Correlation Functions in AdS/CFT

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Chapter 1

Introduction

As far as we know today the dynamics of the fundamental objects of nature is described by four kinds of interactions. Three of them, the electromagnetic as well as strong- and weak force can be described by the standard model of particle physics (SM) which is a quantum field theory with local (gauge) symmetries. The fourth and weakest force, namely gravity, is described by Einstein's theory of general relativity (GR). The latter is a classical theory, that means not taking into account the effects of quantum mechanics and up to now it is not known how the quantum completion of Einstein's theory looks like. The most promising candidates are string theories, in which the fundamental objects are not any longer point-like particles but one-dimensional, extended objects, called strings. In this way string theory provides a natural cut-off, given by the string length, which makes it possible to overcome the difficulties in the renormalization of quantum gravity. String theories as fundamental theories have numerous appealing features both, from a phenomenological, as well as from a conceptual point of view. Originally, however, string theories were proposed in a different framework. In the 1960's they were used to describe hadrons and could explain for instance the observed relation between the mass and the spin of the lightest hadronic resonances. Later it was discovered that strong interactions are much better described by a quantum field theory with gauge group SU(3), called quantum chromodynamics (QCD) and in which the fundamental objects are point like objects called quarks. It was also observed that string theories naturally incorporate a massless spin-2 particle in the spectrum and should therefore rather be used as a description of gravity. Still, it was conjectured that, since string theories described some features of hadrons very well, they should in some sense be a dual description to QCD. This duality conjecture gained some support when people studied the dynamics of D-branes in string theory. D-branes are solitonic solutions of string theory and correspond to higher dimensional surfaces on which open strings can end. In the low-energy limit of string theory, given by supergravity, they correspond to higher-dimensional generalizations of Reissner-Nordström black holes and their dynamics is described by gravity, while their string theoretical description is governed by a gauge theory on the brane world-volume induced by the ends of open strings. It was realized that the results of calculations done independently in the two descriptions agreed. In 1997 Juan Maldacena then proposed the famous *AdS/CFT conjecture*. It relates type IIB superstring theory or M-theory on an $\mathrm{AdS}_d \times \mathrm{M}_{D-d}$, with $D = 10, 11$ and M a compact manifold, background to a conformal quantum field theory living on the boundary of AdS_d. Since its proposal the conjecture has experienced enormous attention and was applied to various problems in gauge- and gravity-theories but also in condensed matter physics. The appealing feature of the correspondence is, that the perturbative regimes of the theories on the two sides of the duality are exactly contrary, making it possible to learn about strongly coupled gauge theory using perturbative techniques on the gravity side but also to learn about quantum gravity using weakly-coupled gauge theory. In this way it was possible to gain insight into the dynamics of black holes and learn about confinement in QCD. Being such a useful tool it is important to know whether the conjecture is actually true. Since there is no proof yet, the only way to get at least hints on the answer to that question is, to test the correspondence by comparison of results calculated independently on the two sides of the duality. This is in general a hard task, since - as pointed out above - the regimes of validity of perturbative approaches to the two theories is exactly contrary. There are some special cases though in which quantities are protected against quantum corrections and therefore the validity of results for these quantities extends to the non-perturbative regime. Such special quantities can then be used to test the correspondence. This thesis deals with the calculation of correlation functions in the AdS/CFT context. Correlation functions in general and a class of special protected quantities, namely correlators of *spinning folded string solutions* will be investigated from the string theory perspective.

In the first chapter we will present the Maldacena conjecture connecting type IIB superstring theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background to $\mathcal{N} = 4$ SU(N) super-Yang-Mills theory on the boundary of AdS_5 and discuss the theories on the two sides of the duality. Afterwards we discuss the quantization of the type IIB superstring on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background using a Wess-Zumino-Witten-like non-linear sigma-model. In the second chapter we discuss the calculation of correlation functions in general, present the spinning folded string solutions and examine correlation functions in the AdS/CFT context.

Chapter 2

The AdS/CFT Correspondence

In this chapter, the AdS/CFT duality conjecture as it was proposed by Juan Martin Maldacena 1998 [1] will be reviewed. The focus will be on the duality between the type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and a $\mathcal{N} = 4$ SU(N) super Yang Mills theory on the boundary of that space, as this is the best understood example of the dualities that were conjectured. In order to be able to give a detailed description of the conjecture, the two sides of the duality will be reviewed first.

2.1 Type IIB Superstring Theory

2.1.1 The Green–Schwarz action

As a first step to understand the AdS/CFT correspondence we will review type IIB superstring theory on flat space in the Green–Schwarz formalism. The arguments presented here, follow in parts those in [4] and [6].

We start from the $\mathcal{N} = 2$ spacetime supersymmetric[1] string action in D dimensions which was first proposed by Green and Schwarz in 1984, [3].

$$\begin{aligned} S = -\frac{1}{2\pi} \int \mathrm{d}\sigma^0 \mathrm{d}\sigma^1 \Big\{ \sqrt{h}\, h^{\alpha\beta} \Pi^\mu_\alpha \Pi^\nu_\beta - 2\mathrm{i}\epsilon^{\alpha\beta} \partial_\alpha X^\mu \left(\bar\theta^1 \Gamma^\nu \partial_\beta \theta^1 - \bar\theta^2 \Gamma^\nu \partial_\beta \theta^1 \right) \\ + \epsilon^{\alpha\beta} \bar\theta^1 \Gamma^\mu \partial_\alpha \theta^1 \bar\theta^2 \Gamma^\nu \partial_\beta \theta^2 \Big\} G_{\mu\nu} \,, \end{aligned} \tag{2.1}$$

with

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \sum_{A=1}^{2} \mathrm{i}\bar\theta^A \Gamma^\mu \partial_\alpha \theta^A \,, \qquad \alpha \in \{0, 1\} \,,$$

[1]Note that in principle one can write down a supersymmetric string action for $\mathcal{N}$ supersymmetries in general. However, imposing κ-symmetry requires $\mathcal{N} \leq 2$.

where $X^\mu(\sigma^0,\sigma^1)$ ($\mu = 0 \dots D-1$) are bosonic worldsheet fields, that map the worldsheet into the D dimensional target space. $\theta^{Aa}(\sigma^0,\sigma^1)$ ($A = 1,2$, $a = 1,\dots,2^{\lfloor D/2 \rfloor}$) denote two additional target-space spinorial scalar worldsheet fields. The index a is that of a spinor in D dimensions. Γ^μ are the D-dimensional Gamma matrices that satisfy the Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}$ and $G_{\mu\nu}(X)$ is the metric on the target space.

To check whether the given action is a consistent action for a supersymmetric string we consider the following limits.

- Set $\theta^1 = \theta^2 = 0$. This corresponds to vanishing supersymmetry. The action reduces to

$$S_{\text{bos}} = -\frac{1}{2\pi}\int \mathrm{d}\sigma^0 \mathrm{d}\sigma^1 \sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} \,, \tag{2.2}$$

  which is the σ-model action for the bosonic string.

- Discard the σ^1 coordinate, such that $X(\sigma^0,\sigma^1) \to X(\sigma^0)$, and similarly for $\theta(\sigma^0,\sigma^1)$. This corresponds to the point particle limit. Then the action goes to

$$S_{\text{p}} = -\int \mathrm{d}\sigma^0 \underbrace{\sqrt{h}h^{00}}_{\equiv e^{-1}} \sum_{A,B=1}^{2} \left(\dot{X}^\mu - \mathrm{i}\bar{\theta}^A\Gamma^\mu\dot{\theta}^A\right)\left(\dot{X}^\nu - \mathrm{i}\bar{\theta}^B\Gamma^\nu\dot{\theta}^B\right) G_{\mu\nu} \,, \tag{2.3}$$

  where $\dot{\mathcal{O}} = \partial_0\mathcal{O}$. This is the straightforward supersymmetric extension of the massive point particle action.

The action (2.1) therefore yields the results expected from an action describing the propagation of a supersymmetric string and hence passes this consistency check. Now let us explore it further by discussing its symmetries.

- **Global space-time Lorentz symmetry** Since all Lorentz indices are contracted and $\bar{\theta}^A\Gamma^\mu\theta^A$ transforms as a space-time vector under Lorentz transformations, the action is manifestly invariant under global Lorentz transformations. Furthermore the action contains only derivatives of the space-time coordinates X^μ such that, in the case of a flat metric, the Lorentz symmetry extends trivially to invariance under global Poincare transformations

$$X^\mu(\sigma^0,\sigma^1) \to a^\mu{}_\nu X^\nu(\sigma^0,\sigma^1) + b^\mu \,,$$

  with $a_{\mu\nu} = -a_{\nu\mu}$.

- **Local reparametrisation symmetry** Local reparametrisations of the form

$$(\sigma^0, \sigma^1) \rightarrow (f^0(\sigma^0, \sigma^1), f^1(\sigma^0, \sigma^1))$$

can be viewed as general coordinate transformations on the worldsheet manifold. Then the fields transform as

$$\begin{aligned} \partial_\alpha \mathcal{O}(\sigma^0, \sigma^1) &\rightarrow \frac{\partial f^\beta}{\partial \sigma^\alpha} \frac{\partial}{\partial f^\beta} \mathcal{O}(f^0, f^1)\,, \\ h_{\alpha\beta}(\sigma^0, \sigma^1) &\rightarrow \frac{\partial f^\gamma}{\partial \sigma^\alpha} \frac{\partial f^\delta}{\partial \sigma^\beta} h_{\gamma\delta}(f^0, f^1)\,, \\ \sqrt{h} &\rightarrow \det \left| \frac{\partial f^\alpha}{\partial \sigma^\beta} \right| \sqrt{h} \end{aligned}$$

and $\sqrt{h}\, \mathrm{d}\sigma^0 \wedge \mathrm{d}\sigma^1$ defines an invariant volume element. The worldsheet tensor density of weight -1, $\epsilon^{\alpha\beta}$, transforms as

$$\epsilon^{\alpha\beta} \rightarrow \det \left| \frac{\partial f^\alpha}{\partial \sigma^\beta} \right|^{-1} \frac{\partial \sigma^\alpha}{\partial f^\gamma} \frac{\partial \sigma^\beta}{\partial f^\delta} \epsilon^{\gamma\delta}\,,$$

such that the action (2.1) is invariant.

- **Local Weyl invariance** The action is invariant under rescalings of the form

$$h_{\alpha\beta} \rightarrow \mathrm{e}^{\phi(\sigma^0, \sigma^1)} h_{\alpha\beta}$$

and all other fields left invariant. Then $\sqrt{h} \rightarrow \mathrm{e}^{\phi}\sqrt{h}$ such that $\sqrt{h}\, h^{\alpha\beta}$ is invariant. Analogously to bosonic string theory this symmetry implies that the worldsheet energy momentum tensor is traceless.

- **Global $\mathcal{N} = 2$ supersymmetry** Global supersymmetry transformations are given by

$$\begin{aligned} \delta\theta^A &= \epsilon^A\,, \\ \delta\bar{\theta}^A &= \bar{\epsilon}^A\,, \\ \delta x^\mu &= \mathrm{i} \sum_{A=1}^{2} \bar{\epsilon}^A \Gamma^\mu \theta^A\,, \end{aligned}$$

where ϵ^A is a (τ, σ)-independent spinor. The combination $\Pi^\mu_\alpha = \partial_\alpha X^\mu - \sum_{A=1}^{2} \mathrm{i}\bar{\theta}^A \Gamma^\mu \partial_\alpha \theta^A$ is easily shown to be invariant under this transformation. With a bit more effort it can also be shown that the other terms in the action are invariant up to total derivatives in the following cases:

1. $D = 3$ and θ is Majorana
2. $D = 4$ and θ is Majorana or Weyl
3. $D = 6$ and θ is Weyl
4. $D = 10$ and θ is Majorana–Weyl

So even the existence of a *classical* superstring theory in this formalism restricts the possible number of dimensions.[2]

- **Local κ-symmetry** The action exhibits an additional local fermionic symmetry given by

$$\begin{aligned} \delta X^\mu &= \mathrm{i} \sum_{A=1}^{2} \bar{\theta}^A \Gamma^\mu \delta\theta^A \, , \\ \delta\theta^A &= 2\mathrm{i}\Gamma^\mu \Pi_\alpha^\nu G_{\mu\nu} \kappa^{A\alpha} \, , \\ \delta\left(\sqrt{h} h^{\alpha\beta}\right) &= -16\sqrt{h} \left(P_-^{\alpha\gamma} \bar{\kappa}^{1\beta} \partial_\gamma \theta^1 + P_+^{\alpha\gamma} \bar{\kappa}^{2\beta} \partial_\gamma \theta^2 \right) \, , \end{aligned}$$

where the $\kappa^{A\alpha}$ are (τ, σ) independent space-time spinors and worldsheet vectors which are restricted to be anti-self-dual ($A = 1$) or self-dual ($A = 2$):

$$\begin{aligned} \kappa^{1\alpha} &= P_-^{\alpha\beta} \kappa_\beta^1 \, , \\ \kappa^{2\alpha} &= P_+^{\alpha\beta} \kappa_\beta^2 \, . \end{aligned}$$

Here we have used the projection tensors

$$P_\pm^{\alpha\beta} = \frac{1}{2} \left(h^{\alpha\beta} \pm \frac{1}{\sqrt{h}} \epsilon^{\alpha\beta} \right) \, ,$$

that project a vector onto its self-dual or anti-self-dual part respectively. They satisfy the conditions

$$\begin{aligned} P_\pm^{\alpha\beta} h_{\beta\gamma} P_\pm^{\gamma\delta} &= P_\pm^{\alpha\delta} \, , \\ P_\pm^{\alpha\beta} h_{\beta\gamma} P_\mp^{\gamma\delta} &= 0 \, . \end{aligned}$$

The contributions to the variation of the action coming from the different terms cancel exactly in the same cases in which the global supersymmetry is manifest. Note that, as we will see later, local κ-symmetry decouples half of the components of θ.

[2] Note that in the cases when θ is Weyl but not Majorana $\bar{\theta}\Gamma^\mu \partial_\alpha \theta$ must be replaced by $(\bar{\theta}\Gamma^\mu \partial_\alpha \theta - \partial_\alpha \bar{\theta}\Gamma^\mu \theta)/2$.

- **Local bosonic $\boldsymbol{\lambda}$ symmetry** The closure of the algebra of κ transformations requires an additional local bosonic symmetry which is given by

$$
\begin{aligned}
\delta\theta^1 &= \sqrt{h}P_-^{\alpha\beta}\partial_\beta\theta^1\lambda_\alpha\,,\\
\delta\theta^2 &= \sqrt{h}P_+^{\alpha\beta}\partial_\beta\theta^2\lambda_\alpha\,,\\
\delta X^\mu &= \mathrm{i}\sum_{A=1}^{2}\theta^A\Gamma^\mu\delta\theta^A\,,\\
\delta\left(\sqrt{h}h^{\alpha\beta}\right) &= 0\,,
\end{aligned}
$$

  where λ_α describes a (τ,σ) independent worldsheet vector. Note that this symmetry has no additional implications for the on-shell theory.

To determine the classical theory described by the action (2.1) one has to look at the set of equations of motion it implies. On a **flat** target space they are given by

$$
\begin{aligned}
\Pi_\alpha\cdot\Pi_\beta &= \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\Pi_\gamma\cdot\Pi_\delta\,,\\
0 &= \Gamma\cdot\Pi_\alpha P_-^{\alpha\beta}\partial_\beta\theta^1\,,\\
0 &= \Gamma\cdot\Pi_\alpha P_+^{\alpha\beta}\partial_\beta\theta^2\,,\\
0 &= \partial_\alpha\left[\sqrt{h}\left(h^{\alpha\beta}\partial_\beta X^\mu - 2\mathrm{i}P_-^{\alpha\beta}\bar\theta^1\Gamma^\mu\partial_\beta\theta^1 - 2\mathrm{i}P_+^{\alpha\beta}\bar\theta^2\Gamma^\mu\partial_\beta\theta^2\right)\right]\,,
\end{aligned}
\tag{2.4}
$$

where "$\cdot$" refers to the target space scalar product built with the metric $G_{\mu\nu}$. These equations have a quite complicated nonlinear structure which makes the quantization of the theory in a covariant way hard. Also the local κ-symmetry imposes a mixture of first and second class constraints on the canonical variables which turn out to be hard to disentangle even in flat space. In the following we will therefore discuss the quantization on flat space in light-cone gauge. Later we will give a nonlinear coset σ-model for type IIB superstring theory on an $\mathrm{AdS}_5\times\mathrm{S}^5$ background and present an attempt to quantize it perturbatively.

2.1.2 Light Cone Gauge Quantization

In the previous section we saw that the classical superstring theory in the Green–Schwarz (GS) formulation exists only in 3, 4, 6 or 10 dimensions. As in the bosonic case, quantum effects fix the dimension to a specific value, which is ten in the case of superstrings. One way to argue that this has to be the case is demanding the absence of negative norm states, which is easily done in the Ramond-Neveu-Schwarz (RNS) formulation of superstring theory. Another way to find the critical dimension is by requiring that the Lorentz algebra is satisfied in the light-cone gauge. In the following we will just assume that $D=10$.

As stated before, in order to have a supersymmetric and κ-symmetric action of the form (2.1) in ten dimensions, the θ coordinates must be chosen to be Majorana–Weyl spinors, which implies that both θ^1 and θ^2 must be assigned a handedness. This gives rise to two physically distinct possibilities. Either θ^1 and θ^2 are of the same, or of opposite handedness. In the case of open strings the two spinorial coordinates have to be equated at the ends of the open strings which fixes them to have the same handedness. This theory is called "type I superstring theory". In a theory of closed strings on the other hand, the spinorial coordinates are not related and both possibilities can be realised. This leads to "type IIA superstring theory" in case θ^1 and θ^2 are of opposite and to "type IIB superstring theory" in case they are of same handedness. In the following we will discuss the light-cone quantization of type IIB superstring theory.

In order to simplify the superstring action we can exploit the diverse symmetries the action possesses and make some appropriate gauge choices. We start by using the local reparametrisation invariance to choose a conformally flat gauge and bring the metric to the form

$$h_{\alpha\beta} = \mathrm{e}^{\phi}\eta_{\alpha\beta}\,. \tag{2.5}$$

After imposing this gauge choice there is still a residual symmetry given by conformal transformations which leads to the appearance of a Virasoro algebra. In terms of $\sigma^{\pm} \equiv \sigma^0 \pm \sigma^1$ the symmetry transformations are given by $\sigma^{\pm} \to \xi^{\pm}(\sigma^{\pm})$, which corresponds to

$$\begin{aligned}
\tilde{\sigma}^0 &= \frac{1}{2}\left[\xi^+(\sigma^+) + \xi^-(\sigma^-)\right]\,,\\
\tilde{\sigma}^1 &= \frac{1}{2}\left[\xi^+(\sigma^+) - \xi^-(\sigma^-)\right]\,,
\end{aligned}$$

which is solved by any $\tilde{\sigma}^0$ respecting

$$\left(\left(\frac{\partial}{\partial\sigma^0}\right)^2 - \left(\frac{\partial}{\partial\sigma^1}\right)^2\right)\tilde{\sigma}^0 = 0\,. \tag{2.6}$$

In the case of the bosonic string a straightforward solution to this equation is obvious. It is given by setting the σ^0 direction equal to one of the X^μ. Here, since the equations of motion of the X^μ coordinates are far more complicated, we cannot impose such a simple gauge condition for the residual symmetry. Luckily there is another symmetry of the action which we can exploit to simplify the equations of motion, namely κ-symmetry. After imposing a gauge fixing condition for this symmetry we will come back to the residual conformal symmetry.

A convenient gauge choice for κ-symmetry is given by

$$\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0\,, \tag{2.7}$$

where $\Gamma^\pm \equiv \frac{1}{\sqrt{2}}\left(\Gamma^0 \pm \Gamma^9\right)$ and in the following we will use the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^9\right) .$$

Note that while both matrices Γ^+ and Γ^- are nilpotent, their sum is nonsingular. Therefore half of the eigenvalues of each of them is zero, which implies that half of the components of both θ^1 and θ^2 are fixed to zero by the gauge choice (2.7).
Having imposed the gauge choices (2.5) and (2.7) the equations of motion (2.4) simplify a lot. The crucial point to realize is, that the κ-gauge choice implies that $\bar{\theta}\Gamma^\mu\partial_\alpha\theta$ vanishes unless $\mu = -$. This can be seen by using $\Gamma^0\Gamma^\pm = \Gamma^\mp\Gamma^0$, $\left(\Gamma^\pm\right)^\dagger = \Gamma^\mp$ and $\mathbb{1} = \frac{1}{2}\left(\Gamma^+\Gamma^- + \Gamma^-\Gamma^+\right)$. Now, using $\Gamma_\mu\Pi^\mu_\alpha = \Gamma_-\Pi^- + \Gamma_+\Pi^+_\alpha + \Gamma_i\Pi^i_\alpha$ and $\Gamma_+ = -\Gamma^-$, the second equation in (2.4) can be multiplied by Γ^+ to give

$$2\Pi^+_\alpha P_-^{\alpha\beta}\partial_\beta\theta^1 = 0 \,,$$

which, using $\Pi^+_\alpha = p^+\delta_{\alpha,0}$ and the gauge choice (2.5), simplifies further to give

$$\left(\partial_0 + \partial_1\right)\theta^1 = 0 \,.$$

Similarly one one can derive the equation of motion for θ^2, given by

$$\left(\partial_0 - \partial_1\right)\theta^2 = 0 \,.$$

Finally from the last equation in (2.4) one gets

$$\left(\partial_0^2 - \partial_1^2\right)X^i = 0 \,. \tag{2.8}$$

From these equations we see that θ^1 and θ^2 describe waves propagating in opposite directions on the worldsheet. Since we are considering type IIB string theory here, they are not related via boundary conditions.

Using (2.8) one can find a straightforward solution of equation (2.6) on a flat spacetime background. It is, similarly to the bosonic case, given by $\tilde{\sigma}^0 = X^+/p^+ + \text{const.}$, where p^+ is a constant. Usually, omitting the tildes, this is rewritten as

$$X^+(\sigma^0, \sigma^1) = x^+ + p^+\sigma^0 \tag{2.9}$$

and called the "light-cone gauge fixing" condition. Note that, since $\delta_{\text{SUSY}}X^+ = \sum_{A=1}^2 \bar{\epsilon}^A\Gamma^+\theta^A$, this gauge choice is consistent with the κ-gauge (2.7). After imposing (2.9) the reparametrisation freedom is fixed completely.

