# WIGNER 3j-SYMBOLS AND THE LORENTZ GROUP

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Summary Use of a group $\mathcal{L}^{\uparrow}(C_2)$ which double covers $\mathcal{L}^{\uparrow}$, and of the spanning property of the spinor light cone, leads to a rapid derivation of the properties of Wigner's 3j-symbols. No obscure computations, choices of phase, or the like, are needed. The reality of the 3j-symbols follows from their invariance under the antilinear space reversal $\in \mathcal{L}^{\uparrow}_{-}(C_2)$. Some (possibly) new recursion relations are established. It is noted that classical invariant theory made use of the spinor light cone a century ago, and that the classical 3j-symbol takes integer values.

1. Introduction For some years now the author has been engaged in writing a unified coordinate-free account of what may be described briefly as "the mathematics of Minkowski space", or in a little more detail as "the linear, multilinear and antilinear algebra of Minkowski space M, the Lorentz group $\mathcal{L}$, and of associated spaces and groups". Hopefully, in the near future, the complete work will be published in book form. The present talk will describe a rather small subset of this work, namely that dealing with the 3j-symbols for the $D^j$ representations of $\mathcal{L}$.

Traditional accounts[1,2] of the Wigner 3j-symbols and Clebsch-Gordan coefficients for the 3-dimensional rotation group can be criticized in that they make many of the important properties of the coefficients appear in a far from clear light. This lack of clarity is produced chiefly by (i) weakly-motivated choices of phase (ii) proofs involving computations of a somewhat complicated and murky nature (iii) treating the less symmetrical CG-coefficients before the more symmetrical 3j-symbols (iv) dealing with components of an object rather then the object itself. To correct these defects, the present account deals first with a certain trilinear invariant, then with its components (the 3j-symbols); next the trilinear invariant is used to introduce certain linear maps, and only then do we deal with the matrices of these maps (the CG-coefficients).

Important, but rather less traditional, accounts of the subject have been given by Schwinger[3], using certain operator methods, and by Bargmann[4], using function space methods. In contrast with these contributions the present account is mathematically simpler to the extent that it uses nothing more than the linear (and antilinear) algebra of finite-dimensional vector spaces. (However, at certain points, the present account would appear to be quite close to that of Bargmann.)

An essential ingredient of the present account is to treat the 3j-symbols as belonging to the representation theory of $SL(C_2)$ - which we view in its metrical guise of $Sp(C_2)$ - rather than that of $SU(C_2)$; moreover we use as well the antisymplectic transformations $ALSp(C_2)$, which adjoin to $Sp(C_2)$ to form a group $\mathcal{L}^{\uparrow}(C_2) = ALLSp(C_2)$ which (see Eq. (2.7)) double covers the orthochronous Lorentz group $\mathcal{L}^{\uparrow}$. Each choice of time-axis in Minkowski space defines a space inversion operator $\mathcal{P} \in ALSp(C_2)$, and picks out a corresponding $SU_{\mathcal{P}}(C_2)$ subgroup of $Sp(C_2)$, with inner product $(\ ,\ )_{\mathcal{P}}$ defined, as in Eq. (2.12), by

$$(\xi_1, \xi_2)_{\mathcal{P}} = [\mathcal{P}\xi_1, \xi_2], \quad \xi_i \in C_2 \quad . \tag{1.1}$$

Starting from the one frame-independent bilinear form $[\ ,\ ]$ on $C_2$, we are of course at liberty at any stage to specialize our considerations to any one of the host of frame-dependent hermitian forms $(\ ,\ )_{\mathcal{P}}$.

Traditional accounts, which start out from SU(2), proceed in the opposite direction. They make the belated discovery of an antilinear operator $\mathcal{P}$ commuting with SU(2)-transformations, and so can introduce a bilinear form by $[\xi_1, \xi_2] = (\mathcal{P}^{-1}\xi_1, \xi_2)$. However, having started out from a particular hermitian form ( , ), they tend to concentrate upon it and to play down the role of the bilinear form $[\ ,\ ]$. But in fact it is the latter which is of paramount importance — even if in the end we specialize to SU(2) — since it is required in the definition of the fundamental trilinear invariant (in Section 4.1).

As a lead-in to the treatment of 3j-symbols in Section 4, several useful theorems concerning the space $V^j$ of (2j+1)-component spinors will be stated (Theorems 3.2 - 3.6). The last of these is particularly noteworthy, in that it demonstrates that the crucial structure inherited by $V^j$ from $C_2$ is not (in general) the metrical structure but is the <u>spinor light cone structure</u> $N^j$, defined in Section 3.1.

Most of the methods employed in this account are in essence far from modern, and many go back more than a century ago! In particular we do not scorn the use of classical bases, and draw the reader's attention to the fact that the <u>classical 3j-symbols take integer values</u>! Possibly more use could be made of this — compare, for example, the simplicity of the classical recursion relation (4.18) to its standard from (4.19).

Next, a brief word concerning notation. We use the logograms L, AL to denote maps which are, respectively, linear, antilinear; their combination ALL is used to denote "all" such maps, i.e. linear <u>and</u> antilinear ones.

Finally we point out that, due to lack of space, several proofs, including that of Theorem 3.6, have had to be omitted.

## 2. <u>The multiantilinear algebra of the Lorentz group</u>.

Only an abbreviated account of this topic will be given, tailored to the needs of the intended applications.

**2.1** <u>The space $C_2$ of Lorentz 2-component spinors</u>. Let $C_2$ denote a complex 2-dimensional vector space which is equipped with symplectic geometry by means of a (non-degenerate) skew-symmetric bilinear form $[\ ,\ ]$:

$$[\xi_1, \xi_2] = -[\xi_2, \xi_1] \in \mathbb{C}, \qquad \xi_i \in C_2 \tag{2.1}$$

Let $\mathcal{L}_+^{\uparrow}(C_2) \equiv Sp(C_2)$ and $\mathcal{L}_-^{\uparrow}(C_2) \equiv ALSp(C_2)$ denote the sets of isometries and anti-isometries of $C_2$. Together they form a subgroup $\mathcal{L}^{\uparrow}(C_2) \equiv ALLSp(C_2) \equiv Sp(C_2) \cup ALSp(C_2)$ of the group $GALL(C_2) \equiv GL(C_2) \cup GAL(C_2)$ the latter group consisting of all the linear and antilinear automorphisms of $C_2$, while the mappings belonging to the subgroup satisfy in addition the invariance property

$$[a\xi_1, a\xi_2] = [\xi_1, \xi_2]^a, \quad a \in \mathcal{L}^{\uparrow}(C_2). \tag{2.2}$$

Here $\lambda^a$, for $\lambda \in \mathbb{C}$, denotes $\lambda$ or $\bar{\lambda}$ according as <u>a</u> is a linear or antilinear map.

