ANGULAR ANALYSIS OF ELEMENTARY PARTICLE REACTIONS*

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INTRODUCTION

Most current experiments in High Energy Physics aim at measuring some matrix element of the S-matrix. Symmetry laws are of great help in analysing such quantities in simpler terms. Among them, a symmetry group which has so far withstood all fluctuations of theoretical understanding and experimental checking, is the inhomogeneous, proper, orthochronous Lorentz group. Space and time reflections, on the other hand, seem to be only approximately good symmetries, for some classes of phenomena. We shall primarily discuss here consequences of relativistic invariance, following a line of reasoning due to E.P. Wigner, which is, without any doubt, the most penetrating, and leads in a completely rational fashion to all known results concerning the phenomenological angular analysis of particle collisions. It has the further advantage that it is applicable to any symmetry group, as we shall see on some particular examples.

We shall assume here that the reader is reasonably familiar with non relativistic angular momentum theory, as it is expounded in many text books, as well as with basic facts concerning the inhomogeneous Lorentz group. Relevant results and pertaining references will however either be quoted in the body of the text or collected in appendices for the reader's convenience.

In Chapter O, we shall set the general framework in which we are working, namely,

a) the formulation of symmetry laws in quantum mechanics and the basic theorem of Wigner,

b) its implication on S matrix.

In Chapter I, we shall describe "particles" of arbitrary "masses" and "spins", following Wigner's analysis. Assemblies of identical particles subject to Fermi or Bose statistics will then be described in terms of the corresponding fields. Close contact will be made with the perhaps known descriptions of spin 0 (Klein Gordon), spin $\frac{1}{2}$ (Dirac), spin 1 (Maxwell, Proca), particles.

In Chapter II, we shall analyse "two particle" states, in terms of the so-called ℓ-s , multipole, helicity couplings, and write down the correspending analysis of a reaction involving two incoming, two outgoing particles. Contact will be made with reactions involving particles of low spins and a collection of, may be familiar looking, formulae will be found in an appendix.

Unitarity of the S matrix will be seen to allow the definition of phase shifts under special conditions whereas analyticity assumptions will be shown to yield threshold behaviours of familiar looking types. Such assumptions are of course either subject to controversy, or consequences of somewhat sophisticated models, and go far beyond the exactness of a symmetry law.

In Chapter III, we examine three particle states for which relativistic effects are slightly more subtle than for two particle states. ℓ-s or helicity couplings in cascade are compared to a symmetric coupling which one might pictorially call Casimir-Dalitz coupling.

Among technical appendices, one is devoted to "reduction formulae" pertaining to a field theory of the L S Z type, in order to make a bridge with a still current description of reactions between particles with spins.

The content of these notes is a reexposition of material which has so far been presented to assemblies of theorists or mathematicians. It is hoped that it will not sound too abstract to experimentalists. The line of argument is of a highly theoretical nature, and is due to Wigner and coworkers or followers. The outcome, though, is impressively close to phenomenology : no reference is ever made to configuration space in any doubtful way - at least at this level where the particle wave duality is not yet expressed in terms of space time localized fields, the necessity of which is of a highly controversial nature - ; only such things as energy momenta are ever mentioned, namely only quantities which are actually measured in experiments*. Thus, the reader will judge whether it is worth paying the price of an abstract approach.

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* One might however like to be able and describe what actually happens in a finite region of space and time. It turns out that such a description requires quite a bit more than mere kinematics.

CHAPTER 0

a) Wigner's formulation of symmetry laws in quantum mechanics.

Quantum mechanical states of a physical system are described in a Hilbert space $\mathcal{H}$ by rays, namely, two normalized vectors $|\Psi\rangle$, $e^{i\alpha}|\Psi\rangle$ (any α real) of a Hilbert space represent the same state. This is so because observed quantities are of the type $\langle\Psi|A|\Psi\rangle$ where A is a hermitian operator (e.g. $A = |\Phi\rangle\langle\Phi| \rightarrow \langle\Psi|A|\Psi\rangle = |\langle\Phi|\Psi\rangle|^2$. i.e. the transition probability between state $|\Phi\rangle$ and state $|\Psi\rangle$).

We shall say that a symmetry law $\mathscr{S}$ holds, if, given any ray $|\underline{\Psi}\rangle = \{e^{i\alpha}|\Psi\rangle \quad 0<\alpha\leq 2\pi\}$, there exists a ray $|{}^{\mathscr{S}}\underline{\Psi}\rangle$ (Ψ transformed by $\mathscr{S}$) such that

$$|\langle\underline{\Phi}|\underline{\Psi}\rangle|^2 = |\langle{}^{\mathscr{S}}\underline{\Phi}|{}^{\mathscr{S}}\underline{\Psi}\rangle|^2 \quad ,$$

(i.e. the transformation $|\underline{\Psi}\rangle \rightarrow |{}^{\mathscr{S}}\underline{\Psi}\rangle$ preserves transition probabilities) and $|{}^{\mathscr{S}}\underline{\Psi}\rangle$ spans all $\underline{\mathcal{H}}$ when $|\underline{\Psi}\rangle$ does.

Then, one has the fundamental

Wigner's theorem[1]

To every symmetry law $\mathscr{S}$: $|\underline{\Psi}\rangle \rightarrow |{}^{\mathscr{S}}\underline{\Psi}\rangle$ ($|\langle\underline{\Phi}|\underline{\Psi}\rangle|^2 = |\langle{}^{\mathscr{S}}\underline{\Phi}|{}^{\mathscr{S}}\underline{\Psi}\rangle|^2$), there corresponds an additive operator $U(\mathscr{S})$, namely, such that $U(\mathscr{S})\,(|\Phi\rangle + |\Psi\rangle) = U(\mathscr{S})|\Phi\rangle + U(\mathscr{S})|\Psi\rangle$ for all $|\Phi\rangle$ and $|\Psi\rangle$ in $\mathcal{H}$, which is either unitary or antiunitary, namely, $UU^+ = U^+U = \mathbb{1}$, where U^+ is defined by $\langle\Phi|U^+|\Psi\rangle = \langle U\Phi|\Psi\rangle$ in the unitary case and $\langle\Phi|U^+|\Psi\rangle = \langle U\Phi|\Psi\rangle^*$ in the antiunitary case, and such that $U(\mathscr{S})\,|\Phi\rangle$ belongs to the ray $|{}^{\mathscr{S}}\underline{\Phi}\rangle$ whenever $|\Phi\rangle$ belongs to the ray $|\underline{\Phi}\rangle$.

The operator $U(\mathscr{S})$ is uniquely defined up to a multiplicative phase factor.

Let now G be a symmetry group, namely a set containing a unit element e, endowned with a (non necessarily commutative) product

$$g_1 , g_2 \in G \rightarrow g_1 \cdot g_2 \in G \qquad ,$$

such that 1) $e \cdot g = g \cdot e = g$

2) $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

3) for all $g \in G$ there is an inverse g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$,

each element of which defines a symmetry operation $|\underline{\Psi}\rangle \rightarrow |{}^{g}\underline{\Psi}\rangle$ which is transition probability preserving and "onto".

First of all $U(e) = \omega \mathbb{1}$, where ω is a phase factor, because of the uniqueness of $U(e)$ up to a phase and the fact that $\mathbb{1}$ is a solution. We shall choose $\omega = 1$.

We next prove that there is a system of phase factors $\omega(g_1 , g_2)$ such that

$$U(g_1)\, U(g_2) = \omega(g_1 , g_2)\, U(g_1\, g_2) \qquad (0,a,1) \quad .$$

Let $|\Phi\rangle \in |\underline{\Phi}\rangle$

Then $U(g_1 g_2)\, |\Phi\rangle \in |{}^{g_1 g_2}\underline{\Phi}\rangle$

$$U(g_1)\, U(g_2)\, |\Phi\rangle = U(g_1) \Big(U(g_2)\, |\Phi\rangle \Big)$$

$$U(g_2)|\Phi\rangle \in |{}^{g_2}\underline{\Phi}\rangle \quad , \quad \text{thus :} \quad U(g_1)\, U(g_2)\, |\Phi\rangle \in |{}^{g_1 g_2}\underline{\Phi}\rangle \quad .$$

Thus, one may write

$$U(g_1)\, U(g_2)\, |\Phi\rangle = \omega(g_1 , g_2 , \Phi)\ U(g_1 g_2)\, |\Phi\rangle \qquad ,$$

where $\omega(g_1,g_2,\Phi)$ is a phase factor ; but, using additivity :

$$U(g_1)\,U(g_2)\left(|\Phi\rangle+|\Psi\rangle\right) = \omega\left(g_1,g_2,|\Phi\rangle+|\Psi\rangle\right)\,U(g_1g_2)\left(|\Phi\rangle+|\Psi\rangle\right)$$
$$= \omega\left(g_1,g_2,|\Phi\rangle\right)U(g_1g_2)|\Phi\rangle + \omega\left(g_1,g_2,|\Psi\rangle\right)U(g_1g_2)|\Psi\rangle \quad .$$

If Φ and Ψ are taken linearly independent, one obtains

$$\omega(g_1,g_2,\Phi+\Psi) = \omega(g_1,g_2,\Phi) = \omega(g_1,g_2,\Psi) \quad .$$

Thus $\omega(g_1,g_2,\Phi)$ does not depend on Φ, and we shall write it $\omega(g_1,g_2)$.

The operators $U(g)$ fulfilling eq. (O,a,1) are said to form a representation of G up to a phase.

Two representations $U(g)$, $U'(g)$, where both $U(g)$ and $U'(g)$ are either unitary or antiunitary, with factor systems $\omega(g_1,g_2)$, $\omega'(g_1,g_2)$, such that

$$\omega(g_1,g_2) = \omega'(g_1,g_2)\;\;\theta(g_1\cdot g_2)\;\;[\theta(g_2)]^{-\varepsilon(g_1)}\;\;[\theta(g_1)]^{-1} \quad ,$$

where $\theta(g)$ is a phase factor, are said to belong to the same <u>type</u> ; here, $\varepsilon(g) = +1$ if $U(g)$ is unitary, $\varepsilon(g) = -1$ if $U(g)$ is antiunitary. For instance, $\mathcal{U}(g)$ and $\theta^{-1}(g)\,\mathcal{U}(g)$ belong to the same type.

The classification of physical systems under a symmetry group G depends, thus, on

- $\varepsilon(g)$
- a type of factor system $\omega(g_1,g_2)$
- given $\varepsilon(g)$, $\omega(g_1,g_2)$, a class of representation up to unitary equivalence.

<u>Remarks</u>.

I - if G is a connected Lie group $\varepsilon(g) = +1$ for all g because the product of two unitary operators is unitary, whereas the product of a unitary and an antiunitary operator is antiunitary.

II - if G contains an element ρ whose square is the identity **e**, and $U(\rho)$ is antiunitary

$$U(\rho)\,U(\rho) = \omega(\rho,\rho)\,U(e) = \omega(\rho,\rho)\mathbb{1} \quad \text{for a possible choice of } U(e)$$

$$U(\rho)\,U(\rho)\,U(\rho) = U(\rho)\,\omega(\rho,\rho)\mathbb{1} = \omega^*(\rho,\rho)\;U(\rho)$$

$$= \omega(\rho,\rho)\,U(e)\,U(\rho) = \omega(\rho,\rho)\;U(\rho) \qquad ,$$

thus $\omega^2(\rho,\rho) = \mathbb{1} \quad \rightarrow \quad \omega(\rho,\rho) = \pm 1$.

b) Collision theory : the S matrix

We now have a description of a physical system in a Hilbert space $\mathcal{H}$ in which there is given a representation up to a factor of the symmetry group G.

A collision theory is set up if $\mathcal{H}$ can be spanned by a collection of states $|\alpha \text{ in}\rangle$ representing possible incoming beams and targets, as well as states $|\beta \text{ out}\rangle$ representing states possibly produced in a collision experiment*. The reasonable assumption that every incoming state can be produced as the product of a collision experiment and that, given two orthogonal incoming states, collision prepares two orthogonal states, implies that there are in $\mathcal{H}$ two basis $|\alpha \text{ in}\rangle$, $|\alpha \text{ out}\rangle$ which can be put in coïncidence by action of a unitary operator S :

$$\langle\alpha \text{ out}| = \langle\alpha \text{ in}|\, S$$

$$S\,S^+ = S^+ S = \mathbb{1} \qquad .$$

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* α and β represent collections of quantum numbers characterizing the possible prepared or produced states, e.g. momentum, spin, etc...

The transition amplitude for an incoming state $|\alpha \ \text{in}\rangle$ to yield an outgoing state $|\beta \ \text{out}\rangle$ is

$$\langle\beta \ \text{out}|\alpha \ \text{in}\rangle = \langle\beta \ \text{in}|S|\alpha \ \text{in}\rangle = \langle\beta \ \text{out}|S|\alpha \ \text{out}\rangle$$

which we shall call in short the transition amplitude from initial state $|\alpha\rangle$ to final state $|\beta\rangle$.

Assume now that the transition probabilities are invariant under G :

$$|\langle\beta|S|\alpha\rangle|^2 = |\langle{}^g\beta|S|{}^g\alpha\rangle|^2 \quad ;$$

one gets

$$\langle\beta|S|\alpha\rangle = \langle{}^g\beta|S|{}^g\alpha\rangle\,\omega(g,\alpha,\beta)$$

$$\text{or} \quad \langle{}^g\beta|S|{}^g\alpha\rangle^*\,\omega(g,\alpha,\beta) \quad ,$$

where ω is a phase factor (this argument starts as the proof of Wigner's theorem, cf. ref.[0] Chap. XV , I , 2, Th II and III).

Using the linearity of S one easily finds that :

1) $\omega(g,\alpha,\beta)$ is independent of α , β within any subspace of $\mathcal{H}$ where linear superposition of vectors makes any physical sense (e.g. in particular not a mixture of states with different spin types, i.e. integer versus half integer, or different charges) which we shall call coherent subspace.

2) the first alternative holds if U(g) is unitary, the second one when U(g) is antiunitary*. The symmetry group G is indeed compatible with collision processes if there is no transition between subspaces labelled by different eigenvalues of observables invariant under G , if some such exist. Each such observable is said to be associated with a super selection rule(I,II)

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* In practical cases U(g) is usually unitary ; however some transformations are such that $|\langle\beta|S|\alpha\rangle| = |\langle{}^g\alpha|S|{}^g\beta\rangle|$ where U(g) is antiunitary (e.g. time reversal). Then $\langle\beta|S|\alpha\rangle = \langle{}^g\alpha|S|{}^g\beta\rangle\,\omega(g)$.

and one of its eigenvalues to label a superselection sector. The intersections of all superselection sectors corresponding to all possible commuting superselection observables* are just the coherent subspaces mentioned above.

Next, one proves that $\omega(g)$ is a representation of G in the sense that

$$\omega\,(g_1 g_2) = \omega^{\varepsilon(g_2)}(g_1)\ \omega(g_2) \quad .$$

If G is connected, then, $\omega(g)$ is a true one dimensional unitary representation.

At any rate, within a coherent subspace, one has

$$U^{-1}(g)\ S\ U(g) = \omega(g)\ S \quad .$$

If $\omega(g)$ is the trivial representation $\omega(g) = 1$; if such is the case, $U(g)$ commutes with S . This is enforced** by some other property of the S matrix, derivable for instance from axiomatic field theories, known as "cluster decomposition property" which states that if two experiments are carried out far apart, the overall S matrix element factorizes into the product of the two matrix elements, which implies $\omega^2(g) = \omega(g)$ i.e. $\omega(g) = 1$.

If the one dimensional representations of G consist of the identity representation exclusively (which is the case for relativistic invariance), then commutation of $U(g)$ with S is automatically insured.

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* If all superselection observables are associated with the symmetry group, they commute ; if they do not commute, one is in trouble!

** This argument was first given to us by M. Froissart.

CHAPTER I

ONE PARTICLE STATES.[2]

a) The inhomogeneous proper orthochronous Lorentz group $\mathcal{P}_+^{\uparrow}$ and its covering group $\bar{\mathcal{P}}_+^{\uparrow}$

Unless otherwise specified, G will be $\mathcal{P}_+^{\uparrow}$, the inhomogeneous proper orthochronous Lorentz group , i.e. the group of transformations $\{a,\Lambda\}$ where a is a four vector $a = \left(a^o, \vec{a} = \{a^1, a^2, a^3\}\right)$ and Λ a four by four matrix, pseudo-orthogonal for the metric $G = \{g^{\mu\nu} = 0 \; , \mu \neq \nu \; ; g^{oo} = - g^{11} = - g^{22} = - g^{33} = 1\}$: $\Lambda^T \, G \, \Lambda = G$, with determinant + 1 , $\Lambda^o_o > 0$. This is the group of changes of Lorentz frames, preserving the direction of time flow and the orientation in three dimensional space like planes. Let x^{μ} be the coordinates of a four-vector in frame $\mathcal{F}$, x'^{μ} its coordinates in frame $\mathcal{F}'$, then

$$x'^{\mu} = a^{\mu} + \Lambda^{\mu}_{\;\nu} \, x^{\nu}$$

$$(x' - y')^2 = (x - y)^2 = (x - y) \, G \, (x - y) = (x^o - y^o)^2 - (\vec{x} - \vec{y})^2 \; .$$

The group law is $\{a,\Lambda\} \, \{a',\Lambda'\} = \{a + \Lambda a' \, , \Lambda\Lambda'\}$.

In this case, Wigner's analysis shows[II)] that unitary representations of $\mathcal{P}_+^{\uparrow}$ up to a phase are true representations of a group $\bar{\mathcal{P}}_+^{\uparrow}$ constructed as follows (its universal covering group) :

Let

$$\sigma_{\mu} = \left\{ \sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

be the set of Pauli matrices.

With each four vector a , associate the hermitian 2×2 matrix $\underset{\sim}{a} = a^{\mu} \sigma_{\mu}$. $[\, a^{\mu} = \frac{1}{2} \operatorname{tr} \underset{\sim}{a}\, \sigma_{\mu} \,]$.

Let $SL(2,C)$ be the group of 2×2 complex matrices of determinant 1 .

With each vector a and matrix $A \in SL(2,C)$, associate the vector $A.a$ defined by

$$\underset{\sim}{A.a} = A \underset{\sim}{a} A^{+} \quad ;$$

one finds

$$\det \underset{\sim}{A.a} = \det \underset{\sim}{a} = a^2 = a.a$$

$$\frac{\partial (A.a)^{o}}{\partial a^{o}} = \operatorname{tr} A A^{+} > 0$$

$$A.(a+b) = A.a + A.b \quad ;$$

$A.a$ is thus obtained from a by an orthochronous Lorentz transformation which can furthermore be proved to be proper $(\det = +1)$. Two elements $+A$ and $-A$ correspond to the same Lorentz transformation.

$\overline{\mathcal{P}}{}^{\uparrow}_{+}$ is thus the set of elements $\{\underset{\sim}{a}, A\}$ with the group law

$$\{\underset{\sim}{a}, A\} \{\underset{\sim}{a}', A'\} = \{\underset{\sim}{a} + \underset{\sim}{A.a'} \,,\, AA'\} \quad .$$

We shall see shortly that this distinction between $\mathcal{P}^{\uparrow}_{+}$ and $\overline{\mathcal{P}}{}^{\uparrow}_{+}$ is precisely what allows for the existence of half integer spins.

We now proceed to look for inequivalent unitary representations of $\overline{\mathcal{P}}{}^{\uparrow}_{+}$. It can be shown[II)] that any "continuous" unitary representation can be decomposed into a sum - or rather an integral - of irreducible ones. (i.e. schematically any Hilbert space of representation can be ordered so that

$$U(g) = \begin{pmatrix} U_1(g) & 0 & \cdots\cdots & 0 \\ 0 & U_2(g) & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & U_n(g) \end{pmatrix}$$

, each constituant not being decomposable in this fashion).

We shall now construct in a heuristic way those irreducible representations which we shall need for physical applications.

b) <u>The Lie Algebra of</u> $\mathcal{P}_+^\uparrow$ <u>and the physical representations</u>

Let $U(a,\Lambda) = e^{iP.a} \, e^{\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$ be the exponential form of the representation. P is the hermitian generator of translations, interpreted as the energy momentum operator, $-J^{\nu\mu} = J^{\mu\nu}$ is the hermitian generator of the Lorentz transformation in the (μ,ν) two-plane (six such), $\omega_{\mu\nu}$ being the angle (either circular if $\mu,\nu \neq 0$, or hyperbolic if μ or $\nu = 0$).

The group law implies

$$e^{-iPa'} \, e^{iPa} \, e^{iPa'} = e^{iPa}$$

$$e^{-i\omega'_{\mu\nu} J^{\mu\nu}} \, e^{iPa} \, e^{i\omega'_{\mu\nu} J^{\mu\nu}} = e^{iP.\Lambda^{-1}(\omega')a} \qquad \text{(I,b,1)}$$

$$e^{-i\omega'_{\mu\nu} J^{\mu\nu}} \, e^{i\omega_{\mu\nu} J^{\mu\nu}} \, e^{i\omega'_{\mu\nu} J^{\mu\nu}} = e^{iJ^{\mu\nu} \Lambda_\mu{}^{\mu'}(-\omega') \Lambda_\nu{}^{\nu'}(-\omega') \omega_{\mu'\nu'}}$$

whose infinitesimal versions are (a' and ω' small)

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, J_{\nu\lambda}] = i\,[g_{\lambda\mu} P_\nu - g_{\nu\mu} P_\lambda] \qquad \text{(I,b,2)}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\,[g_{\mu\rho} J_{\nu\sigma} + g_{\nu\sigma} J_{\mu\rho} - g_{\mu\sigma} J_{\nu\rho} - g_{\nu\rho} J_{\mu\sigma}] \; .$$

The last two commutation relations just say that P_μ , $J_{\mu\nu}$ behave respectively as a vector and a tensor operator under Lorentz transformations.

As usual we shall look for a maximal set of commuting observables.

The four P's commute ; let p be an "eigenvalue"* Let us now look for Lorentz transformations Λ which leave p invariant, and write $\Lambda = e^{\frac{1}{2}\omega_{\mu\nu}I^{\mu\nu}}$ where $I^{\mu\nu}$ are the four by four matrices which generate the $\mu\nu$ transformations in p space :

$$(I^{\mu\nu})^{k\lambda} = -i\,(g^{\mu k}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu k}) \quad .$$

The invariance of p implies $\omega_{\mu\nu}\,p^\nu = 0$ which is solved according to $\omega_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\,s^\rho\,p^\sigma$ where $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor $\varepsilon_{0123} = +1$, $\varepsilon_{\mu\nu\rho\sigma} = \pm 1$ according as $\mu\nu\rho\sigma$ is an even/odd permutation of (0123) , and s a four vector whose component along p is irrelevant. Thus the $U(\Lambda(p))$'s representing Lorentz transformations which leave p invariant are :

$$U(\Lambda(p)) = e^{\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}s^\rho p^\sigma} \quad .$$

Let then $|p\rangle$ be a vector of representation space such that $P|p\rangle = |p\rangle$

$$U(\Lambda(p))|p\rangle = e^{\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}s^\rho p^\sigma}\,|p\rangle$$

$$= e^{-\frac{1}{2}W.s}\,|p\rangle \quad ,$$

where $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,J^{\nu\rho}\,p^\sigma$.

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* Wherever "eigenvalue" is written, it is implied that continuous spectrum may lead to the consideration of improper eigenvectors.

One easily finds from (I,b,2) :

$$[W_\mu, P_\nu] = 0 \quad ,$$

$$[W_\mu, W_\nu] = -i\,\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \quad ,$$

$$[J_{\mu\nu}, W_\lambda] = i\,(g_{\mu\lambda} W_\nu - g_{\nu\lambda} W_\mu) \quad . \qquad \text{(I,b,3)}$$

(The latter indicating that W behaves as a four vector under Lorentz transformations).

One furthermore notes the identity $W.P = 0$ (I,b,4)
which stems from the antisymmetry of $\varepsilon_{\mu\nu\rho\sigma}$ (this was indeed contained in the construction of $W(p)$ as generating Lorentz transformations leaving p unchanged).

From these commutation rules one observes that $P.P = P^2$, $W.W = W^2$ commute with all P^μ's and $J^{\mu\nu}$'s ; their "eigenvalues" label the representation (one can indeed show that no other function of P , J commutes with all P's , J's).

Then, P being taken diagonal with eigenvalue p such that $p^2 = m^2$, m^2 fixed, since W commutes with P , $W|p\rangle$ is still an eigenvector of P with eigenvalue p ; we shall call $W(p)$ the "restriction" of the operator W to the eigenspace of P corresponding to the eigenvalue p . We have of course $W(p).p = 0$ so that $W(p)$ can be expanded along a basis of independent vectors orthogonal to p . At this point, we have to make distinctions according as p is time-like ($p^2 > 0$), light-like ($p^2 = 0$) , space-like ($p^2 < 0$), or identically zero.

① $p^2 > 0$ $p^o > 0$ or $p^o < 0$ *

One can attach to each p three space-like vectors $n_i(p)$:

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* In physics, one shall always have to deal with representations with $p^o > 0$, some formal manipulations however use those with $p^o < 0$.

$$n_i(p).n_j(p) = -\delta_{ij} \qquad i,j = 1,2,3$$

$$p.n_i(p) = 0 \qquad \det(p, n_1(p)\, n_2(p)\, n_3(p)) = \pm 1$$

according as $p^o \gtrless 0$ (I,b,4),

and expand : $$W(p) = \sum_{i=1}^{3} W_i(p)\, n_i(p) \qquad \text{(I,b,5)}$$

where $W_i(p) = -W(p).n_i(p)$.

From (I,b,3), one finds for the commutation rules between

$$S_i(p) = \frac{W_i(p)}{m} \qquad \text{(I,b,6)}$$

$$[S_i(p), S_j(p)] = i\, \varepsilon_{ijk}\, S_k(p) \qquad \text{(I,b,7)}$$

$\varepsilon_{ijk} = \pm 1$ according as ijk is an $\left(\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right.$ permutation of 1,2,3 .

$S_i(p)$ are generators of the SU(2) subgroup of $\mathcal{P}_+^\uparrow$ which leaves p invariant. Thus,

$$\sum_1^3 S_i^2(p) = -\frac{W^2}{m^2}$$ has possible eigenvalues $s(s+1)$,

s integer or half integer.

Together with each p one can diagonalize $S_3(p)$ whose eigenvalues s_3 range by integer steps from $-s$ to $+s$. In other words, we have now defined basis vectors

$$|[p], s_3\rangle$$

fulfilling $$P|[p], s_3\rangle = p|[p], s_3\rangle$$

$$S_3(p)|[p], s_3\rangle = s_3|[p], s_3\rangle \qquad \text{(I,b,8)}$$

$$(S_1 \pm iS_2)(p)|[p]\ s_3\rangle = \sqrt{s(s+1) - s_3(s_3 \pm 1)}\ |[p]\ s_3 \pm 1\rangle \quad ; \tag{I,b,8}$$

$[p]$ reminds one that the corresponding state is defined relative to a set p , $n_i(p)$ subject to (I,b,5). $[p]$ will be understood as an element of SL(2,C) corresponding to the Lorentz transformation which transforms

$$\overset{o}{p} = (\pm m,0,00)\ ,\ n_1(\overset{o}{p}) = (0,1,00)\ ,\ n_2(\overset{o}{p}) = (0,0,1,0)\ ,\ n_3(\overset{o}{p}) = (0,0,0,1)$$

into p , $n_1(p)$, $n_2(p)$, $n_3(p)$ respectively.

The choice of $[p]$, which is arbitrary, first makes precise on which four vector $n_3(p)$ orthogonal to p the spin operator W/m is measured, and how the phase of the corresponding state is chosen (choice of $n_1(p)$, $n_2(p)$ and sign of $[p]$ in SL(2,C)).

The commutation rules (I,b,1 and 2) show that :

$$|[p],s_3\rangle = U([p])\ |[\overset{o}{p}],s_3\rangle \quad . \tag{I,b,9}$$

One sees easily that $S_i(\overset{o}{p}) = -\frac{1}{2}\,\varepsilon_{ijk}\,J_{jk}$ are just the generators of the SU(2) subgroup of SL(2,C) , which corresponds to space rotations.

