iF^{-1}(p,\omega,s)\langle p,\omega,|B_\ell|p',\omega'\rangle.$$

Operator products are defined by

$$\langle p,\omega|AB|p',\omega'\rangle = \int_0^\infty dp''\int_{-\infty}^{+\infty} d\omega''\langle p,\omega|A|p'',\omega''\rangle\langle p'',\omega''|B|p',\omega'\rangle.$$

We make the extension to non-integral ℓ from Eq. (2). We note that in the iteration solution to Eq. (2) ℓ enters only in the function Q_ℓ. The function Q_ℓ is analytic in the entire ℓ-plane except for fixed poles at the negative integers. Wherever the iteration solution converges we may immediately conclude that the scattering amplitude is analytic in the ℓ plane. The Regge poles, of course, reflect the divergence of this series. To prove a domain of meromorphy in the ℓ-plane we use instead the development of the solution of Eq. (2) as the ratio of two series via the Fredholm method. We write formally

$$T_\ell = B_\ell(1-K_\ell)^{-1} \tag{3}$$

and use the identity

$$(1-K_\ell)^{-1} = -(\delta/\delta K^T)D_\ell/D_\ell$$

where

$$D_\ell = D\ell + (1-K_\ell) = \exp[\mathrm{Tr}\log(1-K_\ell)]. \tag{4}$$

This will be a useful representation wherever the perturbation series for D_ℓ converges. As in the case of potential scattering, it can be shown that for $\mathrm{Re}\,\ell > -3/2$ the integrals of the form $\mathrm{Tr}\,K_\ell^n$ in the expansion for D converge and also that the perturbation series for D (and for N) converges absolutely. The proof, which is somewhat lengthy and will not be presented here, involves finding changes of variable which reduce the integral equation (3) to one with a bounded kernel and a finite range of integration. Standard methods are now applicable to show the convergence of N_ℓ and D_ℓ in this region, $\mathrm{Re}\,\ell > -3/2$.

It follows that T_ℓ is meromorphic in the half-plane $\mathrm{Re}\,\ell > -3/2$. The Regge poles are zeros of D_ℓ. Note that N_ℓ and D_ℓ may have fixed singularities at $\ell = -1$. These are both, in fact, simple poles. The proof is analogous to that in the case of potential scattering.

Before proceeding to the Regge trajectories and to their applications, we note some properties of our ND^{-1} factorization. First we discuss the singularities of D_ℓ as a function of s.

A typical trace in the evaluation of D_ℓ has the form:

$$\mathrm{Tr}\,K^n = \int \cdot dp_1\, d\omega_1\, F^{-1}(p_1,\omega_1,s) Q_\ell\left[\frac{p_1^2+p_2^2+k^2-(\omega_1-\omega_2)^2}{2p_1p_2}\right]$$

$$\cdot dp_2\, d\omega_2\, F^{-1}(p_2,\omega_2,s)\ \ldots$$

$$\cdot dp_n\, d\omega_n\, F^{-1}(p_n,\omega_n,s)\, Q_\ell\left[\frac{p_n^2+p_1^2+k^2-(\omega_n-\omega_1)^2}{2p_np_1}\right].$$

First we consider the region $|\mathrm{Re}\sqrt{s}| < \alpha m$. In this region we may perform a transformation, following Wick, which considerably simplifies the problem. In this region the zeros of $F(p,\underline{\omega},s)$ in the $\underline{\omega}$ plane (giving the masses a negative imaginary part) are located as shown in Fig. 3.

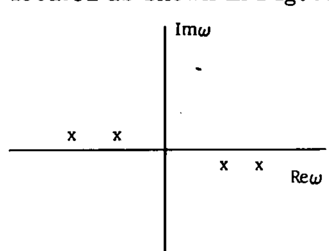

Fig. 3

The Wick trick is now to rotate all the ω_i integration contours simultaneously counterclockwise into the imaginary axis. No singularities of the Q_ℓ functions in (5) are encountered in this rotation. Now we look at the new denominators in Eq. (5), $F(p_i, i\omega_i, s)$ and see that for real p_i and ω_i, they are non-vanishing in the region $|\operatorname{Re}\sqrt{s}| < 2m$. $D_\ell(s)$ is therefore free of singularities in this region of the s-plane. We anticipate that $D_\ell(s)$ has only the right hand cut beginning at $s = 4m^2$. We must still show, however, that no spurious singularities at complex s were introduced by our ND^{-1} factorization. This can be shown by some further distortions of contour which will not be gone into here.

