# A CONTRIBUTION TO THE EIGHTH-ORDER ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON


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## ABSTRACT

The contribution to the anomalous magnetic moment of the electron from the eighth-order triple-bubble diagram is evaluated. The value obtained is

$$\left(\frac{\alpha}{\pi}\right) \left[-4\zeta(4) + \frac{32}{63}\zeta(3) + \frac{151849}{40824}\right]$$

One of the most critical tests of quantum electrodynamics[1] is the comparison between theory and experiment of the anomalous magnetic moment of the electron. The current situation[2] is as follows:

$$a_e^{exp} = (1159656.7 \pm 3.5) \times 10^{-9} \tag{1}$$

$$a_e^{theory} = (1159651.9 \pm 2.5) \times 10^{-9} \tag{2}$$

The theoretical result was obtained using $\alpha^{-1} = 137.03608\ (26)$ and the Levine and Wright result.[3] With the likelihood of continued improvement in the experimental accuracy, as well as on the theoretical side, the reduction in numerical integration errors provided by the on-going program[4] to analytically calculate previously numerically-determined sixth-order contributions, it is not unreasonable to suggest that, in the future, eighth-order calculations will be necessary.

In the case of the muon, estimates of the eighth-order contributions have been made.[5,6] But for the 891 diagrams (this includes mirror graphs) whose mass-independent contributions yield the eighth-order electron magnetic moment, no calculations have been made. Presumably, there will be tremendous cancellation among many contributions yielding a value[7] of order $(\frac{\alpha}{\pi})^4$, but, of course, no one really knows.

To show that eighth-order computations are not completely unreasonable and to make a small beginning at such a vast program, we outline here the analytic calculation of the contribution to the electron magnetic moment from the triple-bubble diagram. Admittedly, this is not one of the more complicated contributions to be evaluated; but, nevertheless, it represents a start.

We begin with the following parametric expression for the triple-bubble contribution.[8]

$$a_{e3} = (\frac{\alpha}{\pi})^4 \int_0^1 dx(1-x) \{\int_0^1 \frac{dy x^2 y^2 (1-y^2/3)}{x^2(1-y^2)+4(1-x)}\}^3 \tag{3}$$

The y-integral is easily done, leaving only the one-dimensional integral over x.

$$a_{e3}^{(8)} = \frac{1}{27} \left(\frac{\alpha}{\pi}\right)^4 \int_0^1 dx(1-x) \left\{-\frac{5}{3} + \frac{4(1-x)}{x^2} + \frac{(x-2)(x^2+2x-2)}{x^3} \ln (1-x)\right\}^3$$

$$= \frac{1}{27} \left(\frac{\alpha}{\pi}\right)^4 \{I_{111} + I_{222} + I_{333} + 3I_{112} + 3I_{113} + 3I_{122} + 3I_{133} + 3I_{223} + 3I_{233} + 6I_{123}\} \quad (4)$$

where

$$\{I_{111}, I_{222}, I_{333}, I_{112}, I_{113}, I_{122}, I_{133}, I_{223}, I_{233}, I_{123}\} = \int_0^1 dx(1-x) \{a^3, b^3, c^3, a^2b, a^2c, ab^2, ac^2, b^2c, bc^2, abc\} \quad (5)$$

with

$$a = -\frac{5}{3}$$

$$b = 4(1-x)/x^2$$

and $$c = \frac{(x-2)(x^2+2x-2)}{x^3} \ln (1-x) \quad (6)$$

Although the sum of the terms in eq. (4) is, of course, finite, by splitting up $a_{e3}^{(8)}$ in this way, terms which diverge have been introduced. Specifically, terms occur for which

$$\lim_{\varepsilon\to 0} \int_\varepsilon^1 dx(1-x)\, a^i b^j c^k \quad \text{diverges.}$$

By very carefully keeping track of these quantities, it will be shown that the divergences cancel in the sum, and the finite quantity, $a_{e3}^{(8)}$ will be obtained. This approach was first checked out by evaluating the fourth-order single-bubble and the sixth-order double-bubble contributions to $a_e$. The divergences cancel and the well-known results are obtained.