For any operation u of SU(2) , one has the usual spin s representation :

$$U(u)\ |[\overset{o}{p}],s_3\rangle = |[\overset{o}{p}],s_3'\rangle\ \mathcal{D}^{(s)}_{s_3's_3}(u) \quad .$$

Hence, according to (I,b,8) :

Firstly *

$$|[p]',s_3\rangle = |[p],s_3'\rangle\ \mathcal{D}^{(s)}_{s_3's_3}\left([p]^{-1}\ [p]'\right) \quad , \tag{I,b,10}$$

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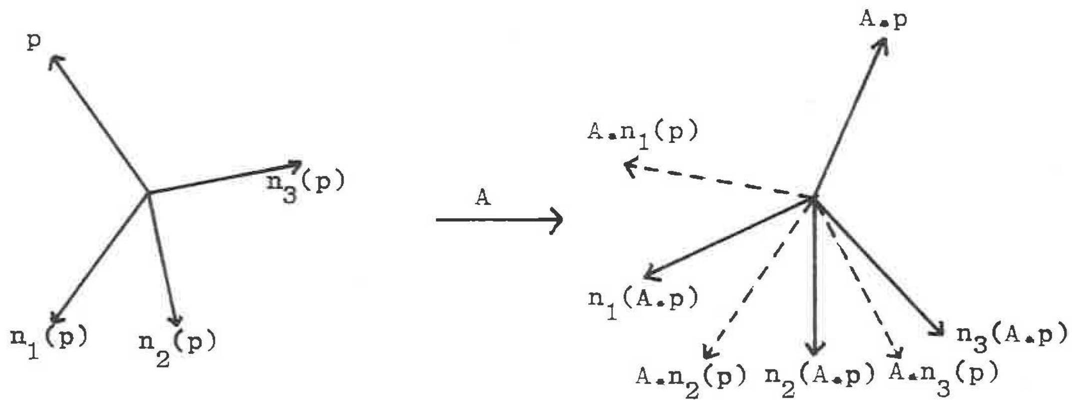
* $|[p]',s_3\rangle = U([p]')\ |[\overset{o}{p}],s_3\rangle = U([p])\ U([p]^{-1})\ U([p]')\ |[\overset{o}{p}],s_3\rangle$
$= U([p])\ |[\overset{o}{p}],s_3'\rangle\ \mathcal{D}^{(s)}_{s_3's_3}\left([p]^{-1}\ [p]'\right)$

which shows how a spin state changes when the frame $\{n_i(p)\}$ is changed (note that $[p]^{-1}[p]'$ take $\overset{o}{p}$ to p, p back to $\overset{o}{p}$ and is thus an SU 2 operation) ;

Secondly :

$$\begin{aligned} U(a,A)\,|[p],s_3\rangle &= U(a)\,U(A)\,|[p],s_3\rangle \\ &= U(a)\,U\big([A.p]\big)\,U\big([Ap]^{-1}A[p]\big)\,|[\overset{o}{p}]\,s_3\rangle \\ &= U(a)\,U\big([A.p]\big)\,|[\overset{o}{p}],s_3'\rangle\,\mathcal{D}^{(s)}_{s_3's_3}\big([Ap]^{-1}A[p]\big) \\ &= U(a)\,|[A.p],s_3'\rangle\,\mathcal{D}^{s}_{s_3's_3}\big([Ap]^{-1}A[p]\big) \qquad \text{(I,b,11)} \\ &= e^{i(A.p).a}\,|[A.p]\,s_3'\rangle\,\mathcal{D}^{s}_{s_3's_3}\big([Ap]^{-1}A[p]\big) \quad , \end{aligned}$$

which yields the transformation of a state under inhomogeneous Lorentz transformation. This transformation law is intuitive if one is willing to represent a state by a four vector p (its momentum) and a four vector $n_3(p)$ together with a number s_3 : its spin s_3 along $n_3(p)$. This state seen in a different frame obtained from the latter by a "Lorentz transformation" A , will have momentum $A.p$ and spin s_3 along $A.n_3(p)$ which is a priori different from $n_3(A.p)$ since for each value p the attached frame was arbitrarily chosen.



The Wigner rotation is just the one which expresses

$$|A[p],s_3\rangle = |[A.p],s_3'\rangle \; \mathcal{D}^{s}_{s_3' s_3}\left([A.p]^{-1} A[p]\right)$$

according to (I,b,10) .

It now remains to ensure the unitarity of $U(a,A)$ for a scalar product which expresses the orthogonality of states with different momenta which we write in a Lorentz invariant way* :

$$\langle[p'],s_3'|[p],s_3\rangle = 2\,\omega_p \;\; \delta(\vec{p} - \vec{p}\,') \;\; \delta_{s_3 s_3'} \qquad \text{(I,b,12)}$$

$$\omega_p = \sqrt{\vec{p}^{\,2} + m^2} \; .$$

Any state $|\Psi\rangle$ of representation space can be expanded in the basis $|[p],s_3\rangle$ according to

$$|\Phi\rangle = \int \frac{d^3p}{2\omega_p} \sum_{s_3} \varphi_{s_3}([p]) \; |[p],s_3\rangle \qquad \text{(I,b,13)}$$

$$\varphi_{s_3}([p]) = \langle[p],s_3|\Phi\rangle \; . \qquad \text{(I,b,14)}$$

The scalar product being

$$\langle\Psi|\Phi\rangle = \int \frac{d^3p}{2\omega_p} \sum_{s_3} \psi^*_{s_3}([p]) \;\; \varphi_{s_3}([p]) \; . \qquad \text{(I,b,15)}$$

Calling $\;{}^{(a,A)}\varphi_{s_3}([p]) = \langle[p],s_3|U(a,A)|\Phi\rangle \qquad$ (I,b,16)

the wave function of a transformed state, one finds ;

$${}^{(a,A)}\varphi_{s_3}([p]) = e^{ip.a} \; \mathcal{D}_{s_3 s_3'}\left([p]^{-1}A[A^{-1}p]\right) \varphi_{s_3'}\left([A^{-1}p]\right) \; . \qquad \text{(I,b,17)}$$

---

* The Lorentz invariance can be checked by $\int f(p)\theta_+(\overset{o}{p})\,\delta(p^2 - m^2)\,d^4p = \int \frac{d^3p}{2\omega_p} \; f(\omega_p, \vec{p})$

Note that for $|\Phi\rangle$ in the Hilbert space, normalized to 1 ,

$$\langle\Phi|\Phi\rangle = 1 = \int \frac{d^3p}{2\omega_p} \sum_{s_3} |\varphi_{s_3}([p])|^2 \quad .$$

States $|[q],\sigma\rangle$ with wave functions

$$\langle [p],s_3|[q],\sigma\rangle = 2\omega_q \;\; \delta(\vec{q} - \vec{p}) \; \delta_{s_3\sigma}$$

are thus not in the Hilbert space, as is usual for plane wave states.

2) $p^2 = 0 \quad p^o > 0 \; , \; p^o < 0$

Two cases : either the spin quantum number W^2 is different from zero,but in this case the representation does not correspond to an object seen in physics,

or , $W^2 = 0$.

Now,if $p^2 = 0 \quad W(p)^2 = 0 \quad p.W(p) = 0$ one has necessarily $W(p) = \lambda(p)p$ (this is simply geometry).

Let us again attach to p , two only, in this case,space-like vectors : $n_i(p) \quad i = 1,2 \quad p.n_i(p) = 0 \quad n_1(p).n_2(p) = 0 \quad \det(t,n_1(p)\; n_2(p),p) \gtrless 0$ (according to $p^o \gtrless 0$) ,

Let $\overset{o}{p} = (\pm 1,0,0,1) \quad n_1(\overset{o}{p}) = (0,1,0,0) \quad n_2(\overset{o}{p}) = (0,0,1,0)$ and let $[p]$ be the Lorentz transformation which takes $\overset{o}{p}$, $n_1(\overset{o}{p})$, $n_2(\overset{o}{p})$ into p , $n_1(p)$ $n_2(p)$.

The Lorentz transformations which leave $\overset{o}{p}$ invariant have the form

$$\begin{pmatrix} e^{\frac{i\Phi}{2}} & Z \\ 0 & e^{\frac{-i\Phi}{2}} \end{pmatrix} \quad *$$

where Φ is the rotation parameter, x , y $(Z = x + iy)$ the translation parameter of a two dimensional euclidian group (a degenerate form of the previous rotation group).

Indeed $W(p).p = 0$ implies $W(p) = \lambda(p)\, p + \sum_{1}^{2} W_i(p)\, n_i(p)$ (I,b,18)

with the commutation rules

$$\begin{aligned} [W_1(p)\ ,\ W_2(p)] &= 0 \\ [\lambda(p)\ ,\ W_1(p)] &= W_2(p) \\ [\lambda(p)\ ,\ W_2(p)] &= -\ W_1(p) \end{aligned} \qquad \text{(I,b,19)}$$

$$W^2(p) = W_1^2(p) + W_2^2(p) = 0 \quad \Longrightarrow \quad W_1(p) = W_2(p) = 0 \quad ,$$

i.e. one represents the translation of this euclidean group by $\mathbb{1}$ (gauge invariance). The rotation Φ is thus represented by $e^{i\lambda\Phi}$ λ integer of half integer (again because of the two fold covering coming from SL(2,C) versus Lorentz).

---

* $A\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} A^+ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \underset{\sim}{\overset{o}{p}} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\rightarrow |a|^2 = 1$, $ca^* = ac^* = cc^* = 0$, $ad - bc = 1$

yields the result.
(similarly in the $\overset{o}{p} = (1,000)$ case, $A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{\overset{o}{p}} \rightarrow AA^+ = 1$
$A \in SU(2)$).

We thus have basis vectors $|[p],\lambda\rangle$ defined by

$$\lambda(p)\ |[p],\lambda\rangle = \lambda\ |[p],\lambda\rangle ,$$

$$W_1(p)\ |[p],\lambda\rangle = 0 \quad . \tag{I,b,20}$$

λ , integer or half integer, is the helicity of the massless particle. The second equation expresses gauge invariance : if $[p]$ and $[p]'$ are two frames corresponding to the same momentum, one has

$$n_1'(p) = n_1(p)\cos\varphi - n_2(p)\sin\varphi + \alpha_1 p$$

$$n_2'(p) = n_1(p)\sin\varphi + n_2(p)\cos\varphi + \alpha_2 p$$

and
$$|[p]',\lambda\rangle = |[p],\lambda\rangle\ e^{i\lambda\varphi}$$

$$= |[p],\lambda\rangle\ \mathcal{D}^{|\lambda|}_{\lambda\lambda}\left([p]^{-1}[p]'\right) \quad , \quad ^{*} \tag{I,b,21}$$

which shows that $|[p],\lambda\rangle$ is independent of α_1 , α_2 and , as before

$$U(a,A)\ |[p],\lambda\rangle = e^{i(A.p).a}\ |[A.p],\lambda\rangle\ \mathcal{D}^{|\lambda|}_{\lambda\lambda}\left([A.p]^{-1}A[p]\right)^{*} \tag{I,b,22}$$

with the scalar product

$$\langle[p]',\lambda|[p],\lambda\rangle = 2\omega_p\ \delta(\vec{p} - \vec{p}\,') \tag{I,b,23}$$

and the expansion

$$|\Phi\rangle = \int \frac{d^3p}{2\omega_p}\ \varphi([p],\lambda)\ |[p],\lambda\rangle \quad . \tag{I,b,24}$$

---

* At this point we need know that $\mathcal{D}(A)$ is defined for A arbitrary and $\mathcal{D}^{|\lambda|}_{\lambda\lambda}(A) = (a)^{2\lambda}$ or $(d)^{2\lambda}$ according as $\lambda \gtrless 0$ (see footnote, previous page, and Appendix one).

Remarks

i) We shall not describe here the representations corresponding to $p^2 < 0$, which are as unphysical as those corresponding to $p^2 = 0$ $W^2 \neq 0$, although they may be of formal use, as well as those for which $p = 0$, except for the trivial one $U(a,\Lambda)\,|0\rangle = |0\rangle$, which is called the vacuum representation.

ii) As far as massless particles are concerned, the presence of a single helicity state is familiar in the neutrino case for which parity is not an operation of the symmetry group ; in the photon case however, the presence of both helicities is due to the fact that parity is included in the symmetry group, as we shall see later.

We shall now open a parenthesis which takes us back to every day's theoretical life or text-book treatments, but we insist that from the physical point of view this parenthesis is completely redundant and may be harmlessly skipped.

c) Theoretical Constructions

They are mainly based on the observation that the quantities $\mathcal{D}^s_{\sigma\sigma'}(u)$, usually defined for $u \in SU(2)$ $(u\,u^\dagger = 1)$ and fulfilling

$$\sum_{\sigma'} \mathcal{D}^s_{\sigma\sigma'}(u)\; \mathcal{D}^s_{\sigma'\tau}(u') = \mathcal{D}^s_{\sigma\tau}(u\,u') \qquad \text{(group law)}$$

$$\mathcal{D}^s_{\sigma\sigma'}(u) = \mathcal{D}^s_{\sigma'\sigma}(u^T) = \mathcal{D}^{s\,*}_{\sigma'\sigma}(u^\dagger) ,$$

are still defined for any two by two complex matrix and fulfill the same identities.

We shall occasionally need properties of the matrix $C = i\sigma_2$

$$C\,A\,C^{-1}\,A^{T} = \det A$$

$C = -C^{T} = C^{*}$, $C^{2} = -1$; $\mathcal{D}^{s}(C)$ will still be called C when no confusion may arise.

- Spinor amplitudes.

i) $p^{2} = m^{2} > 0$.

From $$\varphi_{s_3}([p]') = \mathcal{D}_{s_3 s'_3}\left([p]'^{-1}[p]\right)\varphi_{s'_3}([p]) ,$$

one observes that

$$\varphi_A(p) = \mathcal{D}_{A\,s_3}([p]')\;\varphi_{s_3}([p]') = \mathcal{D}_{A\,s'_3}([p])\,\varphi_{s'_3}([p]) \qquad (I,c,1)$$

is independent of $[p]$, i.e. only depends on the four vector p .
Similarly, using the unitarity of $[p]'^{-1}[p] = \left([p]'^{-1}[p]\right)^{\dagger\,-1} = [p]'^{\dagger}\,[p]^{\dagger\,-1}$

$$\hat{\varphi}_A(p) = \mathcal{D}_{A\,s_3}\left([p]'^{\dagger -1}\right)\varphi_{s_3}([p]') = \mathcal{D}_{A\,s'_3}\left([p]^{\dagger -1}\right)\varphi_{s'_3}([p])$$
$$(I,c,2)$$

enjoys the same property (note that $[p]^{\dagger -1} \neq [p]$ since $[p]$ is not unitary).

These are called the spinor amplitudes associated with state $|\Phi\rangle$.

They transform simply according to

$$^{(a,A)}\varphi(p) = \mathcal{D}(A)\;\varphi(A^{-1}.p)\quad e^{ip.a} \qquad (I,c,3)$$

$$^{(a,A)}\hat{\varphi}(p) = \mathcal{D}(A^{\dagger -1})\;\varphi(A^{-1}.p)\quad e^{ip.a} \qquad (I,c,4)$$

$$\langle\Psi,\Phi\rangle = \int \frac{d^3p}{2\omega_p}\sum_A \hat{\psi}^{*}_A(p)\;\varphi_A(p) = \int \frac{d^3p}{2\omega_p}\sum_A \psi^{*}_A(p)\;\hat{\varphi}_A(p) \qquad (I,c,5)$$

$$\langle \Psi, \Phi \rangle = \int \frac{d^3p}{2\omega_p} \sum_{A,A'} \psi^*_A(p) \; \mathcal{D}_{AA'}\left(\frac{\tilde{p}}{m}\right) \; \varphi_A(p)$$

$$= \int \frac{d^3p}{2\omega_p} \sum_{A,A'} \hat{\psi}^*_A(p) \; \mathcal{D}_{AA'}\left(\frac{p}{m}\right) \; \hat{\varphi}_A(p) \qquad (I,c,5)$$

where we have used

$$[p][p]^\dagger = [p] \, \mathbb{1} \, [p]^\dagger = [p] \frac{\overset{o}{p}}{m} [p]^\dagger = \frac{p}{m} = \frac{p^o + \vec{p}\,\vec{\sigma}}{m}$$

$$[p]^{\dagger -1}[p]^{-1} = [p]^{\dagger -1} \mathbb{1} \, [p]^{-1} = [p^{\dagger -1}] \frac{\overset{\sim}{\overset{o}{p}}}{m} [p]^{-1} = \frac{\tilde{p}}{m} = \frac{p^o - \vec{p}.\vec{\sigma}}{m} \qquad (I,c,6)$$

and the "Dirac equations" :

$$\varphi(p) = \mathcal{D}\left(\frac{p}{m}\right) \hat{\varphi}(p) \qquad\qquad \hat{\varphi}(p) = \mathcal{D}\left(\frac{\tilde{p}}{m}\right) \varphi(p) \qquad (I,c,7)$$

ii) $p^2 = 0$.

From $\varphi\Big([p]',\lambda\Big) = \mathcal{D}^{|\lambda|}_{\lambda\lambda}\Big([p]'^{-1}[p]\Big)\,\varphi\Big([p],\lambda\Big)$,

we have*

$$\hat{\varphi}^-_A(p) = \mathcal{D}^{|\lambda|}_{A\lambda}\Big([p]'^{\dagger -1}\Big) \; \varphi\Big([p]',\lambda\Big) = \mathcal{D}^{|\lambda|}_{A\lambda}\Big([p]^{\dagger -1}\Big) \; \varphi\Big([p],\lambda\Big) ,$$

for $\lambda < 0$, where we used

$$\mathcal{D}^{|\lambda|}_{A\lambda}\Big([p]'^{\dagger -1}\Big) \; \mathcal{D}_{\lambda\lambda}\Big([p]'^{-1}[p]\Big) = \mathcal{D}^{|\lambda|}_{A\lambda}\Big([p]'^{\dagger -1} \frac{1-\sigma_3}{2} [p]'^{-1}[p]\Big)$$

$$= \mathcal{D}^{|\lambda|}_{A\lambda}\Big(\frac{\tilde{p}}{2} [p]\Big) = \mathcal{D}^{|\lambda|}_{A\lambda}\Big([p]^{\dagger -1}\Big) ,$$

---

* We need the following results : $\mathcal{D}^j_{mm'}\left(\frac{1+\sigma_3}{2}\right) = \delta_{j,m}\,\delta_{j,m'}$ and $\mathcal{D}^j_{mm'}\left(\frac{1-\sigma_3}{2}\right) = \delta_{j,-m}\,\delta_{j,-m'}$

similarly, for $\lambda > 0$

$$\varphi^+_A(p) = \mathcal{D}^\lambda_{A\lambda}\left([p']\right) \varphi\left([p]',\lambda\right) = \mathcal{D}^\lambda_{A\lambda}\left([p]\right) \varphi\left([p],\lambda\right)$$

where we used

$$[p]' \frac{1+\sigma_3}{2} [p]'^\dagger = [p] \frac{1+\sigma_3}{2} [p]^\dagger = \frac{p}{2} .$$

<u>Exercise</u>.

Construct $\varphi^-(p)$, $\hat{\varphi}^+(p)$.

<u>Remarks</u>.

In this case, spinor amplitudes have but one independent component, i.e. they are subject to constraint equations (see ref. 4).

The uncapped (resp. capped) ones still transform according to $\mathcal{D}(A)$, $\mathcal{D}(A^{\dagger -1})$.

In the scalar product one just has to replace $\frac{p}{m}$ $\left(\frac{\tilde{p}}{m}\right)$ of the massive case by $\frac{p}{2}$, $\frac{\tilde{p}}{2}$.

d) <u>Examples</u>

1) $m^2 > 0$, $s = 0$, $p^0 > 0$.

One has wave functions $\varphi(p)$ defined on the positive hyperboloïd, with the scalar product

$$\langle \Phi,\Psi \rangle = \int \frac{d^3p}{2\omega_p} \varphi^*(\vec{p}) \psi(\vec{p}) .$$

Given $\varphi(\vec{p})$ sufficiently regular, one can construct the distribution $\varphi(\vec{p})\, \theta(p^0)\, \delta(p^2 - m^2) = \tilde{\Phi}(p)$ whose Fourier transform $\Phi(x) = \int e^{ipx}\, \tilde{\Phi}(p)\, d^4p$

is a positive frequency solution of the Klein Gordon equation.

The scalar product then reads

$$\langle \Phi,\Psi \rangle = \frac{1}{i(2\pi)^3} \int_{x^o = t} \Phi^*(x) \overleftrightarrow{\partial}^o \Psi(x)\, d^3x \quad ,$$

and the transformation law is

$$^{(a,A)}\varphi(x) = \varphi(\Lambda x + a) \quad .$$

One could argue similarly for the negative energy solutions ; both of them span representation spaces with characteristics $m^2 > 0$ $s = 0$ $p^o \gtrless 0$.

One can set a one to one correspondence between negative energy solutions and positive energy ones by

$$\Psi^-(x)^* = \Phi^+(x) \quad .$$

Then from the point of view of relativity, solutions of the Klein Gordon equation can be interpreted as describing two kinds of particles with the same mass and zero spin.

2) $m^2 > 0$, $s = \frac{1}{2}$, $p^o > 0$

We write the Dirac equations

$$\varphi(p) = \frac{\underset{\sim}{p}}{m}\, \hat{\varphi}(p) \qquad \hat{\varphi}(p) = \frac{\tilde{p}}{m}\, \varphi(p) \quad ,$$

let

$$\Phi(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi(p) \\ \hat{\varphi}(p) \end{pmatrix} \quad .$$

Then one has $(\not{p} - m)\, \Phi(p) = 0$

$$\not p = \gamma^o p^o - \vec{\gamma}\cdot\vec{p} \quad ,$$

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad ,$$

with the transformation law

$$\{a,A\}\,\Phi(p) = e^{ip.a}\; S(A)\;\; \Phi(A^{-1}.p)$$

where

$$S(A) = \begin{pmatrix} \mathcal{D}(A) & 0 \\ 0 & \mathcal{D}(A^{-1\dagger}) \end{pmatrix}$$

if $\quad A = e^{i\frac{\vec{\sigma}}{2}\vec{n} + \frac{\vec{\sigma}}{2}\vec{v}} \qquad S(\Lambda) = e^{\frac{i}{2}\varepsilon_{ijk}\frac{\sigma^{ij}}{2}n^k + \frac{\sigma_{oi}}{2}v^i}$

$$= e^{\frac{i}{2}\omega_{\mu\nu}\frac{\sigma^{\mu\nu}}{2}}$$

where $\qquad \omega_{ij} = \varepsilon_{ijk}\, n^k \qquad \omega_{oi} = v_i$

$$\sigma^{\mu\nu} = \frac{1}{2i}\;\{\gamma^\mu,\gamma^\nu\} \quad .$$

The scalar product is given by

$$\langle\Phi,\Psi\rangle = \int \Phi^*(p)\;\;\gamma^o\;\Psi(p)\;\;\frac{d^3p}{2\omega_p} = \int \bar{\Phi}(p)\;\;\Psi(p)\;\;\frac{d^3p}{2\omega_p} \quad ,$$

where $\bar{\Phi}(p) = \Phi^*(p)\,\gamma^o$. This scalar product stays positive as long as $p^o > 0$.

We may as well write

$$\langle\Phi,\Psi\rangle = \int \bar{\Phi}(p)\;\gamma^o\;\Psi(p)\;\;\frac{d^3p}{2\;m} \quad ,$$

which is sometimes used and is positive for both $p^o \gtrless 0$.

Further Dirac matrices are defined as

$$\beta = \gamma^o \quad , \quad \alpha^1 = \gamma^o\gamma^1 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = -\alpha_1$$

$$\gamma^5 = \gamma^o\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} .$$

The spin operator is $W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\frac{\sigma^{\nu\rho}}{2}\frac{p^\sigma}{m}$;

it allows one to define a basis for solutions of the Dirac equation through

$$W(p).n_3(p)\, U([p],s_3) = s_3\, U([p],s_3)$$

$$W(p).(n_1 \pm in_2)(p)\, U([p],s_3) = \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - s_3(s_3 \pm 1)}\;\; u([p],s_3 \pm 1)$$

$$W.n_1(p) = -\gamma^5 \not{n}_1(p) \qquad ,$$

which yields

$$U.([p],s_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D}_{.s_3}([p]) \\ \mathcal{D}_{.s_3}([p]^{\dagger -1}) \end{pmatrix}$$

with the normalization $\bar{u}([p],s_3')\; u([p],s_3) = \delta_{s_3 s_3'}$.

The solutions of the Dirac equations for $p^o > 0$ can thus be expanded according to

$$\Phi(p) = \sum_{s_3} f([p],s_3)\;\; u([p],s_3) \qquad ,$$

and one finds the f's transform according to (I,b,17) and their scalar product is again given by (I,b,15) .

For $p^o < 0$ similar results hold and lead to the complex conjugate representation.

3) $m^2 > 0$, $s = 1$, $p^o > 0$.

$\varphi_A(p)$ and $\hat{\varphi}_A(p)$ can be considered as complex vectors defined on the upper sheet $p^o > 0$ of the hyperboloïd $p^2 = m^2$. We may express them in cartesian form $\vec{\varphi}(p)$, using

$$\frac{1}{\sqrt{2}} \; (\vec{\sigma}.\vec{\varphi})_{mm'} = \begin{pmatrix} 1/2 & 1/2 & \big| & 1 \\ m & -m' & \big| & A \end{pmatrix} (-1)^{1/2 - m'} \quad \varphi_A$$

then, (I,c,3) and (I,c,4) can be written as* :

$$\left(\vec{\sigma}.\{a,A\} \; \vec{\varphi}\right) = A \; (\vec{\sigma}.\vec{\varphi}) \; A^{-1} \qquad ,$$

the Dirac equation :

$$\begin{cases} \varphi(p) = \mathcal{D}\left(\dfrac{p}{m}\right) \hat{\varphi}(p) \\ \hat{\varphi}(p) = \mathcal{D}\left(\dfrac{\tilde{p}}{m}\right) \varphi(p) \end{cases} ,$$

then reads :

$$\begin{cases} \dfrac{p^o - \vec{p}\,\vec{\sigma}}{m} \; (\vec{\sigma}.\vec{\varphi}) = (\vec{\sigma}.\vec{\hat{\varphi}}) \; \dfrac{p^o - \vec{p}\,\vec{\sigma}}{m} \\ (\vec{\sigma}.\vec{\varphi}) \; \dfrac{p^o + \vec{p}\,\vec{\sigma}}{m} = \dfrac{p^o + \vec{p}\,\vec{\sigma}}{m} \; (\vec{\sigma}.\vec{\hat{\varphi}}) = \left(\vec{\sigma}\;\vec{\hat{\varphi}}^* \; \dfrac{p^o + \vec{p}\,\vec{\sigma}}{m}\right)^{\dagger} \end{cases} .$$

---

* This transformations law can be compared with the transformation under rotation group of $\vec{\sigma}\,\vec{u}$, $\vec{u}$ real trivector, $\vec{\sigma}.R\vec{u} = \mathcal{U}(R) \; \vec{\sigma}.\vec{u} \; \; \mathcal{U}^{+}(R)$, and allows us to consider A as a complex rotation (element of the complex orthogonal group).

If we introduce antisymmetric tensors $F_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$ by : $F^{oi} = E^i$, $F^{ij} = \varepsilon^{ijk} H^k$, where $\vec{E}$ and $\vec{H}$ are real vectors defined by $\vec{\varphi} = \vec{E} + i\vec{H}$ (similarly $\hat{\vec{\varphi}} = \vec{\mathcal{E}} - i\vec{\mathcal{H}}$ defines $\mathcal{F}$). The Dirac equation then reads :

$$p_\mu \left(F^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) = p_\mu \left(\mathcal{F}^{\mu\nu} - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma} \right)$$

i.e.

$$p_\mu \, G^{\mu\nu} = 0 \text{ , with } G^{\mu\nu} = F^{\mu\nu} - \mathcal{F}^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (F_{\rho\sigma} + \mathcal{F}_{\rho\sigma})$$ an arbitrary complex antisymmetric tensor .

One can thus find A_μ such that

$$G^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} p_\rho A_\sigma \quad ,$$

and the scalar product can be rewritten

$$\langle \mathcal{G}, G \rangle = \int d\Omega(p) \, \bar{\mathcal{G}}^{\mu\nu}(p) \, G_{\mu\nu}(p) \, \frac{1}{m^2}$$

$$= - \int d\Omega(p) \bar{\mathcal{A}}^\mu(p) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right) A^\nu(p) \quad ,$$

so that one can choose $p_\mu A^\mu = 0$ with the scalar product

$$\langle \mathcal{A}, A \rangle = - \int d\Omega(p) \, \bar{\mathcal{A}}_\mu(p) \, A^\mu(p)$$

This is the usual representation of spin 1 particles by vector fields transverse to the momentum. One can recover the Wigner amplitude by writing

$$A(p) = \sum_1^3 f_i([p]) \, n_i([p])$$

$$p.n_i([p]) = 0 \qquad n_i([p]).\, n_j([p]) = - \delta_{ij} \quad ,$$

and from the transformation law

$$\{a,\Lambda\}A_{\mu}(p) = e^{ip.a} \Lambda_{\mu}^{\ \nu} A_{\nu}(\Lambda^{-1} p) ,$$

recover the Wigner transformation for $f_{i}([p])$.

4) $m^2 > 0 \qquad s = \frac{3}{2}$

One may transform the spinor amplitudes $\varphi_A(p)$ $\left(\hat{\varphi}_A(p)\right)$, $-\frac{3}{2} \leq A < +\frac{3}{2}$, into totally symmetric spinor amplitudes $\varphi_{\alpha_1\alpha_2\alpha_3}$ $(\alpha_1,\alpha_2,\alpha_3 = \pm 1/2)$, the relevant $\mathcal{D}_{AA'}$ matrices being replaced by $\mathcal{D}_{\alpha_1\alpha_1'}$ $\mathcal{D}_{\alpha_2\alpha_2'}$ $\mathcal{D}_{\alpha_3\alpha_3'}$.