**2.2.** <u>The space $V^{j,j'}$ of (j,j')-spinors</u>. Let $V^j$ denote the 2j th symmetrized tensorial power $\vee^{2j}C_2$ of $C_2$, and let $D^j(a)$, for $a \in GALL(C_2)$, denote the restriction to $V^j$ of $\otimes^{2j}a$. Define also the spaces $V^{j,j'}$, and corresponding corepresentations $D^{j,j'}$ of $GALL(C_2)$, by

$$V^{j,j'} = V^j \otimes \overline{V^{j'}}, \quad D^{j,j'}(a) = D^j(a) \otimes \overline{D^{j'}(a)}, \quad a \in GALL(C_2), \tag{2.3}$$

where $\bar{V}$ denotes the antispace of V. In the case $j = j'$ we write $R^{j,j}$ for the real vector space consisting of those elements of $V^{j,j}$ which are real under the natural conjugation $\phi_1 \otimes \phi_2 \to \phi_2 \otimes \phi_1$, $\phi_i \in V^j$, and note that $D^{j,j}(a)$ can be thought of as a real operator upon $R^{j,j}$.

In particular we mention (i) the space $C_3$ of Lorentz complex 3-vectors (ii) real 4-dimensional Minkowski space M (iii) its complexification $M^{\mathbb{C}}$ (iv) the space of Weyl (5-component) spinors (v) the (real) space of trace-free Ricci tensors, given respectively by

$$\begin{aligned}
&\text{(i)} \quad C_3 = V^1 = C_2 \vee C_2 \cong Sk(C_2,C_2) \cong L_0(C_2,C_2),\\
&\text{(ii)} \quad M = R^{\frac12,\frac12} \cong ALSk(C_2,C_2)^5,\\
&\text{(iii)} \quad M^{\mathbb{C}} = V^{\frac12,\frac12} = C_2 \otimes \bar{C}_2 \cong AL(C_2,C_2)^5,\\
&\text{(iv)} \quad V^2 = \vee^4 C_2 \cong S_0(C_3,C_3) \cong (C_3 \vee C_3)_0 \cong \text{the space of binary quartics}^{6,7},\\
&\text{(v)} \quad R^{1,1} \cong ALS(C_3,C_3) \cong S_0(M,M) \cong (M \vee M)_0^{7,12}.
\end{aligned} \tag{2.4}$$

2.3 Induced scalar products and isometries. The space $V^{j,j'}$ inherits a non-degenerate scalar product $[\ ,\ ]$ from that on $C_2$. In particular that on $V^j$ is defined to be the restriction to $\vee^{2j}C_2$ of the usual induced scalar product upon $\otimes^{2j}C_2$; thus if $\phi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_{2j}$ and $\psi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_{2j}$ are two (j,0)-spinors, their scalar product involves the permanent of the $(2j+1) \times (2j+1)$-matrix whose ik-element is the scalar product $[\xi_i, \eta_k]$:

$$(2j)!\,[\phi,\psi] = \text{permanent}\,\big([\xi_i,\eta_k]\big). \tag{2.5}$$

The scalar product upon $V^j$ clearly satisfies

$$[\phi,\psi] = (-)^{2j}[\psi,\phi]\,, \quad \phi,\psi \in V^j; \tag{2.5a}$$

in particular the geometry on $C_3 = V^1$ is complex orthogonal. That[10] on $M = R^{\frac12,\frac12}$ is real orthogonal, with signature (+ − − −) —as can be checked using the basis (2.10) below— so that M is indeed a Minkowski space.

If $a \in \mathcal{L}^{\uparrow}(C_2)$, then $D^{j,j'}(a)$ is clearly an isometry, or antiisometry, of $V^{j,j'}$. In particular

$$[D^j(a)\phi\,,\,D^j(a)\psi] = [\phi,\psi]^a\,, \quad a \in \mathcal{L}^{\uparrow}(C_2). \tag{2.6}$$

However in general the group homomorphism $D^{j,j'}$ has image only some "small" subgroup of the isometry group of $V^{j,j'}$. As will be noted below, in Section 3.1 and footnote 16, the cases $(j,j') = (\frac12,\frac12)$, or = (1,0), are exceptional in that the homomorphisms $D^{\frac12,\frac12}: a \mapsto a \otimes \bar{a}$, and $D^1: a \mapsto a \vee a$, give rise to group isomorphisms

$$\mathcal{L}^{\uparrow}(C_2)/Z_2 \cong \mathcal{L}^{\uparrow}, \text{ and } \mathcal{L}^{\uparrow}(C_2)/Z_2 \cong ALLO_+(C_3). \tag{2.7}$$

2.4 Product bases. Each symplectic basis $\{\xi,\eta\}$ for $C_2$, satisfying that is $[\xi,\eta] = 1$, gives rise to an associated product basis for $V^{j,j'}$. In particular the associated standard basis for $V^j$ is $\{e^j_m \,;\, m = -j,\ldots,+j\}$, where $e^j_m$ is defined by

$$e^j_m = \sqrt{\binom{2j}{m}}\;\xi^{j+m}\eta^{j-m}. \tag{2.8}$$

(Here and below we use the abbreviation $\xi^{j+m}\eta^{j-m}$ to denote the symmetrized product $\xi \vee \xi \vee \cdots \vee \xi \vee \eta \vee \cdots \vee \eta$ of $j + m$ factors $\xi$ and $j - m$ factors $\eta$.) The associated metric tensor in the space $V^j$ is the standard 1j-symbol[11]:

$$\begin{pmatrix} j \\ m\ m' \end{pmatrix} \equiv [e^j_m, e^j_{m'}] = (-)^{j-m}\delta_{m,-m'}. \tag{2.9}$$

(The relevant permanent — see Eq (2.5) — contains a $(j+m)\times(j+m)$ block of +1's and a $(j-m)\times(j-m)$ block of -1's).

The associated product basis $\{\xi\otimes\bar{\xi},\ \eta\otimes\bar{\eta},\ \xi\otimes\bar{\eta},\ \eta\otimes\bar{\xi}\}$ in the space $M^{\mathbb{C}}$ is a null tetrad basis, from which we construct an associated (real) orthonormal basis[10] $\{x,y,z,t\}$ for M:

$$\sqrt{2}x = \xi\bar{\eta}+\eta\bar{\xi},\ \sqrt{2}y = -i(\xi\bar{\eta}-\eta\bar{\xi}),\ \sqrt{2}z = \xi\bar{\xi}-\eta\bar{\eta},\ \sqrt{2}t = \xi\bar{\xi}+\eta\bar{\eta}, \tag{2.10}$$

whose metric tensor is diag( - - - +), in confirmation of the signature of M.