$$a_{e1}^{(4)} = \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{119}{36} - \frac{\pi^2}{3}\right) \quad (7)$$

and

$$a_{e2}^{(6)} = (\frac{\alpha}{\pi})^3 \left(- \frac{943}{324} - \frac{4\pi^2}{135} + \frac{8}{3}\zeta(3)\right) \tag{8}$$

We now proceed to evaluate $a_{e3}^{(8)}$ using eqs. (4), (5), and (6). A table of integrals used in the calculation is presented in the appendix. In what follows, it is understood that $\int_0^1 dxf(x)$ represents $\int_\varepsilon^1 dxf(x)$ and the $\varepsilon\to 0$ limit will be taken for appropriate sums of terms as described below.

By making use of the lower-order calculations leading to eqs. (7) and (8), or by a direct computation, it is easy to see that the combination $(I_{111} + 3I_{112} + 3I_{113} + 3I_{122} + 3I_{133} + 6I_{123})$ is finite. Either way, one obtains

$$I_{111} + 3I_{112} + 3I_{113} + 3I_{122} + 3I_{133} + 6I_{123} = -120\zeta(3) + \frac{29\pi^2}{3} + \frac{2485}{54} \tag{9}$$

This means, of course, that the sum of the remaining terms $(I_{222} + I_{333} + + 3I_{223} + 3I_{233})$ must be finite. Evaluation of these terms, however, is much more complicated, and so, in this case, some details of the computation are given.

The four remaining terms are considered individually, in order of increasing complexity. $I_{222}$ is easily evaluated.

$$I_{222} = \frac{-64}{5} + \frac{64}{\varepsilon} \frac{-128}{\varepsilon^2} + \frac{128}{\varepsilon^3} \frac{-64}{\varepsilon^4} + \frac{64}{5\varepsilon^5} \tag{10}$$

In order to evaluate $I_{223}$, we make use of integrals (A1) - (A8) of the appendix. Writing

$$I_{223} = 16\int_o^1 dx \{-\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} - \frac{21}{x^4} + \frac{30}{x^5} - \frac{18}{x^6} + \frac{4}{x^7}\} \ln(1-x) \tag{11}$$

and substituting in the values of the integrals we obtain

$$I_{223} = \frac{8\pi^2}{3} - \frac{14078}{225} + \frac{496}{15}\ln\varepsilon + \frac{96}{5\varepsilon} + \frac{88}{\varepsilon^2} - \frac{1072}{9\varepsilon^3} + \frac{64}{\varepsilon^4} - \frac{64}{5\varepsilon^5} \qquad (12)$$

For $I_{233}$, we have

$$I_{233} = 4\int_0^1 \{1 - \frac{2}{x} - \frac{11}{x^2} + \frac{32}{x^3} + \frac{8}{x^4} - \frac{112}{x^5} + \frac{148}{x^6} - \frac{80}{x^7} + \frac{16}{x^8}\} \ln^2(1-x) \qquad (13)$$

Using eqs. (A9) - (A17), we find

$$I_{233} = 4\ \{-4\zeta(3) - \frac{187\pi^2}{315} + \frac{14543}{450} - \frac{494}{45}\ln\varepsilon - \frac{1027}{45\varepsilon} - \frac{12}{\varepsilon^2} + \frac{248}{9\varepsilon^3} - \frac{16}{\varepsilon^4} + \frac{16}{5\varepsilon^5}\} \qquad (14)$$

The most complicated term is $I_{333}$, which involves integrals (A18) - (A27). It may be written as

$$I_{333} = \int_0^1 dx\ \{-x + 1 + \frac{18}{x} - \frac{30}{x^2} - \frac{96}{x^3} + \frac{252}{x^4} + \frac{24}{x^5} - \frac{600}{x^6} + \frac{720}{x^7} - \frac{352}{x^8} + \frac{64}{x^9}\} \ln^3(1-x) \qquad (15)$$