From the symmetry condition,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \alpha_1 & \alpha_2 & 0 \end{pmatrix} \varphi_{\alpha_1\alpha_2\alpha_3} = 0 .$$

Let

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \alpha_1 & \alpha_2 & M \end{pmatrix} \varphi_{\alpha_1\alpha_2\alpha_3} = \varphi_{M\alpha_3} ,$$

whereby the $\mathcal{D}$'s turn into $\mathcal{D}_{MM'}$, $\mathcal{D}_{\alpha_3\alpha_3'}$. Indices M and α_3 can be capped separately by application of the relevant $\mathcal{D}\left(\frac{p^0 + p\,\vec{\sigma}}{m}\right)$. If $\vec{S}$ are the spin 1 operators, the spin 3/2 condition

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \alpha_1 & M & \beta \end{pmatrix} \varphi_{M\alpha_1} = 0 ,$$

reads $\frac{1 - \vec{S}.\vec{\sigma}}{3} \; \vec{\varphi} = 0$ i.e. $\vec{\varphi} = i\vec{\sigma} \times \vec{\varphi}$

where M is turned into a vector index, α being considered as a spin $\frac{1}{2}$ index.

The simultaneous consideration of $\varphi_{\hat{M}\alpha}$ $\varphi_{M\hat{\alpha}}$ $\varphi_{\hat{\hat{M}}\hat{\alpha}}$ and the corresponding Dirac equations allow to define $\Psi_{\mu\alpha}$ where μ is a four vector index, α a Dirac spinor index, the constraints being :

$$p.\Psi = 0 \qquad (\not{p} - m)\Psi = 0$$

$$\Psi_\mu = \left[\frac{1}{2} (S_{\lambda\kappa} \, \sigma^{\lambda\kappa}) \, \Psi \right]_\mu$$

$$= i \, \sigma_\mu^{\ \nu} \Psi_\nu$$

hence $\gamma^\mu \, \Psi_\mu = 0$.

Note that if Ψ_μ fulfills both the Dirac equation and the transversality condition, $\Phi = \gamma^5 \, \gamma^\mu \, \Psi_\mu$ is just its spin $\frac{1}{2}$ component.

5) $m^2 > 0 \quad s = 2$

One can similarly switch from the Wigner description to one in terms of a symmetric transverse, traceless tensor : $G_{\mu\nu}(p)$:

$$G_{\mu\nu}(p) = G_{\nu\mu}(p)$$

$$p^\mu \, G_{\mu\nu}(p) = 0$$

$$G^\mu_{\ \mu}(p) = 0 \quad ,$$

with the scalar product $(F,G) = \int d\Omega\,(p) \, \bar{F}_{\mu\nu}(p) \;\; G^{\mu\nu}(p)$.

One can expand $G_{\mu\nu}(p) = \sum_A f_A([p])\ Y^A_{\mu\nu}([p])$ where the $Y^A_{\mu\nu}([p])$'s are constructed from $n_i(p)$ as $Y^A(\vec{x})$ is constructed in terms of x_i, $i = 1,2,3$ ($Y^A(\vec{x})$: solid harmonics of order 2) e.g.

$$Y^o_{\mu\nu}([p]) = n^3_\mu\ n^3_\nu - \frac{1}{3}\left(g_{\mu\nu} - \frac{p_\mu\ p_\nu}{m^2}\right) .$$

The transformation law

$$\{a.A\}G_{\mu\nu}(p) = e^{iA.p.a}\ \Lambda_\mu^{\ \mu'}\ \Lambda_\nu^{\ \nu'}\ G_{\mu'\nu'}(A^{-1}.p) ,$$

induces on the f_A's the Wigner rotation.

6) $m = 0$.

The previous constructions have quite decent limits which one may construct starting from the Wigner form ; the selection of one helicity value requires however the use of an extra projector :

i) $\lambda = \pm\frac{1}{2}$.

$$\not{p}\,\Psi = 0$$

$$W_\mu\ \Psi = \pm\frac{1}{2}\,p_\mu\ \Psi = \pm\frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\ \frac{\sigma^{\nu\rho}}{2}\ p^\sigma\ \Psi$$

$$= \pm\frac{1}{2}\,\gamma_5\ \sigma_{\mu\nu}\ p^\nu\ \Psi$$

$$= \pm\frac{1}{2}\,\gamma_5\ \frac{\gamma_\mu\ \gamma_\nu - g_{\mu\nu}}{i}\ p^\nu\ \Psi ,$$

by use of the Dirac equation, we get

$$W_\mu\ \Psi = \pm\frac{i\ \gamma_5}{2}\ p_\mu\ \Psi = \pm\frac{1}{2}\,p_\mu\ \Psi ,$$

hence the projector condition $(1 \pm i\,\gamma_5)\,\Psi = 0$

ii) $\lambda = \pm 1$

Let $F_{\mu\nu}$ be the antisymmetric tensor which describes a spin 1 particle, with $\partial_\mu F^*_{\mu\nu} = 0$ where $F^*_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}$; the projector condition corresponding to helicity ± 1 is then found to be "self or antiself duality" :

$$F_{\mu\nu} \pm i\,F^*_{\mu\nu} = 0 \quad .$$

In terms of the vector description, besides the transversality condition, one finds that $A_\mu(p)$ is defined modulo p , i.e. $A_\mu(p)$, $A_\mu(p) + \lambda(p)\,p_\mu$ represent the same state (as indeed the scalar product is insensitive to a gauge transformation). If $n_1(p)$, $n_2(p)$ are such that $\det(\,n_1(p), n_2(p), p\,, n\,) > 0$ where n is some positive time axis, then helicity states $\lambda = \pm 1$ are described by $\frac{n_1 \pm i\,n_2}{\sqrt{2}}(p)$ (which isotropic vectors are indeed independent of the choice of n_1 , n_2).

e) <u>Discrete Operations</u> [2;5]

If the symmetry group is enlarged so that it includes space and time inversions :

$$x \to \underline{x} = \Pi\,x \qquad\qquad \Pi = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = G \qquad (I,e,1)$$

where G is the matrix of the metric tensor :

$$G\,L\,G^{-1} = L^{-1T}$$

$$x \to -\underline{x} = T\,x \qquad T = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \qquad (I,e,2)$$

$$x \to -x = \Pi\,T\,x \qquad \Pi\,T = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}, \qquad (I,e,3)$$

one has, in order to construct representations up to a phase of the enlarged group, to specify the actions of these discrete operations on $\bar{\mathcal{P}}^{\uparrow}_{+}$.

One has :

$$\begin{aligned} \Pi(a,A)\,\Pi^{-1} &= (\underline{a}\,,\,A^{\dagger\,-1}) \\ T(a,A)\,T^{-1} &= (-\underline{a}\,,\,A^{\dagger\,-1}) \qquad (I,e,4) \\ \Pi\,T\,(a,A)\,(\Pi\,T)^{-1} &= (-a\,,\,A) \quad . \end{aligned}$$

These relationships do not specify the extended "covering group" completely, but hold, whatever the solution, and are sufficient for the construction of representations.

<u>Proof</u>

Every Lorentz transformation Λ can be written $\Lambda = L\,R$ where L is a pure Lorentz transformation and R is a rotation. Correspondingly $A(\Lambda) = H\,\mathcal{U}$ where H is hermitean and $\mathcal{U}$ is unitary.

$$\Lambda^{-1T} = L^{-1T}\,R^{-1T} = L^{-1}\,R$$

$$A(\Lambda^{-1T}) = H^{-1}\,\mathcal{U} = A^{\dagger\,-1} \quad .$$

(There cannot be any arbitrary sign factor in front, because such a sign $\varepsilon_\pi(A)$ should fulfill the group law $\varepsilon_\pi(A)\, \varepsilon_\pi(A') = \varepsilon_\pi(AA')$ and has thus to be unity).

We now look for representations up to a phase of the "extended covering group". Upon restriction to the restricted group, they will yield representations of the latter. There will thus be in representation space, basis of the type $|[p], s_3, \eta\rangle$, $|[p], \lambda, \eta\rangle$ where η are degeneracy labels. We shall furthermore restrict ourselves to representations where $U(\Pi)$ is unitary, $U(\Pi\ T)$, $U(T)$, antiunitary, in order to deal with representations for which the spectrum of the momentum operator contains only time like or light like values.

Indeed, from the group law ;

$$U(\Pi)\ e^{iP.a}\ U^{-1}(\Pi) = e^{iP.\underline{a}} \quad .$$

Hence

$$U(\Pi)\ e^{iP.a}\ U^{-1}(\Pi)\ |p\rangle = e^{ip.\underline{a}}\ |p\rangle$$

$$e^{iP.a}\ U^{-1}(\Pi)\ |p\rangle = e^{\varepsilon\ ip.\underline{a}}\ U^{-1}(\Pi)\ |p\rangle$$

$\varepsilon = +1$ if $U(\Pi)$ is unitary,

$\varepsilon = -1$ if $U(\Pi)$ is antiunitary.

Thus $U^{-1}(\Pi)\ |p\rangle$ has momentum $\varepsilon\ \underline{p}$ $(p.\underline{a} = \underline{p}.a)$

$\varepsilon\ \underline{p} \in \overline{V}^+$ if $p \in \overline{V}^+ \rightarrow \varepsilon = +1$.

A similar argument implies that $U(T)$, $U(\Pi\ T)$ are antiunitary.

Since $\Pi^2 = T^2 = (\Pi\ T)^2 =$ identity, there exist phases ω such that :

$$U(\Pi)^2 = \omega_\Pi$$

$$U(\Pi\ T)^2 = \omega_{\Pi T}$$

$$U(T)^2 = \omega_T \quad .$$

From a previous argument (Chapter 0, § a) Remark II) $\omega_{\Pi T} = \pm 1$, $\omega_T = \pm 1$

i) Representation of space inversions.

We shall only deal with massive particles leaving the case of massless ones as an exercise.

$$U(\Pi)\ U(a,A)\ U^{-1}(\Pi)\ |[p],s_3,\eta\rangle = U(\underline{a},A^{\dagger -1})\ |[p],s_3,\eta\rangle$$

$$= e^{i\underline{a}\ \Lambda^{-1T}p}\ |[\Lambda^{-1T}p],s_3',\eta\rangle\ \mathcal{D}_{s_3's_3}\left([\Lambda^{-1T}p]^{-1}\ A^{\dagger -1}[p]\right)$$

$$= e^{ia\ \Lambda\underline{p}}\ |[\Lambda^{-1T}p],s_3',\eta\rangle\ \mathcal{D}_{s_3't_3'}\left([\Lambda^{-1T}p]^{\dagger}[\Lambda\underline{p}]\right)$$

$$\times\ \mathcal{D}_{t_3't_3}\left([\Lambda\underline{p}]^{-1}\ A[\underline{p}]\right)\ \mathcal{D}_{t_3s_3}\left([\underline{p}]^{-1}\ [p]^{\dagger -1}\right) \quad ,$$

where we have used for the Wigner rotation $\mathcal{D}(R) = \mathcal{D}(R^{\dagger -1})$.

Hence

$$U(a,A)\ U^{-1}(\Pi)\ |[p],s_3,\eta\rangle\ \mathcal{D}_{s_3t_3}\left([p]^{\dagger}\ [\underline{p}]\right)$$

$$= e^{ia\ \Lambda\underline{p}}\ |[\Lambda\underline{p}],s_3',\eta\rangle\ \mathcal{D}_{s_3't_3'}\left([\Lambda\underline{p}]^{\dagger}[\Lambda\underline{p}]\right)\ \mathcal{D}_{t_3't_3}\left([\Lambda\underline{p}]^{-1}\ A[\underline{p}]\right) \quad ,$$

which means that

$$U^{-1}(\Pi)\ |[p],s_3,\eta\rangle\ \mathcal{D}_{s_3t_3}\left([p]^{\dagger}\ [\underline{p}]\right)$$

transforms as a vector $|\underline{p}\ ,\ s_3\rangle$

(note that $[p]^{\dagger}\ [\underline{p}]$ is a Wigner rotation :

$[p]^{\dagger}\ [\underline{p}]\ [\underline{p}]^{\dagger}\ [p] \ = \ [p]^{\dagger}\ \frac{p^o - \vec{p}\,\vec{\sigma}}{m}\ [p] \ = \ [p]^{\dagger}\ [p]^{\dagger -1}\ [p]^{-1}\ [p] \ = \ 1$).

Thus, one may write

$$U^{-1}(\Pi)\ |[p],s_3,\eta\rangle \ = \ |[\underline{p}],s_3',\eta'\rangle\ \mathcal{D}_{s_3's_3}\left([\underline{p}]^{-1}\ [p]^{\dagger -1}\right)\tilde{\Pi}_{\eta'\eta} \quad ,$$

where $\tilde{\Pi}_{\eta'\eta}$ is a numerical matrix in the degeneracy parameters :

$$\tilde{\Pi}\,\tilde{\Pi}^{\dagger} = 1 \qquad\qquad \tilde{\Pi}^2 = \omega_{\Pi}^{-1} \quad .$$

If one wants an irreducible representation, $\tilde{\Pi}$ fulfilling the above requirements has to be one dimensional : $\tilde{\Pi}_{\eta'\eta} \ = \ \eta_{\Pi}$.

The final result is :

$$\begin{aligned} U(\Pi)\ |[p],s_3\rangle \ &= \ \eta_{\Pi}\ |[\underline{p}],s_3'\rangle\ \mathcal{D}_{s_3's_3}\left([\underline{p}]^{-1}\ [p]^{\dagger -1}\right) \\ &= \ \eta_{\Pi}\ |[\underline{p}],s_3'\rangle\ \mathcal{D}_{s_3's_3}\left([\underline{p}]^{\dagger}\ [p]\right) \end{aligned} \qquad (I,e,5)$$

__Exercise__

Repeat the argument for massless particles. Show, as expected, that couples of opposite helicities occur.

__Remarks__

If $[p]$ is taken to be the pure Lorentz transformation transforming the time axis $(1,0,0,0)$ to p :

$$[p] = \frac{m + p^o + \vec{p}\,\vec{\sigma}}{\sqrt{2m\,(m + p^o)}}$$

$[\underline{p}]^{\dagger} = [\underline{p}] = [p]^{-1}$, hence no Wigner rotation appears in the above formula.

ii) Representation of Space-time inversion.

$$U(\Pi\ T)\ U(a,A)\ U^{-1}(\Pi\ T)\ \ |[p],s_3,\eta\rangle$$

$$= U(-a,A)\ |[p],s_3,\eta\rangle$$

$$= e^{-ia.\Lambda p}\ |[\Lambda p],s_3',\eta\rangle\ \mathcal{D}_{s_3's_3}\left([\Lambda p]^{-1}\ A[p]\right) \quad .$$

Hence (watch the effect of antiunitarity of $U(\Pi\ T)$)

$$U(a,A)\ U^{-1}(\Pi\ T)\ |[p],s_3,\eta\rangle = e^{ia.\Lambda p}\ U(\Pi\ T)^{-1}\ |[\Lambda p],s_3',\eta\rangle\ \mathcal{D}^*_{s_3's_3}\left([\Lambda p]^{-1}A[p]\right)$$

$$= e^{ia.\Lambda p}\ U(\Pi\ T)^{-1}\ |[\Lambda p],s_3',\eta\rangle\ \mathcal{D}_{s_3't_3'}(C)\ \mathcal{D}_{t_3't_3}\left([\Lambda p]^{-1}A[p]\right)\ \mathcal{D}_{t_3s_3}(C^{-1})$$

(where we used $C\,A\,C^{-1} = A^{-1T}$, $C = i\,\sigma_2$, and unitarity of the Wigner rotation).

One deduces

$$U(\Pi\ T)^{-1}\ |[p],s_3,\eta\rangle = |[p],s_3',\eta'\rangle\ \mathcal{D}_{s_3's_3}(C)\ \widetilde{\Pi\ T}_{\eta'\eta} \quad .$$

Now we have

$$U(\Pi\ T)^2 = \omega_{\Pi T} = \pm 1 \quad .$$

Hence $$\omega_{\Pi T} = (-)^{2s} \;\widetilde{\Pi\, T}\;\; \widetilde{\Pi\, T}^* \quad ,$$

(where we used $C\, C^* = -1$).

If $\omega_{\Pi T}\,(-)^{2s} = +1$ $\widetilde{\Pi\, T}$, which is unitary, is orthogonal, symmetric and can thus be diagonalized by a real orthogonal matrix, the irreducible version of which is one dimensional. In this case, one has :

$$\omega_{\Pi T} = (-)^{2s} = U(\Pi\; T)^2$$

$$U(\Pi\; T)\, |[p], s_3\rangle = \eta_{\Pi T}\, |[p], s_3'\rangle \;\mathcal{D}_{s_3' s_3}(C) \quad , \qquad \text{(I,e,6)}$$

namely, space-time inversion reverses spin.

We shall not deal here with the so-called abnormal type which arises when $\omega_{\Pi T}(-)^{2s} = -1$; each particle in this case has to have an internal degree of freedom which takes up two values in the case of an irreducible representation.

Exercise

Same problem in the case of massless particles. One helicity is enough.

iii) Representation of Time inversions.

A similar calculation yields for the "normal type" :

$$\omega_T = (-)^{2s} = U(T)^2 \qquad \text{(I,e,7)}$$

$$U(T)\, |[p], s_3\rangle = \eta_T\, |[\underline{p}], s_3'\rangle \;\mathcal{D}_{s_3' s_3}\Big([\underline{p}]^{\dagger}\, [p] C\Big) \quad .$$

(If $[p]$ is the Wigner boost $[p] = \dfrac{m + p^o + \vec{p}\,\vec{\sigma}}{\sqrt{2m\,(m + p^o)}}$, time inversion reverses both momentum and spin !) .

Remark

Note the phase relationship

$$U(\Pi)\, U(\Pi\, T) = \eta_{\Pi}\, \eta_{\Pi T}\, \eta_T^* \; U(T)$$

f) Representation of assemblies of identical particles : Fock space.

i) Once the description of one particle states is accurately known, the construction of many identical particle states fulfilling either Bose or Fermi statistics proceeds in a well known way by applying to a vacuum state $|0\rangle$ products of creation operators labelled according to one particle states : here $a^\dagger([p],s_3)$, which fulfill commutation or anticommutation relations with their adjoints $a([p'],s_3)$ according to the normalization condition :

$$\Big[a([p],s_3) \,,\, a([p'],s_3')\Big]_{\pm} = 2\omega_p\, \delta(\vec{p} - \vec{p}\,')\, \delta_{s_3 s_3'} \quad . \qquad (I,f,1)$$

The vacuum state is further characterized by the property that it is annihilated by all annihilation operators :

$$a([p],s_3\rangle\, |0\rangle = 0 \quad . \qquad (I,f,2)$$

In this formulation, one has :

$$|[p],s_3\rangle = a^\dagger([p],s_3)\, |0\rangle \quad . \qquad (I,f,3)$$

The covariance property

$$|[p],s_3\rangle = |[p]',s_3'\rangle\; \mathcal{D}_{s_3' s_3}\Big([p']^{-1}[p]\Big) \quad ,$$

implies that $a^{\dagger}([p],s_3)$ is defined for all $[p]'$s with the transformation law

$$a^{\dagger}([p],s_3) = a^{\dagger}([p]',s_3') \ \mathcal{D}_{s_3' s_3}\left([p']^{-1}[p]\right) \quad . \tag{I,f,4}$$

If one wants to describe various kinds of non identical particles, one can show[6] that they can be constructed from a unique vacuum by application of creation and annihilation operators $a_i^{(\dagger)}([p])$ which can at will be chosen to commute or anticommute for different i's , without any physical implication.

ii) Additive operators .

We recall a useful construction : let $|i\rangle$ be a basis of a Hilbert space of one particle states, $a_i^{\dagger}$ the corresponding creation operators in Fock space. To every operator $\mathcal{Q}$ in this Hilbert space, with matrix elements $\langle j|\mathcal{Q}|i\rangle$, one can associate a field operator

$$Q = \sum_i a_j^{\dagger} \langle j|\mathcal{Q}|i\rangle \ a_i \quad . \tag{I,f,5}$$

Then, as a consequence of commutation or anticommutation relations

$$[a_i, a_j]_{\pm} = \langle i|j\rangle \quad . \tag{I,f,6}$$

Commutation relations $[\mathcal{P},\mathcal{Q}]_- = \mathcal{R}$ yield $[P,Q] = R$. (I,f,7)

Exercise

Given the momentum P , and angular momentum operators $J_{\mu\nu}$ for one particle states, construct the corresponding field operators.

Remark

The momentum of a state with many particles of definite momenta is the sum of the individual momenta.

This is not true for the "spin" W_μ , as we shall see soon : the spin of a system of particles with definite spins includes an orbital part !

iii) Fields.

This paragraph is a digression from pure phenomenology. Assumptions particular to local field theories yield however sufficiently strong and useful results to be worth mentioning.

a) One type of particles.

Let $|0\rangle$ be the vacuum state and $a^\dagger([p],s_3)$ be the creation operators.

One may construct free fields :

$$\varphi_A(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}^{(s)}_{As_3}([p])\; a([p],s_3)\; e^{-ip.x} + \mathcal{D}^{(s)}_{As_3}([p]C^{-1})\; a^\dagger([p],s_3)\; e^{ip.x} \Big] \qquad (I,f,8)$$

$$\begin{aligned} \tilde{\varphi}_A(x) &= \mathcal{D}^{(s)}_{AA'}(C)\; \varphi^\dagger_{A'}(x) \\ &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}^{(s)}_{As_3}([p]^{\dagger-1})\; a([p],s_3)\; e^{-ip.x} + \mathcal{D}^{(s)}_{As_3}([p]^{\dagger-1}C)\; a^\dagger([p],s_3)\; e^{ip.x} \Big] , \end{aligned} \qquad (I,f,9)$$

which transform respectively under Poincaré transformation according to

$$U(a,A)\ \varphi_A(x)\ U^{-1}(a,A) = \mathcal{D}^{(s)}_{AA'}(A^{-1})\ \varphi_{A'}(A.x + a) \qquad (I,f,10)$$

$$U(a,A)\ \tilde{\varphi}_A(x)\ U^{-1}(a,A) = \mathcal{D}^{(s)}_{AA'}(A^{\dagger})\ \tilde{\varphi}_{A'}(A.x + a) \quad , \qquad (I,f,11)$$

where $U(a,A)$ is the representation in Fock space deduced from the one particle representation :

$$U(a,A)\ a^{\dagger}([p],s_3)\ U^{-1}(a,A) = a^{\dagger}([A.p],s'_3)\ \mathcal{D}_{s'_3 s_3}\left([A.p]^{-1}A[p]\right) e^{iAp.a} \quad ,$$

$$U(a,A)\ |0\rangle = |0\rangle \quad .$$

- The connection between spin and statistics.

If one assumes local commutativity in the form

$$\left[\overset{(\sim)}{\varphi}(x)\ ,\overset{(\sim)}{\varphi}(x')\right]_{\pm} = 0 \quad \text{for} \quad (x - x')^2 < 0$$

then one deduces that + or − sign (Fermi or Bose statistics) have to be used according as s is half-integer or integer.

The same conclusion holds for the more complicated fields to be introduced presently. These fields are free fields (they fulfill in particular the Klein Gordon equation $(\Box_x + m^2)\ \varphi_A(x) = 0$), and the Dirac equation

$$\varphi_A(x) = \mathcal{D}_{AA'}\left(\frac{\partial_o + \vec{\partial}\,\vec{\sigma}}{-\,im}\right)\ \tilde{\varphi}_{A'}(x)$$

$$\tilde{\varphi}_A(x) = \mathcal{D}_{AA'}\left(\frac{\partial_o - \vec{\partial}\,\vec{\sigma}}{-\,im}\right)\ \varphi_A(x) \quad ,$$

and can serve as asymptotic fields in a theory of the L S Z type.

b) Neutral multiplets.

Assume that besides Lorentz invariance, one has an internal compact symmetry group G, so that particles are labelled as basis vectors of a finite dimensional representation $\mathcal{U}$ of G:

we have creation operators $a^\dagger([p],s_3,\mu)$ behaving according to

$$U(g)\, a^\dagger([p],s_3,\mu)\, U^{-1}(g) = a^\dagger([p],s_3,\mu')\, \mathcal{U}_{\mu'\mu}(g) \qquad (I,f,12)$$

for $g \in G$, where $U(g)$ is the Fock space version of $\mathcal{U}$.

If the complex conjugate representation $\mathcal{U}^*$ is equivalent to $\mathcal{U}$, namely, there exists S such that

$$\mathcal{U}(g) = S\, \mathcal{U}^*(g)\, S^{-1} , \qquad (I,f,13)$$

one can construct fields behaving locally under G:

$$U(g)\, \varphi_{A,\mu}(x)\, U^{-1}(g) = \varphi_{A,\mu'}(x)\, \mathcal{U}_{\mu'\mu}(g)$$

$$\varphi_{A,\mu}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}_{As_3}([p])\, a([p],s_3,\mu)\, e^{-ip.x} + \mathcal{D}_{As_3}([p]C^{-1})\, a^\dagger([p],s_3,\mu')\, S_{\mu'\mu}\, e^{-ip.x} \Big] \qquad (I,f,14)$$

(it is in particular so for $G = SU(2)$ and for some representations of SU(3)).

<u>Exercise</u>.

Construct $\tilde{\varphi}_{A,\mu}(x)$

c) Charged multiplets.

If $\mathcal{U}$ and $\mathcal{U}^*$ are not equivalent, it is impossible to construct local fields in terms of creation operators transforming under G according to $\mathcal{U}(G)$. If however one introduces particles transforming according to $\mathcal{U}^*(G)$:

$$U(g)\, b^{\dagger}([p],s_3,\mu)\, U^{-1}(g) \;=\; b^{\dagger}([p],s_3,\mu')\, \mathcal{U}^{*}_{\mu'\mu}(g) \qquad (I,f,15)$$

and identically with the $a^{\dagger}$'s under (a.A) , one can construct :

$$\varphi_{A,\mu}(x) \;=\; \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}_{As_3}([p])\, a([p],s_3,\mu)\, e^{-ip.x} + \mathcal{D}_{As_3}([p]C^{-1})\, b^{\dagger}([p],s_3,\mu)\, e^{ip.x} \Big] \qquad (I,f,16)$$

which transform locally according to

$$U(g)\, \varphi_{A,\mu}(x)\, U^{-1}(g) \;=\; \varphi_{A,\mu'}(x)\; \mathcal{U}^{*}_{\mu'\mu}(g) \quad . \qquad (I,f,17)$$

In order that local commutativity might hold, it is necessary to assume that the a's and b's commute or anticommute according as the spin is integer or half integer - which, from a previous remark has no physical consequence.

$a^{\dagger}(\,p\,,s_3,\mu)$ and $b^{\dagger}(\,p\,,s_3,\mu)$ are said to create charge conjugate particles.

The operation which interchanges them is charge conjugation :

$$\mathcal{C}\; a^{\dagger}([p],s_3,\mu)\, \mathcal{C}^{-1} \;=\; b^{\dagger}([p],s_3,\mu)$$

$$\mathcal{C}\; a\,([p],s_3,\mu)\, \mathcal{C}^{-1} \;=\; b\,([p],s_3,\mu)$$

$$\mathcal{C}\,\mathcal{C}^+ = \mathcal{C}^+\mathcal{C} = \mathbb{1} \quad .$$

The charge conjugate field $\varphi^c_{A,\mu}(x)$ is obtained from $\varphi_{A,\mu}$ by this interchange.

Example.

$G = U(1)$:

$$U(\alpha)\ a^\dagger([p],s_3)\ U^{-1}(\alpha) = e^{i\alpha m}\ a^\dagger([p],s_3) \quad ,$$

where $e^{im\alpha}$ is a one dimensional representation labelled by the integer m (usually $m = 1$!).

Exercise.

Construct $\tilde{\varphi}$ $\tilde{\varphi}^c$

φ and $\tilde{\varphi}^c$ transform according to $\mathcal{U}^*(G)$

$\tilde{\varphi}$ and φ^c transform according to $\mathcal{U}(G)$

φ and φ^c transform according to $\mathcal{D}(A^{-1})$

$\tilde{\varphi}$ $\tilde{\varphi}^c$ transform according to $\mathcal{D}(A^\dagger)$.

The action of space and time inversions is easily deduced from the study of one particle states :

$$U(\Pi)\ \varphi_A(x)\ U^{-1}(\Pi) = \eta^*_\Pi\ \tilde{\varphi}_A(\underline{x})$$

$$U(\Pi)\ \tilde{\varphi}_A(x)\ U^{-1}(\Pi) = \eta^*_\Pi\ \varphi_A(\underline{x})$$

$$(I,f,18)$$

$$U(T)\ \varphi_A(x)\ U^{-1}(T) = \eta^*_T\ \varphi_{A'}(-x^o,\vec{x})\ \mathcal{D}_{A'A}(C)$$

$$U(T)\ \tilde{\varphi}_A(x)\ U^{-1}(T) = \eta^*_T\ \tilde{\varphi}_{A'}(-x^o,\vec{x})\ \mathcal{D}_{A'A}(C)$$

$$U(\Pi\ T)\ \varphi_A(x)\ U^{-1}(\Pi\ T) = \eta^*_{\Pi T}\ \tilde{\varphi}_{A'}(-x)\ \mathcal{D}_{A'A}(C)$$

$$U(\Pi\ T)\ \tilde{\varphi}_A(x)\ U^{-1}(\Pi\ T) = \eta^*_{\Pi T}\ \varphi_{A'}(-x)\ \mathcal{D}_{A'A}(C) \quad , \qquad (I,f,18)$$

and identical equations for the charge conjugate fields.

Remark.

In deriving these transformation laws, do not forget that U(T) and U(Π T) are antiunitary !