Before we solve the equations of motion and get the spectrum of this theory, we have to discuss the physical degrees of freedom, that survive gauge fixing. As discussed before, the action (2.1) possesses manifest ten dimensional Lorentz invariance, but after fixing the light-cone gauge only the $\mathrm{SO}(8) \subset \mathrm{SO}(9,1)$ symmetry of the transverse components is manifest. A generic spinor in eight dimensions has 16 complex components which after imposing Majorana and Weyl conditions reduce to 8 real components. Therefore the transverse components of θ form an eight-dimensional spinor representation of the Spin(8) covering group of SO(8). There are three eight-dimensional real representations of the rank-four Lie algebra $\mathfrak{spin}(8)$, related by an outer automorphism of the SO(8) Dynkin diagram which is called "triality". One is the (fundamental) vector representation $\mathbf{8}_{\mathrm{V}}$ and the other two are spinor representations usually denoted by $\mathbf{8}_{\mathrm{S}}$ and $\mathbf{8}_{\mathrm{C}}$. They describe spinors with opposite eight-dimensional chirality. Following the notation of [4] we will use the letters i, j, ... for $\mathbf{8}_{\mathrm{V}}$ indices, a, b, ... for $\mathbf{8}_{\mathrm{S}}$ and $\dot{a}$, $\dot{b}$, ... for $\mathbf{8}_{\mathrm{C}}$ indices. Denoting now the transverse spinor components by S^a and $S^{\dot{a}}$ respectively, in the case of type IIB string theory, we have

$$\sqrt{p_+}\theta^A \to \mathbf{8}_{\mathrm{S}} + \mathbf{8}_{\mathrm{S}} = \left(S^{1a}, S^{2a}\right) ,$$

where the $\sqrt{p_+}$ factor was introduced for later convenience. In this notation the equations of motion

$$\partial_+\partial_- X^i = 0\,, \qquad \partial_+ S^{1a} = 0\,, \qquad \partial_- S^{2a} = 0\,, \tag{2.10}$$

take a very simple form. This fact suggests that they can be obtained from a much easier action than (2.1). In fact such an action is given by

$$S_{\text{l.c.}} = -\frac{1}{2}\int \mathrm{d}^2\sigma \left(\sqrt{h}\partial_\alpha X^i \partial^\alpha X^j \delta_{ij} - \frac{\mathrm{i}}{\pi}\bar{S}^a \rho^\alpha \partial_\alpha S^b \delta_{ab}\right) , \tag{2.11}$$

where S^a denotes a two-component Majorana worldsheet spinor consisting of the two one-component Majorana–Weyl worldsheet spinors S^{1a} and S^{2a} describing the left- and right moving degrees of freedom respectively and ρ^α are the two-dimensional Dirac matrices obeying $\left\{\rho^\alpha, \rho^\beta\right\} = -2\eta^{\alpha\beta}$. In the representation in which the S^{Aa} are real, they are given by

$$\rho^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \qquad \rho^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} .$$

This action looks pretty much like the light-cone action of superstring theory in the RNS formulation. There is an important difference though. While in the RNS formulation both the worldsheet bosons X^i and the worldsheet fermions Ψ^i transform in the $\mathbf{8}_{\mathrm{V}}$ representation of $\mathfrak{spin}(8)$, in the action (2.11) the X^i transform in the $\mathbf{8}_{\mathrm{V}}$ but the S^a in the $\mathbf{8}_{\mathrm{S}}$ representation. In order to get the action (2.11) from the RNS light-cone gauge action one has to bosonize and then re-fermionize the fermions Ψ^i. This procedure inherits a very important subtlety concerning the bosonization procedure on finite-volume spaces which in the end leads to the need of the GSO projection to re-establish supersymmetry in the RNS formulation.

In order to solve the equations of motion and give the mode expansion of the target space coordinates we have to specify the boundary conditions. Of course in the case of closed strings the only option is periodicity

$$\begin{aligned} X^i\left(\sigma^0,\sigma^1\right) &= X^i\left(\sigma^0,\sigma^1+\pi\right) \text{ and} \\ S^{Aa}\left(\sigma^0,\sigma^1\right) &= S^{Aa}\left(\sigma^0,\sigma^1+\pi\right) . \end{aligned}$$

Then, since both the equations of motion and the boundary conditions looks the same as for bosonic closed strings in flat space, they are solved by the oscillator expansions given by

$$\begin{aligned} S^{1a}\left(\sigma^-\right) &= \sum_{n\in\mathbb{Z}} S^a_n \mathrm{e}^{-2\mathrm{i}n\sigma^-} , \\ S^{2a}\left(\sigma^+\right) &= \sum_{n\in\mathbb{Z}} \tilde{S}^a_n \mathrm{e}^{-2\mathrm{i}n\sigma^+} , \\ X^i\left(\sigma^0,\sigma^1\right) &= X^i_{\mathrm{R}}\left(\sigma^-\right) + X^i_{\mathrm{L}}\left(\sigma^+\right) , \\ X^i_{\mathrm{R}}\left(\sigma^-\right) &= \frac{1}{2}x^i + \frac{1}{2}p^i\sigma^- + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^i_n \mathrm{e}^{-2\mathrm{i}n\sigma^-} , \\ X^i_{\mathrm{L}}\left(\sigma^+\right) &= \frac{1}{2}x^i + \frac{1}{2}p^i\sigma^+ + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^i_n \mathrm{e}^{-2\mathrm{i}n\sigma^+} . \end{aligned} \tag{2.12}$$

Finally, thanks to the simplifications yielding a free action (2.11) and linear equations of motion (2.10) we can quantize the (transverse) theory by imposing the canonical equal-time (anti-) commutation relations

$$\begin{aligned} \left[\dot{X}^i\left(\sigma^0,\sigma^1\right), X^j\left(\sigma^0,\sigma'^1\right)\right] &= -\mathrm{i}\pi\delta^{ij}\delta\left(\sigma^1-\sigma'^1\right) , \\ \left\{S^{Aa}\left(\sigma^0,\sigma^1\right), S^{Bb}\left(\sigma^0,\sigma'^1\right)\right\} &= \pi\delta^{ab}\delta^{AB}\delta\left(\sigma^1-\sigma'^1\right) , \end{aligned} \tag{2.13}$$

such that the oscillators satisfy

$$\begin{aligned} \left[\alpha^i_m,\alpha^j_n\right] = \left[\tilde{\alpha}^i_m,\tilde{\alpha}^j_n\right] &= m\delta_{m+n,0}\delta^{ij} \text{ and} \\ \left\{S^a_m,S^b_n\right\} = \left\{\tilde{S}^a_m,\tilde{S}^b_n\right\} &= \delta^{ab}\delta_{m+n,0} . \end{aligned} \tag{2.14}$$

2.1.3 The Flat Space Spectrum

After quantizing the type IIB superstring theory in light-cone gauge, we are able to write down its spectrum. In order to do that, we will first concentrate on only one of the two sectors, describing left- and right-moving states on the closed string. The mass-shell condition for one of the sectors is basically the same as that of a type I open superstring

$$M^2 \propto \sum_{n>0} \left(\alpha^i_{-n} \alpha^j_n \delta_{ij} + n S^a_{-n} S^a_n \right) .$$

The ground state $|\phi\rangle$ is massless such that, as we have seen before, no GSO projection is needed to truncate the spectrum and eliminate a tachyon. Moreover, since the operator S^a_0 commutes with M^2, the ground state is degenerate and has to form a representation of the Clifford algebra

$$\left\{ S^a_0, S^b_0 \right\} = \delta^{ab}.$$

The 16 dimensional representation respecting this algebra can be shown to decompose as $\mathbf{8}_\mathrm{V} + \mathbf{8}_\mathrm{C}$. Therefore the massless ground state contains the complete multiplet that has to be obtained from two seperate constructions in the RNS formalism. Now at the first excited level, the massive string states can be formed by acting with α^i_{-1} or S^a_{-1} on $|\phi\rangle$.
In order to get the states of type IIB we have to form tensor products of the left- and right-moving sectors. Therefore the massless ground state is given by

$$\begin{aligned}(\mathbf{8}_\mathrm{V} + \mathbf{8}_\mathrm{C}) \otimes (\mathbf{8}_\mathrm{V} + \mathbf{8}_\mathrm{C}) = &(\mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{V} + \mathbf{1} + \mathbf{28} + \mathbf{35}_\mathrm{C})_\mathrm{B} \ , \\ &+ (\mathbf{8}_\mathrm{S} + \mathbf{8}_\mathrm{S} + \mathbf{56}_\mathrm{S} + \mathbf{56}_\mathrm{S})_\mathrm{F} \ ,\end{aligned}$$

where "B" denotes bosonic and "F" denotes fermionic states. Therefore the massless ground state of chiral type IIB superstring theory contains the dilaton Φ ($\mathbf{1}$), the antisymmetric 2-form $B_{\mu\nu}$ in a $\mathbf{28}$ as well as the graviton in the $\mathbf{35}_\mathrm{V}$. Additionally we find the form fields C_0 in a $\mathbf{1}$, C_2 in a $\mathbf{28}$ and C_4 in the remaining $\mathbf{35}_\mathrm{C}$. The fermionic massless spectrum contains two Majorana–Weyl gravitinos $\mathbf{56}_\mathrm{S}$ and two Majorana–Weyl spinors $\mathbf{8}_\mathrm{S}$.
Since we need an equal number of left- and right-moving excitations to form closed string states in order to avoid defining a prefered σ^1 position on the string, the massive type IIB states can, analogously to the massless ground state, be formed of tensor products of massive open-string states with themself at each level.[3]

2.1.4 Low Energy Effective Action

Since the string theory described by the action (2.1) is quite complicated and string theory in general is not understood well enough to write down a string field theory action it is useful to write down a low energy effective field theory in order to better

[3] More details on this procedure can be found in [4].

understand its dynamics. As a first step to go from the full action to a simplified effective one, one has to integrate out all massive fields. In principle, this does not approximate the theory but is a first step in calculating the exact path integral and would still give a very complicated and, in addition, non-local theory. Therefore one systematically expands the action in the number of derivatives, since each derivative corresponds to a suppression by a power proportional to $1/M$, where M is the characteristic mass scale of the string theory. Of course, keeping only the first terms in such an expansion yields a theory describing the low energy dynamics of the theory but lacking a good ultraviolet convergence. In the case of string theory no action of the form $S[\Phi_i]$ with Φ_i being the fields is known, such that a direct expansion of the action is not possible. Instead one has to write down a field theoretic action describing the same particle content and symmetries and that yields the same equations of motion as the string theory in the massless regime. Such actions are known to be given by supergravity actions in ten dimensions.

The bosonic part of the supergravity action (in the *string frame*) describing the classical low energy dynamics of type IIB superstring theory was found to be

$$S = S_{\rm NS} + S_{\rm R} + S_{\rm CS} \, , \tag{2.15}$$

where

$$\begin{aligned}
S_{\rm NS} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{G}\mathrm{e}^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|H_3\right|^2\right) \, , \\
S_R &= -\frac{1}{4\kappa^2}\int \mathrm{d}^{10}x\sqrt{G}\left(\left|F_1\right|^2 + \left|\tilde{F}_3\right|^2 + \frac{1}{2}\left|\tilde{F}_5\right|^2\right) \, , \\
S_{\rm CS} &= -\frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3
\end{aligned}$$

and $F_{n+1} = \mathrm{d}C_n$, $H_3 = \mathrm{d}B_2$, $4\pi\kappa^2 = (2\pi l_{\rm s})^8$, as well as

$$\begin{aligned}
\tilde{F}_3 &= F_3 - C_0 H_3 \qquad \text{and} \\
\tilde{F}_5 &= F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3 \, .
\end{aligned}$$

In addition the self-duality condition for the five-form $\tilde{F}_5$,

$$\tilde{F}_5 = *\tilde{F}_5 \, ,$$

has to be imposed as a constraint.

This action has a property which is quite important in the context of the AdS/CFT duality. Namely it possesses a global SL(2,$\mathbb{R}$) symmetry. In order to see this one can introduce a two-component vector

$$\vec{B}_2 = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \, ,$$

which under an SL(2,ℝ) transformation

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

transforms as $\vec{B} \to \Lambda \vec{B}$. Also one can combine the dilaton Φ and the so-called axion C_0 into the *axio-dilaton* τ as

$$\tau = C_0 + \mathrm{i}\mathrm{e}^{-\Phi} \,,$$

which then transforms under Λ as

$$\tau \to \frac{a\tau + b}{c\tau + d} \,.$$

Note that, since the dilaton transforms nontrivially, the string-frame metric $G_{\mu\nu}$ is not invariant. An invariant combination of $G_{\mu\nu}$ and Φ is given by the *Einstein-frame* metric

$$G^{\mathrm{E}}_{\mu\nu} = \mathrm{e}^{-\Phi/2} G_{\mu\nu} \,.$$

Under this change of variables

$$\int \mathrm{d}^{10}x \sqrt{G} \mathrm{e}^{-2\Phi} R \to \int \mathrm{d}^{10}x \sqrt{G^{\mathrm{E}}} \left(R^{\mathrm{E}} - \frac{9}{2} \partial^\mu \Phi \partial_\mu \Phi \right) \,,$$

where R^{E} refers to the Ricci scalar build using the Einstein-frame metric G^{E}. Using that the four-form C_4 and the self-duality condition for $\tilde{F}_5$ are invariant under SL(2,ℝ) transformations one can show that the action (2.15) is also invariant.[4]

Besides this the type IIB supergravity action incorporates more appealing features which can be used to study some properties of the full string theory. One of those, which is also important in the context of AdS/CFT duality, is the presence of BPS solutions which give insight to the strongly coupled regime of the string theory. Some aspects of these D-Brane solutions of the supergravity theory will therefore be discussed in section 2.3.

2.1.5 Summary

In the last sections we presented the foundations of type IIB superstring theory in the Green–Schwarz formulation. We described the action, its symmetries, a light-cone-gauge quantization on flat space and the low energy effective supergravity theory. Of course the theory has far more features than those we can discuss

[4] More details on this symmetry can be found in chapter 13 of [5].

here. An important property is, that the D-branes, which show up as solutions of the supergravity theory, are, as Polchinksi discovered in 1995, related to the string theory objects that open strings end on. Incorporating these into type IIB gives a theory containing closed strings in the bulk as well as open strings that end on the D-branes. In this way one can incorporate non-abelian gauge theories into the theory. This fact made it possible to study setups that could, in the low energy regime, lead to theories similar to the standard model or its supersymmetric extensions. Note, however, that, since D-branes can be seen as sources of closed strings, the presence of D-branes in type IIB superstring theories breaks half of the supersymmetries, as it relates the string left- and right-movers.

Further, the type IIB theory can, via a web of various dualities, be related to the other string theories, namely type IIA, type I and heterotic string theory, and even to M theory. It is also connected to “F theory” [7] and the IKKT matrix model [8].

Consequently, not least because of its importance for the AdS/CFT duality, type IIB string theory is a broad and still very active research topic.

2.2 $\mathcal{N}=4$ Super-Yang-Mills in four Dimensions

Here, we would like to give a short introduction to $\mathcal{N}=4$ Super-Yang-Mills theory in (3+1) spacetime dimensions. We start by discussing the $\mathcal{N}=4$ supersymmetry algebra in four dimensions and its massive and massless representations. Afterwards we present the supersymmetric Yang-Mills Lagrangian and discuss its features. The resources on this topic, on which this section is based, are [9][10][11][12].

2.2.1 $\mathcal{N}=4$ Supersymmetry in four dimensions

Imposing supersymmetry on flat four dimensional Minkowksi spacetime with metric $\eta_{\mu\nu}=\mathrm{diag}\,(-+++)$ enlarges the Poincare symmetry group $\mathbb{R}^4 \ltimes \mathrm{SO}(1,3)$ with generators P_μ and $M_{\mu\nu}$, by imposing $\mathcal{N}$ spinor supercharges Q^a_α and $\bar{Q}^a_{\dot{\alpha}} \equiv (Q^a_\alpha)^*$. Here $a=1,\dots,\mathcal{N}$ and $\alpha=1,2$ denotes a spinor index. The left handed Weyl spinor Q^a, transforming in the $\left(\frac{1}{2},0\right)$ and the right handed Weyl spinor $\bar{Q}^a$, transforming in the $\left(0,\frac{1}{2}\right)$ of SO(1,3) can be written as a 4-component Dirac spinor

$$Q^a = \begin{pmatrix} Q^a_\alpha \\ \bar{Q}^{\dot{\alpha}}_a \end{pmatrix}.$$

The supersymmetry algebra is then given by

$$\begin{aligned} &\left\{Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{ab}, && \left\{Q^a_\alpha, Q^b_\beta\right\} = 2\epsilon_{\alpha\beta} Z^{ab}\,, \\ &\left[M_{\mu\nu}, Q^a_\alpha\right] = \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^a_\beta, && \left[M_{\mu\nu}, \bar{Q}^{a\dot{\alpha}}\right] = \mathrm{i}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{a\dot{\beta}} \end{aligned} \tag{2.16}$$

and the Q^a and $\bar{Q}^a$ commute with the translation generator. In the above definition we used the σ matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where γ^μ are the four dimensional Dirac matrices in the Weyl representation, in the usual manner. Further

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right),$$
$$\bar{\sigma}^{\mu\nu} = \frac{\mathrm{i}}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right).$$

The Z^{ab} are called central charges, are anti-symmetric in the a, b indices and commute with all other generators. Note that for $\mathcal{N} = 1$ supersymmetry the anti-symmetry implies that $Z = 0$.

The supersymmetry (*susy*) algebra (2.16) is, in the absence of central charges, invariant under U($\mathcal{N}$) transformations U under which

$$Q^a_\alpha \to U^a{}_b Q^b_\alpha \qquad \text{and} \qquad \bar{Q}^a_{\dot{\alpha}} \to U^{*a}{}_b \bar{Q}^b_{\dot{\alpha}}\,. \tag{2.17}$$

These automorphisms of the susy algebra are called *R-symmetries*. The R-symmetry group can be split into global phase transformations of all supercharges forming a $\mathrm{U}(1)_R$ and mixing of the supercharges as in (2.17) forming an $\mathrm{SU}(\mathcal{N})_R$. If the central charges are non-zero the R-symmetry group gets broken to subgroups of U(N).

In order to explore the implications of supersymmetry, it is natural to look at the particle representations. To study the massless representations one can choose a frame in which the particle's momentum takes the form $P^\mu = (E, 0, 0, E)$. Then the susy algebra takes the form

$$\left\{Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\right\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot{\beta}} \delta^{ab}\,. \tag{2.18}$$

If we consider only unitary representations with a positive definite Hilbert space, we can use the above relation for $\alpha = 1$ to get that $Q^a_1 = 0$. Also using (2.18) with $\alpha = 2, \dot{\beta} = \dot{1}$ we get $\bar{Q}^a_{\dot{1}} = 0$ and plugging in $Q^a_1 = 0$ into $\left\{Q^a_1, Q^b_2\right\} = 2Z^{ab}$ yields $Z^{ab} = 0$.
One can show that Q^a_2 raises the helicity of a state by $\frac{1}{2}$ while $\bar{Q}^a_{\dot{2}}$ lowers it by $\frac{1}{2}$. This enables us to construct the multiplets starting from the highest helicity states and applying the lowering operators for all values of a. Note that in CPT invariant theories, such as quantum field and string theories, the spectrum must

be invariant under a helicity sign change. Therefore in $\mathcal{N}=4$ susy the massless multiplet with maximal helicity 1 contains a chiral and an anti-chiral real vector, four left-handed and four right handed spinors and six complex scalars. Note that the fermionic and the bosonic degrees of freedom match as it should be the case for a supersymmetric multiplet.

The case of massive particles is a bit more complicated. There we can choose a rest frame in which the momentum takes the form $P^\mu = (M,0,0,0)$, so that the susy anti-commutator becomes

$$\left\{Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\right\} = 2M\sigma^0_{\alpha\dot{\beta}}\delta^{ab}\,. \tag{2.19}$$

Then one can define a *ground state* via

$$Q^a_\alpha\,|0\rangle = 0 \qquad a=1,\ldots,\mathcal{N} \quad \alpha=1,2\,.$$

Note that such *ground states* can carry momentum as well as spin. As can be seen from the commutator $\left[\bar{Q}^{a\dot{\alpha}}, J^i\right] = \frac{1}{2}\left(\bar{\sigma}^0\right)^{\dot{\alpha}\beta}\sigma^i_{\beta\dot{\gamma}}\bar{Q}^{a\dot{\gamma}}$, acting with a $\bar{Q}^a_{\dot{\alpha}}$ on such a state changes its helicity projection by $\pm\frac{1}{2}$. Therefore, in the case of $\mathcal{N}=1$, from a general massive spin s multiplet, a multiplet of $4(2s+1)$ states can be obtained. For instance when the original multiplet is a spin-1/2 doublet, we can generate two spin-1/2 doublets, a spin-1 triplet and a spin-0 singlet. For higher supersymmetries, in the case of vanishing central charges, this number of states can be obtained from *each* of the independent supercharges to give multiplets of $2^{2\mathcal{N}}(2s+1)$ states. In the presence of central charges, the supersymmetry generators get related and we cannot follow the simple arguments above. In this case the algebra possesses an important feature. To see this we will use the SU($\mathcal{N}$) transformations explained above. Under these transformations the central charges get changed according to $Z^{ab} \to U^a{}_c U^b{}_d Z^{cd}$. This enables us to bring them into block diagonal form

$$Z = \operatorname{diag}\left(\epsilon Z_1, \ldots, \epsilon Z_{\mathcal{N}/2}\right)\,,$$

where ϵ is the two-dimensional anti-symmetric tensor and we restricted ourselves to the case of an even number of supersymmetries. Now, for convenience, splitting the index a into two parts $(\hat{a},\bar{a})$ with $\hat{a}=1,2$, $\bar{a}=1,\ldots,\mathcal{N}/2$ and defining the operators

$$\mathcal{Q}^{\bar{a}}_{\alpha\pm} \equiv \frac{1}{2}\left(Q^{1\bar{a}}_\alpha + \pm\left(\sigma^0\right)_{\alpha\dot{\beta}}\bar{Q}^{2\bar{a}\dot{\beta}}\right)\,,$$

the relation (2.19) can be written as

$$\left\{\mathcal{Q}^{\bar{a}}_{\alpha\pm}, \bar{\mathcal{Q}}^{\bar{b}}_{\pm\dot{\beta}}\right\} = \delta^{\bar{a}\bar{b}}\left(\sigma^0\right)_{\alpha\dot{\beta}}\left(M \pm Z_{\bar{a}}\right)\,.$$

Note that on the right hand side there is no sum over $\bar{a}$. Now, in the same way as in the massless case above, in a unitary representation, the operator on the left hand side has to be positive, which results in the so called *BPS bound* (for Bogomolny-Prasad-Sommerfield)

$$M \geq |Z_{\bar{a}}| \, .$$

If the BPS bound is (partially) saturated, that means $M = |Z_{\bar{a}}|$ for some $\bar{a} = 1, \ldots, \mathcal{N}/2$, the supercharges $\mathcal{Q}^{\bar{a}}_{\alpha+}$ or $\mathcal{Q}^{\bar{a}}_{\alpha-}$ have to vanish. For these *BPS states* the susy multiplets hence are shortened. If $M = |Z_{\bar{a}}|$ is satisfied for r values of $\bar{a}$, the corresponding mulitplets are called $\left(\frac{1}{2}\right)^r$ BPS and have dimension $2^{2\mathcal{N}-2r}$.

2.2.2 SU(N) Super-Yang-Mills

Starting from the Lagrangian of SU(N) $\mathcal{N} = 1$ Super-Yang-Mills (SYM) theory in ten dimensions,

$$\mathcal{L} = \mathrm{tr}\left(-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\bar{\lambda}\Gamma^\mu D_\mu \lambda\right) \, ,$$

with the gauge-covariant derivative $D_\mu \mathcal{O} = \partial_\mu \mathcal{O} + \mathrm{i}g \left[A_\mu, \mathcal{O}\right]$, the ten-dimensional Gamma-matrices Γ and the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g \left[A_\mu, A_\nu\right]$, we can construct the $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions via dimensional reduction. By this procedure, in which the extra gauge field components become scalar fields $A^\mu \to X^i$ for $\mu = 4, \ldots, 9$, one gets

$$\begin{aligned} \mathcal{L}' = \mathrm{tr}\Bigg\{ &-\frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{g^2}\sum_i D_\mu X^i D^\mu X^i + \frac{1}{2}\sum_{ij}\left[X^i, X^j\right]^2 \\ &+\mathrm{i}\sum_a \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - g\sum_{a,b,i}\left(C_i^{-1}\right)^{ab}\lambda_a\left[X^i, \lambda_b\right] + g\sum_{a,b,i}\left(C_i\right)_{ab}\bar{\lambda}^a\left[X^i, \bar{\lambda}^b\right]\Bigg\} \, , \end{aligned} \tag{2.20}$$

with $i = 1, \ldots, 6$ and $a, b = 1, \ldots, 4$ and we have used the anti-symmetric matrices

$$\begin{aligned} &C_1 = \mathrm{i}\gamma_1\gamma_5 C \, , \quad C_2 = \mathrm{i}\gamma_2\gamma_5 C \, , \quad C_3 = \mathrm{i}\gamma_3\gamma_5 C \, , \\ &C_4 = \mathrm{i}\gamma_0\gamma_5 C \, , \quad C_5 = -\mathrm{i}C \, , \quad C_6 = -\mathrm{i}\gamma_5 C \, , \end{aligned}$$

with γ_i ($i = 0, \ldots, 3$) being the usual Dirac matrices in the Weyl representation and $C = -\mathrm{i}\gamma_2\gamma_0$ the four dimensional charge conjugation matrix.
Of course this procedure implies that the scalar fields X^i transform in the adjoint represenation of the SU(N) gauge group. Since the ten dimensional Majorana–Weyl spinors λ belong to the gauge multiplet in the ten dimensional Yang-Mills theory, the fermions λ_a and $\bar{\lambda}^a$ transform in the adjoint as well.