For integral ℓ, $T_\ell(s)$ is defined by Eq. (1) with a left-hand cut beginning at $s = 4m^2 - k^2$ and a right-hand cut beginning at $s = 4m^2$, with branch points at the production thresholds $s = (2m + nk)^2$. For non-integral ℓ there is a further kinematic cut which can be renewed by factoring out a factor$(s - 4m^2)^\ell$. Let us define $n_\ell(s)$ by $N_\ell(s) = (s - 4m^2)^\ell n_\ell(s)$ and investigate the singularities of $n_\ell(s)$. What can be shown is that $n_\ell(s)$ has no cut from $s = 4m^2$ to the first inelastic threshold, $s = (2m + k)^2$. Thus $n_\ell(s)$ has a left hand cut and a right hand cut beginning from $s = (2m + k)^2$. The proof can be done in a compact notation by proving the relation

$$[D_\ell(s - i\epsilon)]/[D_\ell(s + i\epsilon)] = \exp[2i\delta(s, \ell)] \tag{6}$$

in the region $4m < s < (2m + k)^2$.

Using a property of determinants we write:

$$\operatorname{Det}\left[\frac{1 - K_\ell(s - i\epsilon)}{1 - K_\ell(s + i\epsilon)}\right] = \operatorname{Det}\{1 + [K_\ell(s + i\epsilon) - K_\ell(s - i\epsilon)]\,[1 - K_\ell(s + i\epsilon)]^{-1}\}. \tag{7}$$

In the region $4m^2 < s < (2m + k)^2$ Cutkosky's method for evaluating discontinuities is equivalent to the replacement:

$$\langle p, \omega | K_\ell(s + i\epsilon) - K_\ell(s - i\epsilon) | p', \omega' \rangle$$

$$\longrightarrow [g^2/i(2\pi)^3]\,\delta[p^2 + m^2 - (\omega - (1/2)\sqrt{s})^2]\,\delta[p^2 + m^2 - (\omega - (1/2)\sqrt{s})]$$

$$\cdot Q_\ell\left[\frac{p^2 + p'^2 + k^2 - (\omega - \omega')^2}{2pp'}\right]. \tag{8}$$

We may then write Eq. (7) as

$$D(s - i\epsilon)/D(s + i\epsilon) = 1 + \left[i\pi^2/\sqrt{s[(1/4)s - m^2]}\right]\langle p, 0 | B_\ell(1 - K_\ell)^{-1} | p, 0\rangle \tag{9}$$

where the infinite determinant was evaluated by noting that aside from the ones along the diagonal, there is only one non-vanishing column of the matrix of which the determinant is being taken, as indicated by the δ-functions in Eq. (8). With our definitions the right hand side of the Eq. (9) is the S matrix element. Thus Eq. (6) is proved and it is furthermore clear that $n_\ell(s)$ has no cut beginning at $s = 4m^2$.

The Regge trajectories are given by the roots of the function

$$D(\ell, s) \equiv D_\ell(s).$$

The two properties of the trajectories which we shall now discuss are:
(1) The asymptote ($\lim s \to \infty$) of the leading trajectory;
(2) The development of the trajectory, $\alpha(s)$, in a perturbation series.
We shall need two properties of $D_\ell(s)$ which can be proved:

(a) $D_\ell(s)$ has a simple pole at $\ell = -1$. This follows, as in potential scattering, from the dependence of the singular (at $\ell = -1$) part of the kernel, K_ℓ, on only the first indices:

$$\langle p, \omega | K_{sing} | p', \omega' \rangle = -[i/(\ell + 1)] F^{-1}(p, \omega, s)$$

(b) For $\ell \neq -1$, $D_\ell \longrightarrow 1$ as $s \longrightarrow \infty$ with the remainder terms approaching zero at least as fast as $s^{-1/2}$. This follows from the transformed form of the kernel, $\tilde{K}_\ell$, which we referred to above, in which $\tilde{K}_\ell$ is bounded and the range of indices finite. We find, for this new kernel, $\tilde{K}_\ell < (1/\sqrt{s})n$, where n is some finite constant. The details are in [1].