- Connection between intrinsic parities of particles and antiparticles

The foregoing equations hold if and only if

$$\eta_\Pi\ \eta_{\overline{\Pi}} = (-)^{2s}$$

$$\eta^*_T = \eta_{\overline{T}}$$

$$\eta^*_{\Pi T} = \eta_{\overline{\Pi T}}\ (-)^{2s}$$

where the $\eta_{\overline{?}}$ are the phase factors pertaining to antiparticles. This is a mere consequence of locality. The first condition is often used : it says that the relative intrinsic parity of a fermion and an antifermion is odd whereas the relative intrinsic parity of a boson and an antiboson is even - taking into account the connection between spin & statistics ! -

g) Application to S matrix theory - density matrices, projectors, transition probabilities.

i) Irreducible systems.

Statistical mixtures of one particle states are, as is well known, represented by density matrices which are positive definite self adjoint operators with trace 1 . In the present case, they are represented by kernels of the type $\rho_{\sigma\sigma'}([p],[p'])$ (or $\rho_{AA'}(p,p')$ in the spinor reoresentation), with

$$\rho_{\sigma\sigma'}([p],[p']) = \rho^*_{\sigma'\sigma}([p'],[p])$$

and

$$\int \frac{d^3p}{2\omega_p} \sum_{\sigma} \rho_{\sigma\sigma}([p],[p]) = 1 .$$

They behave under Lorentz transformations just as $|[p],\sigma\rangle\langle[p'],\sigma'|$ would do.

Projectors on sets of states are on the other hand similarly represented by self adjoint positive definite, bounded operators represented by kernels of the type $P([p],\sigma\ [p'],\sigma')$ which fulfill the reproducing property $P^2 = P$:

$$\int \frac{d^3p''}{2\omega''} \sum_{\sigma''} P\Big([p],\sigma\ [p''],\sigma''\Big)\Big([p''],\sigma''\ [p]',\sigma'\Big)$$

$$= P\Big([p],\sigma\ [p'],\sigma'\Big)$$

ii) Many particle systems.

Same as in i) where $[p],\sigma$ reoresents a collection

$$\Big\{[p_1],\sigma_1, \ldots , [p_n],\sigma_n\Big\}$$

iii) Transition probabilities.

Let T be the transition operator defined in terms of the S matrix by $S = 1 - iT$.

The density matrix of the final state produced from initial state ρ_i is

$$\rho_f = S\,\rho_i\,S^{\dagger} .$$

The average value of any observable $\mathcal{F}$ in this state is :

$$\langle \mathcal{F} \rangle = \mathrm{tr}\, \rho_f \, \mathcal{F} \quad .$$

In particular, if $\mathcal{F}$ is a hermitean mixture of projectors on a set of states, which annihilates the subspace in which ρ_i is non zero ($\mathcal{F}\rho_i = \rho_i \mathcal{F} = 0$), one obtains in this way the transition probability from the state ρ_i to the states $\mathcal{F}$:

$$W_{\mathcal{F},\rho_i} = \mathrm{tr}\, S \, \rho_i \, S^{\dagger} \mathcal{F} = \mathrm{tr}\, T \, \rho_i \, T^{\dagger} \mathcal{F} \quad .$$

The density matrix restricted to states selected by $\mathcal{F}$ is

$$\rho_{f/\mathcal{F}} = \frac{\mathcal{F} \, \rho_f \, \mathcal{F}}{\mathrm{tr}\, \mathcal{F} \, \rho_f \, \mathcal{F}} = \frac{\mathcal{F} \, S \, \rho_i \, S^{\dagger} \mathcal{F}}{\mathrm{tr}\, \mathcal{F} \, S \, \rho_i \, S^{\dagger} \mathcal{F}} = \frac{\mathcal{F} \, T \, \rho_i \, T^{\dagger} \mathcal{F}}{\mathrm{tr}\, \mathcal{F} \, T \, \rho_i \, T^{\dagger} \mathcal{F}} \quad ,$$

the last equality holding when $\mathcal{F} \rho_i = \rho_i \mathcal{F} = 0$.

CHAPTER II

ANALYSIS OF TWO PARTICLE STATES

The space of two particle states, a basis of which can be labelled as $|[p_1]\,\sigma_1\,,\,[p_2]\,\sigma_2\,\rangle$, where $\sigma_1\,,\,\sigma_2$ are spin projections (or helicities, in the massless case) on axis determined by the $[p_i]$'s is acted upon by a representation of the Poincaré group (or rather its covering) which is the tensor product of the two one particle representations of masses $m_{1,2}$ $(p_i^2 = m_i^2)$ and spins $s_{1,2}$ $(-s_i \leq \sigma_i \leq +s_i)$:

$$U(a,A)\ |[p_1]\,\sigma_1\ ,\ [p_2]\,\sigma_2\,\rangle = e^{ia.A.(p_1+p_2)}\ |[A.p_1]\,\sigma_1'\ ;\ [A.p_2]\,\sigma_2'\,\rangle$$

$$\times\ \mathcal{D}^{(s_1)}_{\sigma_1'\sigma_1}\left([Ap_1]^{-1}\,A[p_1]\right)\ \mathcal{D}^{(s_2)}_{\sigma_2'\sigma_2}\left([Ap_2]^{-1}\,A[p_2]\right) \qquad \text{(II.1)}$$

(if one particle is massless and has helicity σ, replace $\mathcal{D}_{\sigma'\sigma}$ by $\mathcal{D}_{\sigma\sigma}$, without any summation).

Such a representation is not irreducible. It is the purpose of the present chapter to reduce it, namely, to define in its Hilbert space basis vectors $|[P],\,\mu\,;\,J,\eta\rangle$ which behave irreducibly :

$$U(a,A)\ |[P],\mu;\ J,\eta\rangle\ =\ e^{iP.a}\ \ |[A.P],\mu'\ ;\ J,\eta\rangle\ \mathcal{D}^{J}_{\mu'\mu}\left([A.P]^{-1}\,A[P]\right) \quad \text{(II.2)}$$

where :

P^2 is the mass of this state, J its total spin, η a Poincaré invariant set of degeneracy parameters (we shall find indeed that the $[P^2,J]$ component of the overall representation occurs several times), and to compute the Clebsch-

Gordan coefficients :

$$\langle [p_1], \sigma_1 \; ; \; [p_2], \sigma_2 \; | [P], \mu \; ; \; J, \eta \rangle$$

which occur in the expression of the new basis vectors in terms of the former ones :

$$|[P], \mu \; ; \; J, \eta\rangle = \int \frac{d^3 p_1}{2\omega_1} \frac{d^3 p_2}{2\omega_2} \sum_{\sigma_1, \sigma_2} |[p_1], \sigma_1 \; ; \; [p_2], \sigma_2\rangle$$

$$\times \; [p_1], \sigma_1 \; ; \; [p_2], \sigma_2 \; | [P], \mu \; ; \; J, \eta\rangle \quad . \tag{II.3}$$

The new basis will be so defined that the Clebsh Gordon coefficients are matrix elements of a unitary matrix :

$$|[p_1], \sigma_1 \; ; \; [p_2], \sigma_2\rangle = \int dP^2 \int \frac{d^3 P}{2\omega_{P^2}(\vec{P})} \sum_{J, \eta} |[P], \mu \; ; \; J, \eta\rangle$$

$$\langle [P], \mu \; ; \; J, \eta \; | [p_1], \sigma_1 \; ; \; [p_2], \sigma_2\rangle \; . \tag{II.4}$$

The initial basis already makes the translation operator $U(a)$ diagonal, i.e. state $|[p_1]\sigma_1 \; [p_2]\sigma_2\rangle$ has total momentum $P = p_1 + p_2$.

One sees thus that the total mass P^2 may range from $(m_1 + m_2)^2$ up to ∞ . P obviously lies in the future light cone.

For fixed P , we may parametrize p_1 , p_2 such that $p_1 + p_2 = P$ through the "relative barycentric momentum" four vector :

$$q_{1,2} = - q_{2,1} = \frac{1}{2} \left(p_1 - p_2 - \frac{(m_1^2 - m_2^2)}{P^2} P \right)$$

$$= p_1 - \frac{p_1 . P}{P^2} P = - \left(p_2 - \frac{p_2 . P}{P^2} P \right) \qquad \text{(II.5)}$$

which is so constructed that it lies in the 2-plane defined by p_1 and p_2 and is orthogonal to P :

$$q_{1,2} . P = 0 \; . \qquad \text{(II.6)}$$

Note for future reference $q_{1,2}^2 = - \dfrac{\lambda(m_1^2, m_2^2, P^2)}{4 P^2}$

a standard function of P^2 ,

where $\lambda(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2(z_1 z_2 + z_1 z_3 + z_2 z_3)$.

P being fixed, $q_{1,2}$ depends thus on two polar angles, on the sphere

$$q_{1,2} . P = 0 \; , \; q_{1,2}^2 = - \frac{\lambda(m_1^2 \; m_2^2 \; P^2)}{4 P^2} \; .$$

The inverse formulae are :

$$p_1 = \frac{P^2 + m_1^2 - m_2^2}{2 P^2} P + q_{1,2}$$

$$p_2 = \frac{P^2 - m_1^2 + m_2^2}{2 P^2} P - q_{1,2} \; .$$

It is intuitive that the spin (total intrinsic angular momentum) of the compound system is made up with the angular momentum carried by the orbital motion, associated with $q_{1,2}$ and the individual spins of the two particles.

Before one can quietly add up these together in the rest frame of the compound system (barycentric frame : P,time axis), one has to be abble and compare the "spins" of particles 1 and 2 which are not at rest in this frame. The matter will then turn out to be reduced to a composition of angular momenta which will be exact.

a) ℓ-s coupling.

Let us indeed reexpress the transformation law of $|[p_1],\sigma_1[p_2]\,\sigma_2\rangle$ in terms of center of mass variables :

$$[p_i] = [p_i,P]\,[P] \qquad (m_i \neq 0) \qquad \text{(II.a.0)}$$

where $[P]$ defines a frame attached to P, $[p_i,P]$ is some Lorentz transformation taking the unit vector of P into the unit vector of p_i .(The case when p_i is the momentum of a massless particle will be dealt with later, as it is slightly more delicate to be interpreted in terms of addition of angular momenta).

Let us now define the symbolic vector

$$\underset{\sim}{q}_{1,2} = \{q_i = - q_{1,2}\cdot n_i(P)\} \qquad \text{(II.a.1)}$$

$$\text{where } n_i(P) = [P]\, n_i(\overset{\circ}{P})$$

where $\overset{\circ}{P} = (1,0,0,0)$, some laboratory time axis. We shall denote by $\underset{\sim}{\hat{q}}$ the corresponding unit vector $\underset{\sim}{\hat{q}} = \frac{\underset{\sim}{q}}{\sqrt{\underset{\sim}{q}^2}}$.

If $q_{1,2} \to A.\, q_{1,2}$ (which results from $p_1 \to A.p_1$, $p_2 \to A.p_2$),

$$\underset{\sim}{q}_{1,2} \to \{-(A.q_{1,2}) \cdot n_i(AP)\} = \left\{-q_{1,2}[P].\left[[P]^{-1}A^{-1}[AP]\right] n_i(\overset{o}{P})\right\}$$

$$= \{q_j \, R^{-1j}{}_i \, (A,P)\} \quad ,$$

where $R(A,P)$ is the Wigner rotation $[AP]^{-1} \; A[P]$.

Let us thus define :

$$|[P].\, \underset{\sim}{q},\, \sigma_1\sigma_2\rangle \equiv |[p_1]\, \sigma_1,[p_2]\sigma_2\rangle \qquad \text{(II.a.2)}$$

(a simple change of variables).

$$U(a,A)\,|[P],\, \underset{\sim}{q},\, \sigma_1,\sigma_2\rangle = e^{i(A.P).a}\, |[AP],\, R(A,P)\, \underset{\sim}{q},\, \sigma_1'\, \sigma_2'\rangle$$

$$\mathcal{D}_{\sigma_1'\sigma_1'}\left([AP]^{-1}[Ap_1\,,AP]^{-1}\; A\; [p_1,P][P]\right) \mathcal{D}_{\sigma_2'\sigma_2}\left([AP]^{-1}[Ap_2,AP]^{-1}A[p_2,P][P]\right) .$$

We see that if we choose $[p_i,P] = [p_i \leftarrow P]_\varphi$, the product of the pure Lorentz transformation in the 2-plane (p_i,P), which takes $\hat{P}$, to $\hat{p}_i$, followed (or preceded !) by a rotation of angle φ independent of p_i,P , around this two plane, the spin rotations indicated above reduce to Wigner rotations $R(A,P)$ because :

$$A[p_1 \leftarrow P]_\varphi = [Ap_1 \,,\, AP]_\varphi\, A \; .$$

We next analyze the $\underset{\sim}{q}$ dependence into spherical waves :

$$|[P]\, \underset{\sim}{q},\ \sigma_1,\sigma_2\rangle = \sum_{\ell,m} \sqrt{\frac{2\ell+1}{4\pi}}\ \mathcal{D}^{\ell}_{m,o}\ (\Omega_{\underset{\sim}{q}})\ |[P]\ \ell,\ m,\ \sigma_1,\ \sigma_2\rangle$$

i.e. (II.a.3)

$$|[P]\ \ell,\ m,\sigma_1,\sigma_2\rangle = \int d\Omega_{\underset{\sim}{q}}\ \mathcal{D}^{\ell *}_{m,o}\ (\Omega_{\underset{\sim}{q}}) \sqrt{\frac{2\ell+1}{4\pi}}\ |[P],\underset{\sim}{q},\ \sigma_1,\sigma_2\rangle\ .$$

Here :
$$\Omega_{\underset{\sim}{q}} = [P]^{-1}\ \Omega(q_{1,2} \leftarrow n_3(P))_{\varphi}[P]$$

where $\Omega(q_{1,2} \leftarrow n_3(P))$ is the pure "rotation" preserving $[P]$, transforming $n_3(P)$ into $\hat{q}_{1,2}$, followed by a rotation φ around the two plane $P,\ q_{1,2}$ to which $\mathcal{D}^{\ell}_{m,o}$ is of course insensitive.

We shall occasionally note $\sqrt{\frac{2\ell+1}{4\pi}}\ \mathcal{D}^{\ell *}_{m,o}\ (\Omega_{\underset{\sim}{q}}) = Y^{\ell}_{m}\ (\underset{\sim}{q})$.

The transformation law then becomes :

$$U(a,A)\ |[P]\ \ell,\ m,\ \sigma_1,\sigma_2\rangle = e^{i(A.P).a}\ |[AP]\ \ell,\ m',\ \sigma_1'\ ,\ \sigma_2'\rangle$$
$$\times\ \mathcal{D}^{\ell}_{m'm}\left([AP]^{-1}\ A[P]\right)\ \ \mathcal{D}^{s_1}_{\sigma_1'\sigma_1}\left([AP]^{-1}\ A[P]\right)\ \ \mathcal{D}^{s_2}_{\sigma_2'\sigma_2}\left([AP]^{-1}\ A[P]\right)$$

where we have used

$$\mathcal{D}^{\ell *}_{m,o}\ \left([P]^{-1}\ \Omega(q_{1,2} \leftarrow n_3(P))[P]\right) =$$
$$\mathcal{D}^{\ell *}_{m,o}\ \left(\left\{[P]^{-1}\ A^{-1}[AP]\right\}.\left\{[AP]^{-1}\Omega(Aq \leftarrow n_3(AP))[AP]\right\}\right.$$
$$\left. .\left\{[AP]^{-1}\ \Omega^{-1}(Aq \leftarrow n_3(AP))\ A\ \Omega(q \leftarrow n_3(P))[P]\right\}\right)$$

and the fact that the last expression between curly brackets is a rotation around the third axis and can thus be dropped, since we are computing $\mathscr{D}^{\ell}_{m,\underline{o}}$.

So, by usual addition of angular momenta :

$$|[P], \mu, j ; \ell, s\rangle = |[P]\ \ell, m\ \sigma_1\sigma_2\rangle \begin{pmatrix} s_1 & s_2 & s \\ \sigma_1 & \sigma_2 & \sigma \end{pmatrix} \begin{pmatrix} s & \ell & j \\ \sigma & m & \mu \end{pmatrix}$$

transforms irreducibly according to

$$U(a,A)\ |[P], \mu, j ; \ell, s\rangle = e^{i(AP).a}\ |[AP]\ \mu' ; j, \ell, s\rangle \; \mathscr{D}^{j}_{\mu'\mu}\left([AP]^{-1}\ A[P]\right) . \tag{II.a.4}$$

One can further see that if one requires these states to behave canonically under space reflections the rotation angles φ which might accompany $[p_i \leftarrow P]$ must be taken equal to zero or π . The intrinsic parity of such a state is then :

$$\eta_{\pi}^{(1)}\ \eta_{\pi}^{(2)}\ (-)^{\ell} .$$

We collect now the final formula, including normalizations :

$$\langle [p_1], \sigma_1 ; [p_2], \sigma_2\ |[P], \mu , j , \ell , s , P^2\rangle =$$

$$\frac{2\sqrt{P^2}\sqrt{2}}{\lambda^{1/4}(P^2, m_1^2, m_2^2)}\ \delta^4(P-p_1-p_2) \sum_{\substack{\sigma_1', \sigma_2', \\ \sigma = \sigma_1' + \sigma_2' \\ m = \mu - \sigma}} \mathscr{D}^{(s_1)}_{\sigma_1\sigma_1'}\left([p_1]^{-1}[p_1 \leftarrow P][P]\right)$$

$$\times\ \mathscr{D}^{(s_2)}_{\sigma_2\sigma_2'}\left([p_2]^{-1}[p_2 \leftarrow P][P]\right) \begin{pmatrix} s_1 & s_2 & s \\ \sigma_1' & \sigma_2' & \sigma \end{pmatrix} \begin{pmatrix} \ell & s & j \\ m & \sigma & \mu \end{pmatrix} Y^{\ell}_{m}(\hat{\underset{\sim}{q}}) . \tag{II.a.5}$$

b) Multipole coupling.

We have remarked that, when one, or both particles are massless , it becomes impossible to transform them to rest in the barycentric frame. Also, we know that a massless particle has only one "spin" (helicity) state. In order to reduce the situation to one where the rotation group in the barycentric frame is involved, it is convenient to choose a frame attached to p_1 such that $n_1(p_1).P = 0$, which, with analogy with the photon case, we shall call a radiation gauge :

$$[p_1]_r = \Omega[q_{1,2} \leftarrow n_3(P)]\ [\omega\hat{P} + |q_{1,2}|\ n_3(P) \leftarrow \hat{P}+n_3(P)]\ [P] \quad \text{,(II.b.1)}$$

where ω is the massless particle barycentric energy, and

$[\omega \hat{P} + |q_{1,2}|\ n_3(P) \leftarrow \hat{P} + n_3(P)]$ is the pure Lorentz transformation, acting in the 2-plane $(\hat{P}, n_3(P))$, and transforming $\hat{P} + n_3(P)$ (light-like) into $\omega \hat{P}+|q_{1,2}|\ n_3(P)$, and $\Omega[q_{1,2} \leftarrow n_3(P)]_\varphi$ is again a rotation taking $n_3(P)$ to $\hat{q}_{1,2}$, in the 2-plane $(n_3(P), q_{1,2})$.

In the case of a massive particle, the frame which plays a similar role is the helicity frame

$$[p_1]_h = [p_1 \leftarrow P]\ \Omega(q_{1,2} \leftarrow n_3(P))\ [P] \qquad \text{(II.b.2)}$$

whose third axis :

$$n_3(p_1) = [p_1 \leftarrow P]\ q_{1,2} = \frac{P - \frac{p_1 . P}{m_1^2}\ p_1}{\sqrt{- P^2 + \frac{(p_1 . P)^2}{m_1^2}}} \qquad \text{(II.b.3)}$$

is the so called helicity axis in the barycentric frame (orthogonal to p_1, of course, in the two plane (P, p_1)).

In the following, $[p_1]$ will be taken as $[p_1]_r$ or $[p_1]_h$ according as $m_1^2 = 0$, $m_1^2 > 0$, and in the case of a massless particle no spin index summation will be allowed.

Let us then compute $\mathscr{D}^{s_1}_{\sigma_1' \sigma_1}\left([Ap_1]^{-1} \; A[p_1]\right)$.

From the commutability of A across pure Lorentz transformations or rotations, the argument reduces, in both cases to

$$\varphi = [AP]^{-1} \; \Omega\left(Aq_{1,2} \leftarrow n_3(AP)\right)^{-1} A \; \Omega\left(q_{1,2} \leftarrow n_3(P)\right)[P]$$

which, not only is a rotation, but leaves $n_3(\overset{\circ}{P})$ invariant, so that $\mathscr{D}_{\sigma_1'\sigma_1}(\varphi)$ is diagonal ; hence the formal identity between calculations with a massless particle in the radiation gauge and a massive particle described in terms of helicity.

Assuming that $[p_2] = [p_2 \leftarrow P]\,[P]$, we thus get :

$$U(a,A)\;\left|[P], \underset{\sim}{q}, \lambda_1 \sigma_2\right\rangle = e^{i(A.P)a}\;\left|[AP], R(A.P)\,\underset{\sim}{q}, \lambda_1 \sigma_2'\right\rangle$$

$$\mathscr{D}^{(s_1)}_{\lambda\lambda_1}\left([AP]^{-1}\,\Omega\,(Aq_{1,2} \leftarrow n_3(AP))\,A\;\Omega\;(q_{1,2} \leftarrow n_3(P))\;[P]\right)$$

$$\times\;\mathscr{D}^{(s_2)}_{\sigma_2'\sigma_2}\left([AP]^{-1} \;\; A[P]\right) .$$

Now remark that :

$$\sqrt{\frac{2k+1}{4\pi}}\;\mathscr{D}^{k}_{m\lambda_1}\left([P]^{-1}\;\Omega(q_{1,2} \leftarrow n_3(P))[P]\right) = \sqrt{\frac{2k+1}{4\pi}}\;\mathscr{D}^{k}_{m\lambda_1}(\Omega_{\underset{\sim}{q}})$$

form, for fixed λ_1, $k-\lambda_1$ integer, varying, just as good a complete set of functions of $\underset{\sim}{q}$ on the unit sphere, as did $\mathcal{D}^{\ell}_{mo}(\underset{\sim}{q})$. We may thus write :

$$|[P], \underset{\sim}{q}, \lambda_1, \sigma_2\rangle = \sum_{k\ m} \sqrt{\frac{2k+1}{4\pi}}\ \mathcal{D}^{k}_{m\lambda_1}\Big([P]^{-1}\Omega\ (q_{1,2} \leftarrow n_3(P))[P]\Big)|[P],k,m,\lambda_1,\sigma_2\rangle$$

i.e. (II.b.4)

$$|[P],k,\ m,\lambda_1,\sigma_2\rangle = \int d\Omega_{\underset{\sim}{q}} \sqrt{\frac{2k+1}{4\pi}}\ \mathcal{D}^{k*}_{m\lambda_1}\Big([P]\ \Omega(q_{1,2} \leftarrow n_3(P))[P]\Big)|[P],\underset{\sim}{q},\lambda_1\sigma_2\rangle \ .$$

The transformation law then becomes :

$$U(a,A)|[P],k,m,\lambda_1,\sigma_2\rangle = e^{i(AP).a}\ |[AP],k,m',\lambda_1,\sigma_2'\rangle$$

$$\mathcal{D}^{k}_{m'm}\Big([AP]^{-1}\ A[P]\Big)\ \mathcal{D}^{s_2}_{\sigma_2'\sigma_2}\Big([AP]^{-1}\ A[P]\Big)$$

where we have used :

$$\mathcal{D}^{k*}_{m_1\lambda_1}\Big([P]^{-1}\Omega(q_{1,2} \leftarrow n_3(P))[P]\Big)\ \mathcal{D}^{s_1}_{\lambda_1\lambda_1}\Big([AP]^{-1}\Omega^{-1}(Aq_{1,2} \leftarrow n_3(AP))A\ \Omega(q_{1,2} \leftarrow n_3(P))[P]\Big)$$

$$= \mathcal{D}^{k*}_{m\lambda_1}\Big([P]^{-1}\Omega(q_{1,2} \leftarrow n_3(P))[P][P]^{-1}\Omega^{-1}(q_{1,2} \leftarrow n_3(P))\ A^{-1}\Omega(Aq_{1,2} \leftarrow n_3(AP))[AP]\Big)$$

$$= \mathcal{D}^{k*}_{m\lambda_1}\Big([P]^{-1}\ A^{-1}\ [AP]\ [AP]^{-1}\ \Omega(Aq_{1,2} \leftarrow\ n_3(AP))[AP]\Big)$$

$$= \mathcal{D}^{k*}_{mm'}\Big([P]^{-1}\ A^{-1}\ [AP]\Big)\ \ \mathcal{D}^{k*}_{m'\lambda_1}\Big([AP]^{-1}\ \Omega(Aq_{1,2} \leftarrow n_3(AP))[AP]\Big)$$

$$= \mathcal{D}^{k*}_{m'\lambda_1}\Big([AP]^{-1}\ \Omega(Aq_{1,2} \leftarrow n_3(AP))[AP]^{-1}\Big)\mathcal{D}^{k}_{m'm}\Big([AP]^{-1}\ A\ [P]\Big) \ .$$

One achieves the reduction with the construction of

$$\left|[P],\mu\ ;\ j,k,\lambda\right\rangle = \left|[P]\ k,m,\lambda_1\sigma_2\right\rangle \begin{pmatrix} k & s_2 & \big| & j \\ m & \sigma_2 & \big| & \mu \end{pmatrix} \qquad \text{(II.b.5)}$$

which transforms according to

$$U(a,A)\ \left|[P],\mu\ ;\ j,\ell,\lambda_1\right\rangle = e^{i(AP).a}\ \left|[AP],\mu';j,k,\sigma_1\right\rangle \mathcal{D}^{j}_{\mu'\mu}\left([AP]^{-1}\ A[P]\right). \text{(II.b.6)}$$

Such a state will be called a multipole state, with total angular momentum of particle 1 equal to k, helicity of particle 1 equal to λ_1.

The corresponding Clebsch Gordan coefficient is

$$\left\langle [p_1]\ \sigma_1[p_2]\ \sigma_2\ \middle|[P],\mu\ ;\ j,k,\lambda_1,P^2\right\rangle = \frac{2\sqrt{P^2}\sqrt{2}}{\lambda^{1/4}(P^2,m_1^2,m_2^2)} \sqrt{\frac{2k+1}{4\pi}}\ \delta^4(P-p_1-p_2)$$

$$\times \sum_{\substack{\sigma_2' \\ m=\mu-\sigma_2'}} \mathcal{D}^{s_1}_{\sigma_1\lambda_1}\left([p_1]^{-1}[p_1]_{\substack{h\\r}}\right) \mathcal{D}^{s_2}_{\sigma_2\sigma_2'}\left([p_2]^{-1}[p_2\leftarrow P][P]\right)$$

$$\begin{pmatrix} k & s_2 & \big| & j \\ m & \sigma_2' & \big| & \mu \end{pmatrix} \mathcal{D}^{k*}_{m\lambda_1}\left([P]^{-1}\ \Omega(q_{1,2}\leftarrow n_3(P))[P]\right) , \qquad \text{(II.b.7)}$$

where h means helicity ($m_1^2 \neq 0$)

r " radiation ($m_1^2 = 0$), in which case $\sigma_1 = \lambda_1$.

Note the useful formula, for k integer

$$\sqrt{\frac{2\ell+1}{4\pi}}\ \mathcal{D}^{k*}_{m,\varepsilon|\lambda|}\ (\Omega_{\underset{\sim}{q}}) = \frac{[\vec{e}(q)]^{\otimes|\lambda|}.\ (\vec{\ell}_q)^{\otimes|\lambda|}}{\sqrt{(k-\lambda+1)\ \ldots\ (k+\lambda)}}\ Y^k_m\ (\underset{\sim}{q})$$

where $\vec{e}(q) = \vec{e}_1(q) + i\,\varepsilon\,\vec{e}_2(q)$

$$e_1(q) = \Omega_{\underset{\sim}{q}}(1,0,0)$$

$$e_2(q) = \Omega_{\underset{\sim}{q}}(0,1,0) \qquad \text{with} \quad \Omega_{\underset{\sim}{q}} = [P]^{-1}\,\Omega(q_{1,2} \leftarrow n_3(P))[P]$$

$$\underset{\sim}{q} = \Omega_{\underset{\sim}{q}}(0,0,1)$$

$$\vec{\ell}_q = \frac{1}{i}\,\underset{\sim}{q} \times \nabla_{\underset{\sim}{q}}$$

which takes us back to familiar formulae for the photon multipoles.

c) Helicity coupling.