Four dimensional spacetime is special for one-form gauge potentials, since for this number of dimensions a two-form field-strength tensor can be self dual. This gives rise to an additional term in the Super-Yang-Mills action, such that the full lagrangian is given by

$$\mathcal{L} = \mathcal{L}' + \mathrm{tr}\left(\frac{\theta}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}\right) . \tag{2.21}$$

The supersymmetry transformations that leave the corresponding action invariant can, via dimensional reduction, be obtained from the susy variations of the ten-dimensional theory given by

$$\delta A_\mu = \mathrm{i}\bar{\zeta}\Gamma_\mu\lambda \,, \qquad \delta\lambda = -\frac{\mathrm{i}}{2}\Gamma^{\mu\nu}\zeta F_{\mu\nu} \,, \tag{2.22}$$

with ζ being a Majorana–Weyl spinor in ten-dimensions and $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$.

The field content of $\mathcal{N} = 4$ SYM in four dimensions is given by the gauge bosons A_μ, six massless real scalars X^i as well as four chiral- and four anti-chiral fermions λ^a_α and $\bar{\lambda}^a_{\dot{\alpha}}$. All of the fields, as pointed out before, transform in the adjoint representation of the SU(N) gauge group. Studying the one-loop β-function for the gauge coupling g one finds a remarkable feature of the theory. For any SU(N) gauge theory it is given by

$$\beta_1(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N - \frac{1}{6}\sum_i C_i - \frac{1}{3}\sum_j \tilde{C}_j\right) ,$$

where the sum over i is over all real scalars with quadratic Casimir C_i and the j sum is over all Weyl fermions with quadratic Casimir $\tilde{C}_j$. Since all fields transform in the adjoint, all Casimir eigenvalues are N. From this one directly sees that the one-loop beta function of the gauge coupling vanishes. By using superspace arguments and light-cone gauge calculations it can be further shown that it vanishes to all loop orders [13][14][15]. This important statement implies that the theory is conformal and does not need to be renormalized.

The conformal symmetry enlarges the Poincare supersymmetry that we described in the previous section to a superconformal symmetry. Namely, the lagrangian (2.21) possesses a global, continuous $\mathrm{PSU}(2,2|4)$ symmetry. This group has several ingredients which are given by

- **Conformal symmetry** generated by translations P^μ, Lorentz transformations $M_{\mu\nu}$, dilations D and special conformal transformations K^μ, which form the group SO(2,4),

- **R-symmetry** generated by the fifteen SO(6) generators T^a,

- **Poincare Supersymmetries** generated by four supercharges Q^a_α and their complex conjugates $\bar{Q}^{a\dot{\alpha}}$ (with vanishing central charges),
- **Conformal Supersymmetries** generated by another set of four supercharges S^a_α and their complex conjugates $\bar{S}^{a\dot{\alpha}}$.

The presence of the additional supercharges S^a_α is needed for the closure of the algebra, because the special conformal symmetries do not commute with the Poincare supersymmetry generators. The conformal symmetries and the R-symmetries combine to the bosonic subalgebra $\mathrm{SO}(2,4) \times \mathrm{SO}(6) \cong \mathrm{SU}(2,2) \times \mathrm{SU}(4)$ of $\mathrm{PSU}(2,2|4)$. The full algebra is given by

$$
\begin{aligned}
[D, P^\mu] &= -\mathrm{i}P^\mu\,, \qquad [D, K^\mu] = \mathrm{i}K^\mu\,,\\
[M_{\mu\nu}, P_\rho] &= -\mathrm{i}\left(\eta_{\mu\rho}P_\nu - \eta_{\rho\nu}P_\mu\right)\,, \qquad [M_{\mu\nu}, K_\rho] = -\mathrm{i}\left(\eta_{\mu\rho}K_\nu - \eta_{\rho\nu}K_\mu\right)\,,\\
[P_\mu, K_\nu] &= 2\mathrm{i}\left(M_{\mu\nu} - \eta_{\mu\nu}D\right)\,,\\
\\
\left\{Q^a_\alpha, \bar{Q}^b_{\dot\beta}\right\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{ab}\,, \qquad \left\{Q^a_\alpha, Q^b_\beta\right\} = \left\{\bar{Q}^a_{\dot\alpha}, \bar{Q}^b_{\dot\beta}\right\} = 0\,,\\
[M_{\mu\nu}, Q^a_\alpha] &= \mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q^a_\beta\,, \qquad \left[M_{\mu\nu}, \bar{Q}^{a\dot\alpha}\right] = \mathrm{i}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{a\dot\beta}\,,\\
[D, Q^a_\alpha] &= -\frac{\mathrm{i}}{2}Q^a_\alpha\,, \qquad \left[D, \bar{Q}^a_{\dot\alpha}\right] = -\frac{\mathrm{i}}{2}\bar{Q}^a_{\dot\alpha}\,,\\
[K^\mu, Q^a_\alpha] &= (\sigma^\mu)_{\alpha\dot\beta}\,\bar{S}^{a\dot\beta}\,, \qquad \left[K^\mu, \bar{Q}^a_{\dot\alpha}\right] = (\bar\sigma^\mu)^{\dot\alpha\beta}\,S^a_\beta\,,\\
\\
\left\{S^a_\alpha, \bar{S}^b_{\dot\beta}\right\} &= 2\sigma^\mu_{\alpha\dot\beta}K_\mu\delta^{ab}\,, \qquad \left\{S^a_\alpha, S^b_\beta\right\} = \left\{\bar{S}^a_{\dot\alpha}, \bar{S}^b_{\dot\beta}\right\} = 0\,,\\
[D, S^a_\alpha] &= \frac{\mathrm{i}}{2}S^a_\alpha\,, \qquad \left[D, \bar{S}^a_{\dot\alpha}\right] = \frac{\mathrm{i}}{2}\bar{S}^a_{\dot\alpha}\,,\\
\left\{Q^a_\alpha, S^b_\beta\right\} &= \epsilon_{\alpha\beta}\left(\delta^{ab}D + T^{ab}\right) + \frac{1}{2}\delta^{ab}\sigma^{\mu\nu}_{\alpha\beta}M_{\mu\nu}\,,\\
\left\{\bar{Q}^a_{\dot\alpha}, \bar{S}^b_{\dot\beta}\right\} &= \epsilon_{\dot\alpha\dot\beta}\left(-\delta^{ab}D + T^{ab}\right) + \frac{1}{2}\delta^{ab}\bar\sigma^{\mu\nu}_{\dot\alpha\dot\beta}M_{\mu\nu}\,,\\
\left\{Q^a_\alpha, \bar{S}^b_{\dot\beta}\right\} &= \left\{\bar{Q}^a_{\dot\alpha}, S^b_\beta\right\} = 0
\end{aligned}
\tag{2.23}
$$

and all other commutators vanish. Note that while Q^a_α transforms in the $\mathbf{6}$ and $\bar{Q}^a_{\dot\alpha}$ in the $\bar{\mathbf{6}}$ of SO(6), S^a_α transforms in the $\bar{\mathbf{6}}$ and $\bar{S}^a_{\dot\alpha}$ in the $\mathbf{6}$. The action of $\mathcal{N} = 4$ SU(N) SYM in four dimensions, in fact, is conjectured to have an additional global discrete symmetry. In order to see this one can combine the gauge coupling constant g and the instanton-angle θ into a complex parameter

$$
\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g^2}\,. \tag{2.24}
$$

Clearly the theory is invariant under $\theta \to \theta + 2\pi$ corresponding to $\tau \to \tau + 1$. The *Montonen-Olive conjecture* states, that the full quantum theory is also invariant under $\tau \to -\frac{1}{\tau}$ which, when $\theta = 0$ corresponds to $g \to \frac{1}{g}$. The combination yields the so called *S-duality* group $\mathrm{SL}(2,\mathbb{Z})$ which is important in the context of the AdS/CFT duality as we will see later.
More mathematical aspects of the superconformal group $\mathrm{PSU}(2,2|4)$ will be studied in more detail in the context of a string σ-model later. For now we will focus on its representations, since in a physical theory the states should come in unitary representations of the underlying symmetry algebra. The unitary representations of the superconformal algebra can be labelled by the quantum numbers of its bosonic subgroup which we will denote by

$$(s_+, s_-),\ \Delta,\ [r_1, r_2, r_3]\ ,$$

with $s_\pm \in \frac{\mathbb{Z}_+}{2}$ labelling the $\mathrm{SO}(1,3)$ representation, $\Delta \in \mathbb{Z}_+$ the label of the $\mathrm{SO}(1,1)$ irreducible representation (irrep) and $[r_1, r_2, r_3]$ are the Dynkin labels of the $\mathrm{SO}(6)_\mathrm{R}$ representation. As can be seen from the algebra (2.23) the supercharges S^a_α have scaling dimension $-\frac{1}{2}$. Consequently, the application of S^a_α on an operator lowers its dimension by $\frac{1}{2}$. For operators in a unitary representation this must at some point yield zero, because in such representations operators of negative scaling dimension are not possible. Therefore one defines *superconformal primary operators* $\mathcal{O}$ by the relation

$$[S^a_\alpha, \mathcal{O}]_\pm = 0 \qquad \forall \alpha, a\,,$$

where $[.,.]_\pm$ denotes the commutator in the case of a bosonic and the anti-commutator in the case of a fermionic $\mathcal{O}$. Clearly these are the operators of lowest scaling dimension in given multiplets. Their *(superconformal) descendants* $\mathcal{O}'$ are defined via

$$\mathcal{O}'^a_\alpha = [Q^a_\alpha, \mathcal{O}]_\pm\ .$$

From the fact that a superconformal primary operator cannot be the Q-commutator of another operator and the susy transformations of the fields, one can see that superconformal primaries can only involve the scalar fields X^i in a symmetrized way. The simplest of those operators are the *single trace* operators of the form

$$\mathrm{symtr}\left(X^{i_1} \dots X^{i_n}\right)\ ,$$

where symtr(.) denotes the symmetrized trace over the gauge algebra. Since $\mathrm{tr}\left(X^i\right) = 0$ the simplest operator is given by

$$\sum_i \mathrm{tr}\left(X^i X^i\right)$$

and is called *Konishi operator*. More complicated superconformal primaries are the *multiple trace* operators which are obtained from products of single trace operators.

The scaling dimension Δ of any operator in a unitary representation can be shown to be bounded from below by the spin and $\mathrm{SO}(6)_\mathrm{R}$ quantum numbers. Since the operators with lowest dimensions were shown to be scalars, such that the spin quantum numbers are zero, in this case only the $\mathrm{SO}(6)_\mathrm{R}$ quantum numbers are important. By systematically analyzing the possible situations, four series of scaling dimensions of the superconformal primaries were found (for details see [9] and references therein)

- $\Delta = r_1 + r_2 + r_3$
- $\Delta = \frac{3}{2}r_1 + r_2 + \frac{1}{3}r_3 \geq 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3$ for $r_1 \geq r_3 + 2$
- $\Delta = \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \geq 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3$ for $r_3 \geq r_1 + 2$
- $\Delta \geq \mathrm{Max}\left(2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3; 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3\right)$

The first three cases are discrete series for which at least one of the supercharges Q^a_α commutes with the superconformal primary. Hence, as explained before, these multiplets are shortened BPS multiplets. The fourth case corresponds to a continuous series of non-BPS states. In table 2.1 the operators and some of their properties are summarized.

Remarkably it is possible to explicitly write down the various BPS operators in a

	max. spin	$\mathrm{SO}(6)_\mathrm{R}$ rep.	Δ
1/2-BPS	2	$[0,k,0]$, $k \geq 2$	k
1/4-BPS	3	$[l,k,l]$, $l \geq 1$	$k+2l$
1/8-BPS	7/2	$[l,k,l+2m]$, $m \geq 1$	$k+2l+3m$
non-BPS	4	any	any

Table 2.1: Four types of operators in 4D $\mathcal{N}=4$ SU(N) SYM. The only 1/1-BPS operator is the identity.

general fashion. We will focus on the 1/2-BPS operators here, since these will be important later. The simplest series is given by the single trace operators

$$\mathcal{O}_k \equiv \frac{1}{n_k}\mathrm{str}\left(X^{\{i_1}\dots X^{i_k\}}\right) , \tag{2.25}$$

where $\{i_1 \dots i_k\}$ denotes the $\mathrm{SO}(6)_\mathrm{R}$-traceless part of the tensor and n_k is a normalization factor. One can build multiple trace operators out of these by taking the product of single trace operators and projecting onto the $[0, \sum_i k_i, 0]$ representation inside the $[0, k_1, 0] \otimes \cdots \otimes [0, k_n, 0]$ tensor product. This is denoted by

$$\mathcal{O}_{(k_1,\ldots,k_n)} \equiv (\mathcal{O}_{k_1} \ldots \mathcal{O}_{k_n})_{[0,k,0]} \ ,$$

with $k = k_1 + \cdots + k_n$.

In the above analysis we set aside an important subtlety. Namely, since the lagrangian (2.21) describes an interacting field theory, the scaling dimension of any operator in principle depends on the gauge coupling g. At zero coupling the *bare* dimensions Δ_0 have the values we gave here and the corrections due to non-vanishing coupling are called *anomalous dimensions*. Note that, first, the anomalous dimensions within the same representation are equal and second, the anomalous dimensions of BPS operators vanish.

2.3 The Maldacena Conjecture

Having reviewed the most important aspects of type IIB superstring theory in ten dimensions and $\mathcal{N} = 4$ SU(N) super-Yang-Mills theory in four dimensions, we are in position to discuss the Maldacena conjecture and some rather heuristic argument for why it has a chance to be true at least in the low energy regime. This presentation follows the ones in [16][17].

2.3.1 Classical Black p-Brane Solutions of Supergravity

First we discuss a specific class of solutions of the low energy effective theory of type IIB string theory corresponding to black brane solutions. We start by rewriting the (bosonic) supergravity action (2.15) in the following way

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{G} \left(\mathrm{e}^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi \right) - \frac{1}{2} \sum_n \frac{1}{n!} F_n^2 + \ldots \right) ,$$

where the dots represent the NS-NS 3-form (formerly called H_3) field strength term as well as the Chern-Simons term. As stated before, this action cannot describe the theory on its own, since the self-duality constraint for the RR 5-form field strength F_5 has still to be imposed. It however turns out to be possible to use the action above in order to derive the equations of motion and impose the self-duality constraint afterwards.

For convenience we perform a Weyl rescaling,

$$\begin{aligned} G_{\mu\nu} &\to \mathrm{e}^{-\frac{1}{2}\Phi} G_{\mu\nu} \ , \\ \sqrt{G}\mathrm{e}^{-2\Phi} R &\to \sqrt{G} \left(R - \frac{9}{2} \frac{1}{\sqrt{G}} \partial_\mu \left(\sqrt{G} \partial^\mu \Phi \right) - \frac{9}{2} \partial_\mu \Phi \partial^\mu \Phi \right) , \end{aligned}$$

to put the action into the Einstein frame

$$S_{\mathrm{E}} = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{G} \left(R - \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \sum_n \frac{1}{n!} \mathrm{e}^{a_n \Phi} F_n^2 + \dots \right) , \tag{2.26}$$

with $a_n = -\frac{1}{2}(n-5)$. For simplicity we will focus on the case where only one of the F_n is non-vanishing. Then the equations of motion are given by

$$\begin{aligned} R^\mu{}_\nu &= \frac{1}{2}\partial^\mu \Phi \partial_\nu \Phi + \frac{1}{2n!} \mathrm{e}^{a_n \Phi} \left(n F^{\mu\xi_2 \dots \xi_n} F_{\nu \xi_2 \dots \xi_n} - \frac{n-1}{8} \delta^\mu_\nu F_n^2 \right) , \\ \nabla^2 \Phi &= \frac{1}{\sqrt{G}} \partial_\mu \left(\sqrt{G} \partial^\mu \Phi \right) = \frac{a_n}{2n!} \mathrm{e}^{a_n \Phi} F_n^2 , \\ 0 &= \partial_\mu \left(\sqrt{G} \mathrm{e}^{a_n \Phi} F^{\mu \nu_2 \dots \nu_n} \right) \end{aligned} \tag{2.27}$$

and F_n satisfies the corresponding Bianchi identity. Note that the equations of motion are invariant under the duality transformations

$$a_n \Phi \to -a_n \Phi, \quad n \to D - n , \quad F_n \to \tilde{F}_{D-n} = \mathrm{e}^{a_n \Phi} \star F_n , \tag{2.28}$$

where $\star$ denotes the Hodge star operator.
Next we consider a diagonal metric

$$\mathrm{d}s^2 = -B^2(r)\mathrm{d}t^2 + C^2(r) \sum_{i=1}^{p} \left(\mathrm{d}x^i \right)^2 + F^2(r)\mathrm{d}r^2 + G^2(r) r^2 \mathrm{d}\Omega_{8-p}^2 , \tag{2.29}$$

where we used the coordinates (t, x^i, y^a), $i = 1, \dots, p$, $a = 1, \dots, 9-p$ and $\mathrm{d}\Omega_d^2$ is the metric on the d-dimensional unit sphere. This corresponds to a (p+1) dimensional hypersurface with $\mathbb{R}^{p+1} \times \mathrm{SO}(1,p)$ symmetry and a $(D-p-1)$-dimensional transverse space with $\mathrm{SO}(10-p-1)$ symmetry in a ten dimensional spacetime. Then $r^2 = \sum_{a=1}^{9-p} (y^a)^2$ corresponds to the (quadratic) distance from the hypersurface, x^i denote the coordinates on the surface and y^a the coordinates in the transverse space. The functions B, C, F, G are restricted to tend to one in the large r limit.

Now we can make the *electric* Ansatz,

$$F_{ti_1 \dots i_p r} = \epsilon_{i_1, \dots i_p} k(r) ,$$

for the field strength. The equation of motion for F_n then yields

$$k(r) = \mathrm{e}^{-a_n \Phi} B(r) C^p(r) F \frac{Q}{(G(r) r)^{8-p}} ,$$

with Q being a constant. This Ansatz corresponds to an electrically charged p-brane with the charge density

$$\mu_p = \frac{1}{\sqrt{s\kappa^2}} \int_{S^{8-p}} \tilde{F}_{8-p} = \frac{\Omega_{8-p} Q}{\sqrt{2\kappa^2}},$$

where Ω_{8-p} is the volume of the $(8-p)$-dimensional unit sphere and the *magnetic* field strength $\tilde{F}_n$ can be obtained from F_n via the duality (2.28).
The form of the functions B, C, F and G can be obtained from the equations of motion. Since the calculation is rather tedious we just give solutions representing a two-parameter subset of the most general ones.

$$\mathrm{d}s^2 = H^{-2\frac{7-p}{\Delta}}(r)\left(-f(r)\mathrm{d}t^2 + \sum_{i=1}^{p}\left(\mathrm{d}x^i\right)^2\right) + H^{2\frac{p+1}{\Delta}}(r)\left(f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right), \tag{2.30}$$

$$F_{ti_1\ldots i_p r} = \epsilon_{i_1,\ldots i_p} H^{-2}(r)\frac{Q}{r^{8-p}},$$

where

$$H(r) = 1 + \left(\frac{h}{r}\right)^{7-p}, \qquad \Delta = (p+1)(7-p) + 4a_n^2,$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{7-p}, \qquad \frac{\Delta Q^2}{16(7-p)} = h^{2(7-p)} + r_0^{7-p} h^{7-p}.$$

The two parameters are the charge density Q of the brane and r_0. For $r_0 \neq 0$ the solution develops a horizon ar $r = r_0$.

We can obtain the *extremal p-Brane* solution by setting $r_0 = a_n = 0$. In this case the solution (2.30) simplifies further in the case $p = 3$, since then

$$\Delta = 16, \qquad h^4 = \frac{Q}{2}$$

and the *extremal 3-brane* solution is given by

$$\mathrm{d}s^2 = H^{-\frac{1}{2}}(r)\left(-\mathrm{d}t^2 + \sum_{i=1}^{p}\left(\mathrm{d}x^i\right)^2\right) + H^{\frac{1}{2}}(r)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right), \tag{2.31}$$

with $H(r) = 1 + \frac{Q}{4r^4}$. It was further shown [18] that for a single Dp-brane of type IIB superstring theory

$$\mu_p = \frac{\sqrt{2\pi}}{(2\pi l_\mathrm{s})^{p-3}},$$

such that

$$Q = g_\mathrm{string}\frac{(2\pi l_\mathrm{s})^{7-p}}{\Omega_{8-p}}.$$

2.3.2 Anti de Sitter Space

Here, we want to briefly review the geometric properties of anti de Sitter (AdS) spaces. They arise as maximally symmetric solutions of Einstein's vacuum equations with cosmological constant

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = \Lambda G_{\mu\nu} \,.$$

Maximal symmetry implies that not only the Ricci tensor is proportional to the metric, $R_{\mu\nu} = \frac{\Lambda}{2-D} G_{\mu\nu}$, but also

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)} \left(G_{\nu\sigma} G_{\mu\rho} - G_{\nu\rho} G_{\mu\sigma} \right) \,.$$

There are three classes of solutions of this kind, namely

- $\Lambda < 0$ anti de Sitter space,
- $\Lambda = 0$ flat space $\mathbb{R}^D$,
- $\Lambda > 0$ de Sitter space.

The most pictorial way to describe AdS_{D+1} is, as a hyperboloid embedded in a $(D+2)$-dimensional pseudo-euclidean space with coordinates y^a $a = 0, \ldots, D+1$, metric
$G_{ab} = \mathrm{diag}\,(+, -, \ldots, -, +)$ and isometry group $\mathrm{SO}(2, D)$. Then AdS_{D+1} is defined as

$$y^2 \equiv \left(y^0\right)^2 - (\vec{y})^2 + \left(y^{D+1}\right)^2 \stackrel{!}{=} R^2 = \text{const.} \tag{2.32}$$

and the induced metric clearly respects the full isometry group $\mathrm{SO}(2, D)$.
The defining equation (2.32) is solved by the coordinates

$$\begin{aligned} y^0 &= R \cosh \rho \cos \tau \,, \qquad & y^{D+1} &= R \cosh \rho \sin \tau \,, \\ y^i &= R \sinh \rho \Omega^i \,, \qquad & \text{with} \quad & \sum_{i=1}^{D} \left(\Omega^i\right) = 1 \,, \end{aligned} \tag{2.33}$$

yielding the induced metric

$$\mathrm{d}s^2 = R^2 \left(- \cosh^2 \rho \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \mathrm{d}\Omega^2 \right) \,. \tag{2.34}$$

Since $\rho \geq 0$ and $0 \leq \tau < 2\pi$, such that the hyperboloid is covered once, these coordinates are called *global coordinates*. Near $\rho = 0$ the metric looks loke the one of $S^1 \times \mathbb{R}^D$, with S^1 representing the compact time direction as depicted in

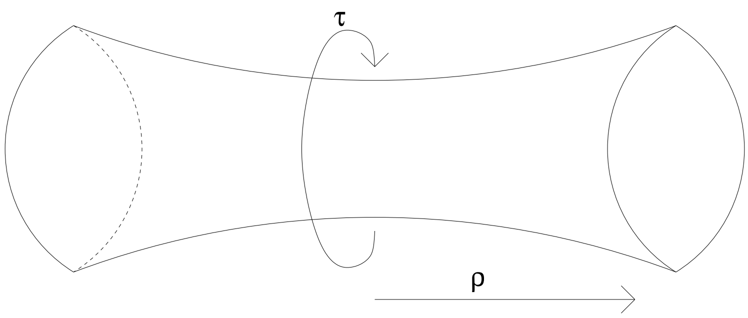

Figure 2.1: AdS_{D+1} as a hyperboloid embedded in $\mathbb{R}^{2,D}$. To obtain a causal space one has to unwrap the τ-circle.

figure 2.1. The universal covering of AdS_{D+1}, however, contains no closed timelike curves and is therefore causal. A causal spacetime can be described using global coordinates and letting $-\infty < \tau < \infty$, which corresponds to simply unwrapping the S^1.