After a good deal of algebraic manipulation, it is found that

$$I_{333} = -108\zeta(4) + \frac{1272}{7}\zeta(3) - \frac{369\pi^2}{35} - \frac{558163}{4200} + \frac{488}{15}\ln\varepsilon + \frac{2284}{15\varepsilon} + \frac{8}{\varepsilon^2} - \frac{304}{3\varepsilon^3} + \frac{64}{\varepsilon^4} - \frac{64}{5\varepsilon^5} \qquad (16)$$

where $\zeta(4) = \frac{2}{5}\zeta^2(2) = \frac{\pi^4}{90}$ (17)

From eqs. (10), (12), (14), and (16) we find

$$I_{222} + I_{333} + 3I_{223} + 3I_{233} = -108\zeta(4) + \frac{936}{7}\zeta(3) - \frac{29\pi^2}{3} + \frac{3047}{56} \qquad (18)$$

All of the divergences have indeed cancelled out, and, as well, a curious tremendously-large cancellation among the finite parts has occured.

The final result is now obtained from eqs. (4), (9), and (18).

$$a_{e3}^{(8)} = \left(\frac{\alpha}{\pi}\right)^4 \{-4\zeta(4) + \frac{32}{63}\zeta(3) + \frac{151849}{40824}\} \qquad (19)$$

Interestingly, the term proportional to $\zeta(2)$ (or $\pi^2$) has cancelled out in the final result. The numerical value[9]

$$a_{e3}^{(8)} = .000876866 \left(\frac{\alpha}{\pi}\right)^4 \tag{20}$$

is of the order of magnitude expected on the basis of the estimate method described in ref. 6 (classes I and J).

ACKNOWLEDGMENT

The author would like to thank S. D. Drell for the kind hospitality extended at SLAC, where this work was initiated.

APPENDIX

We present here a table of integrals used in the calculation. The meaning of the divergent integrals is as described in the text.

$$\int_0^1 dx \ln(1-x) = -1 \tag{A1}$$

$$\int_0^1 \frac{dx}{x} \ln(1-x) = \frac{-\pi^2}{6} \tag{A2}$$

$$\int_0^1 \frac{dx}{x^2} \ln(1-x) = \ln\varepsilon - 1 \tag{A3}$$

$$\int_0^1 \frac{dx}{x^3} \ln(1-x) = \frac{\ln\varepsilon}{2} - \frac{1}{\varepsilon} + \frac{1}{4} \tag{A4}$$

$$\int_0^1 \frac{dx}{x^4} \ln(1-x) = \frac{\ln\varepsilon}{3} - \frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} + \frac{7}{18} \tag{A5}$$

$$\int_0^1 \frac{dx}{x^5} \ln(1-x) = \frac{\ln\varepsilon}{4} - \frac{1}{3\varepsilon^3} - \frac{1}{4\varepsilon^2} - \frac{1}{3\varepsilon} + \frac{19}{48} \tag{A6}$$

$$\int_0^1 \frac{dx}{x^6} \ln(1-x) = \frac{\ln\varepsilon}{5} - \frac{1}{4\varepsilon^4} - \frac{1}{6\varepsilon^3} - \frac{1}{6\varepsilon^2} - \frac{1}{4\varepsilon} + \frac{113}{300} \tag{A7}$$

$$\int_0^1 \frac{dx}{x^7} \ln(1-x) = \frac{\ln\varepsilon}{6} - \frac{1}{5\varepsilon^5} - \frac{1}{8\varepsilon^4} - \frac{1}{9\varepsilon^3} - \frac{1}{8\varepsilon^2} - \frac{1}{5\varepsilon} + \frac{127}{360} \tag{A8}$$

$$\int_0^1 dx(1-x) \ln^2(1-x) = \tfrac{1}{4} \tag{A9}$$

$$\int_0^1 \frac{dx}{x} \ln^2(1-x) = 2\zeta(3) \tag{A10}$$

$$\int_0^1 \frac{dx}{x^2} \ln^2(1-x) = \frac{\pi^2}{3} \tag{A11}$$

$$\int_0^1 \frac{dx}{x^3} \ln^2(1-x) = -\ln\varepsilon + \frac{\pi^2}{6} + \frac{3}{2} \tag{A12}$$

$$\int_0^1 \frac{dx}{x^4} \ln^2(1-x) = -\ln\varepsilon + \frac{1}{\varepsilon} + \frac{\pi^2}{9} + \frac{5}{6} \tag{A13}$$