It is not difficult to find a set of canonical forms for $\mathcal{L}_-^{\uparrow}(C_2)$, as well as for $\mathcal{L}_+^{\uparrow}(C_2)$. One can then use the group isomorphisms[12] (2.7) to deduce a set of canonical forms for $\mathcal{L}_-^{\uparrow}$ and $ALO_+(C_3)$, as well as for $\mathcal{L}_+^{\uparrow}$ and $O_+(C_3)$. We content ourselves here with just two simple instances. Given the basis $\{\xi,\eta\}$, the simplest $\mathcal{L}_-^{\uparrow}(C_2)$-transformation is the conjugation[8] defined by $\xi\mapsto\xi,\ \eta\mapsto\eta$. It follows from Eq. (2.10) that at the M-level this transformation is space reversal with respect to the y-axis: $x\mapsto x,\ y\mapsto -y,\ z\mapsto z,\ t\mapsto t$. At the $V^j$-level, this space reversal is the antilinear[8] map given simply by $e_m^j\mapsto e_m^j$, for each $m = -j,\ldots,j$.

Another simple $\mathcal{L}_-^{\uparrow}(C_2)$-transformation is that (antilinear[8]) map defined by its effect[15] $\xi\mapsto -\eta,\ \eta\mapsto\xi$ on the basis $\{\xi,\eta\}$. At the M-level it is space inversion $\{x,y,z,t\}\mapsto\{-x,-y,-z,t\}$. Noting that the basis $\{-\eta,\xi\}$ is left dual to the basis $\{\xi,\eta\}$, observe that at the $V^j$-level space inversion $\mathcal{P}$ is the antilinear[8] map given by[15] $\mathcal{P}e_m^j = e_j^m$, where $\{e_j^m\}$ is left dual to $\{e_m^j\}$ — i.e. $e_j^m$ is defined by

$$\left[e_j^m,\ e_{m'}^j\right] = \delta_{m'}^m, \tag{2.11}$$

or equivalently by replacing $\{\xi,\eta\}$ by $\{-\eta,\xi\}$ in the definition of $e_m^j$ in Eq. (2.8). We can use $\mathcal{P}$ to define a hermitian form $(\ ,\ )_{\mathcal{P}}$ on $V^j$ by

$$(\phi_1,\phi_2)_{\mathcal{P}} = [\mathcal{P}\phi_1,\phi_2],\ \phi_i\in V^j, \tag{2.12}$$

w.r.t. which $e_j^m$ is an orthonormal basis (in the strict sense). Thus each choice of space inversion and hence of frame (time-axis), in Minkowski space M results in a choice of positive definite unitary geometry for $V^j$ via the inner product $(\ ,\ )_{\mathcal{P}}$.

2.5 Classical bases and components. While standard bases for $V^j$ possess simple normalization properties, for many purposes — as was realized a century ago (bearing in mind Theorem 3.2 below) — it is better to avoid irrationalities and use instead classical bases of the type $\{E_\lambda^j;\ \lambda = 0,1,\ldots,2j\}$ where

$$E_\lambda^j = \binom{2j}{\lambda}\xi^{2j-\lambda}\eta^\lambda,\quad \lambda = 0,1,\ldots,2j. \tag{2.13}$$

The components of a general element $\phi\in V^j$ relative to the two types of basis will be denoted $(\phi^m)$ and $(\Phi^\lambda)$:

$$\phi = \sum_{m=-j}^{j}\phi^m e_m^j = \sum_{\lambda=0}^{2j}\Phi^\lambda E_\lambda^j. \tag{2.14}$$

The relation between the two sets of bases and components is thus

$$E_\lambda^j = \sqrt{\binom{2j}{\lambda}}\,e_m^j,\quad \phi^m = \sqrt{\binom{2j}{\lambda}}\ ,\ \text{where } \lambda = j-m. \tag{2.15}$$

## 3. The space $V^j = \vee^{2j}C_2$ of (j,0) - spinors.

3.1 The spinor light cone $N^j$. An element $\phi\in V^j$ which is of the highly special form $\phi = \xi^{2j}$, for some non-zero $\xi\in C_2$, will be termed a nil spinor[16]. The spinor light cone[17,18] $N^j$ of $V^j$ is defined to consist of all the nil spinors of $V^j$. (These definitions can be generalized[16] in an obvious fashion to $(j_1,j_2)$-spinors.)

Clearly the image $T = D^j(a)$ of a "Lorentz transformation" $a \in \mathcal{L}^{\uparrow}(C_2)$ has the property of preserving the cone $N^j$, since $T\xi^{2j} = \eta^{2j}$, where $\eta = a\xi$. Conversely, if $T \in \mathrm{GALL}(V^j)$ preserves $N^j$ then Theorem 3.6 below implies that T is a scalar multiple of $D^j(a)$ for some $a \in \mathcal{L}^{\uparrow}(C_2)$. Consequently <u>the crucial structural carried by the space $V^j$ is the spinor light cone</u> $N^j$, and not (at least when $j > 1$ ← see footnote 16) the metrical structure $[\ ,\ ]$.

3.2 <u>Theorem</u> Each choice of basis $\{\xi, \eta\}$ for $C_2$ gives rise to an isomorphism of $V^j$ with the space of binary 2j-ics (i.e. the space of polynomials over $\mathbb{C}$ of homogeneous degree 2j in two indeterminates $\xi, \eta$).

3.3 <u>Binomial theorem</u> : $(\xi + z\eta)^{2j} = \sum_{\lambda=0}^{2j} z^{\lambda} E_{\lambda}^{j}$, $(z \in \mathbb{C},\ \xi, \eta \in C_2)$.

3.4 <u>"Penrose's[19] theorem"</u> (=[20] Fundamental theorem of algebra). Every element $\phi \in V^j$ is decomposable; that is there exist $\xi_1, \xi_2, \ldots, \xi_{2j} \in C_2$ such that

$$\phi = \xi_1 \xi_2 \cdots \xi_{2j} \quad (\equiv \xi_1 \vee \xi_2 \vee \cdots \vee \xi_{2j}).$$

Moreover, if $\phi \neq 0$, the factors $\xi_1, \ldots, \xi_{2j}$ are subject to permutations and to rescalings of the type $\xi_i \mapsto \lambda_i \xi_i$, with $\lambda_1 \lambda_2 \ldots \lambda_{2j} = 1$, but are otherwise uniquely determined by $\phi$.