In a physical context, it is often more convenient to look at the euclidean continuation of the space-time background. Performing such a continuation for one of the timelike coordinates, that means

$$y_{\mathrm{E}}^{D+1} = \mathrm{i}y^{D+1}\,,$$

one gets the euclidean version of AdS_{D+1}

$$y^2 \equiv \left(y^0\right)^2 - \sum_{\mu=1}^{D+1} \left(y^\mu\right)^2 \stackrel{!}{=} R^2 = \text{const.}\,,$$

with isometry group $\mathrm{SO}(1, D+1)$. This metric is topologically equivalent to that of the unit ball $\sum_{\mu=1}^{D+1} \left(x^\mu\right)^2 \leq 1$.

In the following we will use different coordinate systems for anti de Sitter spaces. One of them is the *light-cone* coordinate system which can be obtained from the embedding coordinates as

$$u \equiv y^0 + \mathrm{i}y^{D+1}\,, \quad v \equiv y^0 - \mathrm{i}y^{D+1}\,, \quad \xi^\alpha \equiv \frac{y^\alpha}{u} \quad \alpha = 1, \ldots, D\,, \tag{2.35}$$

or in case of euclidean signature

$$u \equiv y^0 + y^{D+1}\,, \quad v \equiv y^0 - y^{D+1}\,, \quad \xi^\alpha \equiv \frac{y^\alpha}{u} \quad \alpha = 1, \ldots, D\,, \tag{2.36}$$

Then AdS_{D+1}, in both cases, is defined by $v = \xi^2 u + \frac{R^2}{u}$ and the metric in this coordinates is given by

$$\left(\mathrm{d}s^2\right)_{\mathrm{AdS}_{D+1}} = R^2 \frac{\mathrm{d}u^2}{u^2} + u^2 \mathrm{d}\xi^2 = \mathrm{d}u\mathrm{d}v - \mathrm{d}\vec{y}^2\,. \tag{2.37}$$

If one sets $R = 1$ there is another coordinate system given by $(\xi^0, \xi^\alpha) = (u^{-1}, \xi^\alpha)$ with metric $\mathrm{d}s^2 = \frac{1}{(\xi^0)^2}\left((\mathrm{d}\xi^0)^2 + \mathrm{d}\vec{\xi}^2\right)$.
The light-cone coordinate system is particularly useful since the boundary of AdS_{D+1} can be explored easily in these coordinates. In order to do that we rescale the light-cone coordinates by a constant factor b

$$y^\alpha \equiv b\tilde{y}^\alpha\,, \qquad u \equiv b\tilde{u}\,, \qquad v \equiv b\tilde{v}$$

and take the limit $b \to \infty$. Then, from the definition of anti de Sitter space in the embedding, $y^2 = R^2$, in this limit we obtain

$$\tilde{u}\tilde{v} - \vec{\tilde{y}}^2 = \frac{R^2}{b^2} \to 0\,,$$

which should describe the boundary of AdS_{D+1}. There is some additional condition on the coordinates though. By taking the limit $b \to \infty$ some non-trivial identification of the coordinates, namely $(\tilde{u}, \tilde{v}, \tilde{y}^\alpha) \sim t\,(\tilde{u}, \tilde{v}, \tilde{y}^\alpha)$, arises. Therefore, the boundary is defined as the D-dimensional space

$$\left(y^0\right)^2 + \left(y^{D+1}\right)^2 = \vec{y}^2 = 1\,, \tag{2.38}$$

which topologically is $\frac{S^1 \times S^{D-1}}{\mathbb{Z}_2}$. Alternatively, in the light-cone coordinates, we obtain two different coordinate systems.

- When $v \neq 0$ we can scale v to one and the boundary is described by $u = \vec{y}^2 = 1$ and the single point $v = 0$ which can be thought of as "the point of infinity".
- When $u \neq 0$ we can, analogously, scale u to one and the boundary is described by $v = (\vec{y}')^2 = 1$ and the single point $u = 0$.

Since infinity is included, the boundary is automatically compactified.

It is interesting to investigate how the isometry group of AdS_{D+1} acts on its boundary. Take for convenience the euclidean version of the space. Then an $\mathrm{SO}(1, D+1)$ transformation acts as

$$\Lambda \begin{pmatrix} u \\ v \\ \vec{y} \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ \vec{y}' \end{pmatrix}\,,$$

which clearly preserves the norm and hence the boundary $uv - \vec{y}^2 = 0$, $(u, v, \vec{y}) \sim \lambda\,(u, v, \vec{y})$ is mapped to itself. We can expand the transformation generator Λ as $\Lambda = \mathbb{1}_{D+2} + w + \mathrm{O}(w^2)$. Then the infinitesimal transformation w has to be of the form

$$w = \begin{pmatrix} a & 0 & \vec{\alpha}^{\mathrm{T}} \\ 0 & -a & \vec{\beta}^{\mathrm{T}} \\ \frac{1}{2}\vec{\beta} & \frac{1}{2}\vec{\alpha} & w_D \end{pmatrix}\,,$$

with w_D an anti-symmetric $D \times D$ matrix. Now, as described above, we can choose $v = 1$ (which is possible everywhere but at infinity $v = 0$) and describe the boundary by the coordinates $\vec{y}$. The $\mathrm{SO}(1, D+1)$ transformation gives

$$(\mathbb{1}_{D+2} + w) \begin{pmatrix} u \\ v \\ \vec{y} \end{pmatrix} = \begin{pmatrix} u(1+a) + \vec{\alpha} \cdot \vec{y} \\ v(1-a) + \vec{\beta} \cdot \vec{y} \\ \left(\vec{y} + \frac{u}{2}\vec{\beta} + \frac{v}{2}\vec{\alpha}\right) + w_D \vec{y} \end{pmatrix},$$

but since v changes non-trivially we have to rescale the image coordinates such that $v' = 1$. Using the equivalence scaling we get $(u', v', \vec{y}\prime) \to \left(\frac{u'}{v'}, 1, \frac{\vec{y}'}{v'}\right)$ as needed. This implies

$$\vec{y}' \to \frac{\vec{y}'}{v'} = \vec{y}\left(1 + a - \vec{\beta} \cdot \vec{y}\right) + \frac{\vec{y}^2}{2} + \frac{1}{2}\vec{\alpha} + w_n \vec{y}\,. \tag{2.39}$$

The resulting transformations (2.39) can be obtained as combinations of four special cases, namely

- only $\vec{\alpha} \neq 0$: $\quad \vec{y} \to \vec{y} + \frac{1}{2}\vec{\alpha}$,
- only $w_n \neq 0$: $\quad \vec{y} \to w_n \vec{y}$,
- only $a \neq 0$: $\quad \vec{y} \to \vec{y}(1+a)$,
- only $\vec{\beta} \neq 0$: $\quad \vec{y} \to \vec{y}\left(1 - \vec{\beta} \cdot \vec{y}\right) + \frac{1}{2}\vec{y}^2 \vec{\beta}$,

corresponding to *translations*, *rotations*, *dilations* and *special conformal transformations* respectively. Hence the $\mathrm{SO}(1, D+1)$ isometries of the euclidean form of AdS_{D+1} act as the conformal group on its boundary which is a compactification of flat D-dimensional euclidean space.

2.3.3 The AdS/CFT Duality Conjecture

Finally we are able to discuss the duality between type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SU(N) super-Yang-Mills theory in four dimensions. The presentation here follows [16].

The setup we are going to investigate is given by N coincident D3 branes in type IIB superstring theory on a flat ten-dimensional Minkowski target-space. As we saw in section 2.1 two kinds of perturbative excitations arise in the string theory, closed and open strings. In a low energy regime just the massless modes survive, such that the closed strings give rise to a gravity supermultiplet propagating in the bulk and the open strings ending on the D3-branes give an $\mathcal{N} = 4$ vector supermultiplet on the four-dimensional world-volume of the branes[5] which can be effectively described by $\mathcal{N} = 4$ U(N) super-Yang-Mills theory. The effective action of the massless string modes can be split according to

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int.}}\,, \tag{2.40}$$

where

- S_{bulk} is the action of ten-dimensional supergravity and some higher order corrections,
- S_{brane} describes the dynamics on the four-dimensional world-volume of the D3-branes and contains the action of $\mathcal{N} = 4$ U(N) super-Yang-Mills theory and some corrections which are of higher order in derivatives of the fields and
- $S_{\text{int.}}$ describes the interactions of the brane and the bulk modes.

The higher order corrections in the action (2.40) appear, because the effects of integrating out the massive fields are taken into account.

Now we take the limit in which l_s and therefore α' is sent to zero while the dimensionless parameters g_{string} and N are kept fixed. Since all higher corrections to the brane action come with positive powers of α' they vanish in this limit, leaving just the pure $\mathcal{N} = 4$ super-Yang-Mills theory. But also the rest of the action simplifies since the ten-dimensional Newton's constant is proportional to κ^2 and therefore also to l_s^8 and all interaction terms in $S_{\text{int.}}$ as well as the higher order corrections in S_{bulk} come with positive powers of the ten-dimensional Newton's constant. Therefore in this specific limit the theory describes two decoupled systems, given by

- free supergravity in the ten-dimensional bulk and
- $\mathcal{N} = 4$ U(N) pure super-Yang-Mills on the four-dimensional brane world-volume.

There is also a different way to describe the above setup. D3-branes are massive, charged objects and therefore they are sources for the supergravity fields and, as we saw in section 2.3.1, there are solutions to the type IIB supergravity equations

[5]Dp-branes are 1/2-BPS objects and therefore break the $\mathcal{N} = 2$ supersymmetry of type IIB to $\mathcal{N} = 1$. Dimensional reduction to the D3-brane world-volume yields $\mathcal{N} = 4$ supersymmetry as discussed in section 2.2. For details on the gauge theories of open strings ending on Dp-branes see for instance chapter 2 of [19].

of motion corresponding to (extremal) 3-Branes given by

$$\begin{aligned}
\mathrm{d}s^2 &= H^{-\frac{1}{2}}(r)\left(-\mathrm{d}t^2+\sum_{i=1}^{3}\left(\mathrm{d}x^i\right)^2\right)+H^{\frac{1}{2}}(r)\left(\mathrm{d}r^2+r^2\mathrm{d}\Omega_5^2\right)\,,\\
F_{ti_1i_2i_3r} &= \epsilon_{ti_1i_2i_3r}H^{-2}(r)\frac{16N\pi g_{\mathrm{string}}l_{\mathrm{s}}^4}{r^5}\,,\\
\text{with } H(r) &= 1+\frac{4N\pi g_{\mathrm{string}}l_{\mathrm{s}}^4}{r^4}=1+\frac{Q}{4r^4}\,.
\end{aligned}$$

Here we have introduced a factor of N in the D3-brane charge, since this solution is supposed to correspond to a stack of N D3 branes.

The metric in the solution has an important property, namely the time-component is non-constant. This implies a redshift of the energy that is measured by an observer at infinity as compared to the energy measured by an observer at a position r given by

$$E_{r=\infty}=H^{-\frac{1}{4}}(r)E_r\,. \tag{2.41}$$

This means that the energy of an object brought closer to $r=0$ and measured by an observer at infinity decreases. Therefore an observer at infinity sees, in the low energy limit, two kinds of excitations which are

- massless particles in the bulk and
- any excitations near $r=0$.

Now in [20] and [21] it was found that in the case of 3-Branes the cross section for the absorption of all massless particles in the closed string sector goes like

$$\sigma\propto\omega Q^8\,,$$

with ω being the energy. That means that in the low energy sector the bulk fields decouple from the near horizon ($r=0$) region. On the other hand, the energy measured from infinity goes like

$$E_{r=\infty}=\left(\frac{\sqrt{2}r}{Q^{\frac{1}{4}}}+\mathcal{O}(r^5)\right)E_r$$

and therefore demanding $E_{r=\infty}$ to be smaller and smaller the excitations are more and more confined to the region near $r=0$. Therefore the low energy limit of the setup again contains two decoupled systems, one of which is free bulk supergravity and the other one is the dynamics of the near horizon region of the brane stack. In the near horizon region $r \ll \sqrt{Q}$ the metric is approximated by

$$\mathrm{d}s^2 \simeq \frac{2r}{Q}\left(\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right) + \frac{Q}{2r^2}\mathrm{d}r^2 + \frac{Q}{2}\mathrm{d}\Omega_5^2\,,$$

which seems to be singular for $r \to 0$. Therefore we perform a change of variables

$$u \equiv \frac{Q}{2r}, \qquad \mathrm{d}u = -\frac{Q}{2r^2}\mathrm{d}r\,,$$

such that the metric becomes

$$\mathrm{d}s^2 \simeq \frac{Q}{2}\left[\frac{1}{u^2}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \frac{\mathrm{d}u^2}{u^2} + \mathrm{d}\Omega_5^2\right]\,.$$

Now we can blow up the region $r \to 0$ by taking the limit $l_s \to 0$ and, in that way, obtain the metric of an $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. This means that the near-horizon geometry of the D3-brane stack is given by a direct product of AdS_5 and S^5 both having the radius $R \equiv \frac{\sqrt{Q}}{2}$.
In both ways of describing the setup we considered, in the low energy regime and taking the limit $l_\mathrm{s} \to 0$, two decoupled systems appeared. One of these systems was in both cases given by free ten-dimensional supergravity. Hence it is natural to identify the other two systems, which brings us to the conjecture

$\mathcal{N} = 4$ $U(N)$ SYM in four dimensions is dual to type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

This statement is of course much stronger than what is motivated by the discussion above. Actually it is the strongest form to put the conjecture by Maldacena, claiming that the two theories are exactly the same for all values of g_string and N. One can formulate weaker forms of the conjecture that are valid only in specific limits of the theories. The supergravity approximation on the string theory side for instance is valid when the radius of curvature of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is large compared to the string length

$$\frac{R^4}{l_\mathrm{s}^4} \propto g_\mathrm{string}N \propto g_\mathrm{YM}^2 N \gg 1$$

and the string coupling g_string is small.[6] Therefore we need $N \gg 1$. The perturbative description of Yang-Mills theory, on the other hand, is valid when $g_\mathrm{string}N \propto g_\mathrm{YM}^2 N \ll 1$. The regimes of validity therefore are perfectly incompatible

[6]The relation between the string coupling g_string and the Yang-Mills coupling g_YM arises from the gauge theory on the brane stack.

and we can use weakly coupled supergravity to learn something about strongly coupled gauge theory and vice versa.[7]
There is more evidence supporting the validity of the conjecture stated above which comes from the global symmetries of the theories. As we saw in section 2.2.2 the global symmetry group of super-Yang-Mills theory in four dimensions is $\mathrm{PSU}(2,2|4)$ whose bosonic subgroup contains the four dimensional conformal group and an SO(6) R-symmetry. Additionally there is the conjectured $\mathrm{SL}(2,\mathbb{Z})$ Montonen-Olive symmetry. On the string theory or supergravity side, the $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the axion-dilaton pair are present.[8] We saw in section 2.3.2 that the $\mathrm{SO}(2,4)$ isometry acts on the four-dimensional boundary of AdS_5 as the conformal group. The (bosonic) global symmetries of the two theories, summarized in table 2.2, therefore match perfectly when the U(N) gauge theory lives on the boundary of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. In section 2.4 we will see that the superstring theory on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background actually possesses the whole $\mathrm{PSU}(2,2|4)$ symmetry. As stated before we concen-

$\mathrm{AdS}_5 \times \mathrm{S}^5$ Type IIB	$\mathcal{N} = 4$ SYM in (3+1)D
• SO(2,4) isometry of AdS_5 • SO(6) isometry of S^5 • SL(2,$\mathbb{Z}$) of axion-dilaton pair	• conformal symmetry in (3+1)D • SU(4) R-symmetry • SL(2,$\mathbb{Z}$) Montonen-Olive symmetry $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\mathrm{YM}}^2}$

Table 2.2: Match of the (bosonic) global symmetries on the two sides of the duality.

trated on a special case of a family of dualities. Indeed in the original work by Maldacena[1] more dualities were conjectured. They relate M theory on $\mathrm{AdS}_7 \times S^4$ to a 6D (0,2) super-conformal field theory (SCFT) or M theory on $\mathrm{AdS}_4 \times S^7$ to a three dimensional $\mathcal{N} = 8$ SCFT. The most general form of this kind of duality conjectures can be formulated as

String-/M-theory on $AdS_d \times M_{D-d}$ is dual to a conformal QFT on boundary of AdS_d.

[7]Actually this reasoning is only valid for $g_{\mathrm{string}} < 1$. In the other case we can perform an $\mathrm{SL}(2,\mathbb{Z})$ transformation and get similar relations.

[8]Actually in section 2.1.4 we described an $\mathrm{SL}(2,\mathbb{R})$ symmetry of the low energy effective supergravity theory. However not the full symmetry is shared by the full string theory. By some stringy and quantum effects, which can be understood in terms of the Dirac quantization condition, it gets broken to its subgroup $\mathrm{SL}(2,\mathbb{Z})$.

2.3.4 Mapping of Physical Quantities

In [1] no precise identification of the observables on the two sides of the duality was given. However, a proposal was given shortly afterwards by Witten in [2]. The main idea is to equate the generating functional for the correlators of an operator $\mathcal{O}$ in the field theory sourced by a field Φ_0, i.e. $S \supset \int_{S^D} \Phi_0 \mathcal{O}$, to the string (or supergravity) partition function of the field Φ, which at the boundary approaches Φ_0. The Ansatz therefore is

$$\left\langle \exp \int_{S^4} \Phi_0 \mathcal{O} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{string}}\left(\Phi\right) \ . \tag{2.42}$$

If the operator $\mathcal{O}$ has scaling dimension Δ, then the field Φ is supposed to have scaling dimension $4-\Delta$ in order to make the source term scale invariant. In a similar fashion the quantum numbers of the operator $\mathcal{O}$ dictate the quantum numbers of the source Φ_0 and therefore the field Φ. In the regime in which the supergravity approximation is valid one can replace the string partition function by $\exp\left(-I_{\mathrm{S}}\left(\Phi\right)\right)$ with I_{S} being the classical supergravity action. In order to make the Ansatz more clear we will sketch (following [2]) a sample supergravity calculation in which Φ_0 is a massless scalar field and therefore sources an operator of scaling dimension $\Delta = 4$. Afterwards we will present a semi-classical expansion for correlators in the supergravity approximation that was given by Witten.

As a matter of fact anti de Sitter space has a nice feature concerning scalar fields. Namely, any function Φ_0 that satisfies the Laplace equation $\mathrm{D}_i\mathrm{D}^i\Phi_0 = 0$ on the boundary has a *unique* extension Φ to the bulk that obeys the field equation. The reason for this statement to be true is, that there is no non-zero square-integrable solution of the Laplace equation that vanishes on the boundary.
We will use the coordinates ξ^μ with metric

$$\mathrm{d}s^2 = \frac{1}{\left(\xi^0\right)^2} \sum_{\mu=0}^{4} \left(\mathrm{d}\xi^\mu\right)^2 \ ,$$

in which the boundary is given by $\xi^0 = 0$ and the single point at infinity $\xi^0 \to \infty$. We will further assume the bulk action to be given by

$$I_{\mathrm{S}}(\Phi) = \frac{1}{2} \int_{\mathrm{AdS}_5} \mathrm{d}^5\xi \sqrt{G} \left|\mathrm{d}\Phi\right|^2 \ ,$$

because for two-point functions only the quadratic part of the action in the fields is needed. Note that the integral is just over the AdS_5 part of the space. This is justified since the S^5 part of the geometry is compact and we can therefore decompose the fields living on the product space into Kaluza-Klein towers on the

S^5, yielding fields that effectively live on AdS_5. Since the bulk field Φ is determined by its values on the boundary, we can write

$$\Phi(\xi) = \int_{S^4} \mathrm{d}^4\vec{\xi'} K\left(\xi^0, \vec{\xi}, \vec{\xi'}\right) \Phi_0(\vec{\xi'})\,, \tag{2.43}$$

with the propagator K fulfilling the Laplace equation

$$\frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi^\mu} \left(\sqrt{G} \frac{\partial}{\partial \xi_\mu} K(\xi, \vec{\xi'}) \right) = 0$$

and having delta function support on the boundary. Using some clever line of argumentation[9] one gets the solution

$$K\left(\xi^0, \vec{\xi}, \vec{\xi'}\right) = c \frac{\left(\xi^0\right)^4}{\left(\left(\xi^0\right)^2 + \left| \vec{\xi} - \vec{\xi'} \right|^2 \right)^4}\,, \tag{2.44}$$

called *massless scalar boundary-to-bulk propagator*. Note that, indeed, for $\xi^0 \to 0^+$: $K\left(\xi^0, \vec{\xi}, \vec{\xi'}\right) \to \delta^{(4)}\left(\vec{\xi} - \vec{\xi'}\right)$. Plugging (2.44) into (2.43) and the resulting expression for $\Phi(\xi)$ into the action $I(\Phi)$ one gets

$$I(\Phi) = 2c \int \mathrm{d}^4\xi \mathrm{d}^4\xi' \frac{\Phi_0(\vec{\xi})\Phi_0(\vec{\xi'})}{\left| \vec{\xi} - \vec{\xi'} \right|^8}\,,$$

which for the operator $\mathcal{O}$ sourced by Φ_0 gives the two-point function

$$\left\langle \mathcal{O}(\vec{\xi})\mathcal{O}(\vec{\xi'}) \right\rangle \sim \frac{1}{\left| \vec{\xi} - \vec{\xi'} \right|^8}\,,$$

as it should be the case for an operator of scaling dimension $\Delta = 4$ in a conformal field theory.

One can do similar calculations for massive scalars, vectors, spinors and p-forms in the bulk. Then a relation between their masses m and the scaling dimension Δ of the corresponding boundary fields can be obtained. The generic results for a D-dimensional boundary read

- scalars: $\Delta_\pm = \frac{1}{2}\left(D \pm \sqrt{D^2 + 4m^2}\right)$,
- spinors: $\Delta = \frac{1}{2}\left(D + 2\,|m|\right)$,

[9] For details on this calculation see [2].

- vectors: $\Delta_\pm = \frac{1}{2}\left(D \pm \sqrt{(D-2)^2 + 4m^2}\right)$,

- p-forms: $\Delta_\pm = \frac{1}{2}\left(D \pm \sqrt{(D-2p)^2 + 4m^2}\right)$.

In the regime in which the supergravity approximation is valid, one can do a semi-classical expansion of the correlation functions. Since the effective supergravity action is proportional to $\frac{1}{\kappa_5^2}$ with κ_5 being related to the ten-dimensional Newton's constant, after integrating out the S^5 degrees of freedom, as $\kappa_5^2 = \frac{8\pi G_{10}}{\mathrm{Vol}(\mathrm{S}^5)}$, the action is proportional to $\frac{1}{N^2}$ which allows an expansion in terms of $\frac{1}{N}$. In this way one obtains a diagrammatic prescription, similar to the Feynman rules in quantum field theory, on how to calculate a given amplitude. The diagrams, sometimes called *Witten diagrams*, are given by a disc whose interior represents the bulk of AdS while the boundary circle of the disc corresponds to the boundary of AdS. Some examples of those diagrams are depicted in figure 2.2. They are accompanied by a set of rules stated below.

