Comparison of analytical solutions of the coupled DGLAP equations for F_2^P at small *x*

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Coupled DGLAP equations involving singlet quark and gluon distributions are explored by a Taylor expansion at small *x* as two first order partial differential equations in two variables: Bjorken *x* and *t* ($t = ln\frac{Q^2}{\Lambda^2}$). The system of equations are then solved by the Lagrange's method and Method of Characteristics. We obtain the proton structure function F_2^P by combining the corresponding non-singlet and singlet structure functions by both the methods. Analytical solutions for F_2^P thus obtained are compared with the data available and their compatibility is checked. Data favours the analytical solution by Lagrange's method.

1. Introduction

Solutions of DGLAP [1, 2, 3, 4] evolution equations give quark and gluon structure functions which produce ultimately proton, neutron and deuteron structure functions. The standard program to study the x dependence of quark and gluon PDFs is to compare the numerical solutions of the DGLAP equations with the data and so to fit the parameters of the x profiles of the PDFs at some initial factorization scale Q_0^2 and the asymptotic scale parameter Λ. However, for analyzing exclusively the small-x region, there exists alternative simpler analysis, some of which are the existing analytical solutions of the DGLAP equations in the small-x limit with considerable phenomenological success [5, 6, 7, 8, 9]. In this work, we make an extensive comparative study on the applicability of two analytical methods: Lagrange's method [10] and method of characteristics [11, 12, 13] in context of the unpolarized proton structure function F_2^P. This suggests utility of such approaches in understanding the dynamics of evolution of quarks and gluons at small *x*.

In Section 2 we describe the formalism, Section 3 is devoted to testing our prediction's comparison with the data, while in Section 4, we give our conclusion.

2. Formalism

2.1 Singlet coupled DGLAP equations in Taylor approximated form

In order to get the F_2^P we need both singlet and non-singlet structure functions. In our earlier work [5] we reported our analytical solutions by the two analytical methods . Hence we discuss only the solution for the quark singlet part here. The coupled DGLAP equations for quark singlet ($\Sigma(x,Q^2)$) and gluon ($G(x,Q^2)$) densities are [1, 2, 3, 4],

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix} \Sigma\left(x,Q^2\right) \\ G\left(x,Q^2\right)\end{pmatrix} = \frac{\alpha_s\left(Q^2\right)}{2\pi}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg}\end{pmatrix} \otimes \begin{pmatrix} \Sigma\left(x,Q^2\right) \\ G\left(x,Q^2\right)\end{pmatrix}. \tag{1}$$

Introducing the variable $t = \ln\frac{Q^2}{\Lambda^2}$ and using the explicit forms of the splitting functions $P_{i,j}(i,j = q,g)$, the evolution equation for singlet distribution in LO can be written as

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}\left[\{3+4\ln(1-x)\}F_2^S(x,t) + 2\int_x^1 \frac{dz}{(1-z)}\left\{(1+z^2)F_2^S\left(\frac{x}{z},t\right)\right.\right.$$
$$\left.\left. -2F_2^S(x,t)\right\} + \tfrac{3}{2}n_f\int_x^1 dz\left(z^2+(1-z)^2\right)G\left(\tfrac{x}{z},t\right)\right] = 0. \tag{2}$$

Here $A_f = \frac{4}{3\beta_0}$, $\beta_0 = 11 - \frac{2}{3}n_f$ and $\alpha_s(t) = \frac{4\pi}{\beta_0 t}$. $F_2^S(x,t)$ is the singlet structure functions of the proton. Introducing a variable u defined as $u = 1 - z$ and doing Taylor approximation we can express equation (2) in a more precise form as

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}\left[3+4\ln(1-x)F_2^S(x,t)+(2x-3)F_2^S(x,t)+\left(x+2x\ln\frac{1}{x}\right)\frac{\partial F_2^S(x,t)}{\partial x}\right]$$
$$-\frac{A_f}{t}\left[n_f\left(1-\frac{3}{2}x\right)G(x,t)-\frac{n_f}{2}\left(5x-3x\ln\frac{1}{x}\right)\frac{\partial G(x,t)}{\partial x}\right]=0. \quad (3)$$