Now $D_\ell(s)$ may be written in the form

$$D_\ell(s) = 1 + [f(s)/(\ell + 1)] + g(s, \ell) \tag{10}$$

where $f(s)$ and $g(s, \ell)$approach zero at infinite s and $g(s, \ell)$ is regular in ℓ for Re $\ell > -3/2$. For large s there may be a root, $D_\ell(s) = 0$, only near $\ell = -1$. The Regger trajectory is given by the solution to

$$\ell = -1 - f(s) - g(s, \ell)(\ell + 1) \tag{11}$$

and the asymptote is clearly $\ell = -1$.

The lowest order Regge trajectory follows from computing the lowest order $f(s)$(which we call $f_1(s)$) in Eq. (10). Using the expansion of $D_\ell(s)$:

$$D_\ell(s) = 1 - \mathrm{Tr}\, K + \frac{(\mathrm{Tr}\, K)^2 - \mathrm{Tr}\, K^2}{2} + \dots$$

we see that $[f_1(s)/(\ell + 1)]$ is given by the singular part of $-\mathrm{Tr}\, K$,

$$-\mathrm{Tr}\, K_{sing} = \frac{i g^2}{(2\pi)^3(\ell + 1)} \int \frac{dp\, d\omega}{F(p, \omega, s)} = \frac{g^2}{8\pi^2(\ell + 1)} \int \frac{ds'}{(s' - s - i\epsilon)\sqrt{s'(s' - 4m^2)}}.$$

Therefore $\alpha(s)$ is given in lowest order by:

$$\alpha(s) = -1 + \frac{g^2}{8\pi^2} \int \frac{ds'}{(s' - s - i\epsilon)\sqrt{s'(s' - 4m^2)}} + 0(g^4) \dots \tag{12}$$

This is a valid expansion for sufficiently weak coupling except near threshold, $s = 4m^2$. It is straightforward but tedious to work out the next order in the expansion of $\alpha(s)$ and we have not done it. In the next order the formula should show some effect of the production threshold.

Now we shall use our first order expression for $\alpha(s)$ to determine the asymptotic behaviour of the sum of the graphs (Fig. 4) in the $\lambda\varphi^3$ theory.

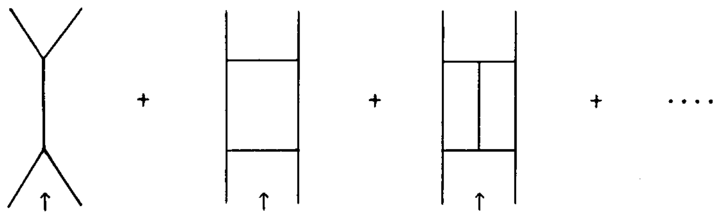

Fig. 4

To be able to make this connection we need, of course, to open up the Watson contour as in the work of Regge. That is, we need convergence of an integral along a line Re $\ell = \ell_0$ and the vanishing of an integral along an infinite semicircle. For our amplitude we have proved these properties only for $\ell_0 > -1/2$, but we shall need them for ℓ_0 slightly less than -1. So let us simply assume for the moment that everything is all right. In addition we require a modification of the Regge formula, due to Mandelstam. to take account of poles to the left of $\ell = -1/2$. The outcome is simply that the leading asymptotic term in the sum of the above graphs is simply:

$$\tilde{T}(s,t) \underset{s\to\infty}{\longrightarrow} \beta(t) s^{\alpha(t)} \tag{13}$$

where $\tilde{T}(s,t)$ is the T-matrix element for the above "crossed ladder graphs", $\alpha(t)$ is the trajectory we just computed (s and t were interchanged when the graphs were crossed).