For both particles, 1 and 2 , one chooses either the radiation gauge or the helicity frame, but following Jacob and Wick [17] , we shall reverse for particle 2, the vectors $n_3(p_2)$ and $n_1(p_2)$ $(m_2^2 \neq 0)$ or the vector $n_1(p_2)$ $(m_2 = 0)$; this can be done by writing :

$$\begin{cases} [p_1] = [p_1 \leftarrow P]\,\Omega(q_{1,2} \leftarrow n_3(P))[P] & \text{for } m_1 \neq 0 \\ [p_2] = [p_2 \leftarrow P]\,\Omega(q_{1,2} \leftarrow n_3(P))[P]\,Y & \text{for } m_2 \neq 0 \end{cases}$$

$$\begin{cases} [p_1]_r = \Omega(q_{1,2} \leftarrow n_3(P))\left[\omega_1\hat{P} + |q_{1,2}|\,n_3(P) \leftarrow \hat{P} + n_3(P)\right][P] & \text{for } m_1 = 0 \\ [p_2]_r = \Omega(q_{1,2} \leftarrow n_3(P))\left[\omega_2\hat{P} + |q_{1,2}|\,n_3(P) \leftarrow \hat{P} + n_3(P)\right][P]\,Y & \text{for } m_2 = 0 \end{cases}$$

Y is a rotation through an angle $(+\pi)$ about the Y axis. We shall use

$\mathcal{D}^{s}_{mm'}(Y) = \delta_{m,-m'}\,(-1)^{s+m}$, and :

$$\mathcal{D}^{j*}_{\mu\lambda_1-\lambda_2}\left([P]^{-1}\Omega(q_{1,2}\leftarrow n_3(P))[P]\right)\mathcal{D}^{s_1}_{\lambda_1\lambda_1}\left([AP]^{-1}\Omega^{-1}(Aq_{1,2}\leftarrow n_3(AP))\,A\Omega(q_{1,2}\leftarrow n_3(P))\,[P]\right)$$

$$\times\ \mathcal{D}^{s_2}_{\lambda_2\lambda_2}\left(Y^{-1}[AP]^{-1}\ \Omega^{-1}(Aq_{1,2}\leftarrow n_3(AP))\ A\ \Omega(q_{1,2}\leftarrow n_3(P))[P]Y\right)$$

$$= \mathcal{D}^{j*}_{\mu\lambda_1-\lambda_2}\left([P]^{-1}\Omega(q_{1,2}\leftarrow n_3(P))[P][P]^{-1}\Omega^{-1}(q_{1,2}\leftarrow n_3(P))A^{-1}\Omega(Aq_{1,2}\leftarrow n_3(AP))[AP]\right)$$

$$= \mathcal{D}^{j*}_{\mu\mu'}\left([P]^{-1}\ A^{-1}[AP]\right)\ \mathcal{D}^{j*}_{\mu'\lambda_1-\lambda_2}\left([AP]^{-1}\Omega(Aq_{1,2}\leftarrow n_3(AP))\right)$$

$$= \mathcal{D}^{j*}_{\mu',\lambda_1-\lambda_2}\left([AP]^{-1}\ \Omega(Aq_{1,2}\leftarrow n_3(AP))[AP]\right)\ \mathcal{D}^{j}_{\mu'\mu}\left([AP]^{-1}A[P]\right)\ .$$

We find now that the state :

$$|[P],\mu;j,\lambda_1\lambda_2\rangle = \int d\Omega_{\underset{\sim}{q}}\sqrt{\frac{2j+1}{4\pi}}\ \ \mathcal{D}^{j*}_{\mu\lambda_1-\lambda_2}\left([P]^{-1}\Omega(q_{1,2}\leftarrow n_3(P))[P]\right)(-1)^{s_2-\lambda_2}\ |[P],\underset{\sim}{q},\lambda_1,\lambda_2\rangle \qquad \text{(II.c.1)}$$

Transforms according to :

$$U(a,A)|[P]\mu;j\ \lambda_1\lambda_2\rangle = e^{i(AP).a}\ |[AP],\mu',j,\lambda_1\lambda_2\rangle\ \mathcal{D}^{j}_{\mu'\mu}\left([AP]^{-1}\ A[P]\right). \qquad \text{(II.c.2)}$$

Remark :

For both particles we have used the same rotation

$\Omega(q_{1,2} \leftarrow n_3(P))$. The additional phase factor $(-1)^{s_2-\lambda_2}$ in (II.c.1) is necessary if one wants to get good parity transformation properties, as pointed out by Jacob and Wick.

The Clebsch Gordan coefficient is :

$$\langle [p_1]\sigma_1 [p_2]\sigma_2 | [P],\mu;\ j,\lambda_1,\lambda_2,P^2\rangle = \frac{2\sqrt{P^2}\sqrt{2}}{\lambda^{1/4}(P^2,m_1^2,m_2^2)} \sqrt{\frac{2j+1}{4\pi}}\ \delta^4(P-p_1-p_2)$$

$$(-1)^{s_2-\lambda_2}\ \mathcal{D}^{s_1}_{\sigma_1\lambda_1}\left([p_1]^{-1}[p_1]_{\frac{h}{r}}\right)\ \mathcal{D}^{s_2}_{\sigma_2\lambda_2}\left([p_2]^{-1}[p_2]_{\frac{h}{r}}\right)\ \mathcal{D}^{j*}_{\mu\lambda_1-\lambda_2}(\Omega_{\underset{\sim}{q}}) .$$

d) <u>Relationship between various coupling schemes.</u>

The states $|[P],\mu;j,\eta\rangle$ defined in preceding sections according to the various coupling schemes, are always normalized to :

$$\langle [P],\mu;j,\eta | [P'],\mu';j',\eta'\rangle = \delta(P-P')\ \delta_{jj'}\ \delta_{\mu\mu'}\ \delta_{\eta\eta'} .$$

Thus, they form an orthonormal basis. Furthermore, they are complete . $\delta_{\eta\eta'}$ means $\delta_{\ell\ell'}\delta_{ss'}$ for ℓs coupling, $\delta_{kk'}\delta_{\lambda_1\lambda_1'}$ for multipole coupling and $\delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'}$ for helicity coupling. It is useful to write down, the unitary transformation which relates two different coupling schemes corresponding to basis $|[P],\mu;j,\eta\rangle$ and $|[P],\mu;j,\varphi\rangle$. Using Clebsh-Gordan coefficients formulae, one gets :

$$\langle [P'],\mu';j',\eta | [P],\mu;j,\varphi\rangle = \delta(P-P')\ \delta_{jj'}\ \delta_{\mu\mu'}\ S^j_{\eta\varphi} \quad ;$$

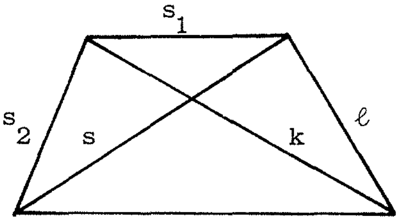
the calculation of $S^j_{\eta\varphi}$ in the various cases offers no difficulty although it is rather lengthy. One gets :

$$S^j_{\langle\lambda_1\lambda_2;\ell,s\rangle} = \sqrt{\frac{2\ell+1}{2j+1}} \begin{pmatrix} s_1 & s_2 & \big| & s \\ \lambda_1 & -\lambda_2 & \big| & \lambda \end{pmatrix} \begin{pmatrix} \ell & s & \big| & j \\ 0 & \lambda & \big| & \lambda \end{pmatrix}$$

$$S^j_{\langle k,\lambda_1';\lambda_1\lambda_2\rangle} = \delta_{\lambda_1\lambda_1'} \sqrt{\frac{2k+1}{2j+1}} \begin{pmatrix} k & s_2 & \big| & j \\ \lambda_1 & -\lambda_2 & \big| & \lambda \end{pmatrix}$$

$$S^j_{\langle k,\lambda_1',\ell,s\rangle} = \sqrt{(2\ell+1)(2s+1)} \begin{pmatrix} \ell & s_1 & \big| & k \\ 0 & \lambda_1' & \big| & \lambda_1 \end{pmatrix} W(s_1 s_2\ \ell,j|s,k) \quad ;$$

W is the Racah coefficient (see for example reference O, Appendix c), associated with the following diagram :



<u>Remark</u> :

All theses $S_{\eta\varphi}$ are real . This is of course due to the conventions used here (one can easily see that they are taken such that η,φ are time reversal invariant : then the $S^j_{\eta\varphi}$'s have to be real).

e) <u>Angular analysis of reactions involving two incoming and two outgoing particles</u>.

Consider a reaction

$$A_1 + A_2 \rightarrow A_3 + A_4$$

masses	m_1	m_2	m_3	m_4	
spins	s_1	s_2	s_3	s_4	(II.e.1)
momenta	p_1	p_2	p_3	p_4	
spin projections	σ_1	σ_2	σ_3	σ_4 .	

Let $S - \mathbb{1} = iT$ be the transition operator.

We recall that relativistic invariance imposes

$$U(a,A)\, S\, U^{-1}(a,A) = S \quad . \tag{II.e.2}^{*}$$

This allows us to use fruitfully an intermediate state expansion in terms of states which reduce products of representations occuring in the initial as well as in the final state :

$$\langle [p_3]\sigma_3 , [p_4]\sigma_4 \mid T \mid [p_1]\sigma_1 , [p_2]\sigma_2 \rangle$$

$$= \int_{(m_3+m_4)^2}^{\infty} d\, M_{34}^2 \int_{(m_1+m_2)^2}^{\infty} d\, M_{12}^2 \int \frac{d^3 P_{34}}{2\omega_{34}} \int \frac{d^3 P_{12}}{2\omega_{12}} \sum_{\substack{\eta_{34}, \eta_{12} \\ J_{34}, J_{12} \\ \mu_{34}, \mu_{12}}} \cdots\cdots$$

---

* Invariance under reflections reads :

$$U(\Pi)\, S\, U^{-1}(\Pi) = S \quad . \quad U(T)\, S\, U^{-1}(T) = S^{+} \quad , \quad U(\Pi T)\, S\, U^{-1}(\Pi T) = S^{+}$$

where phases have been put equal to one by virtue of the cluster decomposition property quoted in chapter 0,ℓ b.

$$\ldots\ldots \langle [p_3],\sigma_3,[p_4],\sigma_4 \, | [P_{34}]\mu_{34}; J_{34}, \eta_{34}, M^2_{34} \rangle$$

$$\langle [P]_{34}, \mu_{34} \, ; J_{34}, \eta_{34}, M^2_{34} \, |T| \, [P]_{12}, \mu_{12} \, ; J_{12}, \eta_{12}, M^2_{12} \rangle \qquad \text{(II.e.3)}$$

$$\langle [P]_{12}, \mu_{12} \, ; J_{12}, \eta_{12}, M^2_{12} \, | [p_1] \, \sigma_1, [p_2] \, \sigma_2 \rangle .$$

Poincaré invariance (Wigner[IV]) Eckart theorem) shows that :

$$\langle [P]_{34}, \mu_{34} \, ; J_{34}, \eta_{34}, M^2_{34} \, |T| \, [P]_{12}, \mu_{12} \, ; J_{12}, \eta_{12}, M^2_{12} \rangle$$

has the form

$$\delta^4(P_{34} - P_{12}) \quad \delta_{J_{12},J_{34}} \quad \delta_{\mu_{12},\mu_{34}} \quad T^{J_{12}}_{\eta_{34},\eta_{12}} \, (P^2_{12}) \qquad \text{(II.e.4)}$$

$$= \delta(M^2_{34} - M^2_{12}) \quad 2\omega_{34} \quad \delta^3(\vec{P}_{34} - \vec{P}_{12}) \, \delta_{J_{12},J_{34}} \quad \delta_{\mu_{12},\mu_{34}} \quad T^{J_{12}}_{\eta_{34},\eta_{12}} \, (P^2_{12})$$

where $T^{J}_{\eta_{34},\eta_{12}} \, (P^2)$ is a reduced matrix elements.

We further define partially reduced Clebsch Gordan coefficients through(*)

$$\langle [p]\sigma, [p']\sigma' \, | [P],\mu \, ; J,\eta,P^2 \rangle = \delta^4(P-p-p') \, \mathcal{D}^{s}_{\sigma\tau} \, ([p]^{-1})$$

$$\mathcal{D}^{s'}_{\sigma'\tau'} \, ([p']^{-1}) \qquad \langle\langle \, p,\tau,p',\tau' \, | \, [P],\mu \, ; J, \eta, P^2 \, \rangle\rangle \qquad \text{(II.e.5)}$$

(*) In other words we express the components of $|P,\mu;j,\eta\rangle$ on a "spinor" basis.

with the result :

$$\langle [p_3]\ \sigma_3, [p_4], \sigma_4\ |T|\ [p_1]\ \sigma_1, [p_2], \sigma_2 \rangle$$

$$= \delta^4(p_1+p_2-p_3-p_4) \sum_{\substack{J,\mu \\ \eta_{12},\eta_{34}}} \mathcal{D}^{s_3}_{\sigma_3\tau_3}([p_3]^{-1})\ \mathcal{D}^{s_4}_{\sigma_4\tau_4}([p_4]^{-1}) \times \qquad \text{(II.e.6)}$$

$$\langle\langle p_3,\tau_3,p_4,\tau_4\ |[P],\mu,\ J,\ \eta_{34},\ P^2\rangle\rangle\ \ T^J_{\eta_{34},\eta_{12}}(P^2)$$

$$\langle\langle [P],\mu\ ;\ J,\ \eta_{12},\ P^2|\ p_1,\tau_1,p_2,\tau_2\rangle\rangle\ \ \mathcal{D}^{s_1}_{\tau_1\sigma_1}([p_1]^{\dagger-1})\ \ \mathcal{D}^{s_2}_{\tau_2\sigma_2}([p_2]^{\dagger-1})$$

where $P = p_1+p_2 = p_3+p_4$, (where, of course spin index summations have to be omitted when massless particles are involved).

If the $[p_i]$'s are chosen adapted to the kind of coupling in which particle i is involved [e.g. $[p_i]_\ell = [p_i \leftarrow P][P]$ in the case of an ℓ,s coupling], the expansion has a form which has a complete non relativistic looking appearance.

For example :

$$\langle [p_3]_\ell\sigma_3[p_4]_\ell\sigma_4\ |T|\ [p_1]_\ell\ \sigma_1[p_2]_\ell\ \sigma_2\rangle = \frac{8P^2\ \ \delta^4(p_1+p_2-p_3-p_4)}{\lambda^{1/4}(P^2,m_1^2,m_2^2)\ \lambda^{1/4}(P^2,m_3^2,m_4^2)}$$

$$\sum_{\substack{J,\mu \\ s_{12}\ell_{12} \\ s_{34}\ell_{34}}} \begin{pmatrix} s_3 & s_4 & | & s_{34} \\ \sigma_3 & \sigma_4 & | & \sigma_{34}\end{pmatrix} \begin{pmatrix} \ell_{34} & s_{34} & | & J \\ m_{34} & \sigma_{34} & | & \mu\end{pmatrix} Y^{\ell_{34}}_{m_{34}}(\hat{\underset{\sim}{q}}_{34})\ T^J_{\substack{\ell_{34},\ell_{12} \\ s_{34},s_{12}}}(P^2)\ Y^{\ell_{12}\,*}_{m_{12}}(\hat{\underset{\sim}{q}}_{12})$$

$$\begin{pmatrix} J & | & \ell_{12} & s_{12} \\ \mu & | & m_{12} & \sigma_{12}\end{pmatrix} \begin{pmatrix} s_{12} & | & s_1 & s_2 \\ \sigma_{12} & | & \sigma_1 & \sigma_2\end{pmatrix}. \qquad \text{(II.e.7)}$$

Such sums over μ will be evaluated in appendix II in familiar operator form in the case where the spins involved are low. Consequences of space and time reflections (connecting time reversed reactions) are easily derived.

One may wonder at this point why we indulged into such a luxury of details concerning the specification of spin basis. The reasons for such care will become apparent in the study of three particle systems, a corner-stone of which is the precise construction of two particle states.

Also, when one studies reactions occurring in cascade, it is convenient to be able to shift reference frames, convenient frames not being necessarily identical for two successive reactions.

We end up with a last important example : the expansion in terms of helicity amplitudes :

$$\langle [p_3]_h,\lambda_3,[p_4]_h,\lambda_4 \; |T| \; [p_1]_h,\lambda_1,[p_2]_h,\lambda_2 \rangle = \frac{8P^2 \quad \delta^4(p_1+p_2-p_3-p_4)}{\lambda^{1/4}(P^2,m_1^2,m_2^2)\;\lambda^{1/4}(P^2,m_3^2,m_4^2)}$$

$$\sum_J \frac{2J+1}{4\pi} \; \mathcal{D}^J_{\lambda_3-\lambda_4,\lambda_1-\lambda_2}(\theta) \; T^J_{\lambda_3,\lambda_4,\lambda_1,\lambda_2}(P^2) \qquad \text{(II.e.8)}$$

with $\cos\theta = -\hat{q}_{1,2}\cdot\hat{q}_{3,4}$

where the summation over μ : $\sum_\mu \mathcal{D}^j_{\lambda_3-\lambda_4,\mu}(\Omega^{-1}_{\underset{\sim}{q}_{3,4}}) \; \mathcal{D}^j_{\mu,\lambda_1-\lambda_2}(\Omega_{\underset{\sim}{q}_{1,2}})$

is easily performed provided all helicity frames (i.e. the $\Omega_{\underset{\sim}{q}}$'s) are so defined that $n_2^\mu(p_i) = \epsilon^{\mu\nu\rho\sigma} p_{1\nu}p_{2\rho}p_{3\sigma}$, which is orthogonal to all p_i's . since then $\Omega^{-1}_{\underset{\sim}{q}_{3,4}} \; \Omega_{\underset{\sim}{q}_{1,2}}$ leaves $n_2(P) = \epsilon_{\mu\nu\rho\sigma}\, p_1^\nu p_2^\rho p_3^\sigma$ invariant.

Exercise.***

Express unitarity of the S matrix, below three particle production threshold, in terms of the reduced matrix $T^{J}_{\eta_{34},\eta_{12}}(P^2)$, using

$$\mathbb{1} = \sum_{\substack{2\ \text{particle}\\ \text{states}}} \int \frac{d_3p_1}{2\omega_1} \frac{d_3p_2}{2\omega_2} \sum_{\sigma_1,\sigma_2} |[p_1],\sigma_1[p_2],\sigma_2\rangle \langle [p_1]\ \sigma_1 [p_2]\sigma_2|$$

$$= \int dP^2 \int \frac{d^3P}{2\omega_P} \sum_{J,\eta,\mu} |[P],\mu\ ;\ J,\eta,P^2\rangle \langle [P],\mu\ ;\ J,\eta,P^2| \ .$$

f) Convergence of angular expansions as a consequence of analyticity assumptions.

It is customary (cf. appendix III) to define spinor amplitudes according to :

$$\langle [p_3],\sigma_3,[p_4],\sigma_4\ |T|\ [p_1],\sigma_1,[p_2],\sigma_2\rangle = \delta^4(p_1+p_2-p_3-p_4)\ \mathcal{D}_{\sigma_3A_3}([p_3]^{-1})\mathcal{D}_{\sigma_4A_4}([p_4]^{-1})$$

$$\mathcal{M}_{A_3A_4\dot{A}_1\dot{A}_2}(p_3p_4p_1p_2)\ \mathcal{D}_{\dot{A}_1\sigma_1}([p_1]^{\dagger-1})\ \mathcal{D}_{\dot{A}_2\sigma_2}([p_2]^{\dagger-1})$$

$$= \delta^{(4)}(p_1+p_2-p_3-p_4)\quad \mathcal{D}_{\sigma_3A_3}([p_3]^{-1})\quad \mathcal{D}_{\sigma_4A_4}([p_4]^{-1}) \qquad \text{(II.f.1)}$$

$$\mathcal{M}_{A_3A_4A_1A_2}(p_3p_4p_1p_2)\ \mathcal{D}_{A_1\sigma_1}(C^{-1}[p_1])\ \mathcal{D}_{A_2\sigma_2}(C^{-1}[p_2])\ .$$

Exercise :

Find the relationship between $\mathcal{M}_{A_3A_4\dot{A}_1\dot{A}_2}$ and $\mathcal{M}_{A_3A_4A_1A_2}$.

A current assumption [7] is that $M_{A_3A_4A_1A_2}(p_1p_2p_3p_4)$ is holomorphic in a (complex) Lorentz invariant domain of the mass shell $p_1+p_2 = p_3+p_4$, $p_i^2 = m_i^2$, where it is furthermore covariant under the complex Lorentz group :

$$M_{A_3A_4A_1A_2}(L.p_1, L.p_2, L.p_3, L.p_4) = \mathcal{D}^{s_3}_{A_3A_3'}(A(L))\, \mathcal{D}^{s_4}_{A_4A_4'}(A(L))$$

$$\mathcal{D}^{s_1}_{A_1A_1'}(A(L))\, \mathcal{D}^{s_2}_{A_2A_2'}(A(L))\, M_{A_3' A_4' A_1' A_2'}(p_1,p_2,p_3,p_4) \, . \qquad \text{(II.f.2)}$$

Covariance implies a decomposition :

$$M_{A_3A_4A_1A_2}(p_3\ p_4\ p_1\ p_2) = \begin{pmatrix} s_3 & s_4 & s_{34} \\ A_3 & A_4 & A_{34} \end{pmatrix}\begin{pmatrix} s_1 & s_2 & s_{12} \\ A_1 & A_2 & A_{12} \end{pmatrix}\begin{pmatrix} s_{34} & s_{12} & \Sigma \\ A_{34} & A_{12} & A \end{pmatrix} \times$$

$$\times\ M_A^{\Sigma}(p_1\ p_2\ p_3\ p_4) \qquad , \qquad \text{(II.f.3)}$$

hence it follows :

$$M_A^{\Sigma}(L.p_1, L.p_2, L.p_3, L.p_4) = \mathcal{D}^{\Sigma}_{AA'}(A(L))\ M^{\Sigma}_{A'}(p_1,p_2,p_3,p_4) \, . \qquad \text{(II.f.4)}$$

This decomposition is easily inverted.

Note that from covariance also, putting $A(L) = -\mathbb{1}$, one obtains the result that $s_1+s_2+s_3+s_4$, and consequently Σ , must be integers.

One can show [7,8] that under some technical restrictions M^{Σ} can be decomposed according to :

$$M_A^{\Sigma}(p_1\ p_2\ p_3\ p_4) = \sum_{\kappa_{\pm}} A^{\Sigma}_{\kappa_{\pm}}(s_{12}s_{13}s_{14})\ Y_A^{\Sigma,\kappa_{\pm}}(p_1\ p_2\ p_3\ p_4) \qquad \text{(II.f.5)}$$

where the $A^{\Sigma}_{\kappa_{\pm}}$'s are holomorphic functions of :

$$s_{12} = (p_1+p_2)^2 \ , \quad s_{13} = (p_1-p_3)^2 \ , \quad s_{14} = (p_1-p_4)^2 \left(s_{12}+s_{13}+s_{14} = \sum_1^4 m_i^2 \right)$$

in the image on the space of invariants s_{1j}, $2 \leq j \leq 4$, of the original holomorphy domain, and

$$Y_A^{\Sigma,\kappa_+}(p_1p_2p_3p_4) = \begin{pmatrix} \kappa_+, & \Sigma-\kappa_+, & \Sigma \\ \mu, & \nu, & A \end{pmatrix} Y_\mu^{(\kappa_+)}(p_1,p_2) \ Y_\nu^{(\Sigma-\kappa_+)}(p_1,p_3)$$

(II.f.6)

$$Y_A^{\Sigma,\kappa_-}(p_1p_2p_3p_4) = \begin{pmatrix} \kappa_- , & \Sigma-1-\kappa_-, & \Sigma-1 \\ \mu & \nu & A \end{pmatrix} Y_\mu^{\kappa_-}(p_1,p_2) \ Y_\nu^{(\Sigma-1-\kappa_-)}(p_1,p_3) \ .$$

Here, the relativistic spherical harmonics[III][8]

$Y^L_M(a,b)$ are defined as follows :

consider the "complex semi-bi-vector" associated to a,b :

$$\vec{e}(a,b) = a_o \vec{b} - b_o \vec{a} + i \, \vec{a} \times \vec{b}$$

and write :

$$Y^L_M(a,b) = Y^L_M(\vec{e}(a,b))$$

the right hand side being the solid spherical harmonic (homogeneous harmonic polynomial of degree L) of argument $\vec{e}(a,b)$.

<u>Remarks.</u>

- The above construction works if p_1, $m_1^2 \neq 0$ is replaced by any p_i

such that $p_i^2 = m_i^2 \neq 0$.

- If $a = (1, 0, 0, 0)$ $Y_M^L(a,b) = Y_M^L(\vec{b})$.

- $Y_M^L(\alpha a + \beta b\ , b) = \alpha^L\, Y_M^L(a,b)$; $Y_M^L(b,a) = (-)^L\, Y_M^L(a,b)$.

- For each Σ there are $\Sigma+1 + \Sigma = 2\Sigma + 1$

amplitudes $A^{\Sigma}_{\kappa_\pm}$, hence, a total of $(2s_1+1)(2s_2+1)(2s_3+1)(2s_4+1)$.

Example :

The $A^{\Sigma}_{\kappa_\pm}$'s fulfill the Mandelstam representation.

Theorem 9 :

The natural domain of convergence of multipole expansions, for fixed energy $P^2 = s_{12} = s_{34}$, in terms of the variable

$$z = \cos\theta_{12,34} = - \hat{q}_{1,2} \cdot \hat{q}_{34}$$

is the largest ellipse, with foci ± 1 in which the $A^{\Sigma}_{\kappa_\pm}(s_{12}, \cos\theta_{12,34})$ are holomorphic. The rate of convergence is governed by the large semi axis of ellipse.

- Sketch of the proof: One first proves that $T_{\lambda\mu}(s_{12},z)$ has the form

$$(1-z)^{\frac{\mu-\lambda}{2}}\,(1+z)^{\frac{\lambda+\mu}{2}}\,\tau_{\lambda\mu}(s_{12},z)$$

where $\tau_{\lambda\mu}(s_{12},z)$ has for fixed s_{12} the same z - analyticity properties as the $A^{\Sigma}_{\kappa_\pm}$'s . One furthermore notices that :

$$d^j_{\lambda\mu}(\theta) = \left(\frac{1-z}{2}\right)^{\frac{\mu-\lambda}{2}} \left(\frac{1+z}{2}\right)^{\frac{\mu+\lambda}{2}} \left[\frac{(j+\mu)!\,(j-\mu)!}{(j+\lambda)!\,(j-\lambda)!}\right]^{1/2} P^{\mu-\lambda,\mu+\lambda}_{j-\mu}(z) \ ,$$

where, for fixed λ, μ, the $P_{j-\mu}^{\mu-\lambda,\mu+\lambda}(z)$ are orthogonal polynomials on the interval $(-1,+1)$, for the measure $\left(\frac{1-z}{2}\right)^{\mu-\lambda}\ \left(\frac{1+z}{2}\right)^{\mu+\lambda}\ dz$.

Thus :

$$T_{\lambda\mu}(s_{12},z) = \left(\frac{1-z}{2}\right)^{\frac{\mu-\lambda}{2}} \left(\frac{1+z}{2}\right)^{\frac{\mu+\lambda}{2}} \sum_j T^j_{\lambda\mu}(s_{12})\ P_{j-\mu}^{\mu-\lambda,\mu+\lambda}(z) ,$$

and, by a classical theorem [10], the expansion $\sum_j \ldots$ has just the domain of convergence stated in the theorem.

If one assumes analyticity in a cut plane in z, as stems from the Mandelstam [11] representation, or in the Lehmann ellipse [11] which emerges from postulates of local field theories, one sees that the ellipse of convergence collapses on the interval $(-1, +1)$ at high energy ; hence, as energy increases, the convergence becomes worse and worse.

The convergence of other multipole expansions is of course a consequence of that of the helicity expansion as is seen from the way they are connected.

g) <u>Threshold behaviour of reduced amplitudes</u> [12].

We shall be concerned here with threshold behaviours of reduced matrix elements $T^J_{\substack{\ell_{34}\ell_{12}\\ s_{34}s_{12}}}(s)$ which are also consequences of analyticity in the momentum transfer variables.

Since we shall be concerned with a neighbourhood of :

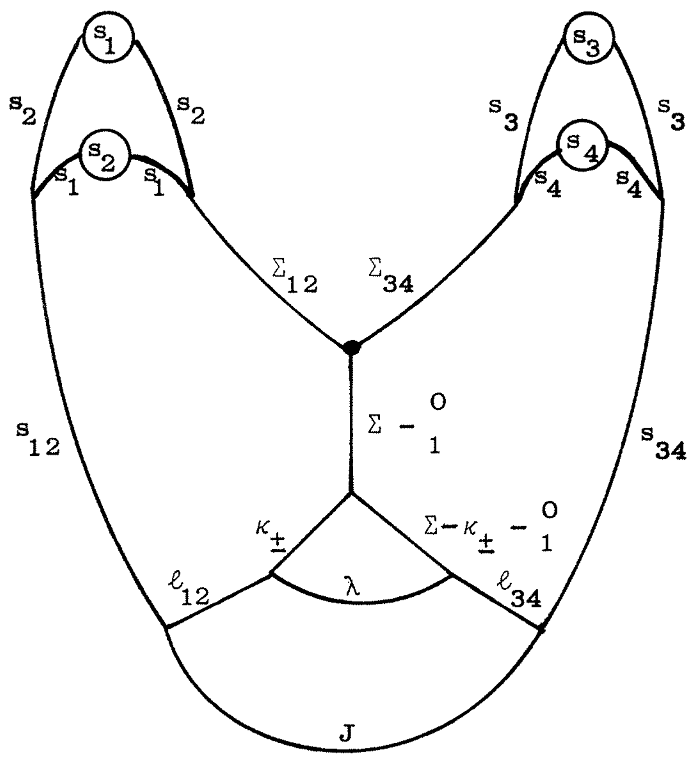
$$s = \max\left\{(m_1+m_2)^2\ ,\ (m_3+m_4)^2\right\} \neq 0$$

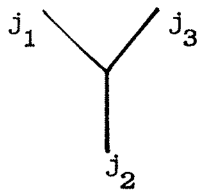
we can use an expansion of the spinor amplitudes in terms of

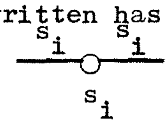
$$Y_A^{\Sigma,\kappa_\pm}(p_1,p_2,p_3,p_4) = \begin{pmatrix} \kappa_\pm, & \Sigma-\kappa_\pm-{0 \atop 1} & \Sigma \\ & & \Sigma-1 \\ \mu & \nu & A \end{pmatrix} Y_\mu^{\kappa_\pm}(P,q_{1,2}) \; Y_\nu^{(\Sigma-\kappa_\pm-{0 \atop 1})}(P,q_{3,4})$$

thus introducing possibly in the coefficients $A^{\Sigma}_{\kappa_\pm}(s_{12},s_{13},s_{14})$ poles at $s_{1,2} = 0$.