The exact relation between the gluon distribution function $G(x,t) = xg(x,t)$ and quark distribution function $F_2^S(x,t) = x\sum_i e_i^2 \{q_i(x,t) + \bar{q}_i(x,t)\}$ is not derivable in QCD even in LO. However, simple forms of such relation are available in literature to facilitate the analytical solution of coupled DGLAP equations [14, 15]. A more rigorous analysis was done by Lopez and Yundurain [16] and they investigated the behaviour of the singlet $F_2^S(x,Q^2)$ and gluon $G(x,Q^2)$ as $x \to 0$. They observed that,

$$F_2^S(x,Q^2)_{x\to 0} = B_s(Q^2)x^{-\lambda_s}, \quad (4)$$

$$G(x,Q^2)_{x\to 0} = B_G(Q^2)x^{-\lambda_s}, \quad (5)$$

where B_s and B_G are Q^2 dependent and λ_s is strictly positive. Thus,

$$\frac{G(x,Q^2)}{F(x,Q^2)}_{x\to 0} \simeq f(Q^2). \quad (6)$$

It suggests a more general form [17],

$$G(x,Q^2) = k(Q^2)F_2^S(x,Q^2). \quad (7)$$

Using above relation given by equation (7), we express equation (3) as

$$\frac{\partial F_2^S(x,t)}{\partial t} - \frac{A_f}{t}\left[\{3+4\ln(1-x)+(2x-3)\}F_2^S(x,t)+n_f\left(1-\frac{3}{2}x\right)K(Q^2)F_2^S(x,t)\right]$$
$$-\frac{A_f}{t}\left[x+2x\ln\frac{1}{x}-\frac{n_f}{2}\left(5x-3x\ln\frac{1}{x}\right)K(Q^2)\right]\frac{\partial F_2^S(x,t)}{\partial x}=0. \quad (8)$$

Which is a partial differential equation for the singlet structure function $F_2^S(x,t)$ with respect to the variables x and t. We solve this PDE equation (8) with the two formalisms described here, the Lagrange's method and Method of Characteristics. In order to do that we express equation (8) as

$$t\frac{\partial F_2^S(x,t)}{\partial t} = \omega_1\frac{\partial F_2^S(x,t)}{\partial x} + \omega_2 F_2^S, \quad (9)$$

where

$$\omega_1 = \frac{4}{3\beta_0}\{x+2x\ln\frac{1}{x}-\frac{n_f}{2}\left(5x-3x\ln\frac{1}{x}\right)K(Q^2), \quad (10)$$

$$\omega_2 = \frac{4}{3\beta_0}\{3+4\ln(1-x)+(2x-3)+n_f\left(1-\frac{3}{2}x\right)K(Q^2). \quad (11)$$

2.2 Solution by the Lagrange's Auxiliary Method

To solve the equation (9) by the Lagrange's Auxiliary method [10], we write the equation in the form,

$$Q(x,t)\frac{\partial F_2^S(x,t)}{\partial t} + P(x,t)\frac{\partial F_2^S(x,t)}{\partial x} = R(x,t,F_2^S(x,t)), \quad (12)$$

where

$$Q(x,t) = t, \quad (13)$$

$$P(x,t) = -\omega_1, \quad (14)$$

and

$$R(x,t,F_2^S(x,t)) = R'(x)F_2^S(x,t) \tag{15}$$

with

$$R'(x) = \omega_2. \tag{16}$$

If $u(x,t,F_2^S) = C_1$ and $v(x,t,F_2^S) = C_2$ are the two independent solutions of equation (12), then in general, the solution of equation (12) is

$$F(u,v) = 0, \tag{17}$$

where F is an arbitrary function of u and v. Using the physically plausible boundary conditions and solving the auxiliary system for u and v, we obtain the solution for equation (12) as

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right)\frac{[X^S(x) - X^S(1)]}{[X^S(x) - (\frac{t}{t_0})X^S(1)]} \tag{18}$$

with the explicit analytical form of $X^S(x)$ in the leading $(\frac{1}{x})$ approximation are,

$$X^S(x) = \exp\left[-\frac{6\beta_0}{4(4+3n_f K(Q^2)}\log[\log x]\right], \tag{19}$$

leading to

$$X^S(1) = 0 \tag{20}$$

which yields,

$$F_2^S(x,t) = F_2^S(x,t_0)\left(\frac{t}{t_0}\right). \tag{21}$$

Equation (21) gives the t evolutions of singlet structure function at LO.