We write:

$$\alpha(t) = -1 + g^2 \alpha_1(t) + g^4 \alpha_2(t) + \dots$$

$$\beta(t) = g^2 \beta_1(t) + g^4 \beta_2(t) + \dots$$

According to our previous calculation:

$$\alpha_1(t) = \frac{1}{8\pi^2} \int_{4m^2}^{\infty} \frac{dt'}{(t'-t)\sqrt{t'(t'-4m^2)}} \, . \tag{14}$$

It turns out that the lowest order β is given by:

$$\beta_1 = (2\pi)^{-4}$$

Now expanding Eq. (13) in powers of g^2 we obtain:

$$\underset{s\to\infty}{T(s,t)} = \frac{g^2\beta_1}{s} + \frac{g^4\beta_1\alpha_1(t)\log s}{s} + \frac{g^4\beta_2(t)}{s} + 0(g^n) \, . \tag{15}$$

Note that the fourth order term which goes as log s/s' can be computed exactly in terms of the second order α and β. We have checked this connection by calculating the logs term in the fourth order box diagram directly. We see also the term in the second order diagram of order $g^{2n}(\log s)^n s^{-1}$ can also be given in terms of the lowest order β_1 and α_1 functions. It is simply:

$$g^{2n}\beta_1(\alpha_1(t))^n (\log s)^n/s.$$

Thus it is seen that the Regge idea coupled with perturbation theory provides a very powerful technique for summing the most divergent parts (as $s \to \infty$) of sets of Feynman graphs. This technique may be useful in field theory whether the entire scattering amplitude is an analytic function of ℓ or not.

One trivial generalization of our model is the inclusion of a mass spectrum for the exchanged particle. We consider a scattering amplitude derived from replacing B_ℓ in Eq. (2) by:

$$\int Q_\ell\left(\frac{p^2+p'^2+y-(\omega-\omega')^2}{2pp'}\right)\sigma(y)\,dy\,.$$

The interesting change in the previous results occurs when $\sigma(y)$, the mass spectral function, goes to zero more slowly than y^{-2}. Let us assume a behaviour:

$$\sigma(y) \longrightarrow y^{-\eta}$$

as

$$y \longrightarrow \infty$$

where

$$0 < \eta < 1.$$

The asymptote of the leading Regge trajectory is now, $\ell = -\eta$. This is the generalization of a result for potential scattering with a potential given by:

$$V(r) = \int_{y_0}^{\infty} \frac{e^{-\sqrt{y}\,r}}{r}\,dy\,\sigma(y).$$

There is one difference; our method in potential scattering fails when $\eta \leq +1/2$, that is, when the potential is more singular than r^{-2} at the origin. In the relativistic theory all $\eta > 0$ are allowed. In relativistic theory, therefore, the asymptote may move as far to the right as $\ell = 0$ in the limit $\eta \to 0$. This would be the case for the $\lambda\varphi^4$ theory, in the sum of the diagrams shown in Fig. 5.

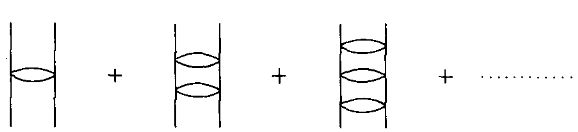

Fig. 5

Here the basic bubble exchange has a mass spectral function which tends to a constant at infinity.

We see that the asymptotes of the trajectories are extremely dependent on the details of the short range force. It is this that makes me pessimistic about the possibility of doing calculations of trajectories in a realistic model using present day techniques. One surely must include, in addition to exchange of pions, nucleon exchanges, hyperon exchanges and exchanges of everything else if one sets out to calculate the asymptotic parts of the trajectories (which would be useful in interpreting high-energy scattering at large momentum transfers). Is there, nevertheless, a reason why the long range terms alone should dominate the trajectories near $\sigma = 0$ (the diffraction region)? Probably there is not. Note that what I am discussing here is not

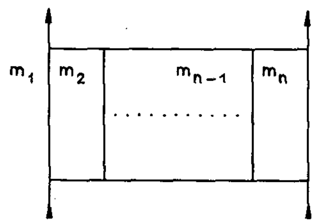

Fig. 6

peripheralism. In the high-energy diagram shown in Fig. 6, it is not a question of what the masses of the horizontal lines should be (the peripheral question). It is rather a question of whether the immensely massive intermediate state, $(m_1, m_2 \ldots\ldots m_n)$, should consist of many particles of low mass or of somewhat fewer particles, some of which are quite massive. The first possibility corresponds to considering only the long range force in the crossed (ladder) channel; the second to including shorter range effects as well.