3.5 <u>Theorem</u>. The spinor light cone $N^j$ spans $V^j$.

3.6 <u>Theorem</u>. If $T \in \mathrm{GALL}(V^j)$, then T preserves the spinor light cone $N^j$ if and only if $T = D^j(a)$ for some $a \in \mathrm{GALL}(C_2)$.

3.7 <u>Remarks</u> (a) Of the above five theorems, the odd one out is Theorem 3.4 is that it is peculiar to the dimension 2 of the base space $C_2$. The other four theorems generalize to dimension $n > 2$ (n-ary quantics, multinomial theorem, etc.); Theorem 3.2 can be paraphrased in the statement "symmetric algebra = coordinate-free polynomial algebra".

(b) Theorems 3.5 and 3.6 readily generalize to the case of $(j_1, j_2)$-spinors.

(c) On account of Theorem 3.5, a multilinear mapping $M : V^{j_1} \times V^{j_2} \times \ldots \to W$ is determined by its values $M(\xi_1^{2j_2}, \xi_2^{2j_2}, \ldots)$ on nil spinors. The values $M(\phi_1, \phi_2, \ldots)$ on general spinors can then be reconstituted by means of <u>polarization</u>, upon using Theorem 3.3. Of course, these are the methods familiar from Classical Invariant Theory (see Ch.8A of Ref.21). Before applying such methods to Wigner's 3j-symbols, let us give a very simple illustration of them.

3.8 <u>Illustration</u>: the bilinear invariant $[\ ,\ ] : V^j \times V^j \longrightarrow \mathbb{C}$.

A bilinear map $[\ ,\ ] : V^j \times V^j \to \mathbb{C}$ is determined by its values upon the nil spinors. The set of values defined by

$$\left[\xi_1^{2j}, \xi_2^{2j}\right] = [\xi_1, \xi_2]^{2j}, \quad \xi_i \in C_2, \tag{3.1}$$

is a possible one; for the degrees on either side tally, thereby guaranteeing the existence of the requisite polarized version of our specialized starting point. (We are supposing, for the sake of this illustration, that we do not already know this completely polarized version — namely that given in Eq.(2.5))

Upon choosing a symplectic basis $\{\xi, \eta\}$ for $C_2$, and writing $\xi_i = \xi + z_i\eta$, $(z_i \in \mathbb{C})$, so that $[\xi_1, \xi_2] = z_2 - z_1$, use of the binomial theorem in Eq.(3.1) yields the value of the <u>classical 1j-symbol</u>

$$\left[\lambda_1 \ {}^{j} \ \lambda_2\right] \equiv \left[E_{\lambda_1}^{j},\ E_{\lambda_2}^{j}\right] \tag{3.2}$$

to be the coefficient of $z_1^{\lambda_1} z_2^{\lambda_2}$ in $(z_2 - z_1)^{2j}$. Thus

$$\left[\lambda_1 \ {}^{j} \ \lambda_2\right] = (-)^{\lambda_1} \binom{2j}{\lambda_1} \delta_{\lambda_1+\lambda_2, 2j} \quad . \tag{3.3}$$

We thus obtain the well-known bilinear invariant

$$[\varphi, \psi] = \sum_{\lambda=0}^{2j} (-)^{\lambda} \binom{2j}{\lambda} \Phi^{\lambda} \Psi^{2j-\lambda} \tag{3.4}$$

of two binary 2j-ics. (The invariance property (2.6) follows from Eq. (3.1) by virtue of the corresponding $\mathcal{L}^{\uparrow}(C_2)$-invariance property (2.2).)

In particular when j = 2, a general element $\phi = \sum \Phi^{\lambda} E^2_{\lambda}$ of $V^2$ is identified (under the isomorphism of Theorem 3.2) with the binary quartic

$$\phi = \Phi^0 \xi^4 + 4\Phi^1 \xi^3 \eta + 6\Phi^2 \xi^2 \eta^2 + 4\Phi^3 \xi \eta^3 + \Phi^4 \eta^4, \tag{3.5}$$

and we obtain the familiar quadratic invariant $\mathcal{J}$ of the binary quartic:

$$\mathcal{J} = \tfrac{1}{2} [\phi, \phi] = \Phi^0 \Phi^4 - 4\Phi^1 \Phi^3 + 3(\Phi^2)^2. \tag{3.6}$$

## 4. Trilinear invariants and the Wigner 3j-symbols

### 4.1 Trilinear invariant $[\ ,\ ,\ ] : V^{j_1} \times V^{j_2} \times V^{j_3} \to \mathbb{C}$.

The definition of this upon nil spinors by

$$\left[ \xi_1^{2j_1}, \xi_2^{2j_2}, \xi_3^{2j_3} \right] = K [\xi_2, \xi_3]^{k_1} [\xi_3, \xi_1]^{k_2} [\xi_1, \xi_2]^{k_3}, \tag{4.1}$$

where K is a normalization constant, will succeed provided only that the "degrees tally": more precisely, non-negative integers[20] $k_1, k_2, k_3$ must exist such that

$$2j_1 = k_2 + k_3, \quad 2j_2 = k_3 + k_1, \quad 2j_3 = k_1 + k_2. \tag{4.2}$$

These equations can be solved, the solution being given uniquely by:

$$\begin{aligned} k_1 &= j_2 + j_3 - j_1, & k_2 &= j_3 + j_1 - j_2, & k_3 &= j_1 + j_2 - j_3, \\ &= J - 2j_1 & &= J - 2j_2 & &= J - 2j_3 \end{aligned} \tag{4.3}$$

where $J = j_1 + j_2 + j_3 = k_1 + k_2 + k_3$, provided only that $j_1, j_2, j_3$ from a triangle of integer perimeter:

$$j_2 + j_3 \leqslant j_1, \quad j_3 + j_1 \leqslant j_2, \quad j_1 + j_2 \leqslant j_3; \quad J = \text{integer}. \tag{4.4}$$

Except for the arbitrariness in the choice of K, no other trilinear invariant exists (see Theorem 2.6A , or 6.1A, of reference 21 — or see Eq. (5.6) below) For a reason given later, we choose $K = K(j_1, j_2, j_3)$ to be

$$K(j_1, j_2, j_3) = \left\{ [2j!] / [k!] \ (J+1)! \right\}^{\frac{1}{2}}, \tag{4.5}$$

where we have used the abbreviation $[p!] \equiv p_1! p_2! p_3!$ .