We then compute the angular integral which yields $T^J_{\substack{\ell_{34},\ell_{12} \\ s_{34},s_{12}}}(s)$ and represent it in terms of a graph [13]:



which has the following meaning : each vertex 

is meant for a Clebsch Gordan coefficient $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and a link between two vertices indicates a summation over the corresponding magnetic index; a necessary orientation at each vertex which should indicate in what order the Clebsch Gordan coefficient has to be written has been omitted as it is of no use for our purpose ; each circle 

is meant for a matrix element $\mathcal{D}^{s_i}_{\sigma_i \sigma_i'}(\hat{p}_i \leftarrow \hat{P})$.

The angular momentum λ comes from assuming that $A^{\Sigma}_{\kappa_\pm}(s_{12}, z)$ has a spectral representation of the type

$$A^{\Sigma}_{\kappa_\pm}(s_{12}, z) = \int_{z_o(s_{12})}^{\infty} \frac{\rho^{\Sigma}_{\kappa_\pm}(s_{12}, z')\, dz'}{z' - z} \; \frac{1}{|q_{12}|\;|q_{34}|}$$

as one can deduce from e.g. the Mandelstam representation, and the use of

$$\frac{1}{z'-z} = \sum_{\lambda} (2\lambda+1)\, P_\lambda(z)\; Q_\lambda(z')$$

together with the addition theorem

$$P_\lambda(z) = \sum_m Y^{\lambda}_m(\hat{\underset{\sim}{q}}_{12})\; Y^{\lambda *}_m(\underset{\sim}{q}_{34}) \, .$$

Vertices $(\ell_{12}, \kappa_\pm, \lambda)$; $(\ell_{34}, \Sigma - \kappa_\pm - {0 \atop 1}, \lambda)$ come from integrations over $d\Omega_{\underset{\sim}{q}_{12}}$, $d\Omega_{\underset{\sim}{q}_{34}}$, respectively, provided the angular dependences of the $\mathcal{D}^{s_i}_{\sigma_i \sigma_i'}(p_i \leftarrow P)$ are neglected, (the regularity of these matrices around threshold makes this possible) ; there emerge, thus, factors

$$|q_{1,2}|^{\kappa_\pm} \;\; |q_{3,4}|^{\Sigma - \kappa_\pm - {0 \atop 1}} \, ,$$

from the solid harmonics in terms of which spinor amplitudes are constructed. Such a graph is finally accompanied by a factor $Q_\lambda(z')$ weighted by the corresponding weight functions.

Now, to a singularity $s_{13} = s_{13}^o$ or $s_{14} = s_{14}^o$, there corresponds, at energy s_{12} a singularity in z of the type :

$$\overset{o}{\zeta}(s_{12}) \Big/ |q_{1,2}|\ |q_{3,4}|$$

where $\overset{o}{\zeta}$ is slowly varying around threshold. Furthermore, at threshold the argument of Q_λ goes to infinity like $\zeta \big/ |q_{1,2}||q_{3,4}|$, so that :

$$Q_\lambda(\zeta' \big/ |q_{1,2}||q_{3,4}|) \sim \frac{|q_{1,2}|^{\lambda+1}\ |q_{3,4}|^{\lambda+1}}{|\zeta'|^{\lambda+1}} .$$

So, provided that the integrals $\int_{z_o}^{\infty} \frac{\rho(s_{1,2}, z')dz'}{z'^{\lambda+1}}$ converge, the threshold behaviour is given by :

$$|q_{1,2}|^{\kappa_\pm+\lambda}\ |q_{3,4}|^{\Sigma-\kappa_\pm - {0 \atop 1} + \lambda}$$

where the exponents have to be minimized according to the triangular inequalities imposed by the non vanishing of Clebsch Gordan coefficients.

a) Elastic scattering : $|q_{1,2}| = |q_{3,4}| = |q|$.

the power to be minimized is $\Sigma - {0 \atop 1} + 2\lambda$. Now :

$$\kappa_\pm + \lambda \geq \ell_{12} \quad , \quad \Sigma - \kappa_\pm - {0 \atop 1} + \lambda \geq \ell_{34} .$$

Hence, in general, the threshold behaviour

$$T^J_{\substack{\ell_{12}\ \ell_{34} \\ s_{12}\ s_{34}}} \sim |q|^{\ell_{12}+\ell_{34}}$$

(unless the coefficient vanishes accidentally).

b) Inelastic scattering .

if $m_1 + m_2 < m_3 + m_4$, the power to be minimized is $\Sigma - \kappa_{\pm} - {0 \atop 1} + \lambda \geq \ell_{34}$,

hence, in general the threshold behaviour :

$$T^J_{\substack{\ell_{12}\ \ell_{34} \\ s_{12}\ s_{34}}} \quad |q_{34}|^{\ell_{34}}$$

(with similar restrictions).

Remark.

This argument is a substitute for classical arguments in potential scattering theory,-singularities in momentum transfers s_{13}, s_{14} essentially play the roles of ranges of direct and exchange potentials - .

CHAPTER III

ANALYSIS OF THREE PARTICLE STATES

a) Couplings in cascade.

Assume we have a representation of the Poincaré group acting on a space with basis $|[p_1],\sigma_1\ ;\ [p_2],\sigma_2\ ;\ [p_3],\sigma_3\rangle$. We may first transform it into : $|[p_{12}],\mu_{12}\ ;\ j_{12},\ \eta_{12}\ ;\ [p_3],\sigma_3\rangle$ where η_{12} is a set of degeneracy parameters which labels the coupling of particles 1 and 2. Since the frame $[p_{12}]$ is arbitrary, it can be chosen appropriately to couple the system (12) with system (3), with a coupling specified by degeneracy parameters $\eta_{12,3}$:

$$|[p_{123}],\sigma_{123}\ ;\ j_{12},\ \eta_{12};\ j_{12,3},\ \eta_{12,3}\rangle \ .$$

It is at this point that the expression of the state $|[p_{12}],\mu_{12};\ j_{12},\eta_{12}\rangle$ for an arbitrary frame $[p_{12}]$ is really useful. Let us for instance write down the Clebsch Gordan coefficient for $\eta_{12},\ \eta_{12,3}$, sets of $\ell - s$ coupling degeneracy parameters :

$$\langle [p_1],\sigma_1,[p_2],\sigma_2,[p_3],\sigma_3 |\ [p_{123}],\mu_{123}\ ;\ p^2_{12},s_{12},\ell_{12},j_{12},p^2_{123},s_{12,3}\ell_{12,3},j_{12,3}\rangle$$

$$= \frac{2\sqrt{p^2_{12}}\sqrt{2}}{\lambda^{1/4}(m^2_1,m^2_2,p^2_{12})}\ \frac{2\sqrt{p^2_{123}}\sqrt{2}}{\lambda^{1/4}(m^2_3,p^2_{12},p^2_{123})}\ \delta^4(p_{123}-p_1-p_2-p_3)\ \delta(p^2_{12}-(p_1+p_2)^2) \times \ldots.$$

$$\times \cdots \sum_{\substack{\sigma_1' \sigma_2' \sigma_3' \\ \sigma_{12} = \sigma_1' + \sigma_2' \\ \mu_{12} = \sigma_{12} + m_{12} \\ \sigma_{123} = \mu_{12} + \sigma_3'}} \mathcal{D}^{s_1}_{\sigma_1 \sigma_1'} \left([p_1]^{-1} [p_1 \leftarrow p_{12}][p_{12} \leftarrow p_{123}][p_{123}] \right) \times$$

$$\mathcal{D}^{s_2}_{\sigma_2 \sigma_2'} \left([p_2]^{-1} [p_2 \leftarrow p_{12}][p_{12} \leftarrow p_{123}][p_{123}] \right) \mathcal{D}^{s_3}_{\sigma_3 \sigma_3'} \left([p_3]^{-1} [p_3 \leftarrow p_{123}][p_{123}] \right)$$

$$\left(\begin{matrix} s_1 & s_2 \\ \sigma_1' & \sigma_2' \end{matrix} \middle| \begin{matrix} s_{12} \\ \sigma_{12} \end{matrix} \right) \left(\begin{matrix} \ell_{12} & s_{12} \\ m_{12} & \sigma_{12} \end{matrix} \middle| \begin{matrix} j_{12} \\ \mu_{12} \end{matrix} \right) Y^{\ell_{12}}_{m_{12}} (\underset{\sim}{q}_{1,2}')$$

$$\left(\begin{matrix} j_{12} & s_3 \\ \mu_{12} & \sigma_3' \end{matrix} \middle| \begin{matrix} s_{123} \\ \sigma_{123} \end{matrix} \right) \left(\begin{matrix} \ell_{12,3} & s_{123} \\ m_{12,3} & \sigma_{123} \end{matrix} \middle| \begin{matrix} j_{123} \\ \mu_{123} \end{matrix} \right) Y^{\ell_{12,3}}_{m_{123}} (\underset{\sim}{q}_{12,3}) \qquad \text{(III,a.1)}$$

$$q_{1,2} = \frac{1}{2} \left(p_1 - p_2 - \frac{m_1^2 - m_2^2}{p_{12}^2} p_{12} \right)$$

$$p_{12} = p_1 + p_2$$

$$q_{12}' = [p_{123} \leftarrow p_{12}] \; q_{12} \qquad (\text{note} : q_{12}' \cdot p_{123} = 0)$$

$$q_{12,3} = \frac{1}{2} \left(p_{12} - p_3 - \frac{p_{12}^2 - m_3^2}{p_{123}^2} p_{123} \right)$$

$$\underset{\sim}{q} = \{ -q \cdot n_i(p_{123}) \} .$$

The occurence of $\underset{\sim}{q}_{12}'$ comes from the fact that $q^i_{12} = - q_{12} . n_i(p_{12})$

occurred in the first coupling. Since $[p_{12}]$ had to be chosen as $[p_{12} \leftarrow p_{123}][p_{123}]$ in order that one might perform the second ℓ - s coupling conveniently, $q^{i}_{12} = - q_{12} \cdot [p_{12} \leftarrow p_{123}]\, n_i(p_{123}) = - q'_{12} \cdot n_i(p_{123})$.

This is a relativistic effect as well as the cascade of Lorentz transformations which occur in the spin matrices.

Such reductions may be of some use in the study of isobar productions. When isobars corresponding to different couplings ([(12)3],[(13)2] say) are simultaneously produced, it is interesting to know the recoupling coefficients. The solution to this problem will be postponed till later since the symmetrical coupling to be described now makes it essentially trivial.

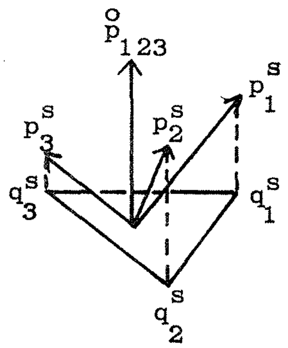
b). The symmetric coupling [15].

We consider states $|[p_1]\sigma_1\ [p_2]\sigma_2\ [p_3]\sigma_3\rangle$ whose total momentum, as we know, is $p_1+p_2+p_3 = p_{123}$; let $[p_{123}]$ be an arbitrary frame attached to p_{123} and let $p^2_{12}, p^2_{23}, p^2_{13}, (p^2_{12} + p^2_{13} + p^2_{23} = m^2_1 + m^2_2 + m^2_3 + p^2_{123})$, be a set of values of invariants formed with the p_i's $(p^2_{ij} = (p_i+p_j)^2)$. p_i's belonging to such a set of invariants are parametrized for a given p_{123} by a rotation R which takes each p_i to a standard position p^s_i such that $p^2_{ij} = (p^s_i + p^s_j)^2$.

(Invariants are fixed by the time components $p^s_i \cdot p_{123}$, and directions by the triangle with vertices :

$$q^s_i = p^s_i - \frac{p^s_i \cdot \overset{o}{p}_{123}}{\overset{o}{p}_{123}}\ \overset{o}{p}_{123}\ ) ;$$

$$p_i = [p_{123}]\ R\ p^s_i \quad .$$



One may then use spin basis $[p_i]$ rigidly attached to the configuration of the p_i's :

$$[p_i] = [p_{123}] \; R[p_i^{s}]$$

and define states

$$|[p_{123}],\mu \; ; \; j_{123}, \; \lambda, \; \lambda_1,\lambda_2,\lambda_3,p_{12}^2,p_{23}^2,p_{13}^2\rangle = \int dR \sqrt{\frac{2j_{123}+1}{4\pi}} \; \mathcal{D}^{j*_{123}}_{\mu\lambda} \quad (R)$$

$$|[p_1] \; (p_{123},R), \; \lambda_1 \; ; \; [p_2] \; (p_{123},R), \; \lambda_2 \; ; \; [p_3] \; (p_{123},R) \; \lambda_3\rangle \quad \text{(III.b.1)}$$

where the λ_i' s are spin projections on the above frames, which transforms according to

$$U(a,A)\,|[p_{123}]\mu \; ; \; j_{123}, \; \lambda,\lambda_1,\lambda_2,\lambda_3, \; p_{12}^2 \; p_{23}^2 \; p_{13}^2\rangle = e^{i\,A.p_{123}.a}$$

(III.b.2)

$$|[p_{123}]\mu'; \; j_{123}, \; \lambda,\lambda_1,\lambda_2,\lambda_3, \; p_{12}^2 \; p_{23}^2 \; p_{13}^2\rangle \; \mathcal{D}^{j_{123}}_{\mu'\mu} \; ([Ap_{123}]^{-1} \; A[p_{123}])$$

[note [16] : $[Ap_i] = [Ap_{123}] \; ([Ap_{123}]^{-1} \; A[p_{123}]) \; R \; [p_i^{s}]$, so that the Wigner rotations computed with the above frames reduce to ± 1, the latter sign being due to the necessity of using SU(2) : j_{123} is thus restricted to be integer or half integer according as $s_1 + s_2 + s_3$ is integer or half integer] .

The quantum number λ can be interpreted as the projection of the total spin on $[p_{123}]R \; n_3(\overset{o}{p}_{123})$, an axis rigidly attached to the three momenta p_i .

The Clebsch Gordan coefficient is

$$\langle [p_1]\sigma_1 \; [p_2]\sigma_2 \; [p_3]\sigma_3 \;\; |[p_{123}],\mu \; ; \; j_{123},\lambda,\lambda_1,\lambda_2,\lambda_3 \; p_{12}^2 \; p_{13}^2 \; p_{23}^2 \rangle \;\; =$$

$$\frac{2\, p_{123}^2}{\sqrt{\pi}} \quad \delta^4(p_{123}-p_1-p_2-p_3) \quad \delta(p_{12}^2-(p_1+p_2)^2) \quad \delta(p_{13}^2-(p_1+p_3)^2)$$

$$\mathcal{D}^{(s_1)}_{\sigma_1\lambda_1}\left([p_1]^{-1}\,[p_1]_r\right) \; \mathcal{D}^{(s_2)}_{\sigma_2\lambda_2}\left([p_2]^{-1}[p_2]_r\right) \; \mathcal{D}^{(s_3)}_{\sigma_3\lambda_3}\left([p_3]^{-1}[p_3]_r\right)$$

$$\sqrt{\frac{2j_{123}+1}{4\pi}} \quad \mathcal{D}^{j_{123}*}_{\mu\lambda} \quad (R\,(p_1,\,p_2,\,p_3)) \qquad \text{(IIIb.3)}$$

where the $[p_i]_r$ are some frames rigidly tied to the system of 3 momenta as described above. A convenient choice is for instance that where $n_3(p_i)$ are the helicity axis in the overall center of mass :

$$n_3(p_i) \;=\; \frac{[p_i \leftarrow p_{123}] \;\; q_i}{\sqrt{-\,q_i^2}}$$

and $n_2(p_i) \;=\; (p_1 \wedge p_2 \wedge p_3)$ (the normal to the plane of the three momenta).

This coupling scheme is appropriate to characterize the decay into three particles of a system of spin j_{123} in terms of the "Dalitz plot variables" p_{ij}^2 and, at each point of the Dalitz plot,of the angles of the normal to the production plane and of the azimuthal angle within this plane, which fixes the orientation of the triangle made up by the three momenta.

One can of course similarly symmetrically couple any number of one particle states.

Remark.

The conditions of convergence of angular expansions of transition amplitudes to final states involving three or more particles is very badly known at the moment because of the lack of information on analyticity properties of such amplitudes.

c) Recoupling coefficient between a state labelled by the symmetric coupling and a state labelled by helicities :

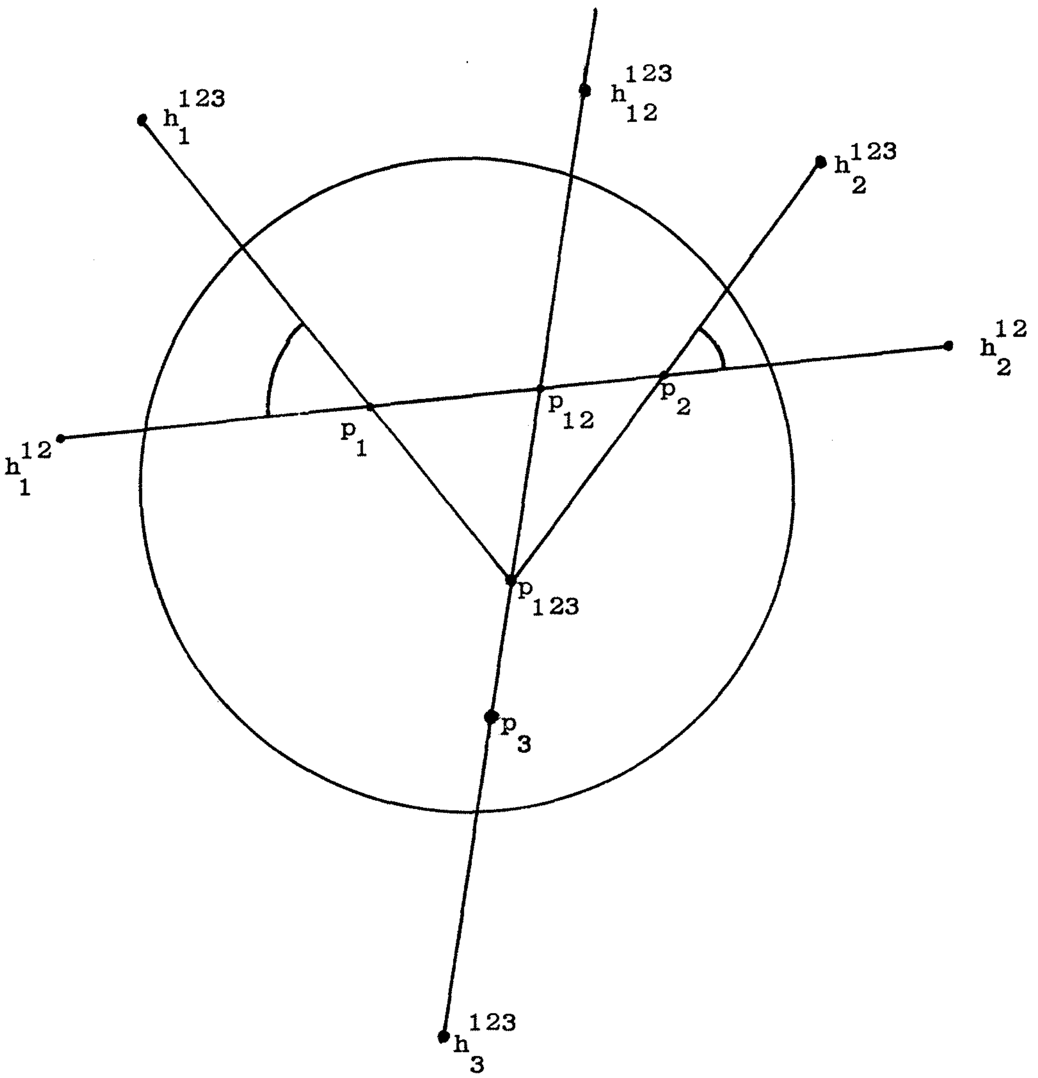
let :

$$\langle [p_{123}],\mu \,;\, j_{123}\; p_{12}^2\; j_{12}\; \lambda_1\; \lambda_2\; \lambda_{12}\; \lambda_3 | [p'_{123}],\mu' \,;\, j'_{123}\; \lambda'_1\; \lambda'_2\; \lambda'_3\; \lambda\; p'^2_{12}\; p'^2_{23}\; p'^2_{13} \rangle \tag{III.c.1}$$

$$= \delta(p_{123} - p'_{123})\;\; \delta_{\mu\mu'}\; \delta_{j_{123}}\;\; \delta_{j'_{123}}\;\; W_{\substack{p^2_{123}\\ j_{123}}}\;\; (p_{12}^2\; j_{12}\; \lambda_1\lambda_2\lambda_{12}\lambda_3 | \;\ldots$$

$$\ldots | p'^2_{12}\;\; p'^2_{23}\;\; p'^2_{13}\;\; \lambda'_1\; \lambda'_2\; \lambda'_3\; \lambda) .$$

All helicity frames are taken to have $(p_1 \wedge p_2 \wedge p_3)$ as axis number 2 . This allows us to make a 2 dimensional drawing of directions of the various vectors involved and read off circular and hyperbolic angles.



The circle represents a curve at infinity on the light cone-time like vectors are represented inside, space like vectors, outside-; Two orthogonal vectors are represented by conjugate points.

h_i^j means the helicity axis of particle i in frame of time axis p^j.

One gets factors

$\delta(p_{12}^2 - p_{12}^{12})$ (obvious)

$\delta_{\lambda_3 \lambda_3'}$ (since particle 3 is described in terms of the same helicity frame in both couplings)

$d^{s_2}_{\lambda_2\lambda_2'}(-\hat{h}^{12}_2\cdot\hat{h}^{123}_2)$ describing the change of helicity basis from the symmetric to the unsymmetric coupling

$d^{s_1}_{\lambda_1\lambda_1'}(-\hat{h}^{12}_1\cdot\hat{h}^{123}_1)$ "

$d^{j_{12}}_{\lambda_{12}\lambda_1-\lambda_2}(\hat{q}_{1,2}\cdot\hat{h}^{123}_{12})$ where $q_{1,2}$ is the relative momentum of particles 1,2 in their barycentric frame (not $q'_{1,2}$!)

$d^{j_{123}}_{\lambda_{12}-\lambda_3\lambda}(-\hat{q}_{12,3}\cdot n)$ where n is the spin axis rigidly tied up with the triangle of momenta. e.g. $q'_{1,2}$.

All scalar products are evaluated in terms of the invariants p'^2_{ij} .

The kinematical factors occurring in each Clebsch Gordan coefficient stay unchanged. Thus we obtain :

$$W_{p^2_{123},j_{123}}(p^2_{12}\ j_{12}\ \lambda_1\ \lambda_2\ \lambda_{12}\ \lambda_3 | p'^2_{12}\ p'^2_{23}\ p'^2_{13}\ \ \lambda'_1\ \lambda'_2\ \lambda'_3\ \lambda) \tag{III.c.2}$$

$$= \frac{4\sqrt{p^2_{12}}\ \sqrt{\pi}\ \delta(p^2_{12}-p'^2_{12})}{\lambda^{1/4}(m^2_1,m^2_2,p^2_{12})\ \lambda^{1/4}(p^2_{12},m^2_3,p^2_{123})}\sqrt{\frac{2j_{12}+1}{4\pi}}\ (-1)^{s_2-\lambda_2+s_3-\lambda_3}\ \delta_{\lambda_3\lambda'_3}\ \times$$

$$\times\ d^{s_2}_{\lambda_2\lambda'_2}\left(-\hat{h}^{12}_2\cdot\hat{h}^{123}_2\right)\ d^{s_1}_{\lambda_1\lambda'_1}\left(-\hat{h}^{12}_1\cdot\hat{h}^{123}_1\right)\ \times\ d^{j_{12}}_{\lambda_{12}\lambda_1-\lambda_2}\left(\hat{q}_{1,2}\cdot\hat{h}^{123}_{12}\right)$$

$$\times\ d^{j_{123}}_{\lambda_{12}-\lambda_3\lambda}\left(-\hat{q}_{12,3}\cdot n\right).$$

The computation of an arbitrary recoupling coefficient [17,18] involves essentially twice as many rotation matrices,as one can see by inserting a complete set of states labelled by the symmetric coupling. Racach coefficients for the rotation group will eventually appear when ℓ-s couplings are considered.

APPENDIX I

BASIC FACTS ABOUT THE LORENTZ GROUP

a) The four sheets of the Lorentz group

Let $x = (x^0, x^1\ x^2\ x^3)$ be a vector in Minkowski's space.

Transformation matrices L :

$$x \rightarrow x' = Lx \quad \therefore \quad x'Gx' = xGx \ , \ G = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} ,$$

such that $L^T G L = G$,

form a group : the homogeneous Lorentz group. This group has four connected components :

$L_+^\uparrow$	$\det L = +1$	$L^0{}_0 > 0$	(a subgroup)
$L_+^\downarrow$	$\det L = +1$	$L^0{}_0 < 0$	
$L_-^\uparrow$	$\det L = -1$	$L^0{}_0 > 0$	
$L_-^\downarrow$	$\det L = -1$	$L^0{}_0 < 0$.	

The last three components are not subgroups (they do not contain the identity).

One has

$$L_+^\downarrow = -\mathbb{1} \times L_+^\uparrow$$

$$L_-^\uparrow = G \times L_+^\uparrow$$

$$L_-^\downarrow = -G \times L_+^\downarrow$$

$L_+^\uparrow$ is not simply connected. Its universal covering group (the smallest Lie group which has the same Lie algebra and is simply connected) is SL(2C), the group of 2×2 complex matrices of determinant one (special linear complex group in two dimensions - or group of 2×2 unimodular complex matrices).

The correspondence is $1 \to 2$:

$$L \to \pm A(L)$$

where $A(L)$ is defined up to a sign by :

$$x' = L\,x \quad \to \quad \underset{\approx}{x}' = A.\underset{\approx}{x}\,A^\dagger$$

($\underset{\approx}{x} = x^\circ + \vec{x}.\vec{\sigma}$ where $\vec{\sigma}$ is the set of Pauli matrices) indeed $\det \underset{\approx}{x} =$ $= x^{\circ 2} - \vec{x}^2 = \det \underset{\approx}{x}'$ (since $\det A = \det A^\dagger = 1$).

b) <u>Subgroups</u>

The little group $L_+^\uparrow(x)$ of a vector x is the subgroup of Lorentz transformations leaving x invariant.

If x is time-like : $x^2 > 0$ $L_+^\uparrow(x)$ is isomorphic with a rotation group SO(3) in three dimensional space : if $\overset{\circ}{x} = (1,0,0,0)$ $L_+(\overset{\circ}{x})$ is just SO(3) in the variables x_i $1 \leq i \leq 3$.

The SL(2,C) representative is SU(2) the group of unitary complex 2×2 matrices of determinant 1 ($A\,\mathbb{1}\,A^\dagger = \mathbb{1}$).

If x is light-like : $x^2 = 0$ $L_+^\uparrow(x)$ is isomorphic with a euclidean group E(2) in two dimensional space (again a three parameter group) :

If $x^\circ = (1,0,0,-1)$

$$L(x^\circ) \begin{pmatrix} 1+\frac{\alpha_1^2+\alpha_2^2}{2} & \alpha_1\cos\varphi+\alpha_2\sin\varphi & -\alpha_1\sin\varphi+\alpha_2\cos\varphi & -\frac{\alpha_1^2+\alpha_2^2}{2} \\ \frac{\alpha_1}{2} & \cos\varphi & -\sin\varphi & -\frac{\alpha_1}{2} \\ \frac{\alpha_2}{2} & \sin\varphi & \cos\varphi & -\frac{\alpha_2}{2} \\ \frac{\alpha_1^2+\alpha_2^2}{2} & \alpha_1\cos\varphi+\alpha_2\sin\varphi & -\alpha_1\sin\varphi+\alpha_2\cos\varphi & 1-\frac{\alpha_1^2+\alpha_2^2}{2} \end{pmatrix}$$

The SL(2C) representative is the subgroup $\begin{pmatrix} e^{\frac{i\Phi}{2}} & z \\ 0 & e^{\frac{-i\Phi}{2}} \end{pmatrix}$ $\overset{\circ}{\underset{\sim}{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

If x is space-like : $x^2 < 0$, $L_+^\uparrow(x)$ is isomorphic with a three dimensional Lorentz group SO(2,1). The SL(2C) representative is isomorphic with SL(2,R) (unimodular real matrices : take $\overset{\circ}{x} = (0,0,1,0)$ and use $A\,\sigma_2\,A^\dagger = \sigma_2$ and $\sigma_2\,A\,\sigma_2^{-1} = A^{-1T}$, a general identity) or SU(1,1) the pseudo unitary group in two dimensions which fulfills $A\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}A^\dagger = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ (pseudo because of the sign!).