2.3 Solution by the method of characteristics

To solve the PDE equation (9) by the method of characteristics, we express it in terms of a new set of coordinates (s,τ), such that equation (9) becomes an ODE w.r.t. one of the new variables. The characteristic equations of equation (9) are given by,

$$\frac{dt}{ds} = t, \tag{22}$$

$$\frac{dx}{ds} = -\omega_1. \tag{23}$$

The left hand side of equation (9) can be expressed as an ordinary derivative with respect to t and the equation becomes an ordinary differential equation:

$$\frac{dF_2^S(s,\tau)}{dt} + c^S(s,\tau)\,F_2^S(s,\tau) = 0, \quad \text{where} \quad c^S(s,\tau) = \omega_2. \tag{24}$$

Integrating equation (24) over t from t_0 to t along the characteristic curve, the solution of the equation for characteristic equations leads us to the solution for $F_2^S(x,t)$ as

$$F_2^S(x,t) = F_2^S(x,t_0)(\frac{t}{t_0})^{n(x,t)} \tag{25}$$

where,

$$
\begin{aligned}
n(x,t) &= \frac{1}{\log(\frac{t}{t_0})}\log\left(\frac{F_2^S(\tau)}{F_2^S(x,t_0)}\right)+\frac{(-\frac{4}{3\beta_0}\xi_1)}{\log\frac{t}{t_0}}, & (26)\\
\xi_1 &= 4\log\left(1-\tau\exp\left[-(\frac{t}{t_0})^{\frac{1}{\alpha_1}}\right]\right)+2\tau\exp\left[-(\frac{t}{t_0})^{\frac{1}{\alpha_1}}\right]\\
&\quad +n_f\left(1-\frac{3}{2}\tau\exp\left[-(\frac{t}{t_0})^{\frac{1}{\alpha_1}}\right]\right)K(Q^2), & (27)\\
\tau &= x\exp\left[(\frac{t}{t_0})^{\frac{1}{\alpha_1}}\right], & (28)\\
\alpha_1 &= \frac{3\beta_0}{4}\{2+K(Q^2)\frac{9}{2}\}. & (29)
\end{aligned}
$$

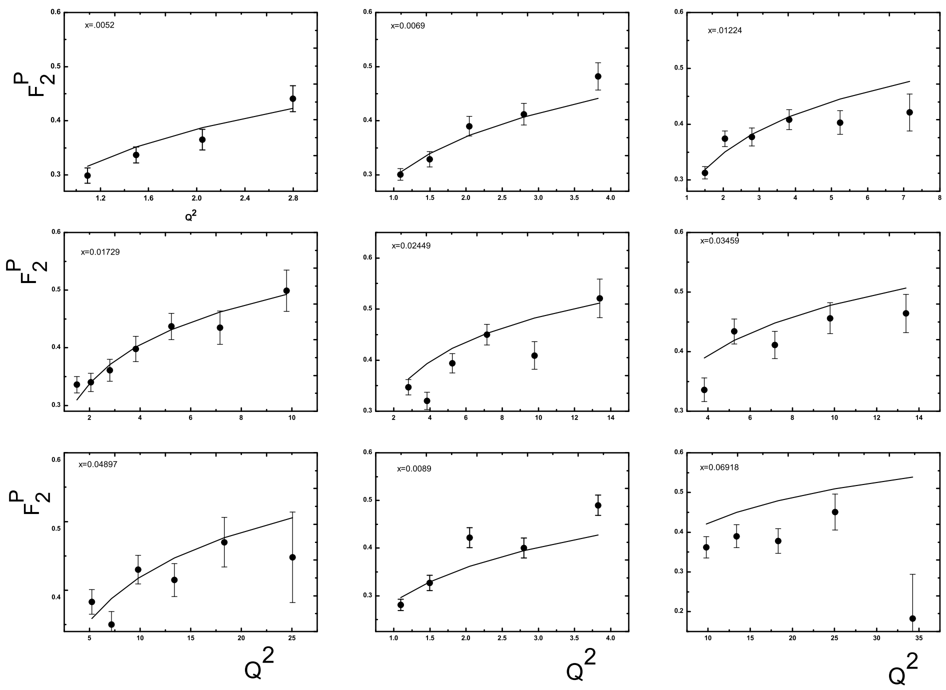

Figure 1. Proton structure function $F_2^P(x,t)$ as function of Q^2 at different x values using Lagrange's method. Data are taken from E665 [19].

2.4 The function $K(Q^2)$

Traditional determination of quark and gluon distribution function includes simultaneous fitting of experimental data (mainly at small x) of the proton structure function $F_2^P(x,Q^2)$ measured in deep inelastic scattering, with a large number of values of x and Q^2. The most appropriate QCD inspired functional form for the function $K(Q^2)$ has to be of the logarithmic form and we consider it to be,

$$K(Q^2)=(\log\frac{Q^2}{\Lambda^2})^{\sigma}, \tag{30}$$

where σ is a parameter to be determined. To determine the 'best-fit' value for σ, we consider the input PDFs at entire x region. Our analysis yields that the best-fit value of σ lies in between 2 and 3 for $Q^2 = 2$ GeV^2. For our further calculations we take the average and fix $\sigma = 2.5$.