### 4.2 Properties.

Eqs (2.5a), (4.1), (2.2) and immediately yield the invariance property

$$\left[ D^{j_1}(a)\phi_1, D^{j_2}(a)\phi_2, D^{j_3}(a)\phi_3 \right] = [\phi_1, \phi_2, \phi_3]^a, \quad a \in \mathcal{L}^{\uparrow}(C_2), \tag{4.6}$$

and also the permutational symmetry property

$$\left[ \phi_{\sigma(1)}, \phi_{\sigma(2)}, \phi_{\sigma(3)} \right] = (\mathrm{sgn}\,\sigma)^J [\phi_1, \phi_2, \phi_3], \quad \phi_i \in V^{j_i}. \tag{4.7}$$

Since $K(0,j,j) = (2j+1)^{-\frac{1}{2}}$, we also have

$$[1, \phi, \psi] = [\phi, \psi] / (2j+1)^{\frac{1}{2}}, \quad \phi, \psi \in V^j. \tag{4.8}$$

### 4.3 Standard and classical 3j-symbols.

These are defined respectively by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left[ e^{j_1}_{m_1}, e^{j_2}_{m_2}, e^{j_3}_{m_3} \right] \quad \text{and} \quad \begin{bmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = K^{-1} \left[ E^{j_1}_{\lambda_1}, E^{j_2}_{\lambda_2}, E^{j_3}_{\lambda_3} \right]. \tag{4.9(a),(b)}$$

By Eqs (2.15), (4.5) they determine each other by means of

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \left\{ \frac{[k!]\,(J+1)!}{[\kappa!][\lambda!]} \right\}^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{4.10}$$

where $\lambda_i = j_i - m_i$, $\kappa_i = j_i + m_i$.

4.4 Properties of 3j-symbols. Various well known properties of the 3j-symbols follow immediately from the general invariance property (4.6). In particular, upon setting $\phi_i = e^{j_i}_{m_i}$ and choosing $a \in \hat{\mathcal{L}}(C_2)$ to be a suitable (a) screw (b) $\pi$-rotation (c) space reversal[21a] (d) space inversion[21a] — namely such that $D^j(a)$ maps $e^j_m$ on to (a) $\alpha^{2m} e^j_m$ (b) $i^{2j} e^j_{-m}$ (c) $e^j_m$ (d) $e^m_j$ — we obtain the results:

(a) $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 0$ if $\sum m_i \neq 0$;

(b) $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^J \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}$;

(c) the 3j-symbols are real;

(d) the fully covariant and fully contravariant[22] forms of the 3j-symbol are equal:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} \left( = \left[ e^{m_1}_{j_1}, e^{m_2}_{j_2}, e^{m_3}_{j_3} \right] \right) .$$

Similarly the permutational properties of the 3j-symbols follow from Eq. (4.7), while (4.8) yields[11]

$$\begin{pmatrix} 0 & j & j \\ 0 & m & n \end{pmatrix} = (2j+1)^{-\frac{1}{2}} \begin{pmatrix} j \\ m\, n \end{pmatrix}.$$

Orthogonality properties are considered in Section 5.

4.5 Computation Setting $\xi_i = \xi + z_i \eta$ in Eq. (4.1) and using the binomial theorem (as in Section 3.8), immediately yields the value of the classical 3j-symbol to be the coefficient of $z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3}$ in $(z_3 - z_2)^{k_1} (z_1 - z_3)^{k_2} (z_2 - z_1)^{k_3}$. Thus the classical 3j-symbol takes integer values:

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \sum (-)^{q_1+q_2+q_3} \binom{k_1}{q_1}\binom{k_2}{q_2}\binom{k_3}{q_3}. \qquad (4.11)$$

Using Eq. (4.10) we deduce "Racah's formula[23]":

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left\{ \frac{[k!]\,[\kappa!]\,[\lambda!]}{(J+1)!} \right\}^{\frac{1}{2}} \sum_{p,q} \frac{(-)^{q_1+q_2+q_3}}{[p!][q!]} , \qquad (4.12)$$

where the summation is over all non-negative integers $p_i, q_i$ satisfying[24]

$$\begin{pmatrix} p_1+q_1 & p_2+q_2 & p_3+q_3 \\ p_2+q_3 & p_3+q_1 & p_1+q_2 \\ p_3+q_2 & p_1+q_3 & p_2+q_1 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} . \qquad (4.13)$$

(The summation in fact reduces to a single summation, the number of terms being $\mathrm{Min}(k_i, \lambda_i, \kappa_i) + 1$)

4.6 Illustration Consider the particular case $j_1 = j_2 = j_3 = 2$; upon adopting the abbreviation $[\lambda_1\lambda_2\lambda_3]$ for the corresponding classical 3j-symbol, we have from Section 4.5:

$$[\lambda_1\lambda_2\lambda_3] = \text{coef of } z_1^{\lambda_1} z_2^{\lambda_2} z_3^{\lambda_3} \text{ in } (z_2-z_3)^2(z_3-z_1)^2(z_1-z_2)^2. \qquad (4.14)$$

Thus $[024] = 1$, $[123] = 2$, $[033] = [411] = -2$, $[222] = -6$, and of course $[\lambda_1\lambda_2\lambda_3] = 0$ if $\sum\lambda_i \neq 6$.

Thus the expression of the trilinear invariant $[\phi_1, \phi_2, \phi_3]$, $\phi_i \in V^2$, in terms of classical components involves (very low-lying) integers, and should be contrasted with the irrationalities of the standard expression. In particular, on setting $\phi_1 = \phi_2 = \phi_3 = \phi$, we derive the well-known expression[25] for the cubinvariant $\mathcal{J}$ of the binary quartic $\phi \in V^2$ of Eq. (3.5):

$$\mathcal{J} = [\phi,\phi,\phi]/6K = \tfrac{1}{6}\sum \begin{bmatrix} 2 & 2 & 2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} \Phi^{\lambda_1}\Phi^{\lambda_2}\Phi^{\lambda_3}$$

$$= \Phi^0\Phi^2\Phi^4 + 2\Phi^1\Phi^2\Phi^3 - \Phi^0(\Phi^3)^2 - (\Phi^1)^2\Phi^4 - (\Phi^2)^3. \tag{4.15}$$