Every Lorentz transformation can be written as the product of a particular transformation sending any four vector on a vector of equal length (and same sign of the time component if the vector is time-like) by an operation of the little group of this vector.

Thus for instance every Lorentz transformation can be decomposed into a rotation and a pure Lorentz transformation :

$AA^\dagger$ is hermitean positive definite, and has therefore a positive square root H which represents the pure Lorentz transformation :

$$H = h^\circ + \vec{h}.\vec{\sigma} \quad , \qquad h \text{ real} \quad h^{\circ 2} - \vec{h}^2 = 1$$

which represents a Lorentz transformation of velocity in the direction $\hat{h}$ with magnitude $v = \dfrac{2|\vec{h}|h_\circ}{h_\circ^2 + |\vec{h}|^2}$.

Then $AA^\dagger = HH^\dagger \rightarrow (H^{-1}A)(H^{-1}A)^\dagger = 1 \Rightarrow A = H\,U$ where $U \in SU(2)$.

c) <u>Representations of SL(2,C)</u>

Let $\begin{pmatrix}\xi \\ \eta\end{pmatrix}$ be a complex two dimensional vector (a "spinor"), transforming according to :

$$A : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = A \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a\xi + b\eta \\ c\xi + d\eta \end{pmatrix} .$$

The 2j+1 - dimensional vector

$$V_{jm}(\xi,\eta) = \frac{(\xi)^{j+m}(\eta)^{j-m}}{\sqrt{(j+m)!\ (j-m)!}}$$

transforms according to

$$\mathcal{D}^j(A) \quad : \quad V_{jm}(\xi,\eta) \rightarrow V'_{jm}(\xi,\eta) = \mathcal{D}^j_{mm'}(A) \quad V_{jm'}(\xi,\eta)$$

$$= V_{jm}\ (a\xi + b\eta,\ c\xi + d\eta) \ .$$

Hence :

$$\mathcal{D}^j_{mn'}(A) = \sum_{\substack{p+q \\ =j-m' \\ 0\leq p\leq j+m \\ 0\leq q\leq j-m}} \sqrt{(j+m')!(j-m')!(j+m)!(j-m)!} \quad \frac{a^{j+m-p}\ b^p\ c^{j-m-q}\ d^q}{(j+m-p)!\ p!\ (j-m-q)!\ q!} \ .$$

This representation is labelled (j,0).

Similarly $\mathcal{D}^j(A^{\dagger-1})$ is labelled (0,j) ; it is not equivalent to $\mathcal{D}^j(A)$.

This formula defines $\mathcal{D}^j(A)$ for A arbitrary !

One can easily see that

$$\mathcal{D}^j_{mm'}(A^*) = \mathcal{D}^{j*}_{mm'}(A)$$

$$\mathcal{D}^j_{mm'}(A^T) = \mathcal{D}^j_{m'm}(A)$$

$$\mathcal{D}^j_{mm'}(\lambda \mathbb{1}) = \delta_{mm'}\, \lambda^{2j} .$$

If $A \in SU(2)$ then $(0,j)$ and $(j,0)$ become identical $(U = U^{\dagger -1})$.

d) Products of representations IV-19) Clebsch Gordan coefficients.

Let (ξ_1,η_1), (ξ_2,η_2) be two spinors. From the identity $A\sigma_2 A^T = \sigma_2 \det A$, one finds that $\xi_1'\eta_2' - \xi_2'\eta_1' = (\xi_1\eta_2 - \xi_2\eta_1)$ (det A); in particular :

$$\frac{(\xi_1\eta_2-\xi_2\eta_1)^{2j}}{2j!} = V_{jm}(\xi_1\eta_1)\,(-)^{j-m}\, V_{j-m}(\xi_2\eta_2)$$

is multiplied by $(\det A)^{2j}$ under the action of A.

The invariant formed with three spinors :

$$I_{j_{12},j_{23},j_{31}} = [\xi_2\eta_3 - \xi_3\eta_2]^{j_{23}}[\xi_3\eta_1-\xi_1\eta_3]^{j_{13}}[\xi_1\eta_2-\xi_2\eta_1]^{j_{12}}\, \sqrt{(j_{12}+j_{23}+j_{31}+1)!}$$

is thus multiplied by $(\det A)^{j_{12}+j_{23}+j_{13}}$. It is homogeneous of degree $2j_1 = j_{12} + j_{13}$ in (ξ_1,η_1), $2j_2 = j_{12} + j_{23}$ in $(\xi_2\eta_2)$, $2j_3 = j_{13} + j_{23}$ in (ξ_3,η_3) and can thus be expanded as :

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} V_{j_1m_1}(\xi_1,\eta_1)\, V_{j_2m_2}(\xi_2,\eta_2)\, V_{j_3m_3}(\xi_3,\eta_3)$$

Clebsch Gordan coefficients are defined by

$$\begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1 & m_2 & \big| & m_3 \end{pmatrix} = \sqrt{2j_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} (-)^{j_3+m_3}$$

they vanish unless $m_1+m_2 = m_3$; $|j_1-j_2|<j_3 < j_1+j_2$, $j_1+j_2+j_3$ integer .

The invariance of $I_{j_{12}j_{23}j_{31}}$ then reads

$$(\det A)^{j_1+j_2-j_3} \begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1 & m_2 & \big| & m_3 \end{pmatrix} \mathcal{D}^{j_3}_{m_3m_3'}(A) = \sum_{m_1'+m_2'=m_3'} \begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1' & m_2' & \big| & m_3' \end{pmatrix} \mathcal{D}^{j_1}_{m_1m_1'}(A)\mathcal{D}^{j_2}_{m_2m_2'}(A).$$

Normalizations have been fixed so that the Clebsch Gordan coefficients fulfill the orthogonality and completeness relations appropriate when $A \in SU(2)$

$$\sum_{j_3,m_3} \begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1 & m_2 & \big| & m_3 \end{pmatrix} \begin{pmatrix} j_3 & \big| & j_1 & j_2 \\ m_3 & \big| & m_1' & m_2' \end{pmatrix} = \delta_{m_1m_1'} \ \delta_{m_2m_2'}$$

$$\sum_{m_1,m_2} \begin{pmatrix} j_3 & \big| & j_1 & j_2 \\ m_3 & \big| & m_1 & m_2 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & \big| & j_3' \\ m_1 & m_2 & \big| & m_3' \end{pmatrix} = \delta_{j_3j_3'} \ \delta_{m_3m_3'}$$

so that one also has : i.e.

$$(\det A)^{j_1+j_2-j_3} \mathcal{D}^{j_3}_{m_3m_3'}(A) = \sum \begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1 & m_2 & \big| & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & \big| & j_3 \\ m_1' & m_2' & \big| & m_3' \end{pmatrix} \mathcal{D}^{j_1}_{m_1m_1'}(A) \ \mathcal{D}^{j_2}_{m_2m_2'}(A) .$$

e) <u>SU(2) versus SL(2,R)</u>

Finite dimensional representations of SL(2,C) obviously restrict to finite dimensional representations both of SU(2) and SL(2,R). The restrictions

to SU(2) yield the well known unitary representations of SU(2). It is not so any more for SL(2,R) because SL(2,R) is not a compact group. Unitary representations of SL(2R) have been classified by V. Bargmann [20].
We shall just say, for the purpose of orientation that a convenient basis of representation space can be labelled by integer or half integer eigenvalues of the generator of rotations (corresponding to the subgroup $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$) which vary by steps of unity over an infinite range either from $-\infty$ to $+\infty$, for some types of representations, or from $(-\infty$ to $-k)$ or from $(+k$ to $+\infty)$ where k is a number which characterizes some other types of representations.

f) Fourier analysis.

Whether it be for SU(2) or SL(2,R) one has the following property : let $f(g)$ be a square integrable function over the group : $\int |f(g)|^2 \, dg$ where dg is the so-called Haar measure on the group ($d \cos\theta \, d\varphi \, d\psi$ for SU(2) , $d \cosh\theta \, d\varphi \, d\psi$ for SL(2,R), then there are sufficiently many unitary representations $\mathcal{D}^j(g)$ with matrix elements $\mathcal{D}^j_{mm'}(g)$ such that :

$$f(g) = \sum_j C^j_{mm'} \, \mathcal{D}^j_{m'm}(g)$$

$$\int |f(g)|^2 dg = \sum_{j,\, m,\, m'} |C^j_{mm'}|^2$$

where the sum may eventually turn into an integral. In the case of SU(2) j just ranges over integers and half integers and the $\mathcal{D}^j(g)$ are just the representations we have previously described.

In the case of SL(2,R) the sum over j becomes an integral plus an infinite discrete sum.

From this, one can deduce the existence of complete sets of functions on "homogeneous spaces" of these groups. For instance, let us rewrite in the case of SU(2) :

$$f(\varphi,\theta,\psi) = \sum_j C^j_{mm'} \mathcal{D}^j_{m'm}(\varphi,\theta,\psi).$$

Consider now functions of φ,θ alone (which label points of the unit sphere. Any function $g(\varphi,\theta)$ can be lifted to a function of (φ,θ,ψ) through $f(\varphi\ \theta\ \psi) = g(\varphi,\theta)\ e^{i\lambda\psi}$.

Thus, for such functions, there only survive in the above sum terms where $m = \lambda$, $j > |\lambda|$ $\left(\mathcal{D}^j_{m'm}(\varphi,\theta,\psi) = \mathcal{D}^j_{m'm}(\varphi,\theta,0)\ e^{im\psi}\right)$. Hence the various complete sets of functions on the sphere $\mathcal{D}^j_{m\lambda}(\varphi,\theta,0)$ for fixed λ , as mentioned in Chapter II § b.

A similar result of course holds in the case of SL(2,R), which has been used in attempts to understand the group theoretical origin of Regge's analysis [14].

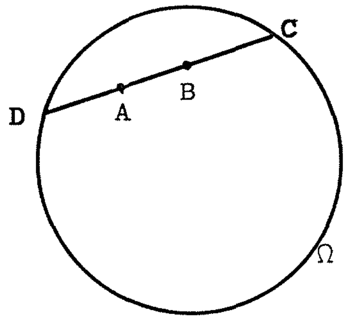
g) Elements of Minkowski geometry

As one knows , the Lorentz group preserves the light cone $x^2 = 0$. It is convenient to associate to each four vector its direction labelled for instance by the 3-vector $\vec{\xi} = \frac{\vec{x}}{x^o}$; when x is transformed under a Lorentz transformation, its direction $\vec{\xi}$ is transformed under a projective transformation $\vec{\xi} \to \vec{\xi}' = \frac{\overrightarrow{L\xi}}{(L\xi)_o}$ which preserves the "sphere" Ω : $\vec{\xi}^2 = 1$.

Time-like, light-like, space-like, vectors are represented by points inside, on, and outside Ω . These three regions are preserved globally under any transformation of the group.

Invariant distance between two points :

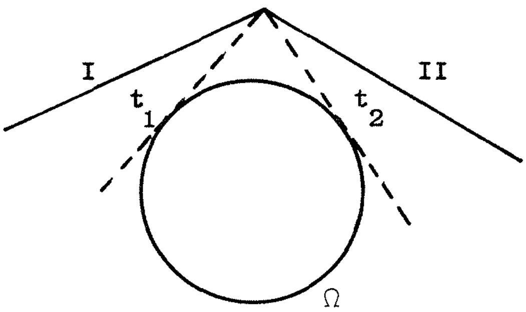
$$\widehat{AB} = \ell n \frac{\overline{AC}}{\overline{AD}} : \frac{\overline{BC}}{\overline{BD}}$$



is a projective invariant,
real if A and B are time-like
pure imaginary if A and B are
space-like (C and D are then complex conjugate of each other).

In particular if the above "anharmonic ratio" is -1 (A and B conjugate with respect to Ω) the corresponding four vectors are orthogonal.

One can similarly define the angle between two two-dimensional planes which intersect, as the logarithm of the anharmonic ratio between their traces on $\vec{\xi}$ space and the tangents to Ω through their intersection in the plane they determine



Invariant volume elements:

if $\xi = \pm 1$ are traces of Ω on a straight line the line element is $\frac{d\xi}{1-\xi^2}$

if $\xi_1^2 + \xi_2^2 = 1$ is the trace of Ω on a two plane, the surface element is

$$\frac{d\xi_1 \; d\xi_2}{(1-\xi_1^2 - \xi_2^2)^2}$$

finally the three dimensional volume element is $\frac{d\xi_1 \; d\xi_2 \; d\xi_3}{(1 - \xi^2)^3}$.

As a result the sum of the angles of a triangle is smaller than π (it is called the defect of the triangle), and is proportional to the surface of the triangle.

This geometry, a hyperbolic analogue of the geometry on the sphere is useful in evaluating recoupling coefficients.

APPENDIX II

ANGULAR EXPANSIONS FOR TWO BODY REACTIONS INVOLVING LOW SPIN PARTICLES.

It was found in Chapter II that angular expansions of transition amplitudes could be expressed in terms of "reduced Clebsch Gordan coefficients" involving spin matrices of a purely relativistic origin ($\mathcal{D}(p_i \leftarrow P)$) multiplying expressions involving SU(2) Clebsch Gordan coefficients and angular functions of three dimensional (symbolic) vectors $\underset{\sim}{q}$. It is the purpose of this appendix to give compact expressions of these "non relativistic" parts in cases where the spins of particles involved are sufficiently low so that the Clebsch Gordanry can be treated easily.

The methods used here allow one to write down fast typical angular distributions or spin correlations. They have a defect, though, that normalization factors, which often are not useful, have to be computed by a separate calculation . So, reduced matrix elements defined here and denoted by small letters will differ from those defined in the body of the notes and denoted by capital letters, within kinematical factors (due to phase space) and $\sqrt{\frac{2j+1}{4\pi}}$ factors due to conventional normalization of Clebsch Gordan coefficients.

We consider transition amplitudes : $\langle \sigma_f\ \sigma_f'\ \underset{\sim}{q_f}\ |T|\ \sigma_i\ \sigma_i'\ \underset{\sim}{q_i} \rangle$ where only spin variables and angular variables (directions of the barycentric momenta $\underset{\sim}{q_i}$, $\underset{\sim}{q_f}$ in initial and final states) are indicated. The spins of particles are s_i , s_i' , s_f , s_f' and the product of relative intrinsic parities of all particles involved is ± 1 . The corresponding reaction is denoted

$$s_i + s_i' \xrightarrow{\pm} s_f + s_f' \quad .$$

1) $0 + 0 \xrightarrow{\pm} 0 + 0$

$$\langle \underset{\sim}{q}_f | T | \underset{\sim}{q}_i \rangle = \sum_{\substack{\ell, m \\ \ell', m'}} \langle \hat{\underset{\sim}{q}}_f | \ell' m' \rangle \langle \ell' m' | T | \ell m \rangle \langle \ell m | \hat{\underset{\sim}{q}}_i \rangle ,$$

where $|\ell m\rangle$ are eigenstates of the total angular momentum operator. From rotation invariance

$$\langle \ell', m' | T | \ell, m \rangle = \delta_{\ell \ell'} \delta_{mm'} t_\ell .$$

From parity conservation $(-)^\ell = \pm (-)^{\ell'}$, so that only even relative intrinsic parity is allowed. Using

$$\langle \ell, m | \hat{q}_i \rangle = Y^*_{\ell m}(\hat{q}_i) ; \quad \langle \hat{q}_f | \ell, m \rangle = Y_{\ell m}(\hat{q}_f)$$

$$\sum_m Y_{\ell m}(\hat{q}_f) Y^*_{\ell m}(\hat{q}_i) = \frac{2\ell + 1}{4\pi} P_\ell (\hat{q}_f \cdot \hat{q}_i) \quad \text{(Legendre polynomial)},$$

we obtain :

$$\langle \underset{\sim}{q}_f | T | \underset{\sim}{q}_i \rangle = \sum_\ell t_\ell \frac{2\ell + 1}{4\pi} P_\ell (\hat{q}_f \cdot \hat{q}_i)$$

where t_ℓ is a function of $|\underset{\sim}{q}_i|$ (or $|\underset{\sim}{q}_f|$) alone.

2$^+$) $0 + \frac{1}{2} \xrightarrow{+} 0 + \frac{1}{2}$

$$\langle \underset{\sim}{q}_f , \sigma_f | T | \underset{\sim}{q}_i , \sigma_i \rangle = \sum_{\substack{\ell m \\ \ell' m' \\ JM \\ J'M'}} \langle \hat{q}_f | \ell' m' \rangle \langle \ell' m' ; \tfrac{1}{2} \sigma_f | \ell', \tfrac{1}{2}, J', M' \rangle \langle \ell', \tfrac{1}{2} ; J', M' | T | \ell, \tfrac{1}{2} ; J, M \rangle \langle \ell, \tfrac{1}{2} ; J, M | \ell\ m\ \tfrac{1}{2}\ \sigma_i \rangle \langle \ell, m | \hat{q}_i \rangle$$

where we have used a slightly more explicit notation for C.G. coefficients.

From rotation and parity invariance

$$\langle \ell', \tfrac{1}{2}; J', M' | T | \ell, \tfrac{1}{2}; J, M \rangle = \underbrace{\delta_{\ell\ell'}}_{\text{parity}} \underbrace{\delta_{JJ'}\, \delta_{MM'}}_{\text{angular momentum}} t_{J,\ell} .$$

For each ℓ, J can take up two values, $\ell \pm \frac{1}{2}$; we call $t_{\ell \pm \frac{1}{2}, \ell} = t_\ell^{\pm}$

$$\sum_M |\ell, \tfrac{1}{2}; J, M\rangle \langle \ell, \tfrac{1}{2}; J, M| = \mathcal{P}_{J,\ell} ,$$

is the projection operator on the manifold of states $|\sigma, \ell\rangle$.

<u>Rule</u>

Let A be a hermitean operator with eigenvalues a_i the projector on the subspace $|a_i\rangle$ is

$$\mathcal{P}_i = \prod_{j \neq i} \frac{A - a_j}{a_i - a_j}$$

$$\mathcal{P}_i |a_i\rangle = |a_i\rangle, \quad \mathcal{P}_i |a_j\rangle = 0 , \quad \mathcal{P}_i \mathcal{P}_j = \delta_{ij} \mathcal{P}_i \quad ; \quad \sum \mathcal{P}_i = \mathbb{1} .$$

Take $A = J^2 = (\vec{\ell} + \frac{1}{2} \vec{\sigma})^2$ where $\vec{\ell}$ is the orbital angular momentum operator, and $\frac{1}{2} \vec{\sigma}$, the spin $\frac{1}{2}$ operator.

Then, $a_{\ell + \frac{1}{2}, \ell} = (\ell + \frac{1}{2})(\ell + \frac{3}{2})$, $a_{\ell - \frac{1}{2}, \ell} = (\ell - \frac{1}{2})(\ell + \frac{1}{2})$,

hence

$$\mathcal{P}_{\ell + \frac{1}{2}, \ell} = \frac{\ell + 1 + \vec{\ell}\cdot\vec{\sigma}}{2\ell + 1} \qquad \mathcal{P}_{\ell - \frac{1}{2}, \ell} = \frac{\ell - \vec{\ell}\cdot\vec{\sigma}}{2\ell + 1}$$

and

$$\langle \ell, m', \tfrac{1}{2}, \sigma_f | \mathcal{P}_{\ell+\frac{1}{2},\ell} | \ell, m, \tfrac{1}{2}, \sigma_i \rangle = \langle \sigma_f | \int d\Omega_q \, Y^*_{\ell m'}(\hat{q}) \, \frac{\ell + 1 + \vec{\ell}_q \cdot \vec{\sigma}}{2\ell + 1} \times Y_{\ell m}(\hat{q}) \, | \sigma_i \rangle$$

and a similar formula for $\mathcal{P}_{\ell-\frac{1}{2},\ell}$.

Thus,

$$\langle \hat{q}_f, \sigma_f | T | \hat{q}_i, \sigma_i \rangle = \langle \sigma_f | \sum_\ell t_{\ell_+} \int d\Omega_q \, \frac{2\ell+1}{4\pi} \, P_\ell(\hat{q}_f \cdot \hat{q}) \, \frac{\ell + 1 + \vec{\ell}_q \cdot \vec{\sigma}}{2\ell + 1} \, \frac{2\ell+1}{4\pi} \times P_\ell(\hat{q}_i \cdot \hat{q}) + t_{\ell_-} \int d\Omega_q \, \frac{2\ell+1}{4\pi} \, P_\ell(\hat{q}_f \cdot \hat{q}) \, \frac{\ell - \vec{\ell}_q \cdot \vec{\sigma}}{2\ell + 1} \, \frac{2\ell+1}{4\pi} \, P_\ell(\hat{q}_i \cdot \hat{q}) \, | \sigma_i \rangle .$$

Using the δ function on the unit sphere :

$$\delta(\hat{q}, \hat{q}') = \sum_\ell \frac{2\ell+1}{4\pi} \, P_\ell(\hat{q} \cdot \hat{q}') \quad ,$$

such that $\int \delta(\hat{q}, \hat{q}') \, f(\hat{q}') \, d\Omega_{q'} = f(\hat{q})$,

and remarking that $\frac{2\ell+1}{4\pi} P_\ell(\hat{q}_f \cdot \hat{q})$ can be replaced by $\sum_\ell \frac{2\ell+1}{4\pi} P_\ell(\hat{q}_f \cdot \hat{q})$ because terms which have been added in cancel out through integration over $\hat{q}$ in view of the orthogonality of spherical harmonics, we get :

$$\langle \underset{\sim}{q_f} \, \sigma_f | T | \underset{\sim}{q_i} \, \sigma_i \rangle = \langle \sigma_f | \sum_{\ell=0}^{\infty} \left[t_{\ell_+} \frac{\ell + 1 + \vec{\ell}_{q_f} \cdot \vec{\sigma}}{2\ell + 1} + t_{\ell_-} \frac{\ell - \vec{\ell}_{q_f} \cdot \vec{\sigma}}{2\ell + 1} \right] \times \frac{2\ell+1}{4\pi} \, P_\ell(\hat{q}_f \cdot \hat{q}_i) \, | \sigma_i \rangle \quad .$$

Note that :

$$\vec{\ell}_{q_f} = \frac{1}{i}\ \vec{q}_f \times \vec{\nabla}_{q_f} \quad ; \quad P_\ell(\hat{q}_f.\hat{q}_i) = P_\ell\left(\frac{\hat{q}_f.\hat{q}_i}{|q_f|\ |q_i|}\right) \quad ,$$

since $\vec{\nabla}_{q_f} \dfrac{1}{|q_f|} \propto \vec{q}_f$,

$$\vec{\ell}_{q_f} P_\ell(q_f.q_i) = \frac{1}{i}\ \hat{q}_f \times \hat{q}_i \ \ P'_\ell\ (\hat{q}_f.\hat{q}_i)$$

$2^-)$ $\quad 0 + \frac{1}{2} \xrightarrow{-} 0 + \frac{1}{2}$

Operating as in the previous case, one has to evaluate "projectors" of the type

$$\sum_M |\ell+1\ ,\ \tfrac{1}{2}\ ;\ J = \ell + \tfrac{1}{2}\ ;\ M\rangle\ \langle \ell\ ,\ \tfrac{1}{2}\ ;\ J = \ell + \tfrac{1}{2}\ ,\ M|$$

which can be converted into projectors by the use of (Kramer's trick) :

$$\langle \underset{\sim}{q},\sigma|\ \ell+1\ ,\ \tfrac{1}{2}\ ;\ J = \ell + \tfrac{1}{2}\ ,M\rangle = \langle \underset{\sim}{q},\sigma|\vec{\sigma}.\hat{q}|\ \ell\ ,\ \tfrac{1}{2}\ ;\ J = \ell + \tfrac{1}{2}\ ,M\rangle$$

($\vec{\sigma}.\vec{q}$ being a scalar under rotation leaves J invariant, but, being a pseudoscalar under space reflections, shifts the parity from $(-)^{\ell+1}$ to $(-)^{\ell}$).

The evaluation of e.g

$$\langle\sigma_f|\int d\Omega_q\ \frac{2\ell+3}{4\pi}\ P_{\ell+1}(\hat{q}_f.\hat{q})\ \vec{\sigma}.\vec{q}\ \frac{\ell+1+\vec{\ell}_q.\vec{\sigma}}{2\ell+1}\ \frac{2\ell+1}{4\pi}\ P_\ell(\hat{q}.\hat{q}_i)\ |\sigma_i\rangle$$

proceeds as before by replacing $\dfrac{2\ell+3}{4\pi} P_{\ell+1}(\hat{q}_f.\hat{q})$ by $\delta(\hat{q}_f.\hat{q})$ which is liable since the presence of $\vec{\sigma}.\hat{q}$ and the projection operator insures that the remainder behaves as a $Y_{\ell+1}(\hat{q})$.

One finally gets

$$\langle \vec{q}_f\ \sigma_f|T|\vec{q}_i\ \sigma_i\rangle = \langle\sigma_f|\ \vec{\sigma}.\hat{q}_f \sum_{\ell=0}^{\infty} t_{\ell_+} \frac{\ell+1+\vec{\ell}_{q_f}.\vec{\sigma}}{2\ell+1} + t_{\ell_-} \frac{\ell-\vec{\ell}_{q_f}.\vec{\sigma}}{2\ell+1}$$

$$\times \frac{2\ell+1}{4\pi} P_\ell(\hat{q}_f.\hat{q}_i)\ |\sigma_i\rangle$$

$3^+)$ $\frac{1}{2}+\frac{1}{2} \xrightarrow{+} 0+0$

$$\langle \underset{\sim}{q_f}|T|\underset{\sim}{q_i}\ \sigma_i,\sigma_i'\rangle = \sum_{\ell m,\vec{\epsilon}} \langle\hat{q}_f|\ell\ m\rangle\ \mathcal{M}_\ell\ \langle \ell,m|\hat{q}_i,\hat{\epsilon}\rangle\ \langle\vec{\epsilon}|\sigma_i,\sigma_i'\rangle$$

$$+ \langle q_f|\ell,m\rangle\ \mathcal{S}_\ell\ \langle\ell,m|\hat{q}_i\rangle\ \langle 0|\sigma_i,\sigma_i'\rangle \qquad ;$$

here we have first coupled the two spin $\frac{1}{2}$ into either a triplet state labelled by the polarization vector $\hat{\epsilon}$, or into the singlet state $\langle 0|$:

$$\langle\hat{\epsilon}|\sigma_i\ \sigma_i'\rangle = \langle\sigma_i'|\hat{\epsilon}\ i\,\sigma_2\vec{\sigma}|\sigma_i\rangle \quad , \quad \langle 0|\sigma_i,\sigma_i'\rangle = \langle\sigma_i'|i\sigma_2|\sigma_i\rangle \quad ,$$

and then used vector and scalar spherical harmonics :

$$\langle\ell,m|\hat{q}_i,\hat{\epsilon}\rangle_{\text{magnetic}} = \frac{\vec{\epsilon}.\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}\ Y^*_{\ell m}(\hat{q}_i)$$ parity $-(-)^\ell$ if $\vec{\epsilon}$ is a true vector

$$\langle\ell,m|\vec{q}_i,\hat{\epsilon}\rangle_{\text{electric}} = i\,\frac{\vec{\epsilon}.\vec{q}_i\times\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}\ Y^*_{\ell m}(\hat{q}_i)$$ parity $(-)^\ell$ (the i factor is useful for time reversal arguments)

$$\langle\ell\ m|\vec{q}_i,\vec{\epsilon}\rangle_{\text{longitudinal}} = \hat{\epsilon}.\hat{q}_i\ \ Y^*_{\ell m}(\hat{q}_i)$$ parity $(-)^\ell$

$$\langle \ell\ m|\vec{q}_i,\epsilon_o\rangle_{\text{scalar}} = \epsilon_o\ Y^*_{\ell m}(\hat{q}_i) \quad \text{parity} \ -(-)^\ell \ \text{if}\ \epsilon_o \ \text{is a pseudoscalar.}$$

Thus :

$$\langle \hat{q}_f|T|\sigma_i\ \sigma'_i\ \hat{q}_i\rangle = \langle\sigma'_i|i\sigma_2 \sum_\ell \left(M_\ell \frac{\vec{\sigma}.\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}} + \delta_\ell \right) \frac{2\ell+1}{4\pi} P_\ell(\hat{q}_f.\hat{q}_i)\ |\sigma_i\rangle$$

Similarly :

$3^-)$

$$\langle \hat{q}_f|T|\sigma_i\ \sigma'_i\ \hat{q}_i\rangle = \langle\sigma'_i|i\sigma_2 \sum_\ell \left[i\ \gamma_\ell \frac{\vec{\sigma}.\vec{q}_i \times \vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}} + \mathcal{L}_\ell\ \vec{\sigma}.\hat{q}_i \right] \frac{2\ell+1}{4\pi}$$

$$\times\ P_\ell(\hat{q}_f.\hat{q}_i)\ |\sigma_i\rangle$$

<u>Exercise</u>

$1 + 0 \rightarrow 0 + 0$

(replace $\langle\sigma'_i|i\sigma_2\ \vec{\sigma}\ |\sigma_i\rangle$ by the polarization vectors of the spin 1 particle).