$$\int_0^1 \frac{dx}{x^5} \ln^2(1-x) = -\frac{11}{12}\ln\varepsilon + \frac{1}{2\varepsilon^2} + \frac{1}{\varepsilon} + \frac{\pi^2}{12} + \frac{59}{144} \tag{A14}$$

$$\int_0^1 \frac{dx}{x^6} \ln^2(1-x) = -\frac{5}{6}\ln\varepsilon + \frac{1}{3\varepsilon^3} + \frac{1}{2\varepsilon^2} + \frac{11}{12\varepsilon} + \frac{\pi^2}{15} + \frac{11}{72} \tag{A15}$$

$$\int_0^1 \frac{dx}{x^7}\ln^2(1-x) = -\frac{137}{180}\ln\varepsilon + \frac{1}{4\varepsilon^4} + \frac{1}{3\varepsilon^3} + \frac{11}{24\varepsilon^2} + \frac{5}{6\varepsilon} + \frac{\pi^2}{18} - \frac{37}{3600} \tag{A16}$$

$$\int_0^1 \frac{dx}{x^8}\ln^2(1-x) = -\frac{7}{10}\ln\varepsilon + \frac{1}{5\varepsilon^5} + \frac{1}{4\varepsilon^4} + \frac{11}{36\varepsilon^3} + \frac{5}{12\varepsilon^2} + \frac{137}{180\varepsilon} + \frac{\pi^2}{21} - \frac{71}{600} \tag{A17}$$

$$\int_0^1 dx(1-x)\ln^3(1-x) = -\frac{3}{8} \tag{A18}$$

$$\int_0^1 \frac{dx}{x}\ln^3(1-x) = -6\zeta(4) \tag{A19}$$

$$\int_0^1 \frac{dx}{x^2}\ln^3(1-x) = -6\zeta(3) \tag{A20}$$

$$\int_0^1 \frac{dx}{x^3}\ln^3(1-x) = -3\zeta(3) - \frac{\pi^2}{2} \tag{A21}$$

$$\int_0^1 \frac{dx}{x^4}\ln^3(1-x) = \ln\varepsilon - 2\zeta(3) - \frac{\pi^2}{2} - \frac{11}{6} \tag{A22}$$

$$\int_0^1 \frac{dx}{x^5}\ln^3(1-x) = \frac{3}{2}\ln\varepsilon - \frac{1}{\varepsilon} - \frac{3\zeta(3)}{2} - \frac{11\pi^2}{24} - \frac{17}{8} \tag{A23}$$

$$\int_0^1 \frac{dx}{x^6}\ln^3(1-x) = \frac{7}{4}\ln\varepsilon - \frac{1}{2\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{6\zeta(3)}{5} - \frac{5\pi^2}{12} - \frac{479}{240} \tag{A24}$$

$$\int_0^1 \frac{dx}{x^7}\ln^3(1-x) = \frac{15}{8}\ln\varepsilon - \frac{1}{3\varepsilon^3} - \frac{3}{4\varepsilon^2} - \frac{7}{4\varepsilon} - \zeta(3) - \frac{137\pi^2}{360} - \frac{169}{96} \tag{A25}$$

$$\int_0^1 \frac{dx}{x^8}\ln^3(1-x) = \frac{29}{15}\ln\varepsilon - \frac{1}{4\varepsilon^4} - \frac{1}{2\varepsilon^3} - \frac{7}{8\varepsilon^2} - \frac{15}{8\varepsilon} - \frac{6\zeta(3)}{7} - \frac{7\pi^2}{20} - \frac{1059}{700} \tag{A26}$$

$$\int_0^1 \frac{dx}{x^9}\ln^3(1-x) = \frac{469}{240}\ln\varepsilon - \frac{1}{5\varepsilon^5} - \frac{3}{8\varepsilon^4} - \frac{7}{12\varepsilon^3} - \frac{15}{16\varepsilon^2} - \frac{29}{15\varepsilon} - \frac{3\zeta(3)}{4} - \frac{363\pi^2}{1120} - \frac{12307}{9600} \tag{A27}$$

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