In terms of standard 3j-symbols and components:[26,27]

$$\mathcal{J} = \frac{(105)^{\frac{1}{2}}}{36}\sum \begin{pmatrix} 2 & 2 & 2 \\ m_1 & m_2 & m_3 \end{pmatrix} \phi^{m_1}\phi^{m_2}\phi^{m_3} . \tag{4.16}$$

4.7 <u>Recursion relations</u>. To derive recursion relations for fixed values of $j_1, j_2, j_3$, use the infinitesimal version of Eq. (4.6). For varying $j_1, j_2, j_3$, adopt factorizations of the kind $[\xi_2,\xi_3]^{k_1} = [\xi_2,\xi_3]^{k_1'}[\xi_2,\xi_3]^{k_1''}$ in Eq. (4.1), set $\xi_i = \xi + z_i\eta$ and use the binomial theorem again to obtain the relations[32]

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \sum_{\lambda_i'+\lambda_i''=\lambda_i} \begin{bmatrix} j_1' & j_2' & j_3' \\ \lambda_1' & \lambda_2' & \lambda_3' \end{bmatrix} \begin{bmatrix} j_1'' & j_2'' & j_3'' \\ \lambda_1'' & \lambda_2'' & \lambda_3'' \end{bmatrix} , \tag{4.17}$$

where $j_i' + j_i'' = j_i$. These are known in one or two special cases, at any rate when expressed in terms of standard symbols. For example, on setting $j_1' = 0$, $j_2' = j_3' = \frac{1}{2}$ we obtain

$$\begin{bmatrix} j_1 & j_2 & j_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \begin{bmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ \lambda_1 & \lambda_2 & \lambda_3-1 \end{bmatrix} - \begin{bmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ \lambda_1 & \lambda_2-1 & \lambda_3 \end{bmatrix} \tag{4.18}$$

which in terms of standard symbols reads

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left\{\frac{(j_2+m_2)(j_3-m_3)}{(J+1)\ (J-2j_1)}\right\}^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} - \left\{\frac{(j_1-m_2)(j_3+m_3)}{(J+1)\ (J-2j_1)}\right\}^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} \tag{4.19}$$

in agreement with Eq.3.7.12 of reference 28. Perhaps[33] the general relations (4.17) are new?

5. <u>The maps $\left\{ {}_{j_1}{}^{j_3}{}_{j_2} \right\}$ and Clebsch-Gordan coefficients</u>.

We content ourselves here with a brief sketch, and will omit certain details[29].

First of all an <u>invariant element</u>[30] $h \in V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$ is defined by

$$[\phi_1,\phi_2,\phi_3] = [h, \phi_1 \otimes \phi_2 \otimes \phi_3], \tag{5.1}$$

and so depends upon the choice of constant K in Eq. (4.1). If K is chosen to be real, then h is invariant under the natural action of $\mathcal{L}_-^{\uparrow}$ as well as $\mathcal{L}_+^{\uparrow}$; consequently $[h, h]$ is real and positive: $[h,h] = [\mathcal{P}h,h] = (h,h)_{\mathcal{P}} > 0$, since $(\ ,\ )_{\mathcal{P}}$ is positive definite. We may therefore fix K — and hence $[\ ,\ ,\ ]$ and h — by demanding $K > 0$ and $[h,h] = 1$. The actual value of K then turns out (after an apparently unavoidable computation) to be that given previously in Eq. (4.5).

Next a linear map $f : V^{j_3} \to V^{j_1} \otimes V^{j_2}$ is defined by

$$A[\phi_1 \otimes \phi_2 , f\phi_3] = [\phi_1, \phi_2, \phi_3] , \tag{5.2}$$

where $A = A(j_1,j_2,j_3)$ is a normalization constant. The invariance properties of $[\ ,\ ]$ and $[\ ,\ ,\ ]$ yield the intertwining property

$$\left[D^{j_1}(a) \otimes D^{j_2}(a)\right] \circ f = f \circ D^{j_3}(a), \quad a \in \mathcal{L}^{\uparrow}(C_2) . \tag{5.3}$$

Schur's lemma now tells us that f is injective. Let us denote Imf by $V^{j_3}_{j_1 j_2} \subset V^{j_1} \otimes V^{j_2}$; by restricting the target space of f we thus obtain a linear isomorphism

$$f_0 \equiv \left\{ {}_{j_1}{}^{j_3}{}_{j_2} \right\} : V^{j_3} \to V^{j_3}_{j_1 j_2} . \tag{5.4}$$

One next shows[29] that $f_o$ is necessarily a scalar multiple of an isometry; consequently we will fix $f_o$, or equivalently the constant A, up to a sign by demanding that $f_o$ equal an isometry. To compute $\pm$ A, note that upon identifying $L(V^{j_3}, V^{j_1}\otimes V^{j_2})$ with $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ in the obvious way, we have $Af = (-)^{2j_3}h$. Hence

$$A^2 = [h,h]/[\tilde{f},f] = 1/\mathrm{tr}(\tilde{f}f) = (2j_3+1)^{-1},$$

(since $\tilde{f}f$ = identity operator $V^{j_3}\to V^{j_3}$). Hence

$$A = \varepsilon\,(2j_3+1)^{-\frac{1}{2}}, \text{ where } \varepsilon = \varepsilon(j_1,j_2,j_3) \text{ is a sign ambiguity.} \tag{5.5}$$

Next (using the inequivalence of $D^j$ with $D^{j'}$ for $j \neq j'$) one shows[29] that the (non-singular) subspaces $V^{j_3}_{j_1 j_2}$ of $V^{j_1}\otimes V^{j_2}$ are mutually orthogonal, and one arrives at the decomposition

$$V^{j_1}\otimes V^{j_2} = \perp_{j_3}\, \delta(j_1,j_2,j_3)\; V^{j_3}_{j_1j_2}, \tag{5.6}$$

after checking that the dimensions tally. Here $\delta(j_1,j_2,j_3)$ is defined to equal 1 if the triangle conditions (4.4) are satisfied, and to equal 0 otherwise. (Of course the fact the multiplicity $\delta$ is 0 or 1 confirms our previous assertions concerning the existence and uniqueness of a trilinear invariant $V^{j_1}\times V^{j_2}\times V^{j_3}\to \mathbb{C}$.) Upon introducing the map

$$\begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix} = \oplus_{j_3}\begin{Bmatrix} & j_3 \\ j_1 & j_2\end{Bmatrix}, \tag{5.7}$$

we obtain the intertwining property

$$\left[D^{j_1}(a)\otimes D^{j_2}(a)\right]\circ \begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix} = \begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix}\circ\left[\oplus_{j_3} D^{j_3}(a)\right], \tag{5.8}$$

it being understood that the summation over $j_3$ is restricted by the condition $\delta(j_1,j_2,j_3) = 1$.