Note: if the spin 1 particle is a real photon, use gauge invariance to eliminate the scalar and longitudinal parts : $\epsilon_o = 0 \quad \hat{\epsilon}.\hat{q}_i = 0$.

$5^+)\quad 1 + \frac{1}{2} \xrightarrow{+} 0 + \frac{1}{2}$

Using all previous ingredients, one obtains :

$$\langle\sigma_f\ \hat{q}_f|T|\sigma_i,\hat{q}_i,\hat{\epsilon}\rangle = \langle\sigma_f| \sum_{\ell}\Big[\mathcal{E}_{\ell_+}\frac{\ell+1+\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ i\ \frac{\vec{\epsilon}\cdot\vec{q}_i\times\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}$$

$$+\ \mathcal{E}_{\ell_-}\frac{\ell-\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ i\ \frac{\vec{\epsilon}\cdot\vec{q}_i\times\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}$$

$$+\ M_{\ell_+}\ \vec{\sigma}\cdot\hat{q}_f\ \frac{\ell+1+\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \frac{\vec{\epsilon}\cdot\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}$$

$$+\ M_{\ell_-}\ \vec{\sigma}\cdot\hat{q}_f\ \frac{\ell-\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \frac{\vec{\epsilon}\cdot\vec{\ell}_{q_i}}{\sqrt{\ell(\ell+1)}}$$

$$+\ \mathcal{L}_{\ell_+}\ \frac{\ell+1+\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \vec{\epsilon}\cdot\vec{q}_i$$

$$+\ \mathcal{L}_{\ell_-}\ \frac{\ell-\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \hat{\epsilon}\cdot\vec{q}_i$$

$$+\ \mathcal{S}_{\ell_+}\ \frac{\ell+1+\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \epsilon_o$$

$$+\ \mathcal{S}_{\ell_-}\ \frac{\ell-\vec{\ell}_{q_f}\cdot\vec{\sigma}}{2\ell+1}\ \epsilon_o\Big]\ \frac{2\ell+1}{4\pi}\ P_{\ell}(\hat{q}_f\cdot\hat{q}_i)\ |\sigma_i\rangle$$

$\{\epsilon_\mu\} = \{\epsilon_o,\vec{\epsilon}\}$ represents the polarization four vector of the spin 1 particle. The transversality condition $\epsilon\cdot k = 0$: $\vec{\epsilon}\cdot\hat{q}_i - \epsilon_o\,\omega_i = 0$ allows one to eliminate the scalar amplitude.

If the spin 1 particle is a real photon, gauge invariance allows one to choose $\epsilon_o = 0$, $\vec{\epsilon}.\hat{q}_i = 0$, which eliminates the longitudinal amplitude. If the spin 1 particle is a virtual photon the scalar part can be eliminated by current conservation (invariance under $\epsilon \to \epsilon + \lambda k$) .

5^-) $1 + \frac{1}{2} \xrightarrow{-} 0 + \frac{1}{2}$

Multiply the whole bracket by $\vec{\sigma}\,\vec{q}_f$

6) $\frac{1}{2} + \frac{1}{2} \xrightarrow{+} \frac{1}{2} + \frac{1}{2}$

a) For each total angular momentum J , the allowed orbital momenta are $J+1$, J , $J-1$. One therefore needs a generalization of Kramer's trick, namely, an operator Q_J such that

$$\langle \sigma_1, \sigma_1', \hat{q} | Q_J | J, M; J+1, 1 \rangle = \langle \sigma_1, \sigma_1', \hat{q} | J, M; J-1, 1 \rangle ,$$

Q_J is the so-called tensor operator. One finds (see e.g. ref [21])

$$Q_J = \frac{2J + 1}{2\sqrt{J(J+1)}} \left(\vec{\sigma}.\hat{q} \ \ \vec{\sigma}'.\hat{q} + \frac{1}{2J+1} \right) ,$$

similarly

$$Q_J^{-1} = \frac{2J + 1}{2\sqrt{J(J+1)}} \left(\vec{\sigma}.\hat{q} \ \ \vec{\sigma}'.\hat{q} - \frac{1}{2J+1} \right) .$$

b) One may couple spins in the initial state and in the final state as in 3) and match parities of various multipoles in the initial and final states.

6^{-}) $\frac{1}{2}+\frac{1}{2} \xrightarrow{-} \frac{1}{2}+\frac{1}{2}$

a) One needs to express now $|J,M;J,1\rangle$ in terms of $|J,M;J \pm 1,1\rangle$ by action of a pseudoscalar operator.

b) Matching parities as simple as in 6^{+})

Exercises

$7^{\pm}$) $1+\frac{1}{2} \xrightarrow{\pm} 1+\frac{1}{2}$.

$8^{\pm}$) $1+1 \rightarrow \frac{1}{2}+\frac{1}{2}$, $1+1 \rightarrow 0+1$ [22]

9) $0+\frac{1}{2} \rightarrow 0+\frac{3}{2} \rightarrow 0+0+\frac{1}{2}$ [23]

Inversion formulae.

We have so far obtained amplitudes in the form

$$\langle f|T|i\rangle = \sum_{J,\alpha,\beta} t^{J}_{\beta,\alpha} \langle f|\mathcal{P}^{J}_{\beta,\alpha}|i\rangle$$

where J is the total angular momentum , β,α, degeneracy parameters labelling final and initial channels respectively, and the $\mathcal{P}^{J}_{\beta,\alpha}$ s', operators enjoying the orthogonality properties :

$$\mathcal{P}^{J}_{\beta\alpha} \quad \mathcal{P}^{J'\dagger}_{\beta'\alpha'} \propto \delta_{J,J'} \quad \delta_{\alpha,\alpha'} \quad \mathcal{P}^{J}_{\beta\beta'}$$

$$\mathcal{P}^{J\dagger}_{\beta\alpha} \quad \mathcal{P}^{J'}_{\beta'\alpha'} \propto \delta_{J,J'} \quad \delta_{\beta,\beta'} \quad \mathcal{P}^{J}_{\alpha,\alpha'}$$

$$\mathrm{Tr} \ \mathcal{P}^{J}_{\alpha\alpha'} \propto \delta_{\alpha\alpha'} \ .$$

Hence, the inversion formula, which may be of some use in studies starting from dispersion relations :

$$t^J_{\beta,\alpha} = \frac{\int df\ di\ \langle f|T|i\rangle \langle i|\mathcal{P}^{J\,\dagger}_{\beta\ \alpha}|f\rangle}{\int df\ di\ \langle f|\mathcal{P}^J_{\beta,\alpha}|i\rangle \langle i|\mathcal{P}^{J\,\dagger}_{\beta\ \alpha}|f\rangle}$$

Example.

photon + 0 $\rightarrow$ 0 + 0

$$\langle \hat{q}|T|\ \hat{\epsilon},\hat{k}\rangle = (\hat{\epsilon}.\hat{q}\times\hat{k})\ f\ (\hat{q}.\hat{k}, |\vec{k}|)$$

$$= -\sum_\ell t_\ell \frac{2\ell+1}{4\pi} \frac{\hat{\epsilon}.\vec{\ell}_q}{\sqrt{\ell(\ell+1)}} P_\ell(\hat{q}.\hat{k})$$

$$= \sum_\ell t_\ell \langle \hat{q}|\mathcal{P}_\ell|\hat{\epsilon}.\hat{k}\rangle ,$$

$$t_\ell = \frac{\int d\Omega_{\hat{q}}\ d\Omega_{\hat{k}} \sum_{\hat{\epsilon}\perp\hat{k}} \langle \hat{\epsilon},\hat{k}|\mathcal{P}_\ell|\ \hat{q}\rangle\ \hat{\epsilon}.\hat{q}\times\hat{k}\ f(\hat{q}.\hat{k},\ |\vec{k}|)}{\int d\Omega_{\hat{q}}\ d\Omega_{\hat{k}} \sum_{\hat{\epsilon}\perp\hat{k}} \langle \hat{\epsilon},\hat{k}|\mathcal{P}_\ell|\ \hat{q}\rangle \langle \hat{q}|\mathcal{P}_\ell|\hat{\epsilon};\hat{k}\rangle}$$

$$= \frac{\int d\Omega_{\hat{q}}\ d\Omega_{\hat{k}}\ \frac{2\ell+1}{4\pi}\ (-\frac{1}{i})\ \frac{\hat{q}\times\hat{k}}{\sqrt{\ell(\ell+1)}}\ P'_\ell(\hat{q}.\hat{k}).\hat{q}\times\hat{k}\ \ f(\hat{q}.\hat{k},\ |\vec{k}|)}{\int d\Omega_{\hat{q}}\ d\Omega_{\hat{k}}\ \frac{2\ell+1}{4\pi}\ P_\ell(\hat{q}.\hat{k})\ \frac{2\ell+1}{4\pi}\ P_\ell(\hat{q}.\hat{k})}$$

$$= \frac{2\pi i}{\sqrt{\ell(\ell+1)}} \int_{-1}^{+1} d(\hat{q}.\hat{k}) \left[1 - (\hat{q}.\hat{k})^2\right] P'_\ell(\hat{q}.\hat{k})\ \ f(\hat{q}.\hat{k},\ |\vec{k}|) ,$$

where we have used $\vec{\ell}^2 = \ell(\ell+1)$.

APPENDIX III

REDUCTION FORMULAE

In this appendix we lay down the L S Z [24] formalism for a quantum field theory for particles with spin and derive familiar looking expressions for matrix elements of the S matrix which will justify the introduction of spinor amplitudes and of their properties stated in Ch.II § e .

We will throughout assume that all particles to be described are massive(*).

Then the L S Z postulates go as follows :

1) The theory is described in a Hilbert space $\mathcal{H}$ in which a unitary, continuous, up to a phase representation of the Poincaré group acts.

There are in $\mathcal{H}$ two Fock basis the so-called $|in\rangle$ and $|out\rangle$ basis constructed from the same vacuum $|0\rangle$ by action of two sets of creation operators labelled by the same quantum numbers (masses, spins, momenta, spin projections, internal quantum numbers) $a^{\dagger}_{\substack{in\\out}}([p],\sigma,\mu)$, which describe observed particles.

The vacuum state is the only Lorentz invariant state.

(*) This restriction has to do with the rather poor understanding one has at present of the meaning of the asymptotic condition stated down below (A.III.13), in the case where there exist massless particles which mediate long range interactions.

The in and out basis are obtained from each other by means of a unitary operator S :

$$a^{\dagger}_{out}([p],\sigma,\mu) \quad = \quad S^{\dagger} \; a^{\dagger}_{in}([p],\sigma,\mu) \; S$$

hence $$a_{out}([p],\sigma,\mu) \quad = \quad S^{\dagger} \; a_{in}([p],\sigma,\mu) \; S \qquad \text{(A.III.1)}$$

and $$\varphi^{out}_{A,\mu}(x) \quad = \quad S^{\dagger} \; \varphi^{in}_{A,\mu}(x) \; S$$

where $\varphi_{A,\mu}(x)$ is any of the local fields constructed from the creation and annihilation operators as in Chapter I.§.f.iii).

The vacuum and one particle states are stable :

$$\langle o_{in}| = \langle o_{\;out}| = \langle o_{in}| \; S$$

$$\text{(A.III.2)}$$

$$\langle [p]\sigma,\mu,in| \;=\; \langle [p]\sigma,\mu,out| \;=\; \langle [p]\sigma,\mu,in| \; S$$
$$= \langle [p]\sigma,\mu,in| \; S^{\dagger} \,.$$

2) The asymptotic local fields $\varphi^{in}_{out\;A,\mu}(x)$ can be interpolated (*) by a local field $\varphi_{A,\mu}(x)$ in a sense to be shortly made precise (A.III.13)

(*) The question whether one interpolating field has to be associated with each kind of observed particle or whether only a set of interpolating fields is necessary, polynomials in which create asymptotically "composite" particles will not be discussed here. We shall stick to the simplifying assumption made in the text.

and with the following properties :

$$[\varphi_{A,\mu}(x) \quad , \quad \varphi_{A'\mu'}(x')]_{\pm} \quad = \quad 0 \qquad (x-x')^2 < 0 \quad \text{(A.III.3)}$$

where the commutator or the anticommutator is taken according as the spin indices A are integer or half integer.

The interpolating fields have the same transformation law as the asymptotic fields, under Poincaré transformation.

This latter property requires a comment. As long as space reflections are not involved, their is no difficulty. But, recall the transformation law under space reflections:

$$U(\Pi) \ \varphi_A^{\substack{in\\out}}(x) \ U(\Pi)^{-1} = \eta_\Pi^{*} \ \tilde{\varphi}_A^{\substack{in\\out}}(\underline{x})$$

$$= \eta_\Pi^{*} \ \mathcal{D}_{AA'}\left(\frac{\partial o - \vec{\partial}\vec{\sigma}}{-im}\right) \ \varphi_{A'}^{\substack{in\\out}}(\underline{x}) \ .$$

This transformation law expressed in terms of φ alone cannot be retained for the interpolating field if the latter does not fulfill the Klein Gordon equation, which is explicitely not assumed, (as, we shall see this would yield a trivial S matrix).

The set of transformations

$$U(\Pi) \ \varphi_A^{\substack{in\\out}}(x) \ U(\Pi)^{-1} = \eta_\Pi^{*} \ \tilde{\varphi}_A^{\substack{in\\out}}(\underline{x})$$

$$U(\Pi) \ \tilde{\varphi}_A^{\substack{in\\out}}(x) \ U(\Pi)^{-1} = \eta_\Pi^{*} \ \varphi_A^{\substack{in\\out}}(\underline{x})$$

is however perfectly admissible when expressed in terms of the Dirac field

$$\Psi_{\alpha,\mu}^{\substack{in\\out}}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_A^{\substack{in\\out}}(x) \\ \tilde{\varphi}_A^{\substack{in\\out}}(x) \end{pmatrix} \qquad \text{(A.III.4)}$$

for which one may therefore assume the existence of an interpolating fields which transforms locally under space reflection :

$$U(\Pi)\ \Psi_{\alpha,\mu}(x)\ U(\Pi)^{-1} = \eta_{\Pi}^{*}\ \gamma_o\ \Psi_{\alpha,\mu}(\underline{x}) \qquad \text{(A.III.5)}$$

where $\gamma_o = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$.

We recall the expressions of various asymptotic fields :

$$\varphi^{as}_{A,\mu}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}_{A\sigma}([p])\ a^{as}([p]\sigma,\mu)\ e^{-ipx}$$

$$+ \mathcal{D}_{A\sigma}([p]\ C^{-1})\ b^{as\dagger}([p]\sigma,\mu)\ e^{ipx} \Big]$$

$$\text{(A.III.6)}$$

$$\tilde{\varphi}^{as}_{\dot{A},\mu}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p} \Big[\mathcal{D}_{\dot{A}\sigma}([p]^{\dagger-1})\ a^{as}([p]\sigma,\mu)\ e^{-ipx}$$

$$+ \mathcal{D}_{\dot{A}\sigma}([p]^{\dagger-1}\ C)\ b^{as\dagger}([p]\sigma,\mu)\ e^{ipx} \Big] .$$

The charge conjugate fields are obtained by the interchange of a's and b's .

We set :

$$\Psi^{as}_{\alpha,\mu}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\omega_p}\ U_{\alpha,\sigma}([p])\ a^{as}([p]\sigma,\mu)\ e^{-ipx}$$

$$+ V_{\alpha,\sigma}([p])\ b^{\dagger}([p],\sigma,\mu)\ e^{ipx} \qquad \text{(A.III.7)}$$

with $\alpha = \begin{pmatrix} A \\ \dot{A} \end{pmatrix}$

$$U_{\alpha\sigma}([p]) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D}_{A\sigma}([p]) \\ \mathcal{D}_{\dot{A}\sigma}([p]^{\dagger-1}) \end{pmatrix} \qquad V_{\alpha,\sigma}([p]) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D}_{A\sigma}([p]\, C^{-1}) \\ \mathcal{D}_{A\sigma}([p]^{\dagger-1} C) \end{pmatrix} \qquad \text{(A.III.8)}$$

with the transformation Law

$$U(a,A) \quad \Psi_{\alpha,\mu}(x) \quad U^{-1}(a,A) = S_{\alpha\alpha'}(A^{-1}) \quad \Psi_{\alpha',\mu}(A.x + a) \qquad \text{(A.III.9)}$$

with

$$S(A) = \begin{pmatrix} \mathcal{D}(A) & \\ & \mathcal{D}(A^{\dagger-1}) \end{pmatrix} . \qquad \text{(A.III.10)}$$

We note $(U^{\dagger})_{\sigma\alpha}([p])\ \gamma^{o}_{\alpha\beta}\ U_{\beta\sigma'}([p]) = \delta_{\sigma\sigma'}$

$$(V^{\dagger})_{\sigma\alpha}([p])\ \gamma^{o}_{\alpha\beta}\ V_{\beta\sigma'}([p]) = (-)^{2s}\ \delta_{\sigma\sigma'}$$

and thus define $\bar{U} = U^{\dagger}\gamma^{o}$, $\bar{V} = V^{\dagger}\gamma^{o}$.

We can thus solve for the asymptotic creation and annihilation operators.

$$a^{as}([p],\sigma,\mu) = \bar{U}_{\alpha}([p],\sigma) \frac{i}{(2\pi)^{3/2}} \int d^3x \quad e^{ipx} \overleftrightarrow{\partial}_{o} \quad \Psi^{as}_{\alpha,\mu}(x)$$

$$a^{\dagger as}([p],\sigma,\mu) = - \frac{i}{(2\pi)^{3/2}} \int d^3x \quad e^{-ip.x} \overleftrightarrow{\partial}_{o} \quad \Psi^{C\,as}_{\alpha,\mu}(x) \quad U^{C}_{\alpha}([p],\sigma)$$

where

$$U^{C}([p],\sigma) = \mathcal{C}\ V([p],\sigma) \qquad \mathcal{C} = \begin{pmatrix} \mathcal{D}(C^{-1}) & \\ & \mathcal{D}(C) \end{pmatrix} . \qquad \text{(A.III.11)}$$

$$b^{as}([p],\sigma,\mu) = \bar{U}_{\alpha}([p],\sigma) \frac{i}{(2\pi)^{3/2}} \int d^3x \quad e^{ipx} \overleftrightarrow{\partial}_{o} \quad \Psi^{C\,as}_{\alpha,\mu}(x)$$

$$b^{\dagger as}([p],\sigma,\mu) = - \frac{i}{(2\pi)^{3/2}} \int d^3x \quad e^{-ipx} \overleftrightarrow{\partial}_{o} \quad \Psi^{as}_{\alpha,\mu}(x) \quad U^{C}_{\alpha}([p],\sigma)$$

where $f \overleftrightarrow{\partial}_o g = f \frac{\partial}{\partial x^o} g - (\frac{\partial}{\partial x^o} f) g$.

The integration over $\vec{x}$ is so arranged that, from each field it only keeps the positive or negative frequency part according to the presence of the plane wave function $e^{\mp ipx}$, and that the integral is independent of x^o.

One may define for the interpolating fields similar expressions e.g. :

$$a(\vec{f},\mu,t) = \int \frac{d^3p}{2\omega p} \; f^*_\sigma([p]) \; \overline{U}_\alpha([p],\sigma)$$

$$\times \frac{i}{(2\pi)^{3/2}} \int_{x^o=t} d^3x \; e^{ipx} \overleftrightarrow{\partial}_o \; \Psi_{\alpha,\mu}(x) \qquad \text{(A.III.12)}$$

which now depend on t and are associated with one particle asymptotic states labelled by the wave function $\vec{f}$ and internal label μ.

The asymptotic condition now states

$$\text{W-lim} \;\; a(\vec{f}, \mu, t) \Big|_{t \to \pm \infty} = a^{\substack{out \\ in}} (\vec{f}, \mu), \qquad \text{(A.III.13)}$$

(and similar conditions on b, $a^\dagger$, $b^\dagger$), where the W-(eak) limit sign means that the equation holds true for all matrix elements of both sides between sufficiently many normalizable states in $\mathcal{H}$.

Remark.

If we had not had to worry about space reflections, we would have constructed a similar formalism in terms of fields φ and φ^c and the first set of components of the U and V spinors.

- Reduction formulae.

We are now ready to express a matrix element

$$S_{FI} = \langle 0| \prod_{\bar{f}} b^{out}([p_{\bar{f}}], \sigma_{\bar{f}}, \mu_{\bar{f}}) \prod_{f} a^{out}([p_f], \sigma_f, \mu_f) \times$$

$$\prod_{i} a^{\dagger in}([p_i], \sigma_i, \mu_i) \prod_{\bar{i}} b^{\dagger in}([p_{\bar{i}}], \sigma_{\bar{i}}, \mu_{\bar{i}}) |0\rangle$$

$$= \langle \bar{f}, f |S| i, \bar{i} \rangle \qquad \text{(A.III.14)}$$

where bars distinguish antiparticles.

From (A.III.11, 12, 13,) S_{FI} can be expressed as a certain limit of the expectation value of a product of interpolating fields.

$$W_{FI} = \langle 0| \prod_{\bar{f}} \Psi^{C}_{\alpha_{\bar{f}}, \mu_{\bar{f}}}(x_{\bar{f}}) \prod_{f} \Psi_{\alpha_f \mu_f}(x_f) \prod_{i} \Psi^{C}_{\alpha_i \mu_i}(x_i) \prod_{\bar{i}} \Psi_{\alpha_{\bar{i}} \mu_{\bar{i}}}(x_{\bar{i}}) |0\rangle \qquad \text{(A.III.15)}$$

(provided some wave packets $\vec{f}$ are added in, which we shall not bother to do).

Given the set of indices (FI), it is convenient to introduce "truncated" products W^{T}_{λ} according to the recursive definition

$$W_{FI} = \sum (-)^{\sigma(\lambda)} W^{T}_{\lambda_1} \dots W^{T}_{\lambda_p} \qquad \text{(A.III.16)}$$

where the sum is taken over all partitions of the set of indices (FI) into subsets $\lambda_1 \dots \lambda_p$ $(\lambda \cup \dots \cup \lambda_p = (FI))$; inside each subset points have to appear in the natural order where they appear in (FI) ; $\sigma(\lambda)$ is the signature of the permutation of fermion operators appearing in $\lambda_1 \dots \lambda_p$ with respect to that appearing in FI .

Accordingly, connected S-matrix elements are defined by :

$$S_{FI} = \sum (-)^{\sigma(\lambda)} S^C_{\lambda_1} \dots S^C_{\lambda_p} . \qquad \text{(A.III.17)}$$

One then uses repeatedly the formula of integration by parts :

$$\int \frac{d^3p}{2\omega_p} \tilde{f}(\vec{p}) \int_{x^o=+\infty} d^3x \; e^{ipx} \overleftrightarrow{\partial}_o \varphi(x) - \int_{x^o=-\infty} d^3x \; e^{ipx} \overleftrightarrow{\partial}_o \varphi(x)$$

$$= \int d^4x \quad f(x) \quad (\Box_x + m^2) \quad \varphi(x) \qquad \text{(A.III.18)},$$

where $f(x) = \int \frac{d^3p}{2\omega_p} \tilde{f}(\vec{p}) \; e^{ipx}$, which holds true if $f(\vec{p})$ decreases fast at infinity in $\vec{p}$, together with the following identity :

Let
$$T(A_1(x_1) \dots A_n(x_n)) = \sum_{P(1\dots n)} (-)^{\sigma(P)} \theta_+(x^o_{P(1)} - x^o_{P(2)}) \dots \theta_+(x^o_{P(n-1)} - x^o_{P(n)})$$

$$A_{P(1)}(x_{P(1)}) \dots A_{P(n)}(x_{P(n)}) , \qquad \text{(A.III.19)}$$

be the chronological product of n local field operators.

$\Big[\theta_+(t) = \begin{matrix} 1 & t>0 \\ 0 & t<0 \end{matrix}$; P = permutation on $(1 \dots n)$

$P(i)$ = transformed of i by $P \Big]$

$\sigma(P)$ = signature of fermion fields permutation. Then one writes .e.g.

$$\int_{x^o=+\infty} f(x) \overleftrightarrow{\partial}_o \varphi(x) \quad T(A_1(x_1)\dots A_n(x_n))$$

$$= \int_{x^o=+\infty} f(x) \overleftrightarrow{\partial}_o \; T(\varphi(x), A_1(x_1) \dots A_n(x_n)) \qquad \text{(A.III.20)}$$

since, in view of the θ_+ functions only terms occuring in the left hand side survive.

Hence

$$\left(\int_{x^o = +\infty} - \int_{x^o = -\infty} \right) d^3x\ f(x) \overleftrightarrow{\partial}_o\ \varphi(x)\ \ T(A_1(x_1)\ \ldots\ A_n(x_n)) \tag{A.III.21}$$

$$= \int d^4x\ \ f(x)\ (\Box_x + m^2)\ \ \ T(\varphi(x)\ \ A_1(x_1)\ \ldots\ A_n(x_n))\ .$$

These elementary steps, together with a recursion argument which uses definition (A.III.17), lead to the result.

$$S^C_{FI} = \prod_{\substack{f\\ \bar{f}}} \frac{i}{(2\pi)^{3/2}}\ \bar{U}_{\alpha_{\substack{f\\ \bar{f}}}}\ ([p]_{\substack{f\\ \bar{f}}}\ \sigma_{\substack{f\\ \bar{f}}}\ \mu_{\substack{f\\ \bar{f}}})\ \mathcal{M}_{\substack{\alpha_f \alpha_{\bar{f}} \alpha_i \alpha_{\bar{i}}\\ \mu_f \mu_{\bar{f}} \mu_i \mu_{\bar{i}}}}\ (p_f,\ p_{\bar{f}},\ p_i,\ p_{\bar{i}})$$

$$\prod_{\substack{i\\ \bar{i}}} \frac{-i}{(2\pi)^{3/2}}\ \ U^C_{\substack{\alpha_i\\ \alpha_{\bar{i}}}}\ \ ([p]_{\substack{i\\ \bar{i}}}\ \ \sigma_{\substack{i\\ \bar{i}}}\ \ \mu_{\substack{i\\ \bar{i}}}) \tag{A.III.22}$$

where

$$\mathcal{M}_{\substack{\alpha_f\ \alpha_{\bar{f}}\ \ \alpha_i\ \ \alpha_{\bar{i}}\\ \mu_f\ \mu_{\bar{f}}\ \ \mu_i\ \ \mu_{\bar{i}}}}\ (p_f\ p_{\bar{f}}\ p_i\ \ p_{\bar{i}}) = \int \prod_{i\bar{i}f\bar{f}} e^{ip_jx_j}\ (\Box_j + m_j^2)\ dx_j \tag{A.III.23}$$

$$\langle o | T \prod_{\bar{f}} (\Psi^C_{\substack{\alpha_{\bar{f}}\\ \mu_{\bar{f}}}}\ (x_{\bar{f}})\ \prod_f\ \Psi_{\substack{\alpha_f\\ \mu_f}}\ (x_f)\ \prod_i\ \Psi_{\substack{\alpha_i\\ \mu_i}}\ (x_i)\ \prod_{\bar{i}}\ \Psi_{\substack{\alpha_{\bar{i}}\\ \mu_{\bar{i}}}}\ (x_{\bar{i}})\ | o \rangle^T$$

and the truncated chronological product vacuum expectation value is defined from truncated vacuum expectation values of products in the same way as the untruncated chronological product was defined from untruncated vacuum expectation values of products.

Remark.

There is one touchy point here, namely the multiplication of field operators which are in general distributions, by θ functions, which are discontinuous ; this defect which may lead to an unproper handling of the high energy behaviour of the theory has been partly got rid of in a recent past, as well as the possible lack of covariance of chronological products.

The definition of the M amplitudes is the starting point of dispersion theory, in this framework, since the locality condition (A.III.3) allows to show [26] that M can be obtained as the boundary value of a function of the p_j's holomorphic in a domain, all of which has not been so far determined.

CONCLUSION

We hope to have conveyed the impression that a detailed analysis of elementary systems according to quantum numbers provided by relativistic invariance (momentum, spin) is worthwhile in so far as it makes perfectly precise the understanding of many particle states and observables structure. If, at times, computations are lengthy, it is hoped that they do not obscure the general idea. The main reason why we have indulged into so much algebra is precisely to show that they are in principle not so dreadful, although numerical work can become fairly abundant.

From a more lofty point of view, it also is apparent that if some day Lorentz invariance is lost at the benefit of another invariance, the frame is all set to deal with the new law, provided representation theory is advanced enough for the new group.

We have to apologize for the arbitrary choice of topics, redundancy as well as omission of some items, and just hope enough of the basic techniques have been exhibited to allow further applications.

ACKNOWLEDGEMENTS

We wish to thank here all contributors to the ideas expressed in these notes. Let our great master Lobatchewsky (*) find here the expression of our gratitude for his useful advice.

(*) cf. Tom Leher, well known song.

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where the so-called Klein transformations, which allow commutation or anticommutation relations to be normal, are constructed.

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This list of references is in no way exhaustive. We have just quoted those references which we thought were relevant to the arguments presented here. In most of them , as well as in the general references, the reader will find an ample supply of references to the original articles dealing with the various subjects under consideration.

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