- Each external source to a field is represented by a point on the boundary circle.

- Each external source is connected to either another external source or an internal interaction point by a *boundary-to-bulk* propagator.

- The structure of the internal interaction points is determined by the bulk supergravity action I_S.

- Two interior points can be connected via *bulk-to-bulk propagators* which are the normal propagators of the given fields in the bulk.

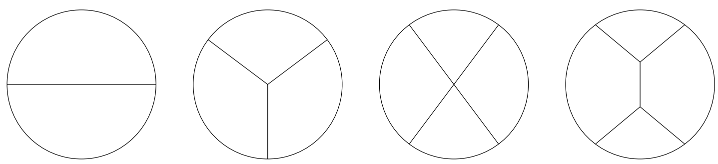

Figure 2.2: *Witten diagrams* for the 2-point and 3-point function as well as 4-point contact and 4-point exchange.

The various propagators for given fields look quite complicated and can be found for instance in [9] and references therein.

2.4 The PSU$(2,2|4)$ Nonlinear Coset σ-Model

In this section we want to discuss the type IIB superstring on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. More details on the calculations can be found in [22], on which this discussion is based.
Though both anti de Sitter space and the sphere are maximally symmetric spaces and therefore probably some of the easiest non-trivial spacetime backgrounds for string theory, it turns out quite hard to quantize the full superstring theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Since the background geometry requires non-vanishing RR five-form flux, the standard RNS formalism cannot be used. Also for the Green–Schwarz formalism in its usual form, in practice, it turns out to be difficult to construct an action. There is an alternative approach though, in which the Green–Schwarz string is defined by a WZW-type non-linear σ-model on the coset superspace of the geometry. In this framework it was possible to write down a consistent, κ-symmetric superstring action [23].

2.4.1 $\mathrm{AdS}_5 \times \mathrm{S}^5$ as a Coset Space

If a Lie group G acts on a manifold M transitively then, under certain technical requirements, the coset space $G/H(p)$, with $H(p)$ being the stabilizer of $p \in M$, can be shown to be homeomorphic to M. In this way we can, for instance, describe the D-dimensional sphere S^D by the cosets [24]

$$S^D \simeq \frac{\mathrm{SO}(D+1)}{\mathrm{SO}(D)} \simeq \frac{\mathrm{O}(D+1)}{\mathrm{O}(D)}\,,$$

or, for odd-dimensional spheres,

$$S^{2D+1} \simeq \frac{\mathrm{SU}(D+1)}{\mathrm{SU}(D)} \simeq \frac{\mathrm{U}(D+1)}{\mathrm{U}(D)}\,.$$

Similarly AdS_{D+1} can be described by the coset

$$\mathrm{AdS}_{D+1} \simeq \frac{\mathrm{SO}(D,2)}{\mathrm{SO}(D,1)}\,,$$

which can be seen by describing the space as

$$\mathrm{AdS}_{D+1} = \Big\{[X]\,|X \in \mathbb{R}^{D,2}, X\cdot X < 0\Big\}\;\text{where}\,[X]=[Y]\;\text{ iff }\;X = zY \quad z \in \mathbb{R}^+\,.$$

Clearly the canonical action of $\mathrm{SO}(D,2)$ on the *rays* $[X]$ is transitive and they are stabilized by an $\mathrm{SO}(D,1)$ subgroup.

Since the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superspace relevant to type IIB superstring theory is an extension of $\mathrm{AdS}_5 \times \mathrm{S}^5$ by 32 fermionic directions the $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ gets enhanced to $\mathrm{PSU}(2,2|4)$ as we already saw in section 2.2.2 for the $\mathcal{N} = 4$ SYM theory. Therefore the coset space we have to consider is given by

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)} . \tag{2.45}$$

To be precise, since we will be dealing with fermions we should replace the orthogonal $\mathrm{SO}(N)$ groups by their double coverings $\mathrm{Spin}(N)$. Furthermore, since, as we saw in section 2.3.2, anti de Sitter space is not causal, one should replace AdS_{D+1} by its universal cover $\widetilde{\mathrm{AdS}}_{D+1}$. Therefore $\mathrm{PSU}(2,2|4)$ should be substituted by its univeral cover $\widetilde{\mathrm{PSU}}(2,2|4)$ to get the coset space

$$\frac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)} .$$

These subtleties will not be important in the following and hence we will use the notation (2.45) for simplicity.

2.4.2 The $\mathfrak{psu}(2,2|4)$ Algebra

We already examined some of the properties of the PSU(2,2|4) group and its representations in section 2.2.2. Here we summarize some important facts about the algebra, the matrix representation and the $\mathbb{Z}_4$-grading.

We start by defining the superalgebra $\mathfrak{sl}(4|4)$ over $\mathbb{C}$. It is spanned by the 8×8 matrices M which can be written in terms of 4×4 blocks as

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \tag{2.46}$$

and fulfill

$$\mathrm{str}\, M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n = 0 .$$

While m and n are real valued 4×4 matrices, θ and η are Grassmann valued.
$\mathfrak{su}(2,2|4)$
is a non-compact real form of $\mathfrak{sl}(4|4)$. It is given by a subset $\mathfrak{su}(2,2|4) = \{M \in \mathfrak{sl}(4|4) | M^\star = M\}$ of $\mathfrak{sl}(4|4)$, where the *Cartan involution* is defined as

$$M^\star \equiv -HM^\dagger H^{-1}$$

and

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.$$

Note that for the Grassmann valued matrices $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$ such that $(M_1M_2)^\dagger = M_2^\dagger M_1^\dagger$. Therefore the elements of $\mathfrak{su}(2,2|4)$ satisfy

$$m^\dagger = -\Sigma m \Sigma, \qquad n^\dagger = -n, \qquad \eta^\dagger = -\Sigma\theta\,, \tag{2.47}$$

such that m and n span the subalgebras $u(2,2)$ and $u(4)$ respectively. Since $i\mathbb{1}$ is also included in $\mathfrak{su}(2,2|4)$, its bosonic subalgebra is given by

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,.$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is obtained by modding this $\mathfrak{u}(1)$ factor out of $\mathfrak{su}(2,2|4)$. Note that $\mathfrak{psu}(2,2|4)$ does not have a realization in terms of 8×8 matrices.

The automorphism group of the $\mathfrak{sl}(4|4)$ algebra contains an order-four generator

$$M \to \Omega(M) \equiv -\mathcal{K}M^{\mathrm{st}}\mathcal{K}^{-1}\,, \tag{2.48}$$

where

$$\mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \qquad K = \begin{pmatrix} -\mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}, \qquad M^{\mathrm{st}} = \begin{pmatrix} m^{\mathrm{t}} & -\eta^{\mathrm{t}} \\ \theta^{\mathrm{t}} & n^{\mathrm{t}} \end{pmatrix}$$

and $\Omega(M_1M_2) = -\Omega(M_2)\,\Omega(M_1)$. This order-four automorphism allows to endow $\mathfrak{sl}(4|4)$ with a $\mathbb{Z}_4$ grading. Abbreviating $\mathcal{G} = \mathfrak{sl}(4|4)$, and defining

$$\mathcal{G}^{(k)} \equiv \left\{ M \in \mathcal{G} | \Omega(M) = \mathrm{i}^k M \right\}\,, \tag{2.49}$$

we can decompose $\mathcal{G}$ as a vector space into a direct sum of subspaces

$$\mathcal{G} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)} \oplus \mathcal{G}^{(2)} \oplus \mathcal{G}^{(3)}\,. \tag{2.50}$$

Then $\left[\mathcal{G}^{(k)}, \mathcal{G}^{(m)}\right] \subset \mathcal{G}^{(k+m)} \bmod \mathbb{Z}_4$ and for every matrix $M \in \mathcal{G}$ we define its projection $M^{(k)}$ to $\mathcal{G}^{(k)}$ by

$$M^{(k)} \equiv \frac{1}{4}\left(M + \mathrm{i}^{3k}\Omega(M) + \mathrm{i}^{2k}\Omega^2(M) + \mathrm{i}^k\Omega^3(M) \right)\,. \tag{2.51}$$

Note that if $M \in \mathfrak{su}(2,2|4)$ then each of its projections is an element of that algebra, $M^{(k)} \in \mathfrak{su}(2,2|4)$.

Since

$$\Omega^\dagger(M) = \Upsilon\Omega(M^\star)\Upsilon = -\left(\Upsilon H\right)\Omega(M)\left(\Upsilon H\right)^{-1}\,,$$

where

$$\Upsilon = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix} \qquad \text{and} \qquad [H, \Upsilon] = 0\,,$$

the Cartan involution acts on $M^{(k)} \in \mathcal{G}^{(k)}$ as

$$\left(M^{(k)}\right)^\star = -\frac{1}{4} H \left[M + \mathrm{i}^k \Upsilon \Omega(M) \Upsilon^{-1} + \mathrm{i}^{2k} \Omega^2(M) + \mathrm{i}^{3k} \Upsilon \Omega^3(M) \Upsilon^{-1}\right] H^{-1}\,. \quad (2.52)$$

In terms of the 4×4 matrices defined in (2.46) the $\mathbb{Z}_4$ decomposition reads as

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} m - K m^{\mathrm{t}} K^{-1} & 0 \\ 0 & n - K n^{\mathrm{t}} K^{-1} \end{pmatrix}, \quad M^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & \theta - \mathrm{i} K \eta^{\mathrm{t}} K^{-1} \\ \eta + \mathrm{i} K \theta^{\mathrm{t}} K^{-1} & 0 \end{pmatrix},$$
$$M^{(2)} = \frac{1}{2}\begin{pmatrix} m + K m^{\mathrm{t}} K^{-1} & 0 \\ 0 & n + K n^{\mathrm{t}} K^{-1} \end{pmatrix}, \quad M^{(3)} = \frac{1}{2}\begin{pmatrix} 0 & \theta + \mathrm{i} K \eta^{\mathrm{t}} K^{-1} \\ \eta - \mathrm{i} K \theta^{\mathrm{t}} K^{-1} & 0 \end{pmatrix}.$$

2.4.3 An $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring Lagrangian

We start by postulating the lagrangian

$$\mathcal{L} = -\frac{g}{2}\left[\gamma^{\alpha\beta} \mathrm{str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa \epsilon^{\alpha\beta} \mathrm{str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right]\,, \quad (2.53)$$

where $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$, $\kappa \in \mathbb{R}$ and

$$A = -\mathfrak{g}^{-1} \mathrm{d}\mathfrak{g} = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}$$

is a one-form with values in $\mathfrak{su}(2,2|4)$ and $\mathfrak{g} \in \mathrm{SU}(2,2|4)$ and $A(i)$ are the components of the $\mathbb{Z}_4$ decomposition (2.50). By construction, A is pure gauge and therefore it satisfies $F = \mathrm{d}A - A \wedge A = 0$ or, in components,

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0\,.$$

Note that, usually, the Wess–Zumino term enters the action as a non-local expression integrated over a three-cycle whose boundary is the string worldsheet. Here it would have the form

$$\int \mathrm{d}^3x\, \theta_3 \equiv \int \mathrm{d}^3x\, \mathrm{str}\left(A^{(2)} \wedge A^{(3)} \wedge A^{(3)} - A^{(2)} \wedge A^{(1)} \wedge A^{(1)}\right)\,.$$

Since A is pure gauge, θ_3 is closed, and moreover, since the cohomology group of $\mathfrak{psu}(2,2|4)$ is trivial, θ_3 is also exact,

$$\theta_3 = \frac{1}{2}\,\mathrm{d}\,\mathrm{str}\left(A^{(1)} \wedge A^{(3)}\right)\,.$$

Hence the Wess–Zumino term can be written as in the lagrangian (2.53).

Now we will discuss the symmetries of the lagrangian we postulated. Besides the obvious reparametrisation- and Weyl-invariance, they are the following:

- **Local U(1).** Since $\mathrm{str}\left(A^{(2)}\right) = \mathrm{str}\,(\mathbb{1}) = 0$, the lagrangian is invariant under
$$A^{(2)} \to A^{(2)} + \mathrm{i}c \cdot \mathbb{1}\,,$$
which corresponds to right multiplication of $\mathfrak{g}$ by a U(1) element.

- **Local SO(4,1) × SO(5).** Consider the right multiplication $\mathfrak{g} \to \mathfrak{g}\mathfrak{h}$ of $\mathfrak{g}$ by an element $\mathfrak{h} \in \mathrm{SO}(4,1) \times \mathrm{SO}(5) \subset \mathrm{SU}(2,2|4)$. Then
$$A \to \mathfrak{h}^{-1} A \mathfrak{h} - \mathfrak{h}^{-1}\mathrm{d}\mathfrak{h}\,,$$
which for the components of A implies
$$A^{(1,2,3)} \to \mathfrak{h}^{-1} A^{(1,2,3)} \mathfrak{h}, \qquad A^{(0)} \to \mathfrak{h}^{-1} A^{(0)} \mathfrak{h} - \mathfrak{h}^{-1}\mathrm{d}\mathfrak{h}\,.$$
Since the lagrangian is invariant under similarity transformations of the $A^{(1,2,3)}$ components, this is a local symmetry of the theory.

- **Global PSU (2,2|4).** The transformation
$$\mathfrak{g} \to G\mathfrak{g}\,,$$
with $G \in \mathrm{PSU}\,(2,2|4)$ being constant, leaves the current A invariant and is therefore a symmetry.

The local $\mathrm{U}(1) \times \mathrm{SO}(4,1) \times \mathrm{SO}(5)$ symmetry shows, that the lagrangian indeed depends on a coset space element from $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$. Since $i\mathbb{1} \in \mathcal{G}^{(2)}$, the local U(1) symmetry allows us to gauge away the trace part of $A^{(2)}$.
Of course from a Green–Schwarz action we expect an additional fermionic symmetry. Indeed the lagrangian possesses

- **κ-symmetry.** Consider the right action of an $\mathfrak{h} = \mathrm{e}^{\epsilon}$ with $\epsilon \in \mathfrak{psu}(2,2|4)$ on the coset representative g. Under this the connection transforms as $A \to A - \mathrm{d}\epsilon + [A, \epsilon]$. Assuming $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$ we get for the $\mathbb{Z}_4$ components
$$\begin{aligned}
\delta A^{(1)} &= -\mathrm{d}\epsilon^{(1)} + \left[A^{(0)}, \epsilon^{(1)}\right] + \left[A^{(2)}, \epsilon^{(3)}\right]\,,\\
\delta A^{(2)} &= -\mathrm{d}\epsilon^{(3)} + \left[A^{(2)}, \epsilon^{(1)}\right] + \left[A^{(0)}, \epsilon^{(3)}\right]\,,\\
\delta A^{(3)} &= \left[A^{(1)}, \epsilon^{(1)}\right] + \left[A^{(3)}, \epsilon^{(3)}\right]\,,\\
\delta A^{(0)} &= \left[A^{(3)}, \epsilon^{(1)}\right] + \left[A^{(1)}, \epsilon^{(3)}\right]\,.
\end{aligned}$$
Make the ansatz
$$\begin{aligned}
\epsilon^{(1)} &= \left\{A^{(2)}_{\alpha-}, \kappa^{(1)\alpha}_{+}\right\}\,,\\
\epsilon^{(3)} &= \left\{A^{(2)}_{\alpha+}, \kappa^{(3)\alpha}_{-}\right\}\,,
\end{aligned}$$

where $V_\pm^\alpha \equiv P_\pm^{\alpha\beta} V_\beta \equiv \frac{1}{2}\left(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta}\right) V_\beta$. The fact that $\epsilon^{(1,3)}$ is in $\mathfrak{su}(2,2|4)$ implies for the parameters κ, that

$$H\kappa^{(1)} = \left(\kappa^{(1)}\right)^\dagger H\,,$$
$$H\kappa^{(3)} = \left(\kappa^{(3)}\right)^\dagger H\,.$$

Now the lagrangian is invariant under the transformation $\mathfrak{g} \to \mathfrak{g}e^\epsilon$ provided that $\kappa = \pm 1$, such that the $P_\pm$ are orthogonal projectors, and, provided that the metric at the same time transforms as

$$\delta\gamma^{\alpha\beta} = \frac{1}{2}\mathrm{tr}\left(\left[\kappa_+^{(1)\alpha}, A_+^{(1)\beta}\right] + \left[\kappa_-^{(3)\alpha}, A_-^{(3)\beta}\right]\right)\,.$$

Having presented the symmetries of the coset non-linear σ-model given by (2.53), we close the discussion of the lagrangian by stating the classical equations of motion that can be obtained from it. The variation of the lagrangian with respect to A yields

$$\delta\mathcal{L} = -\mathrm{str}\left(\mathfrak{g}^{-1}\delta\mathfrak{g}\left(\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha]\right)\right)\,, \tag{2.54}$$

where

$$\Lambda^\alpha = g\left(\gamma^{\alpha\beta}A_\beta^{(2)}\frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(A_\beta^{(1)} - A_\beta^{(3)}\right)\right)\,.$$

Regarding $\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha]$ as an element of $\mathfrak{su}(2,2|4)$ the corresponding equations of motion are given by

$$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = c \cdot \mathbb{1}\,,$$

with c being some constant. Since we are considering $\mathfrak{psu}(2,2|4) = \frac{\mathfrak{su}(2,2|4)}{\mathfrak{u}(1)}$ the relevant equations of motion are given by

$$\partial_\alpha\Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0\,. \tag{2.55}$$

This equation of motion, implying the conservation of the current

$$J^\alpha \equiv \mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}\,,$$

can be projected to the $\mathbb{Z}_4$ components of A to give

$$\partial_\alpha\left(\gamma^{\alpha\beta}A_\beta^{(2)}\right) - \gamma^{\alpha\beta}\left[A_\alpha^{(0)}, A_\beta^{(2)}\right] + \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(\left[A_\alpha^{(1)}, A_\beta^{(1)}\right] - \left[A_\alpha^{(3)}, A_\beta^{(3)}\right]\right) = 0\,,$$
$$P_-^{\alpha\beta}\left[A_\alpha^{(2)}, A_\beta^{(3)}\right] = 0\,,$$
$$P_+^{\alpha\beta}\left[A_\alpha^{(2)}, A_\beta^{(1)}\right] = 0\,,$$

while the projection onto $\mathcal{G}^{(0)}$ vanishes. The equations of motion that come from varying the action with respect to the metric $\gamma^{\alpha\beta}$ are the Virasoro constraints given by

$$\mathrm{str}\left(A_\alpha^{(2)}A_\beta^{(2)}\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\mathrm{str}\left(A_\rho^{(2)}A_\delta^{(2)}\right) = 0\,. \tag{2.56}$$

2.4.4 A Coset Parametrisation

In order to give a more explicit form of the lagrangian (2.53) one has to embed the coset representatives $\mathfrak{g}$ into the SU(2, 2|4) group. This can, of course, be done in different ways but all the embeddings are related by some non-linear field redefinitions and are therefore equivalent. It should be noted however, that in general non-linear field redefinitions can change the boundary conditions of the fermions.

As we have seen before, the bosonic subalgebra of $\mathfrak{psu}(2,2|4)$ is given by $\mathfrak{su}(2,2)\times\mathfrak{su}(4)$. One can show that, regarding both groups as real vector spaces, they are spanned by

$$\begin{aligned}
\mathfrak{su}(2,2) &\sim \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{2}\tilde\gamma^i,\, \frac{1}{2}\tilde\gamma^5,\, \frac{1}{4}\left[\tilde\gamma^i,\tilde\gamma^j\right],\, \frac{\mathrm{i}}{4}\left[\tilde\gamma^5,\tilde\gamma^j\right]\right\} \qquad & i,j = 1,\ldots,4\,,\\
\mathfrak{su}(4) &\sim \mathrm{span}_{\mathbb{R}}\left\{\frac{\mathrm{i}}{2}\tilde\gamma^i,\, \frac{1}{4}\left[\tilde\gamma^i,\tilde\gamma^j\right]\right\} \qquad & i,j = 1,\ldots,5\,,
\end{aligned}$$

with the hermitean matrices

$$\begin{aligned}
\tilde\gamma^1 &\equiv \begin{pmatrix} 0 & \mathrm{i}\sigma_2 \\ -\mathrm{i}\sigma_2 & 0\end{pmatrix}, \qquad \tilde\gamma^2 \equiv \begin{pmatrix} 0 & \mathrm{i}\sigma_1 \\ -\mathrm{i}\sigma_1 & 0\end{pmatrix}, \qquad \tilde\gamma^3 \equiv \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0\end{pmatrix},\\
\tilde\gamma^4 &\equiv \begin{pmatrix} 0 & \mathrm{i}\sigma_3 \\ -\mathrm{i}\sigma_3 & 0\end{pmatrix}, \qquad \tilde\gamma^5 \equiv \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix} = \Sigma,
\end{aligned}$$

fulfilling the SO(5) Clifford algebra

$$\left\{\tilde\gamma^i,\tilde\gamma^j\right\} = 2\delta^{ij} \qquad i,j = 1,\ldots,5\,.$$

Now we choose the parametrisation of a coset representative $\mathfrak{g}$ to be

$$\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})\,. \tag{2.57}$$

Here we have used the $\mathrm{AdS}_5\times\mathrm{S}^5$ coordinates (t, z^i, ϕ, y^i), $i = 1,\ldots,4$, with metric

$$\mathrm{d}s^2 = -G_{\mathrm{tt}}\mathrm{d}t^2 + G_{\phi\phi}\mathrm{d}\phi^2 + G_{\mathrm{zz}}\mathrm{d}z^i\mathrm{d}z^j\delta_{ij} + G_{\mathrm{yy}}\mathrm{d}y^i\mathrm{d}y^j\delta_{ij}\,, \tag{2.58}$$

where

$$G_{\rm tt} = \left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2\,, \quad G_{\phi\phi} = \left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2\,, \quad G_{\rm zz} = \left(\frac{1}{1-\frac{z^2}{4}}\right)^2\,, \quad G_{\rm yy} = \left(\frac{1}{1+\frac{y^2}{4}}\right)^2\,,$$

and $z^2 = z^i z^j \delta_{ij}$ as well as $y^2 = y^i y^j \delta_{ij}$. Further

$$\Lambda(t,\phi) \equiv \exp\begin{pmatrix} \frac{\mathrm{i}}{2}t\tilde{\gamma}^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}\phi\tilde{\gamma}^5 \end{pmatrix}, \quad \mathbb{X} \equiv \begin{pmatrix} \frac{1}{2}z^i\tilde{\gamma}^j\delta_{ij} & 0 \\ 0 & \frac{\mathrm{i}}{2}y^i\tilde{\gamma}^j\delta_{ij} \end{pmatrix}, \quad \chi \equiv \begin{pmatrix} 0 & \theta \\ -\theta^\dagger\Sigma & 0 \end{pmatrix},$$

$\mathfrak{g}(\chi) \equiv \chi + \sqrt{1+\chi^2}$, $\mathfrak{g}(\mathbb{X}) \equiv \sqrt{\frac{1+\mathbb{X}}{1-\mathbb{X}}}$ and $\Lambda(t_1+t_2, \phi_1+\phi_2) = \Lambda(t_1,\phi_1)\Lambda(t_2,\phi_2)$. Note that the coordinate ϕ, $0 \le \phi < 2\pi$, parametrises a big circle in the S^5. Therefore closed strings can have a non-trivial winding number m, such that $\phi\,(\sigma^1 = 2\pi) - \phi\,(\sigma^1 = 0) = 2\pi m$. We will assume $m = 0$ in the following. Note further, denoting $\mathfrak{g}_{\rm b} \equiv \Lambda(t,\phi)\mathfrak{g}(\mathbb{X})$, that the bilinear $\mathrm{str}\,(\mathfrak{g}_{\rm b}\mathrm{d}\mathfrak{g}_{\rm b})^2$ reproduces the metric (2.58). Explicitly the matrix $\mathfrak{g}(\mathbb{X})$ is given by

$$\mathfrak{g}(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{z^2}{4}}}\left[\mathbb{1} + \frac{1}{2}z^i\tilde{\gamma}^j\delta_{ij}\right] & 0 \\ 0 & \frac{1}{\sqrt{1+\frac{y^2}{4}}}\left[\mathbb{1} + \frac{\mathrm{i}}{2}y^i\tilde{\gamma}^j\delta_{ij}\right] \end{pmatrix}.$$

Since $K\,(\tilde{\gamma}^i)^{\rm t}\,K^{-1} = \tilde{\gamma}^i$ for $i = 1,\ldots,5$, the matrices $\mathfrak{g}(\mathbb{X})$ are in the subspace $\mathcal{G}^{(2)} \subset \mathcal{G}$. Therefore the bosonic part of the lagrangian (2.53) is, in this parametrization of the coset representative $\mathfrak{g}_{\rm b}$, given by

$$\mathcal{L}_{\rm bos.} = -\frac{g}{2}\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}\,, \tag{2.59}$$

where $X^M \in \{t, z^i, \phi, y^i\}$ and G_{MN} is the metric (2.58). It therefore looks, as one should expect, like the σ-model lagrangian on a curved spacetime background.