Finally, Clebsch-Gordan coefficients $\langle j_1\, j_2\, m_1\, m_2 | j_3\, m_3\rangle$ and $\langle j_3\, m_3 | j_1\, j_2\, m_1\, m_2\rangle$ are defined to be the matrix elements of the linear isomorphism $\begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix}$ and its inverse $\begin{Bmatrix} j_1 & j_2 \\ & * \end{Bmatrix}$ with respect to standard bases in the relevant spaces. Thus

$$f^{j_3}_{m_3} = \sum_{m_1m_2} e^{j_1j_2}_{m_1m_2}\langle j_1j_2m_1m_2|j_3m_3\rangle,\quad e^{j_1j_2}_{m_1m_2} = \sum_{j_3m_3} f^{j_3}_{m_3}\langle j_3m_3|j_1j_2m_1m_2\rangle, \tag{5.9}$$

where $e^{j_1j_2}_{m_1m_2} \equiv e^{j_1}_{m_1}\otimes e^{j_2}_{m_2}$ and $f^{j_3}_{m_3} \equiv \begin{Bmatrix} & j_3 \\ j_1 & j_2\end{Bmatrix} e^{j_3}_{m_3}$.

Equivalently the CG-coefficients are given by[31]

$$\langle j_1j_2m_1m_2|j_3m_3\rangle = \left[e^{m_1m_2}_{j_1j_2},\, f^{j_3}_{m_3}\right],\quad \langle j_3m_3|j_1j_2m_1m_2\rangle = \left[f^{m_3}_{j_3},\, e^{j_1j_2}_{m_1m_2}\right]. \tag{5.10}$$

Setting $\phi_1 = e^{m_1}_{j_1}$, $\phi_2 = e^{m_2}_{j_2}$, $\phi_3 = e^{j_3}_{m_3}$ in Eq. (5.2) we obtain[22]

$$\frac{\varepsilon(j_1j_2j_3)\,\langle j_1j_2m_1m_2|j_3m_3\rangle}{(2j_3+1)^{\frac{1}{2}}} = (-)^{2j_3}\begin{pmatrix} m_1 & m_2 & j_3 \\ j_1 & j_2 & m_3\end{pmatrix} = (-)^{j_3-m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3\end{pmatrix}. \tag{5.11}$$

Hence we may deduce properties of CG-coefficients from those in Section 4.4 of 3j-symbols. In particular the CG-coefficients are real; consequently upon using the (antilinear) space inversion map $e^j_m \mapsto e^m_j$ and $e^m_j \mapsto (-)^{2j}e^j_m$ in Eq. 5.10, we obtain $\langle j_1j_2m_1m_2|j_3m_3\rangle = \langle j_3m_3|j_1j_2m_1m_2\rangle$. In the case of orthogonality relations, however, it seems best to deduce those for the 3j-symbols from those for the CG-coefficients; for the latter are simply the expression of the fact that the matrix of the map $\begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix}$ is the inverse of the matrix of the map $\begin{Bmatrix} j_1 & j_2 \\ & *\end{Bmatrix} = \begin{Bmatrix} & * \\ j_1 & j_2\end{Bmatrix}^{-1}$.

The standard convention for fixing the sign $\varepsilon(j_1,j_2,j_3)$ is to demand that $\langle j_1j_2j_1(j_3-j_1)|j_3j_3\rangle$ be positive, which leads to

$$\varepsilon(j_1,j_2,j_3) = (-)^{k_2} = (-)^{j_3+j_1-j_2}. \tag{5.12}$$

Footnotes and References.

1. For some standard texts, see the bibliography to ref.2.

2. L.C. Biedenharn and H. van Dam (editors), The Quantum Theory of Angular Momentum, Academic Press 1965

3,4. See the articles by Schwinger and by Bargmann reprinted in ref.2.

5. The linear isomorphism $C_2 \otimes \bar{C}_2 \to AL(C_2,C_2)$ is given by $\xi \otimes \bar{\eta} \mapsto \xi\bar{\eta}$, where the latter denotes the antilinear dyad with effect $\zeta \longmapsto \xi\overline{[\eta,\zeta]}$. The isomorphism $R^{\frac{1}{2},\frac{1}{2}} \to ALSk(C_2,C_2)$ follows upon noting that the adjoint of the dyad $\xi\bar{\eta}$ is $-\eta\bar{\xi}$.

6. See Theorem 3.2 .

7. The linear isomorphisms $\vee^4 C_2 \to (C_3 \vee C_3)_0$ and $R^{1,1} \to (M \vee M)_0$ can be defined – see Theorem 3.5, footnote 16, and Remarks 3.7(b),(c) — by laying down that their effects upon nil spinors are respectively $\xi^4 \mapsto \xi^2 \otimes \xi^2$ and $\xi^2 \otimes \bar{\xi}^2 \longmapsto \xi\bar{\xi} \otimes \xi\bar{\xi}$.

8. Even though $D^j$ represents space inversion antilinearly, the group $\mathcal{L}(C_2)$ is still useful in the construction of manifestly covariant corepresentations of the extended Poincare group P appropriate to the physically relevant UA-decomposition $P^{\uparrow} \cup P^{\downarrow}$; see Section 4.3 of Ref.9.

9. R. Shaw and J. Lever, Commun. Math. Phys. 38, 279 (1974).

10. The action $\Lambda(a) = a \otimes \bar{a}$ upon $R^{\frac{1}{2},\frac{1}{2}}$ is, for $a \in \mathcal{L}^{\uparrow}(C_2)$, isomorphic to the action $p \mapsto a \circ p \circ a^{-1}$ upon $ALSK(C_2,C_2)$ used in Ref.9; (however for $a \in \mathcal{L}^{\downarrow}(C_2)^9$, an extra minus sign has to be introduced in the first action if it is to correspond to the second). The scalar product $[\xi\bar{\xi}, \eta\bar{\eta}] = [\xi,\eta]\overline{[\xi,\eta]}$ upon $R^{\frac{1}{2},\frac{1}{2}}$ corresponds to the scalar product $[p,q] = -\operatorname{tr}(p \circ q)$ upon $ALSk(C_2,C_2)$.