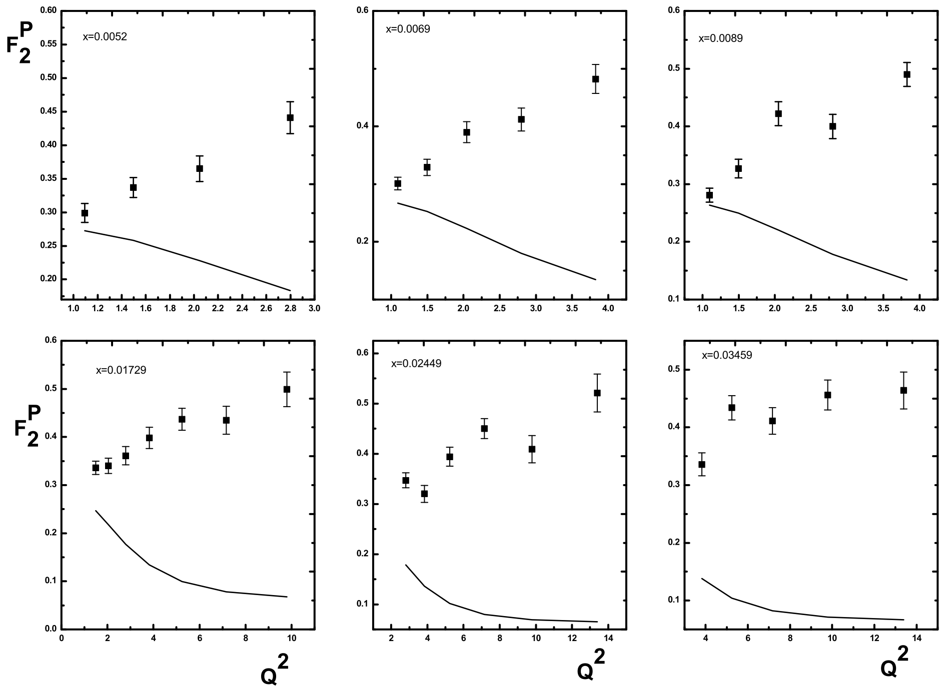

Figure 2. Proton structure function$F_2^P(x,t)$ as function of Q^2 at different x values using Method of characteristics. Data are taken from E665 [19].

2.5 Results and Discussion

We check the compatibility of the analytical methods in terms of proton structure function F_2^P, which we derive combining our analytical solutions for both non-singlet [5] and singlet structure functions to calculate the proton structure function. We use the MSTW 2008 [18] input for evolution and the range of data considered for comparison are $0.0052 \leq x \leq 0.18$ and 1.094 GeV^2 $\leq Q^2 \leq 34.27$ GeV^2 for E665 [19]. Figs. 1 and 2 show comparison of our analytical results for F_2^P obtained by Lagrange's method and method of characteristics respectively, with E665 experimental data within small x and low Q^2 range $0.0052 \leq x \leq 0.04897$ and 1.093 GeV^2 $\leq Q^2 \leq 25$ GeV^2. Though the evolution of analytical result by Lagrange's method conforms well with data, that of method of characteristics decreases with increasing Q^2, which is against the general expectations of pQCD. The Q^2 dependence of the function $K(Q^2)$, given by equation (7) is playing an important role in our analytical solutions. For hence it shows in the solution by equation (25) by method of characteristics, though in case of the solution by Lagrange's method, equation (21), it does not have any visible effect, due to the consideration of $\frac{1}{x}$ limit.

3. Conclusion

This work is an extension of the comparative study of the two important analytical methods, Lagrange's and Method of Characteristics for proton structure function F_2^P, derived from corresponding analytical solutions for F_2^{NS} and F_2^S. For this part we pursue a general form as given by equation (7), relating $F_2^S(x,Q^2)$ and $G(x,Q^2)$ for comparison with theoretical analysis of [16]. The solution by Method of Characteristics has exclusive dependence on the relation. However, data analysed in the range $0.0052 \leq x \leq 0.04897$ and 1.093 GeV^2 $\leq Q^2 \leq 25$ GeV^2 is found to favour the former (Lagrange's method) and not the later (Method of Characteristics).

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