Consider now a local $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$ transformation $\mathfrak{h}$ acting from the left on the coset representative $\mathfrak{g}$. Using the representation (2.57) this means

$$\mathfrak{h}\mathfrak{g} = \mathfrak{h}\Lambda(t,\phi)\mathfrak{h}^{-1}\;\;\mathfrak{h}\mathfrak{g}(\chi)\mathfrak{h}^{-1}\;\;\mathfrak{h}\mathfrak{g}(\mathbb{X})\,,$$

such that the fermions transform under the adjoint of $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$, while the bosons in general transform in a non-linear fashion. For the special case, however, in which $\mathfrak{h}$ corresponds to a global shift in the coordinates (t,ϕ), $\mathfrak{h}$ can be identified with an element $\Lambda(a,b)$, such that

$$\Lambda(a,b)\mathfrak{g} = \Lambda(t+a,\phi+b)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X})\,,$$

that means, both χ and $\mathbb{X}$ remain untouched. Such a parametrisation is useful for a light-cone gauge quantization in which the light-cone directions are t and ϕ.

It is interesting to look at the maximal subgroup of the bosonic symmetries that commutes with the Λ transformations, since this subgroup will be the manifest bosonic symmetry of the light-cone guage fixed lagrangian. The shifts in t and ϕ are generated by the matrices

$$\begin{pmatrix} \mathrm{i}\tilde{\gamma}^5 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{i}\tilde{\gamma}^5 \end{pmatrix}$$

respectively. Since $\frac{1}{4}[\tilde{\gamma}^i, \tilde{\gamma}^j]$ with $i = 1, \ldots, 4$ commute with $\tilde{\gamma}^5$, the centralizer $\mathfrak{C}$ consists of two copies of $\mathfrak{so}(4)$ spanned by these matrices

$$\mathfrak{C} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{so}(4,2) \times \mathfrak{so}(6)\,.$$

Then, for a $\mathfrak{h} \in \mathfrak{C}$, $\mathfrak{h}\Lambda(t,\phi)\mathfrak{h}^{-1} = \Lambda(t,\phi)$ and

$$\mathfrak{h}\mathfrak{g} = \Lambda(t,\phi)\ \ \mathfrak{h}\mathfrak{g}(\chi)\mathfrak{h}^{-1}\ \ \mathfrak{h}\mathfrak{g}(\mathbb{X})\mathfrak{h}^{-1}\ \ \mathfrak{h}\,,$$

where the last $\mathfrak{h}$ acts from the right and can therfore be absorbed by a local $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ transformation which is a symmetry. We hence see, that under an $\mathfrak{h} \in \mathfrak{C}$ the bosons and fermions both transform in the adjoint

$$\mathfrak{g}(\chi) \to \mathfrak{h}\mathfrak{g}(\chi)\mathfrak{h}^{-1}, \qquad \mathfrak{g}(\mathbb{X}) \to \mathfrak{h}\mathfrak{g}(\mathbb{X})\mathfrak{h}^{-1}$$

and therefore these bosonic symmetries are realized linearly.

2.4.5 Light-Cone Gauge

As we saw in section 2.1.2 quantization of a Green–Schwarz action simplifies dramatically in the light-cone gauge, even in flat space. Therefore we will discuss the light-cone gauge of the coset-model action (2.53) here. We closely follow the presentation in [22] and the calculations done in [25]. Since the calculations are rather involved we will not perform them in detail. In section 2.5 we will then use the light-cone gauge to make an attempt to perturbatively quantize the bosonic part of the action.

We start by analyzing the bosonic part of the lagrangian (2.59). Introducing momenta canonically-conjugate to the coordinates

$$P_M \equiv \frac{\delta S}{\delta \dot{X}^M} = -g\gamma^{0\beta}\partial_\beta X^N G_{MN}\,, \tag{2.60}$$

we can rewrite the lagrangian in the form

$$\mathcal{L}_{\text{bos.}} = P_M \dot{X}^M + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2g\gamma^{00}} C_2\,, \tag{2.61}$$

where

$$C_1 \equiv P_M X'^M, \qquad C_2 \equiv G^{MN} P_M P_N + g^2 X'^M X'^N G_{MN}\,.$$

The Virasoro constraints then impose the conditions

$$C_1 = C_2 = 0\,.$$

Now we introduce the light-cone coordinates

$$\begin{aligned} X^- &= \phi - t, \qquad X^+ = (1-a)t + a\phi\,, \\ P_- &= P_\phi + P_t, \quad P_+ = (1-a)P_\phi - aP_t\,, \end{aligned} \tag{2.62}$$

where $a \in \mathbb{R}$. Using these coordinates the lagrangian can be written as

$$\mathcal{L}_{\text{bos.}} = P_- \dot{X}^+ + P_+ \dot{X}^- + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2g\gamma^{00}} C_2\,, \tag{2.63}$$

with

$$\begin{aligned} C_1 &= P_+ X'^- + P_- X'^+ + P_\mu X'^\mu\,, \\ C_2 &= \left(a^2 G^{\phi\phi} - (a-1)^2 G^{tt}\right) P_-^2 + 2\left(aG^{\phi\phi} - (a-1)G^{tt}\right) P_- P_+ + \left(G^{\phi\phi} - G^{tt}\right) P_+^2 \\ &\quad + g^2 \left((a-1)^2 G_{\phi\phi} - a^2 G_{tt}\right) \left(X'^-\right)^2 - 2g^2 \left((a-1)G_{\phi\phi} - aG_{tt}\right) X'^- X'^+ \\ &\quad + g^2 \left(G_{\phi\phi} - G_{tt}\right) \left(X'^+\right)^2 + G^{\mu\nu} P_\mu P_\nu + g^2 G_{\mu\nu} X'^\mu X'^\nu \end{aligned}$$

and $X^\mu \in \{z^i, y^i\}$.
The light-cone gauge we employ is given by

$$X^+ = P_+ \sigma^0, \qquad P_+ = \text{const.}\,. \tag{2.64}$$

Then the vanishing of C_1 implies

$$X'^- = -\frac{1}{P_+} P_\mu X'^\mu\,, \tag{2.65}$$

which can be substituted into $C_2 = 0$ to get an expression for P_-. Then the light-cone gauge Hamiltonian of the system is given by

$$\mathcal{H} = -P_+ P_- \left(P_\mu, X^\mu, X'^\mu\right)\,, \tag{2.66}$$

where we have omitted the total derivative term $P_+ \dot{X}^-$. Setting for simplicity $P_+ = 1$ the result for the Hamiltonian reads

$$\begin{aligned} \mathcal{H} = &\frac{\sqrt{G_{\phi\phi} G_{tt} \left(1 + ((a-1)^2 G_{\phi\phi} - a^2 G_{tt})\right) \mathcal{H}_{\text{x}} + g^2 \left((a-1)^2 G_{\phi\phi} - a^2 G_{tt}\right)^2 (X'^-)^2}}{(a-1)^2 G_{\phi\phi} - a^2 G_{tt}} \\ &+ \frac{(a-1)G_{\phi\phi} - aG_{tt}}{(a-1)^2 G_{\phi\phi} - a^2 G_{tt}}, \qquad \text{with} \qquad \mathcal{H}_{\text{x}} \equiv G^{\mu\nu} P_\mu P_\nu + g^2 G_{\mu\nu} X'^\mu X'^\nu\,. \end{aligned}$$

This expression has a very-complicated and non-linear dependence on the coordinates and momenta which makes it hard to use it for canonical quantization. An effective quantization in a specific limit in which $g \to \infty$ is presented in section 2.5.

There is an important subtlety concerning the light-cone gauge (2.64). We have seen in section 2.1.2 that the freedom to impose the light-cone gauge in flat space comes from a residual symmetry after imposing a conformally flat worldsheet metric. This strongly relied on the fact that the equation of motion for light-cone coordinate X^+ was given by a wave equation. Now, due to the non-constant metric (2.58), the equations of motion for these coordinates are more complicated, which makes imposing a conformally flat world-sheet metric and the light-cone gauge (2.64) at the same time impossible. Still the reparemtrisation invariance makes it possible to impose the light-cone gauge, but then the metric cannot be conformally flat anymore. In the discussion above we started by introducing momenta canonically conjugate to the target-space coordinates. If one wants to include fermions, it becomes difficult to find the conjugate momenta due to non-trivial interactions between the bosonic and fermionic fields. Therefore in this case it is easier to introduce an auxiliary field $\mathfrak{p} \in \mathfrak{su}(2,2|4)$ and rewrite the action in the form [25]

$$\mathcal{L} = -\mathrm{str}\left(\mathfrak{p} A_0^{(2)} + \kappa \frac{g}{2} \epsilon^{\alpha\beta} A_\alpha^{(1)} A_\beta^{(3)}\right) + \frac{\gamma^{01}}{\gamma^{00}} C_1 - \frac{1}{2g\gamma^{00}} C_2 \,, \tag{2.67}$$

where

$$\begin{aligned} C_1 &\equiv \mathrm{str}\left(\mathfrak{p} A_1^{(2)}\right) \,, \\ C_2 &\equiv \mathrm{str}\left(\mathfrak{p}^2 + g^2 \left(A_1^{(2)}\right)\right) \,. \end{aligned}$$

Solving the equations of motion for $\mathfrak{p}$ and plugging the result back into the above expression yields the lagrangian (2.53). The Virasoro constraints are now given by

$$C_1 = 0 \,, \tag{2.68}$$
$$C_2 = 0 \,. \tag{2.69}$$

Without loss of generality one can assume that $\mathfrak{P} \in \mathcal{G}^{(2)}$ and therefore can be written as

$$\mathfrak{p} = \frac{\mathrm{i}}{2} \mathfrak{p}_+ \Sigma_+ + \frac{\mathrm{i}}{4} \mathfrak{p}_- \Sigma_- + \frac{1}{2} \mathfrak{p}_\mu \Sigma_\mu + \mathrm{i} \mathfrak{p}_{\mathbb{1}} \mathbb{1} \,,$$

with the matrices

$$\Sigma_+ = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \quad \Sigma_- = \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \tilde{\gamma}^i & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_{4+i} = \begin{pmatrix} 0 & 0 \\ 0 & i\tilde{\gamma}^i \end{pmatrix}.$$

Analogously to the flat space case the lagrangian and its equations of motions can be further simplified by gauge fixing the κ-symmetry. It can be shown that using this symmetry the fermionic field θ can be brought to the form

$$\theta = \begin{pmatrix} 0 & 0 & \theta_{13} & \theta_{14} \\ 0 & 0 & \theta_{23} & \theta_{24} \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & 0 & 0 \end{pmatrix},$$

which is equivalent to demanding

$$\{\Sigma_+, \chi\} = 0\,, \qquad [\Sigma_-, \chi] = 0\,,$$

where

$$\chi = \begin{pmatrix} 0 & \theta \\ -\theta^\dagger \Sigma & 0 \end{pmatrix},$$

as before. Now one can compute the current $A = -\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} = A_{\text{even}} + A_{\text{odd}}$ to be given by

$$\begin{aligned} A_{\text{even}} &= -\mathfrak{g}^{-1}(\mathbb{X}) \left[\frac{\mathrm{i}}{2} \left(\mathrm{d}X_+ + \left(\frac{1}{2} - a \right) \mathrm{d}X_- \right) \Sigma_+ \left(1 + 2\chi^2\right) + \frac{\mathrm{i}}{4} \mathrm{d}X_- \Sigma_- \right. \\ &\quad \left. + \sqrt{1+\chi^2}\mathrm{d}\sqrt{1-\chi^2} - \chi \mathrm{d}\chi + \mathrm{d}\mathfrak{g}(\mathbb{X})\mathfrak{g}^{-1}(\mathbb{X}) \right] \mathfrak{g}(\mathbb{X})\,, \\ A_{\text{odd}} &= -\mathfrak{g}^{-1}(\mathbb{X}) \left[\mathrm{i} \left(\mathrm{d}X_+ + \left(\frac{1}{2} - a \right) \mathrm{d}X_- \right) \Sigma_+ \chi \sqrt{1+\chi^2} \right. \\ &\quad \left. + \sqrt{1+\chi^2}\mathrm{d}\chi - \chi \mathrm{d}\sqrt{1+\chi^2} \right] \mathfrak{g}(\mathbb{X})\,, \end{aligned}$$

from which one sees that the choice $a = \frac{1}{2}$ simplifies the currents significantly. Using these results we can write the lagrangian (2.67) as

$$\mathcal{L} = P_+ \dot{X}^- + P'_- \dot{X}^+ - \operatorname{str}\left(\mathfrak{p} A'_{\text{even}} + \kappa \frac{g}{2} \epsilon^{\alpha\beta} A^{(1)}_\alpha A^{(3)}_\beta \right),$$

with

$$\begin{aligned} A'_{\text{even}} &= -\mathfrak{g}^{-1}(\mathbb{X}) \left[\sqrt{1+\chi^2}\partial_0\sqrt{1-\chi^2} - \chi\partial_0\chi + \left(\partial_0 \mathfrak{g}(\mathbb{X})\right) \mathfrak{g}^{-1}(\mathbb{X}) \right] \mathfrak{g}(\mathbb{X})\,, \\ p_+ &= \frac{\mathrm{i}}{4} \operatorname{str}\left(\mathfrak{p} \Sigma_- \mathfrak{g}^2(\mathbb{X}) \right) = G_+ \mathfrak{p}_+ - \frac{1}{2} G_- \mathfrak{p}_-, \quad G_\pm \equiv \frac{1}{2} \left(\sqrt{G_{tt}} \pm \sqrt{G_{\phi\phi}} \right), \\ P'_- &= \frac{\mathrm{i}}{2} \operatorname{str}\left(\mathfrak{p} \Sigma_+ \mathfrak{g}(\mathbb{X}) \left(1 + 2\chi^2\right) \mathfrak{g}(\mathbb{X}) \right). \end{aligned}$$

Note that P'_- is not the momentum canonically conjugate to X^+ due to a contribution coming from the Wess–Zumino part of the lagrangian. Now imposing the light-cone gauge (2.64) one can calculate the gauge-fixed lagrangian. The calculations are rather tedious and can be found in the various appendices in [25]. Setting $P_+ = 1$ again, the result reads

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{kin.}} - \mathcal{H}\,, \\ \mathcal{L}_{\text{kin.}} &= P_\mu \dot{X}^\mu - \frac{\mathrm{i}}{2}\mathrm{str}\left(\Sigma_+\chi\partial_0\chi\right) + \frac{1}{2}\mathfrak{g}_\nu\mathfrak{p}_\mu\mathrm{str}\left(\left[\Sigma^\nu,\Sigma^\mu\right]B_0\right) \\ &\quad - \mathrm{i}\kappa\frac{g}{2}\left(G_+^2 - G_-^2\right)\mathrm{str}\left(F_0\mathcal{K}F_1^{\mathrm{st}}\mathcal{K}\right) + \mathrm{i}\kappa\frac{g}{2}G_\mu G_\nu\mathrm{str}\left(\Sigma^\nu F_0\Sigma^\mu\mathcal{K}F_1^{\mathrm{st}}\mathcal{K}\right)\,, \\ \mathcal{H} &= -P'_- - \kappa\frac{g}{2}\left(G_+^2 - G_-^2\right)\mathrm{str}\left(\Sigma_+\chi\sqrt{1+\chi^2}\mathcal{K}F_1^{\mathrm{st}}\mathcal{K}\right) \\ &\quad - \kappa\frac{g}{2}G_\mu G_\nu\mathrm{str}\left(\Sigma_+\Sigma^\nu\chi\sqrt{1+\chi^2}\Sigma^\mu\mathcal{K}F_1^{\mathrm{st}}\mathcal{K}\right)\,, \end{aligned} \tag{2.70}$$

where we have used

$$\mathfrak{g}(\mathbb{X}) = \mathfrak{g}_+\mathbb{1} + \mathfrak{g}_-\Upsilon + \mathfrak{g}_\mu\Sigma^\mu\,, \quad \mathfrak{g}^2(\mathbb{X}) = G_+\mathbb{1} + G_-\Upsilon + G_\mu\Sigma^\mu\,,$$

as well as

$$\begin{aligned} B_\alpha &= -\frac{1}{2}\chi\partial_\alpha\chi + \frac{1}{2}\left(\partial_\alpha\chi\right)\chi + \frac{1}{2}\sqrt{1+\chi^2}\partial_\alpha\sqrt{1+\chi^2} - \frac{1}{2}\partial_\alpha\sqrt{1+\chi^2}\sqrt{1+\chi^2}\,, \\ F_\alpha &= \sqrt{1+\chi^2}\partial_\alpha\chi - \chi\partial_\alpha\sqrt{1+\chi^2}\,, \end{aligned}$$

being the even and odd components of $\mathfrak{g}^{-1}(\chi)\partial_\alpha\mathfrak{g}(\chi)$ respectively. After quantizing, physical states should respect the level matching condition, which is obtained by integrating the Virasoro constraint $C_1 = 0$ over σ^1.

2.5 Quantization at large g

As we have seen in the last section, the kinetic term of the gauge fixed lagrangian (2.70) is highly complicated and yields highly nontrivial Poisson brackets which makes it hard to quantize the theory. Quantization can much easier be performed in various limits. One of these, namely the limit of strong coupling $g \to \infty$ for the bosonic theory, was adressed quite recently in [26]. The main idea the authors of this paper used, is to perform a scaling on the target space coordinates to get a perturbatively accessible theory. Naively starting from the Nambu-Goto action for the bosonic string on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with metric (2.58)

$$S_{\mathrm{NG}} = -g\int \mathrm{d}^2\sigma\sqrt{-\det\left(\partial_r X^\mu\partial_s X^\nu G_{\mu\nu}\right)}$$

and scaling the target space coordinates as $X^\mu \to \sqrt{g} X^\mu$ yields an action that can be expanded in terms of g. It starts with a term quadratic in the world-sheet fields X^μ which is independent of g and all interaction terms come with positive powers of g. If we now split the X^μ into their zero-mode and their oscillator part

$$X^\mu\left(\sigma^0,\sigma^1\right) = x^\mu\left(\sigma^0\right) + \tilde{X}^\mu\left(\sigma^0,\sigma^1\right)\,,$$

we see that a scaling of this form supresses the oscillator parts of the fields in the expansion of the action. Therefore this scaling is not convenient when considering massive fields. The idea of [26] is now to perform a different scaling, namely to scale the phase-space variables according to

$$\begin{aligned} X^\mu\left(\sigma^0,\sigma^1\right) &\to g^{\frac{1}{4}} x^\mu\left(\sigma^0\right) + g^{\frac{1}{2}}\tilde{X}^\mu\left(\sigma^0,\sigma\right)\,, \\ P_\mu\left(\sigma^0,\sigma^1\right) &\to g^{\frac{1}{4}} p_\mu\left(\sigma^0\right) + g^{\frac{1}{2}}\tilde{P}_\mu\left(\sigma^0,\sigma\right)\,, \end{aligned} \tag{2.71}$$

where P^μ are the momenta canonically-conjugate to the coordinates X^μ.
Now starting from the action (2.63) and specifying $a = \frac{1}{2}$ in the light-cone coordinates (2.62) we will perturbatively solve the constraints $C_1 = C_2 = 0$. In order to do that employ the light-cone gauge

$$P_+ = p_+, \qquad X^+ = X^+ + p_+\sigma^0\,,$$

where x^+ and p_+ are (σ^0,σ^1) independent. As before we can solve the $C_1 = 0$ constraint by setting

$$X'^- = -\frac{1}{p_+} P_\mu X'^\mu\,.$$

To make the expressions more explicit we will in the following use a notation in which P_i and Q_i denote the momenta canonically conjuagete to the AdS_5 coordinates Z^i and the S^5 coordinates Y^i respectively. Then the constraint arising from the vanishing of C_2 is given by

$$\begin{aligned} 0 = &\left[\left(\frac{1+\vec{Y}^2/4}{1-\vec{Y}^2/4}\right)^2 - \left(\frac{1-\vec{Z}^2/4}{1+\vec{Z}^2/4}\right)^2\right]\left(\frac{P_-^2}{4} + p_+^2\right) + \left(1-\vec{Z}^2/4\right)^2\vec{P}^2 \\ &+ \left[\left(\frac{1+\vec{Y}^2/4}{1-\vec{Y}^2/4}\right)^2 + \left(\frac{1-\vec{Z}^2/4}{1+\vec{Z}^2/4}\right)^2\right] P_- p_+ + \left(1+\vec{Y}^2/4\right)^2\vec{Q}^2 \\ &+ g^2\left[\left(\frac{1-\vec{Y}^2/4}{1+\vec{Y}^2/4}\right)^2 - \left(\frac{1+\vec{Z}^2/4}{1-\vec{Z}^2/4}\right)^2\right]\frac{1}{4p_+^2}\left(P^i Z'^j\delta_{ij} + Q^i Y'^j\delta_{ij}\right)^2 \\ &+ g^2\left[\frac{\left(\vec{Z}'\right)^2}{\left(1-\vec{Z}^2/4\right)^2} + \frac{\left(\vec{Y}'\right)^2}{\left(1+\vec{Y}^2/4\right)^2}\right]. \end{aligned} \tag{2.72}$$

Next, as discussed above, we rescale the fields according to (2.71) and then calculate P_- as a series in g. In that calculation we will restrict to the case in which

$$q_\phi = \frac{1}{2}p_- + p_+ \propto g^0 \,,$$

with the notation

$$x^\mu = \int \frac{\mathrm{d}\sigma^1}{2\pi} X^\alpha(\sigma^1), \quad p_\alpha = \int \frac{\mathrm{d}\sigma^1}{2\pi} P_\alpha(\sigma^1), \quad \text{with} \quad \alpha \in \{t, \phi, +, -\} \,.$$

This condition implies that writing p_+ and p_- as a power series in g, the powers of g appearing in both series have to be the same.
To leading order, equation (2.72) yields

$$-2P_- p_+ = g\mathcal{M}^2 + \mathcal{O}\left(g^{\frac{1}{2}}\right) \,,$$

where

$$\mathcal{M}^2 = \vec{\tilde{P}}^2 + \vec{\tilde{Q}}^2 + \left(\vec{\tilde{Z}}'\right)^2 + \left(\vec{\tilde{Y}}'\right)^2 \,.$$

From $p_t^2 = q_\phi^2 - 2p_+p_-$ one can also get the leading-order energy

$$p_t^2 = q_\phi^2 + gM^2 + \mathcal{O}\left(g^{\frac{1}{2}}\right) \,,$$

with the *flat-space mass operator*

$$M^2 = \int \frac{\mathrm{d}\sigma^1}{2\pi} \mathcal{M}^2 \,.$$

From the fact that $p_+ = \frac{1}{2}\left(q_\phi - p_t\right)$ and the condition on q_ϕ one can deduce the seperate form of P_- and p_+ to be

$$p_+ = p_+^{(0)} g^{\frac{1}{2}} + p_+^{(2)} + \mathcal{O}\left(g^{-\frac{1}{8}}\right) \,,$$
$$P_- = -g^{\frac{1}{2}} \frac{\mathcal{M}^2}{2p_+^{(0)}} + \mathcal{O}\left(g^{\frac{1}{8}}\right) \,.$$

Using this one can extract the next higher order expression for P_- from (2.72). This can be repeated to give an iterative procedure to calculate p_+, P_- and therefore p_t^2 to any order. In [26] the calculation is performed up to order g^0 and the results can be found there.