11. Our 1j-symbol is the transpose of that employed by Wigner in Ref.2.

12. The vector space isomorphisms (2.4) are also useful for solving canonical form problems. For example one can find canonical forms for an object $T \in ALS(C_3,C_3)$ i.e. for an antilinear map $T : C_3 \to C_3$ which is self-adjoint : $[T\phi, \psi] = \overline{[\phi, T\psi]}$, $\phi,\psi \in C_3$. (One way to do this is to use the "anti-Jordan" canonical form[13] for general antilinear operators.) Use of the isomorphism $ALS(C_3,C_3) \cong (M \vee M)_0$ then enables one to deduce a set of canonical forms for a trace-free Ricci tensor T, as given for example in Section 2 of Ref.14.

   Incidentally, since the square of an antilinear operator is a linear operator, $T \in ALS(C_3,C_3)$ implies $W \equiv T^2 \in S(C_3,C_3)$. Upon extracting the trace, we obtain the "Weyl square" $W_0 \in S_0(C_3,C_3)$ of the Ricci tensor T. The antilinear algebra way of introducing the Weyl square was in fact how the author first encountered it; for a possible use, see Section 5 of Ref.14.

13. R. Shaw (unpublished, 1969).

14. C. D. Collinson and R. Shaw, Intern. J. Theor. Phys., 6, 347 (1972).

15. Using the isomorphism $M \cong ALSk(C_2,C_2)$ again, $\mathcal{P}$ at the $C_2$-level is given by $\mathcal{P} = \sqrt{2}t$ ($= \xi\bar{\xi} + \eta\bar{\eta}$ , as in Eq.(2.10)), and so at the $V^j$-level it is $D^j(\sqrt{2}\, t)$, thus exhibiting clearly the dependence of $\mathcal{P}$ upon a particular time-axis.

16. A spinor $\phi \in V^{j_1,j_2}$ is said to be nil if it is of the form $\phi = \pm\xi^{2j_1} \otimes \bar{\xi}^{2j_2}$ for some non-zero $\xi \in C_2$ (the minus sign being needed only in the case $j_1 = j_2$). The set of nil spinors of $V^{j_1,j_2}$ forms the spinor light cone $N^{j_1,j_2}$.

   The term "nil" is used rather than "null", so as to reserve the latter to refer (as in "null tetrad basis") to a non-zero spinor of zero length: $[\psi, \psi] = 0$. Clearly every nil spinor is null. The cases when $(j_1, j_2)$ equals $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(1,0)$, $(0,1)$ or $(\frac{1}{2},\frac{1}{2})$ are exceptional in that every null spinor is nil in these cases (provided in the $(\frac{1}{2},\frac{1}{2})$ case we restrict our attention to the real space $M = R^{\frac{1}{2},\frac{1}{2}}$). In all the other cases there exist null spinors – for example the basic $(j,0)$-spinors $e^j_m$, with $m \neq 0$, $m \neq \pm j$ – which are null but not nil. Consequently in these other cases the images $D^{j_1,j_2}(a)$ of Lorentz transformations, which clearly preserve the cone $N^{j_1,j_2}$, can not be characterized entirely metrically.

17. I borrow this name from Dowker, J. S., - see Ref.18.
18. J. S. Dowker and M. Goldstone, Proc. Roy. Soc. A, 303, 381 (1968).
19. R. Penrose, Annals of Physics, 10, 171 (1960). 19a. Using Theorem 3.2.
20. In order that the requisite polarized version of the r.h. side should exist.
21. H. Weyl, The Classical Groups, Princeton University Press, 1946. 21a. As. in § 2.4.
22. Caution : in dealing with mixed[30] forms of the 3j-symbols, note, by Eqs. (2.5a), (2.11), that the left dual of the basis $\{e^m_j\}$ is $\{(-)^{2j} e^j_m\}$.
23. This can be traced back, via Van der Waerden (1932) and Weitzenböck (1923) to Clebsch and Gordan (1872).
24. The notation is as in Bargmann's article (Rev.Mod. Phys 34, 829 (1962)), which is reprinted in Ref.2. At this point one can spot the Regge symmetries.
25. See any classical text on invariant theory. Since the corresponding trilinear invariant $\mathcal{J}(\ ,\ ,\ )$ is determined by its values upon nil spinors by $\mathcal{J}(\alpha^4, \beta^4, \gamma^4) = \frac{1}{6}[\beta,\gamma]^2[\gamma,\alpha]^2[\alpha,\beta]^2$, the latter, in the classical literature, is referred to as the "symbolic expression" of the cubinvariant $\mathcal{J} \equiv \mathcal{J}(\phi,\phi,\phi)$, and $\alpha, \beta, \gamma$ are said to be "equivalent symbols".
26. The expression agrees with that in Ref.27, after taking into account a factor $\sqrt{8}$ due to a different normalization.
27. J. A. Roche and J. S. Dowker, J. Phys. A1, 527 (1968).
28. A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press, 1960.
29. In particular we do not state certain useful general theorems concerning representations of groups by means of the isometries of a complex vector space equipped with orthogonal or symplectic geometry. For the most part these follow, by familiar methods, from Schur's Lemma.
30. The fully covariant standard 3j-symbols are the covariant components of h, i.e. with respect to the basis $\left\{ e^{m_1}_{j_1} \otimes e^{m_2}_{j_2} \otimes e^{m_3}_{j_3} \right\}$; the mixed forms of the 3j-symbols are defined to be the corresponding mixed components of h. Consequently[22] take note of results such as
$$\begin{pmatrix} j_1 & j_2 & m_3 \\ m_1 & m_2 & j_3 \end{pmatrix} = (-)^{2j_3} \left[ e^{j_1}_{m_1},\ e^{j_2}_{m_2},\ e^{m_3}_{j_3} \right].$$
31. Since $\left\{ \begin{matrix} j \\ j_1 j_2 \end{matrix} \right\}$ is an isometry, $f^m_j \equiv \left\{ \begin{matrix} j \\ j_1\ j_2 \end{matrix} \right\} e^m_j$ is left dual to $f^j_m = \left\{ \begin{matrix} j \\ j_1 j_2 \end{matrix} \right\} e^j_m$.
32. Similarly, using factorizations into n factors, we obtain corresponding relations involving products of n standard 3j-symbols, whose triples $j^{(s)}_1 j^{(s)}_2 j^{(s)}_3$ satisfy $\sum_{s=1}^{n} j^{(s)}_i = j_i$. (Eq. 4.11 is an $n = 3$ instance!)
33. No! As witness to the success of the colloquium's poster sessions, I was informed by S. Ström of work by Bose and Patera - see Canad. J. Phys. 49, 947 (1971) - who in turn told me that my Eq(4.17) can be found in Vilenkin's book on Special Functions and Group Representations (but with no recognition of the integer-valued nature of the symbols).