Having solved the Virasoro constraints, one can quantize the theory by promoting the fields to operators and replacing the Poisson brackets by commutators in the usual way. The coordinates and momenta fulfill the equal-time Poisson brackets

$$\left\{Z^i\left(\sigma^0,\sigma^1\right),P^j\left(\sigma^0,\sigma^{1\prime}\right)\right\}_{\text{P.B.}}=\left\{Y^i\left(\sigma^0,\sigma^1\right),Q^j\left(\sigma^0,\sigma^{1\prime}\right)\right\}_{\text{P.B.}}=2\pi\delta\left(\sigma^1-\sigma^{1\prime}\right)\delta^{ij}\,,$$

which can be solved by imposing the oscillator expansion

$$\begin{aligned}
\tilde{Z}^i\left(\sigma^0,\sigma^1\right)&=\frac{\mathrm{i}}{2}\sum_{n\neq0}\left[\frac{\alpha_n^i\left(\sigma^0\right)}{n}\mathrm{e}^{-in\sigma^1}+\frac{\tilde{\alpha}_n^i\left(\sigma^0\right)}{n}\mathrm{e}^{in\sigma^1}\right]\,,\\
\tilde{P}^i\left(\sigma^0,\sigma^1\right)&=\frac{1}{2}\sum_{n\neq0}\left[\alpha_n^i\left(\sigma^0\right)\mathrm{e}^{-in\sigma^1}+\tilde{\alpha}_n^i\left(\sigma^0\right)\mathrm{e}^{in\sigma^1}\right]\,,\\
\tilde{Y}^i\left(\sigma^0,\sigma^1\right)&=\frac{\mathrm{i}}{2}\sum_{n\neq0}\left[\frac{\beta_n^i\left(\sigma^0\right)}{n}\mathrm{e}^{-in\sigma^1}+\frac{\tilde{\beta}_n^i\left(\sigma^0\right)}{n}\mathrm{e}^{in\sigma^1}\right]\,,\\
\tilde{Q}^i\left(\sigma^0,\sigma^1\right)&=\frac{1}{2}\sum_{n\neq0}\left[\beta_n^i\left(\sigma^0\right)\mathrm{e}^{-in\sigma^1}+\tilde{\beta}_n^i\left(\sigma^0\right)\mathrm{e}^{in\sigma^1}\right]\,,
\end{aligned}$$

with the non-vanishing equal-time oscillator brackets being given by

$$\begin{aligned}
&\left\{z^i,p^j\right\}_{\text{P.B.}}=\delta^{ij}\,,\quad \left\{\alpha_m^i,\alpha_n^j\right\}_{\text{P.B.}}=-\mathrm{i}m\delta_{m+n}\delta^{ij}\,,\quad \left\{\tilde{\alpha}_m^i,\tilde{\alpha}_n^j\right\}_{\text{P.B.}}=-\mathrm{i}m\delta_{m+n}\delta^{ij},\\
&\left\{y^i,q^j\right\}_{\text{P.B.}}=\delta^{ij}\,,\quad \left\{\beta_m^i,\beta_n^j\right\}_{\text{P.B.}}=-\mathrm{i}m\delta_{m+n}\delta^{ij}\,,\quad \left\{\tilde{\beta}_m^i,\tilde{\beta}_n^j\right\}_{\text{P.B.}}=-\mathrm{i}m\delta_{m+n}\delta^{ij}\,.
\end{aligned}$$

The Virasoro generators are defined as

$$L_n\equiv=\frac{1}{2}\sum_{\substack{m=-\infty\\m\neq0,n}}^{\infty}\left(\vec{\alpha}_{n-m}\cdot\vec{\alpha}_m+\vec{\beta}_{n-m}\cdot\vec{\beta}_m\right)\,,\quad \tilde{L}_n\equiv=\frac{1}{2}\sum_{\substack{m=-\infty\\m\neq0,n}}^{\infty}\left(\vec{\tilde{\alpha}}_{n-m}\cdot\vec{\tilde{\alpha}}_m+\vec{\tilde{\beta}}_{n-m}\cdot\vec{\tilde{\beta}}_m\right)$$

and the flat-space mass operator can be expressed by the oscillators as

$$M^2=2\left(L_0+\tilde{L}_0\right)\,.$$

Defining further $\Phi\equiv2\left(L_0-\tilde{L}_0\right)$ the level matching condition, which comes, as before, from integrating the constraint $C_1=0$, can be written as $\Phi\left|\Psi\right\rangle=0$ for any physical state $\left|\Psi\right\rangle$. Note that in the quantized theory both L_0 and $\tilde{L}_0$ are well-defined up to a normal ordering constant. However, since this constant is the same for both operators, the level matching condition is well-defined. Note further that though the oscillator expansions and brackets look pretty much like in flat space, they are far more complicated due to the non-trivial σ^0 dependence of the oscillators. In section 3.4 we will adress the problem to explicitly calculate it, since it is important in order to define vertex operators and calculate correlators.

Now one can quantize the theory by replacing Poisson brackets by commutators, $\{.,.\}_{\text{P.B.}} \to -i\,[.,.]$ in the usual fashion. The result are the standard commutators

$$\begin{aligned}
\left[\alpha^i_m, \alpha^j_n\right] &= m\delta_{m+n}\delta^{ij}, &\qquad \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] &= m\delta_{m+n}\delta^{ij}\,, \\
\left[\beta^i_m, \beta^j_n\right] &= m\delta_{m+n}\delta^{ij}, &\qquad \left[\tilde{\beta}^i_m, \tilde{\beta}^j_n\right] &= m\delta_{m+n}\delta^{ij}\,, \\
\left[L_m, \alpha^i_n\right] &= -n\alpha^i_{n+m}, &\qquad \left[L_m, \beta^i_n\right] &= -n\beta^i_{n+m}\,, \\
\left[\tilde{L}_m, \tilde{\alpha}^i_n\right] &= -n\tilde{\alpha}^i_{n+m}, &\qquad \left[\tilde{L}_m, \tilde{\beta}^i_n\right] &= -n\tilde{\beta}^i_{n+m}\,.
\end{aligned}$$

Then, after performing a unitary transformation and fixing the constants $p_+^{(0)}$, the spectrum for physical states is of the form[10]

$$p_t^2 = q_\phi^2 + gM^2 + \sqrt{g}\left(\vec{p}^2 + M^2\vec{z}^2 + \vec{q}^2\right) + \mathcal{O}\left(g^{\frac{1}{4}}\right)\,,$$

which looks like the spectrum of a massive particle on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[10] For details on this see [26].

Chapter 3

Correlation Functions

Correlation functions are central objects in quantum field and string theories since they correspond to physical observables and govern the dynamics of the theory. In the AdS/CFT duality context they play a special role since one can obtain insights about the dynamics of strongly coupled field theories by performing calculations on the string theory side, and therefore use string theory to predict measurable quantities. This chapter is devoted to the studies of correlation functions in the AdS/CFT correspondence. We will restrict ourselves to the bosonic sector of the theory in most of the cases.

3.1 Correlation Functions in String Theory

As we have seen in the previous sections, the type IIB Green–Schwarz lagrangian (2.1) as well as the coset-model lagrangian (2.53) possess local reparametrisation and Weyl invariance. This invariance makes the quantum field theory on the worldsheet a conformal field theory in two-dimensions. It is easy to see from the lagrangians that this feature extends also to the purely bosonic sector. Reparametrisation invariance makes it possible to map the worldsheet corresponding to an interaction of N incoming and outgoing strings to a compact space as displayed in figure 3.1. In order to keep the information about the quantum numbers of the particles participating in the interaction, local operators, called *vertex operators*, have to be inserted on the compact space. Due to the existence of two different quantization schemes, namely the *operator* and the *path integral* formalism, there are two different methods to calculate correlation functions. In the path integral formalism a correlator of M states is given by the functional integral

$$A \propto \int \mathcal{D}X\left(\vec{\sigma}\right)\mathcal{D}h_{\alpha\beta}\left(\vec{\sigma}\right)\mathrm{e}^{-S(X)}\prod_{i=1}^{M}V_{\Lambda_i}\left(k_i\right),$$

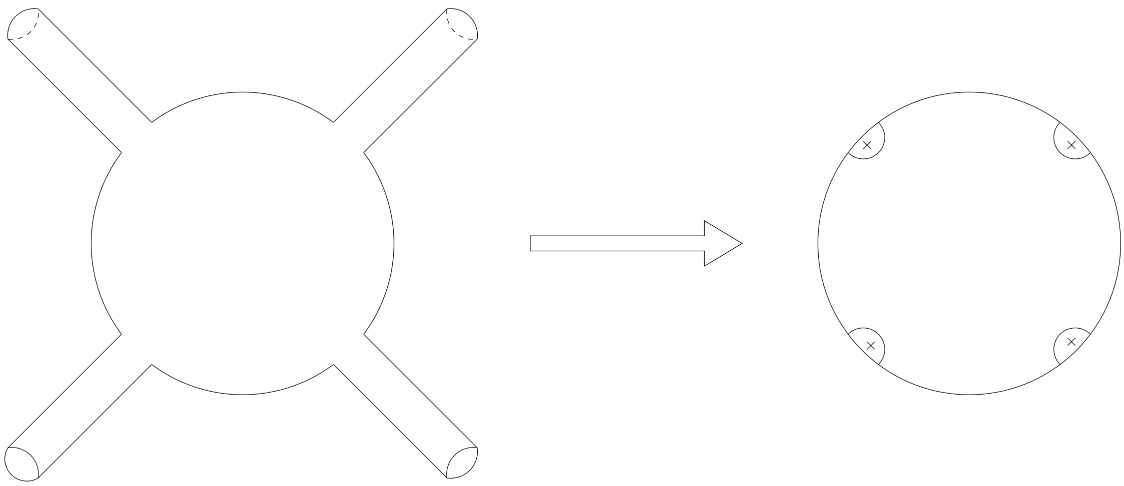

Figure 3.1: The closed string 4-point amplitude can be cast into a sphere with four punctures at which vertex operators get inserted.

where k_i are the momenta of the particles labelled by Λ_i. On a flat target space background the vertex operators are of the form

$$V_\Lambda(k) = \int \mathrm{d}^2\sigma \sqrt{h} W_\Lambda(\vec{\sigma}) \mathrm{e}^{\mathrm{i}k\cdot X},$$

where W_Λ is a local operator that is a worldsheet-scalar and carries the same Lorentz quantum numbers as the state Λ.
In section 3.2.3 we will discuss a saddle-point method used to calculate amplitudes of some class of operators in the path integral formalism on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background.

In the operator formalism we have to make use of the *operator state correspondence* to define the relation between local operators and their corresponding states. In analogy to quantum field theory one defines in- and outgoing states as

$$\begin{aligned} |\Lambda; k\rangle &= \lim_{\sigma^0 \to i\infty} \mathrm{e}^{-\mathrm{i}\sigma^0} V_\Lambda\left(k, \sigma^0\right) |0; 0\rangle\,, \\ \langle \Lambda; k| &= \lim_{\sigma^0 \to -i\infty} \mathrm{e}^{\mathrm{i}\sigma^0} \langle 0; 0| V_\Lambda\left(k, \sigma^0\right). \end{aligned}$$

Using these definitions an amplitude of the form

$$A \propto \langle \Phi_1| V_2(k_2) \Delta V_3(k_3) \dots \Delta V_{M-1}(k_{M-1}) |\Phi_M\rangle\,, \tag{3.1}$$

with Δ being the σ^0-propagators, can be calculated using operator techniques. We will do such a calculation in section 3.3.

3.2 Spinning String Correlation Functions in AdS/CFT

In general, since we are not dealing with free theories on both sides of the AdS/CFT duality, the physical quantities depend in a non-trivial way on the coupling constants, or more precisely on the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N = \frac{R^4}{\alpha'^2}$. As we saw in section 2.2.2, in the case of BPS operators it is possible that the dependence becomes trivial due to constraints coming from supersymmetry. Checking the duality beyond the BPS regime, by comparing the results for physical quantities obtained by calculations on the two sides, remains a challenging task. The reason is that while the perturbative string theory description is valid in the large λ limit, the perturbative regime of the SYM theory is at $\lambda \ll 1$ and therefore one has to compute the full functional dependence of the operators on λ. Analogously to the case of quantization, there are several limits in which some quantum numbers of single-string states scale with $\sqrt{\lambda}$ in the large λ limit and predictions can be made. One example is the BMN limit[27] in which one considers small closed strings whose center of mass is moving along a large circle of S^5 with large angular momentum $J \gg 1$ and finite $\frac{J^2}{\lambda} = \text{const.}$. In this limit one can describe the states as quadratic fluctuations near a point-like string and show that string σ-model corrections of higher than 1-loop order vanish. In that way a precise correspondence between the energies of string states and the scaling dimensions of the dual CFT operators was established.[1] Here, we will consider a different limit, first proposed in [29], describing single-string multispin states with at least one large S^5 spin component J. The classical energy has a regular expansion in $\frac{\lambda}{J^2}$ and the quantum superstring σ-model corrections are suppressed in the $J \gg 1$ limit with $\frac{J^2}{\lambda} = \text{fixed}$ [31]. In this section we first review the classical solutions corresponding to these *spinning strings* and then discuss the calculation of some of the non-BPS correlators of such states.

3.2.1 Classical Spinning String Solutions

The bosonic part of the coset-model lagrangian given by (2.59) can be rewritten in terms of the 6+6 embedding coordinates X_M ($M \in \{1, \ldots, 6\}$) of S^5 and Y_P ($P \in \{0, \ldots, 5\}$) of AdS_5, yielding the action

$$S = \sqrt{\lambda} \int \mathrm{d}\tau \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \left(L_{\mathrm{AdS}} + L_{\mathrm{S}} \right), \tag{3.2}$$

[1] For details see [31] and references therein.

where

$$L_{\mathrm{AdS}} = -\frac{1}{2}\eta^{PQ}\partial_a Y_P \partial^a Y_Q + \frac{1}{2}\widetilde{\Lambda}\left(\eta^{PQ}Y_P Y_Q + 1\right),$$
$$L_{\mathrm{S}} = -\frac{1}{2}\delta^{MN}\partial_a X_M \partial^a X_N + \frac{1}{2}\Lambda\left(\delta^{MN}X_M X_N - 1\right).$$

Λ and $\widetilde{\Lambda}$ are Lagrange multipliers used to impose the constraints

$$\eta^{PQ}Y_P Y_Q = -1\,, \qquad \delta^{MN}X_M X_N = 1$$

and $\eta^{PQ} = \mathrm{diag}\,(-1,+1,+1,+1,+1,-1)$. In these coordinates the conserved charges corresponding to the global $\mathrm{SO}(2,4)\times\mathrm{SO}(6)$ symmetry are given by

$$S_{PQ} = \sqrt{\lambda}\int_0^{2\pi}\frac{\mathrm{d}\sigma^1}{2\pi}\left(Y_P\dot{Y}_Q - Y_Q\dot{Y}_P\right), \tag{3.3}$$
$$J_{MN} = \sqrt{\lambda}\int_0^{2\pi}\frac{\mathrm{d}\sigma^1}{2\pi}\left(X_M\dot{X}_N - X_M\dot{X}_N\right).$$

Now we introduce *global coordinates* given by

$$\begin{aligned}
\mathbf{Y}_0 &\equiv Y_5 \ + \ \mathrm{i}Y_0 = \cosh\rho\, e^{\mathrm{i}t},\\
\mathbf{Y}_1 &\equiv Y_1 \ + \ \mathrm{i}Y_2 = \sinh\rho\sin\theta e^{\mathrm{i}\phi_1},\\
\mathbf{Y}_2 &\equiv Y_3 \ + \ \mathrm{i}Y_4 = \sinh\rho\cos\theta e^{\mathrm{i}\phi_2},\\
\mathbf{X}_1 &\equiv X_1 + \mathrm{i}X_2 = \sin\gamma\cos\psi e^{\mathrm{i}\varphi_1},\\
\mathbf{X}_2 &\equiv X_3 + \mathrm{i}X_4 = \sin\gamma\sin\psi e^{\mathrm{i}\varphi_2},\\
\mathbf{X}_3 &\equiv X_5 + \mathrm{i}X_6 = \cos\gamma e^{\mathrm{i}\varphi_3},
\end{aligned} \tag{3.4}$$

such that the $\mathrm{AdS}_5\times\mathrm{S}^5$ metric is

$$\left(\mathrm{d}s^2\right)_{\mathrm{AdS}_5} = \mathrm{d}\rho^2 - \cosh^2\rho\ \mathrm{d}t^2 + \sinh^2\rho\left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi_1^2 + \cos^2\theta\ \mathrm{d}\phi_2^2\right),$$
$$\left(\mathrm{d}s^2\right)_{\mathrm{S}^5} = \mathrm{d}\gamma^2 + \cos^2\gamma\ \mathrm{d}\varphi_3^2 + \sin^2\gamma\left(\mathrm{d}\psi^2 + \sin^2\psi\ \mathrm{d}\varphi_1^2 + \cos^2\psi\ \mathrm{d}\varphi_2^2\right)$$

and the 3+3 Cartan generators of the $SO(2,4)\times SO(6)$ isometry group of the metric can be chosen to be translations in AdS_5 time t, the two angles ϕ_a and the three S^5 angles φ_i.

The Ansatz for a rotating string solution is given by

$$\begin{aligned}
\mathbf{Y}_0 &\equiv Y_5 \ + \ \mathrm{i}Y_0 = z_0(\sigma)e^{\mathrm{i}\omega_0\sigma^0},\\
\mathbf{Y}_1 &\equiv Y_1 \ + \ \mathrm{i}Y_2 = z_1(\sigma)e^{\mathrm{i}\omega_1\sigma^0},\\
\mathbf{Y}_2 &\equiv Y_3 \ + \ \mathrm{i}Y_4 = z_2(\sigma)e^{\mathrm{i}\omega_2\sigma^0},\\
\mathbf{X}_1 &\equiv X_1 + \mathrm{i}X_2 = \widetilde{z}_1(\sigma)e^{\mathrm{i}\nu_1\sigma^0},\\
\mathbf{X}_2 &\equiv X_3 + \mathrm{i}X_4 = \widetilde{z}_2(\sigma)e^{\mathrm{i}\nu_2\sigma^0},\\
\mathbf{X}_3 &\equiv X_5 + \mathrm{i}X_6 = \widetilde{z}_3(\sigma)e^{\mathrm{i}\nu_3\sigma^0},
\end{aligned} \tag{3.5}$$

where $z_i \in (z_0, z_1, z_2)$ and $\widetilde{z}_a \in (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3)$ are complex variables and can be written as

$$\begin{aligned} \widetilde{z}_a &= \widetilde{r}_a e^{\mathrm{i}\alpha_a}, \qquad && \sum_{a=1}^{3} \widetilde{r}_a^2 = 1, \\ z_r &= r_r e^{\mathrm{i}\beta_r}, \qquad && \eta^{rs} r_r r_s = -r_0^2 + r_1^2 + r_2^2 = -1. \end{aligned} \tag{3.6}$$

Plugging in this ansatz into the expressions for the conserved charges yields

$$\begin{aligned} E = S_0 \equiv S_{50} \qquad &= \sqrt{\lambda}\omega_0 \int_0^{2\pi} \frac{\mathrm{d}\sigma^1}{2\pi} r_0^2(\sigma^1)\,, \\ S_i \equiv S_{(2i)(2i+1)} &= \sqrt{\lambda}\omega_i \int_0^{2\pi} \frac{\mathrm{d}\sigma^1}{2\pi} r_i^2(\sigma^1) \qquad && i \in \{1,2\}\,, \\ J_i \equiv S_{(2i)(2i+1)} &= \sqrt{\lambda}\nu_i \int_0^{2\pi} \frac{\mathrm{d}\sigma^1}{2\pi} \widetilde{r}_i^2(\sigma^1) \qquad && i \in \{1,2,3\}\,, \end{aligned}$$

which, due to the constraints (3.6), fulfill

$$\begin{aligned} &\sum_i \frac{J_i}{\sqrt{\lambda}\nu_i} = 1, \\ &\sum_{r,s} \eta^{sr} \frac{S_r}{\sqrt{\lambda}\omega_s} = -1 \quad \text{i.e.} \quad \frac{E}{\kappa} - \frac{S_1}{\omega_1} - \frac{S_2}{\omega_2} = \sqrt{\lambda}. \end{aligned} \tag{3.7}$$

The equations of motion coming from the variation of the lagrangian (3.2) yield

$$\begin{aligned} \partial_\sigma \left(\delta^{ab} \widetilde{r}_a \widetilde{r}_b \alpha'_a\right) = 0 \quad &\Rightarrow \quad \alpha'_a = \frac{v_a}{\widetilde{r}_a^2} \qquad v_a = \text{const.}, \\ \partial_\sigma \left(\eta^{rs} r_r r_s \beta'_r\right) = 0 \quad &\Rightarrow \quad \beta'_r = \frac{u_r}{r_r^2} \qquad u_r = \text{const.}, \end{aligned} \tag{3.8}$$

as well as

$$\begin{aligned} r_0'^2 + \kappa^2 r_0^2 = \sum_{r=1}^{2} \left(r_r'^2 + \omega_r^2 r_r^2 + \frac{u_r^2}{r_r^2} \right) + \sum_{a=1}^{3} \left(\widetilde{r}_a'^2 + \nu_a^2 \widetilde{r}_a^2 + \frac{v_a^2}{\widetilde{r}_a^2} \right), \\ \sum_{r=1}^{2} \omega_r u_r + \sum_{a=1}^{3} \nu_a v_a = 0. \end{aligned} \tag{3.9}$$

Let now

$$\begin{aligned} k_r = m_a = 0, \\ u_r = v_a = 0, \end{aligned}$$

that means assume that ϕ_r and φ_a do not depend on σ. In terms of global coordinates this ansatz is described by

$$\begin{array}{rclrclrcl} t &=& \kappa\sigma^0, & \phi_r &=& \omega_r\sigma^0, & \varphi_a &=& \nu_a\sigma^0, \\ \cosh\rho &=& r_0, & \sinh\rho\sin\theta &=& r_1, & \sinh\rho\cos\theta &=& r_2, \\ \sin\gamma\cos\psi &=& \widetilde{r}_1, & \sin\gamma\sin\psi &=& \widetilde{r}_2, & \cos\gamma &=& \widetilde{r}_3. \end{array} \tag{3.10}$$

While the periodicity of r_0, $r_0(\sigma^1+2\pi) = r_0(\sigma^1)$, implies that $\rho(\sigma^1) = \rho(\sigma^1+2\pi)$ the angles θ, γ and ψ may be periodic up to a 2π shift, e.g. $\psi(\sigma^1+2\pi) = \psi(\sigma^1) + 2\pi n$. If $n = 0$ the solutions are called *folded*, if $n \neq 0$ they are called *circular*.
A non-trivial two-spin folded string solution called "(S, J)" solution is now given by

$$\kappa, \omega_1, \nu_1 \neq 0, \quad \rho = \rho(\sigma), \quad \theta = 0, \quad \gamma = 0, \quad \psi = 0$$

and $\psi(\sigma^1+2\pi) = \psi(\sigma^1)$, such that

$$\begin{array}{ll} \mathbf{Y}_0 = \cosh\rho e^{\mathrm{i}\kappa\sigma^0} = r_0 e^{\mathrm{i}\kappa\sigma^0}, & \mathbf{X}_1 = e^{\mathrm{i}\nu_1\sigma^0}, \\ \mathbf{Y}_1 = \sinh\rho e^{\mathrm{i}\omega_1\sigma^0}, & \mathbf{X}_2 = 0, \\ \mathbf{Y}_2 = 0, & \mathbf{X}_3 = 0, \end{array} \tag{3.11}$$

with the constraint

$$\rho'^2 - \kappa^2\cosh^2\rho + \omega_1^2\sinh^2\rho = -\nu_1^2 \tag{3.12}$$

coming from the Virasoro constraints. Then the conserved charges are given by

$$\begin{aligned} J \equiv J_1 &= \sqrt{\lambda}\nu_1, \\ S \equiv S_1 &= \sqrt{\lambda}\omega_1 \int_0^{2\pi} \frac{\mathrm{d}\sigma^1}{2\pi}\sinh^2\rho, \\ E &= \sqrt{\lambda}\kappa \int_0^{2\pi} \frac{\mathrm{d}\sigma^1}{2\pi}\cosh^2\rho. \end{aligned}$$

Another non-trivial solution, called "(J, J')" solution

$$\kappa, \nu_2, \nu_3 \neq 0, \quad \rho = 0, \quad \theta = 0, \quad \gamma = \frac{\pi}{2}, \quad \psi = \psi(\sigma^1),$$

is connected to the above solution by the analytic continuation

$$\begin{array}{c} \rho \to \mathrm{i}\psi, \quad \kappa \to -\nu_2, \quad \omega_1 \to -\nu_3, \quad \nu_1 \to -\kappa, \\ E \to -J_2, \qquad S_1 \to J_3, \qquad J_1 \to -E. \end{array}$$

The dependence of the energy E on the other quantum numbers S and J can now be extracted using equation (3.7). The classical expression for the energy for the folded (S, J) solution is a complicated function of S and J which interpolates between the functional behaviour in a variety of limits given by[28]

1. $\rho \approx 0$, $\mathcal{J} \ll 1$, $\mathcal{S} \ll 1$ $\qquad E = \sqrt{J^2 + 2\sqrt{\lambda}S} + \dots,$

2. $\rho \approx 0$, $\mathcal{J} \gg \sqrt{\lambda}$, $\mathcal{J} \gg \mathcal{S}$ $\qquad E = J + S + \frac{\lambda S}{2J^2} + \dots,$

3. $\mathcal{S} \gg \sqrt{\lambda}$, $\mathcal{J} \ll \ln(\mathcal{S})$ $\qquad E = S + \frac{\sqrt{\lambda}}{\pi}\ln\frac{S}{\sqrt{\lambda}} + \frac{\pi J^2}{2\sqrt{\lambda}\ln\frac{S}{\sqrt{\lambda}}} + \dots,$

4. $\ln\left(\frac{S}{J}\right) \ll \frac{J}{\sqrt{\lambda}} \ll \frac{S}{\sqrt{\lambda}}$ $\qquad E = S + J + \frac{\lambda}{2\pi^2 J}\ln^2\frac{S}{J} + \dots$

where we used $\mathcal{S} = \frac{1}{\sqrt{\lambda}}S$ and $\mathcal{J} = \frac{1}{\sqrt{\lambda}}J$. Note that analogous expressions can be calculated in the gauge theory framework, where the corresponding gauge invariant operators are of the form $\operatorname{tr}\left(D^S Z^J\right) + \dots$ with the covariant derivative D and complex scalar field Z. Agreement was found at least for the cases (2) and (4). For the case (1) quantum corrections are expected to still play a role and hence this expression cannot be compared to gauge theory. Case (3) is the limit we will explore in the following chapters. An analysis of this limit on the gauge theory side still has to be performed.

3.2.2 Vertex Operators

The central ingredients needed to compute correlation functions on the string theory side of the duality conjecture (2.42) are, as we saw before, the vertex operators corresponding to physical states in the theory. In particular we would like to know the vertex operators corresponding to the classical spinning folded string solutions we discussed in the last section. Unfortunately the construction of vertex operators in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string theory context is not well understood. Therefore what is usually done, is to make a reasonable guess for an operator respecting the $\mathrm{SO}(4,2) \times \mathrm{SO}(6)$ quantum numbers of the state, and to check whether the flat space limit is obtained correctly when the corresponding string gets very small and hence does not experience the curvature of AdS_5 and S^5. One can then go on by a semi-classical evaluation of the two point function and compute the dependence of the energy on the other quantum numbers, which can then be compared to the relations obtained for the classical solutions.

In the string-gauge duality context as we saw it in section 2.3.4 the operators corresponding to the string states should be sourced by a field on the boundary. Therefore a general vertex operator should be parametrized by a boundary point x, as

$$\mathbf{V}(x) = \int \mathrm{d}^2\sigma\; V\left(x'(\sigma), x, z(\sigma); \dots\right), \tag{3.13}$$

where ... stand for the S^5 coordinates and fermions present in the superstring theory and we used the coordinates

$$Y_m = \frac{x_m}{z}, \ Y_4 = \frac{1}{2z}\left(-1 + z^2 + x^m x_m\right), \ Y_5 = \frac{1}{2z}\left(1 + z^2 + x^m x_m\right), \ m = 0, \ldots, 3,$$

with $x^m x_m = -x_0^2 + \sum_{i=1}^3 x_i x_i$. $\mathbf{V}(x)$ is called *integrated* vertex operator. The *unintegrated* vertex operator V can be split into a propagator part

$$K\left(x'(\sigma) - x, z(\sigma)\right) = c\left(\mathbf{Y}_0\right)^{-E}$$

and the part encoding the information about the other quantum numbers U as

$$V\left(x'(\sigma), x, z(\sigma); \ldots\right) = K\left(x'(\sigma) - x, z(\sigma)\right) U\left(x'(\sigma), z(\sigma); \ldots\right).$$

Let us now discuss the vertex operator corresponding to the spinning folded string solution we described in the previous section. In flat space a bosonic string state with spin S and energy E on the leading Regge trajectory is given by[30]

$$V_S = \mathrm{e}^{-\mathrm{i}Et}\left(\partial X \bar{\partial} X\right)^{S/2},$$

where $X \equiv x_1 + \mathrm{i}x_2$ and $\bar{X} = x_1 - \mathrm{i}x_2$. A straightforward guess for the operator corresponding to a string state with angular momentum J in the S^5 is then

$$U_J = \left(\partial \mathbf{X}_1 \bar{\partial} \mathbf{X}_1\right)^{J/2}, \tag{3.14}$$

with $\mathbf{X}_1$ being defined in the same way as in (3.4). Similarly the operator corresponding to a string state with spin S in AdS_5 is given by

$$U_S = \left(\partial \mathbf{Y}_1 \bar{\partial} \mathbf{Y}_1\right)^{S/2}. \tag{3.15}$$

After imposing these operators one has to check whether they correspond to the expected classical solutions. In case of the spinning folded string solution $(S, 0)$ with $\mathcal{S} \gg \sqrt{\lambda}$, the check was performed in [32] and [30]. We will just summarize the general spirit of the derivation. One starts by writing down the bosonic string action including operator insertions at two distinct points on the worldsheet one of which can be set to zero due to the symmetries present in the theory. The operator insertions are source terms for the fields and therefore this action is the effective action for the two-point function. For the case of two insertions of vertex operators of the form (3.15) V_S at $\vec{\sigma} = 0$ and V_{-S} at $\vec{\sigma} = \vec{\sigma_1}$ the action is given by

$$\begin{aligned} S = S_0 + E \int \mathrm{d}^2\sigma \ln\left(\mathbf{Y}_0\right)\left(\delta^2\left(\vec{\sigma}\right) - \delta^2\left(\vec{\sigma} - \vec{\sigma_1}\right)\right) \\ - \frac{S}{2}\int \mathrm{d}^2\sigma\, \delta^2\left(\vec{\sigma}\right) \ln\left(\partial \mathbf{Y}_1 \bar{\partial} \mathbf{Y}_1\right) \\ - \frac{S}{2}\int \mathrm{d}^2\sigma\, \delta^2\left(\vec{\sigma} - \vec{\sigma_1}\right) \ln\left(\partial \mathbf{Y}_1 \bar{\partial} \mathbf{Y}_1\right). \end{aligned} \tag{3.16}$$

Now one can show that a special case of the spinning folded string solution $(S,0)$ presented in the previous section in which $\rho(\sigma^1)=\mu\sigma^1$, solves the corresponding equations of motion[2], provided

$$\kappa=\frac{E-S}{\sqrt{\lambda}},\qquad \mu=\frac{1}{\pi}\ln\frac{S}{\sqrt{\lambda}}.$$

The marginality condition on the vertex operators then implies[33] $\kappa\approx\mu$, which yields the expected relation between energy and spin

$$E\approx S+\frac{\sqrt{\lambda}}{\pi}\ln\frac{S}{\sqrt{\lambda}}.$$

The states corresponding to the vertex operators (3.14) and (3.15) are non-BPS states however. This means that the vertex operators are not eigenstates of the anomalous dimension operator and they hence will mix with other operators with the same quantum numbers. Ignoring fermions those operators are given for instance for (3.15) by

$$(\mathbf{Y}_0)^{-E-p-q}(\mathbf{Y}_1)^{p+q}(\partial\mathbf{Y}_0)^p(\partial\mathbf{Y}_1)^{\frac{S}{2}-p}\left(\bar\partial\mathbf{Y}_0\right)^q\left(\bar\partial\mathbf{Y}_1\right)^{\frac{S}{2}-q}+\mathcal{O}\left(\partial Y_M\partial Y^M\bar\partial Y_K\bar\partial Y^K\right),$$

with $p,q\in\left\{0,\ldots,\frac{S}{4}\right\}$. Since all those operators have the same classical quantum numbers their mixing is not suppressed and hence it is hard to calculate correlators of such operators. There is a way though to semi-classically calculate at least 3-point functions which we will review in the next section.

Note, that in the following we will sometimes perform a euclidean continuation of the form

$$Y_{0\mathrm{e}}=\mathrm{i}Y_0,$$

such that $Y_MY^M=-Y_5^2+\sum_{i=0}^4Y_i^2=1$ becomes the definition of AdS_5 in the embedding[3], as we saw in section 2.3.2 and the isometry group $\mathrm{SO}(4,2)$ is replaced by $\mathrm{SO}(5,1)$. This group contains the discrete transformation $Y_{0\mathrm{e}}\leftrightarrow Y_4$ under which $E\leftrightarrow\Delta$, with Δ being the eigenvalue of the dilatation operator acting as $\xi\to k\xi$ and we can label the states by the set of quantum numbers (Δ,S_1,S_2) instead of (E,S_1,S_2). In that case the vertex operator contains a factor $(Y_5+Y_4)^{-\Delta}$ which replaces the factor of $(\mathbf{Y}_0)^{-E}$ such that the propagator part of the unintegrated vertex operator becomes

$$K\left(\vec{\xi}'(\sigma)-\vec{\xi},\xi^0(\sigma)\right)=c\,(Y_5+Y_4)^{-\Delta}=c\left[z+z^{-1}x^mx_m\right]^{-\Delta}$$

and in the relations of the energy and the other quantum numbers E gets replaced by Δ.

[2]This is a solution to the equation that approximates (3.12) in the limit $\kappa=\omega_1=\mu\gg1$ on the interval $\sigma^1\in\left(0,\frac{\pi}{2}\right]$. The formal solution on $0<\sigma^1\leq2\pi$ is obtained by combining four of such stretches to get a *sawtooth-wave* like looking curve.

[3]Note that we have set $R=1$.

3.2.3 Semiclassical Computation of Some 3-point Functions

As we saw in the last chapter the direct computation of n-point correlators of non-BPS operators is quite difficult. There was a recent proposal [34] however on how to semiclassically compute at least three-point functions[4]. The general idea is to consider a correlation function of n vertex operators V_{H} that are cosidered *heavy*, that means having $\Delta \sim Q_i \sim \sqrt{\lambda} \gg 1$, and some number m of *light* vertex operators for which $Q_i \sim 1$ and $\Delta \sim \lambda^{1/4}$ or $\Delta \sim 1$,

$$\langle V_{\mathrm{H}1}(x_1) \dots V_{\mathrm{H}n}(x_n) V_{\mathrm{L}n+1}(x_{n+1}) \dots V_{\mathrm{L}n+m}(x_{n+m}) \rangle . \tag{3.17}$$

Then in the large λ expansion the leading order will be given by evaluating the light vertex operators on the semi-classical trajectories that are determined by the heavy operators in the way we saw in (3.16) and below. This can be motivated as follows. In the case in which one wants to semi-classically compute the correlator of $n+m$ heavy operators one has to solve the equations of motion coming from the action with $n+m$ operator insertions, which is quite hard in general. If now the quantum numbers of m of the operators are much smaller than those of the others, the solution to the equations of motion will be dominated by the contribution of the n heavy operators and the effect of the light operators can be included perturbatively.

Here we will analyze three-point functions of two heavy operators corresponding to spinning folded string solutions and one light vertex operator following [34]. Since we are dealing with a conformal field theory the three-point functions are, due to the symmetries, constrained to be of the form[5]

$$\langle V_1(x_1) V_2(x_2) V_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_{123}} |x_1 - x_3|^{\Delta_{132}} |x_2 - x_3|^{\Delta_{231}}} , \tag{3.18}$$

where $\Delta_{abc} \equiv \Delta_a + \Delta_b - \Delta_c$. The constant C_{123} will in general depend on the choice of normalization of all three of the operators. One can define the ratio

$$\frac{\langle V_{\mathrm{H}}(x_1) V_{\mathrm{H}}(x_2) V_{\mathrm{L}}(x_3) \rangle}{\langle V_{\mathrm{H}}(x_1) V_{\mathrm{H}}(x_2) V_{\mathrm{L}'}(x_3) \rangle} \cdot \frac{\langle V_{\mathrm{H}'}(x_1) V_{\mathrm{H}'}(x_2) V_{\mathrm{L}'}(x_3) \rangle}{\langle V_{\mathrm{H}'}(x_1) V_{\mathrm{H}'}(x_2) V_{\mathrm{L}}(x_3) \rangle},$$

in which the normalization constants of both the heavy and the light vertex operators cancel out. Alternatively one can define

$$\tilde{C}_{123} = \frac{\langle V_{\mathrm{H}}(x_1) V_{\mathrm{H}}(x_2) V_{\mathrm{L}}(x_3) \rangle}{\langle V_{\mathrm{H}}(x_1) V_{\mathrm{H}}(x_2) \rangle}, \tag{3.19}$$

[4]This has also been extended to correlators of more than three operators[37].

[5]For simplicity we consider only correlators of scalar operators here. There are similar expressions for operators with spin.

which depends only on the normalization of the light vertex operator. In this way one avoids difficulties coming from infinite symmetry factors which cancel out in the ratio of three- and two-point functions. $\tilde{C}_{123}$ will in general not be a constant since the x-dependence of the operators does not cancel generically. However we will see below that there are cases in which $\tilde{C}_{123}$ becomes independent of all x_i.

Before we can start calculating correlators of the form discussed above we have to discuss some types of operators we can use as the light ones. The simplest candidates are possibly the dilaton operator and its higher Kaluza-Klein harmonics with angular momentum J in the S^5. The corresponding vertex operators are given by[34]

$$U_J^{(\text{dil})} = (\mathbf{X}_1)^J \left(\partial Y_M \bar{\partial} Y^M + \partial X_k \bar{\partial} X^k + \dots\right), \tag{3.20}$$

where the dots represent fermionic terms. They are highest-weight states of the $\mathrm{SO}(2,4) \times \mathrm{SO}(5)$ symmetry as it should be the case and since the dilaton and its harmonics are BPS states they are actual eigenvectors of the anomalous dimension operators and do not mix with other operators.

The 1/2 BPS *superconformal primary scalar* operator is the highest weight state of the SO(6) representation $[0, J, 0]$ $(J \geq 2)$ we saw in section 2.2.2. It is given by [34]

$$U_J^{(\text{dil})} = (\mathbf{X}_1)^J \left(z^{-2}\left(\partial x^m \bar{\partial} x_m - \partial z \bar{\partial} z\right) - \partial X_k \bar{\partial} X^k\right) + \dots, \tag{3.21}$$

where the dots stand for fermions and derivative terms that will not contribute to the computation. The dilaton operator we discussed above is the supersymmetry descendant of this operator.

With these two classes of operators at hand we can now calculate some correlators in the way described above. Since the correlation functions should transform as singlets under the symmetry group of the theory, the two heavy operators should be conjugate to each other. Further, as we have seen in equation (3.18), the x_i dependence of the correlator is determined completely by the conformal invariance which enables us to fix the position of the light vertex operator to $x_3 = (0, 0, 0, 0)$ such that it looks like

$$V_{\text{L}}(0) = c \int \mathrm{d}^2\sigma \, (Y_5 + Y_4)^{-\Delta_{\text{L}}} \, U\left(x(\sigma), z(\sigma); \dots\right).$$

Since the heavy operators will, in our case, correspond to spinning folded string solutions, the semiclassical solutions will fulfill $Y_4 = 0$ and therefore $Y_5 = z^{-1}$ such that the semiclassical expression for the three-point function (in Poincare coordinates) simplifies to

$$\langle V_{\text{H}}(x_1) V_{\text{H}}(x_2) V_{\text{L}}(0)\rangle \sim \int \mathrm{d}^2\sigma \; z_{\text{cl.}}^{\Delta} \, U\left(x_{\text{cl.}}(\sigma), z_{\text{cl.}}(\sigma); \dots\right). \tag{3.22}$$

To simplify the computation of such a correlator further let us investigate the classical solution corresponding to the heavy operators a bit closer. Performing the euclidean continuation of the (S, J) folded spinning string solution (3.11) with $\kappa = \omega_1$ and $\rho = \mu\sigma^1$, that means writing $\sigma^0_{\mathrm{e}} = i\sigma^0$ and $Y_{0\mathrm{e}} = iY_0$ we get

$$
\begin{aligned}
Y_{0\mathrm{e}} &= \cosh\left(\kappa\sigma^0_e\right)\cosh\left(\mu\sigma^1\right), \quad & Y_5 &= \cosh\left(\kappa\sigma^0_e\right)\sinh\left(\mu\sigma^1\right), \\
Y_{1\mathrm{e}} &= \cosh\left(\kappa\sigma^0_e\right)\sinh\left(\mu\sigma^1\right), \quad & Y_2 &= -\mathrm{i}\sinh\left(\kappa\sigma^0_e\right)\sinh\left(\mu\sigma^1\right),
\end{aligned}
$$

which in the (z, x) coordinates looks like

$$
z = \frac{1}{\cosh\left(\kappa\sigma^0_e\right)\cosh\left(\mu\sigma^1\right)}, \qquad x_{0\mathrm{e}} = \tanh\left(\kappa\sigma^0_{\mathrm{e}}\right), \tag{3.23}
$$

$$
x_1 = \tanh\left(\mu\sigma^1\right), \qquad x_2 = -\mathrm{i}\tanh\left(\mu\sigma^1\right)\tanh\left(\kappa\sigma^0_{\mathrm{e}}\right), \tag{3.24}
$$

where $x_{0\mathrm{e}} = \mathrm{i}x_0$ and $\kappa^2 = \mu^2 + \nu^2$. Note that for $\sigma^0_{\mathrm{e}} \to \pm\infty$ the string approaches the boundary $z \to 0$ of AdS_5 and since $z^2 + x^2_{0\mathrm{e}} + x^2_1 + x^2_2 = 1$ in that case $|x| = 1$. Therefore and since the two-point correlator of two operator insertions with scaling dimensios Δ_1 and Δ_2 at x_1 and x_2 respectively behaves like $|x_1 - x_2|^{-2\Delta_1}\,\delta_{\Delta_1,\Delta 2}$, $\tilde{C}_{123}$ becomes a constant that we can extract from the ratio

$$
\tilde{C}_{123} = \frac{\langle V_{\mathrm{H}}(x_1)V_{\mathrm{H}}(x_2)V_{\mathrm{L}}(0)\rangle}{\langle V_{\mathrm{H}}(x_1)V_{\mathrm{H}}(x_2)\rangle} = c_\Delta \int \mathrm{d}^2\sigma\; z^\Delta_{\mathrm{cl.}}\; U\left(x_{\mathrm{cl.}}(\sigma), z_{\mathrm{cl.}}(\sigma); \dots\right), \tag{3.25}
$$

where Δ is the scaling dimension of the light operator and c_Δ depends just on the normalization of this operator.

Using equation (3.25) we are finally in the position to calculate semi-classical three-point functions. First we choose the dilaton-type operator (3.20) as the light vertex operator. In the (z, x) coordinates it looks like

$$
U = (\mathbf{X}_1)^j\left[z^{-2}\left(\partial x_m\bar{\partial}x^m + \partial z\bar{\partial}z\right) + \partial X_k\bar{\partial}X_k\right],
$$

where $\Delta = 4 + j$ and j is the S^5 angular momentum. The normalization constant c_Δ was calculated to be given by[35]

$$
c_\Delta = c_{4+j} = \frac{2^{-j/2}}{2\pi^2}\,(j+3)\,.
$$

Evaluating U on the classical solution (3.23) we get

$$
\mathrm{e}^{j\mu\sigma^0_{\mathrm{e}}}\left(\kappa^2\cosh^2\left(\mu\sigma^1\right) + \mu^2 - \kappa^2\sinh^2\left(\mu\sigma^1\right) - \nu^2\right) = 2\mu^2\mathrm{e}^{j\mu\sigma^0_{\mathrm{e}}},
$$

such that the expression for $\tilde{C}_{123}$ becomes

$$
\tilde{C}_{123} = 4c_\Delta \int_{-\infty}^{\infty}\mathrm{d}\sigma^0_{\mathrm{e}}\int_0^{\frac{\pi}{2}}\mathrm{d}\sigma^1\;\frac{2\mu^2\mathrm{e}^{j\nu\sigma^0_{\mathrm{e}}}}{\left[\cosh\left(\mu\sigma^1\right)\cosh\left(\kappa\sigma^0_{\mathrm{e}}\right)\right]^\Delta},
$$

where the factor 4 comes from the four slices of $\rho = \mu\sigma^1$ as discussed in footnote 2. Performing the integral yields

$$
\begin{aligned}
\tilde{C}_{123} &= c_\Delta \frac{\mu}{\kappa} 2^{j+8} C(j,\mu) B\left(j, \frac{\nu}{\kappa}\right), \\
C(j,\mu) &= \sinh\left(\frac{\pi}{2}\mu\right) \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}(5+j), \frac{3}{2}, -\sinh^2\left(\frac{\pi}{2}\mu\right)\right), \\
B\left(j, \frac{\nu}{\kappa}\right) &= \frac{{}_2F_1(4+j,\, b_+,\, b_+ + 1,\, -1)}{b_+} + \frac{{}_2F_1(4+j,\, b_-,\, b_- + 1,\, -1)}{b_-}, \\
b_\pm &= 4 + j\left(1 \pm \frac{\nu}{\kappa}\right).
\end{aligned}
$$

For the case of the dilaton, that means $j = 0$, we get

$$
\tilde{C}_{123} = \frac{64 c_\Delta (\mathcal{S}-1)(\mathcal{S}^2 + 4\mathcal{S} + 1)\ln\mathcal{S}}{9\pi(\mathcal{S}+1)^3\sqrt{\mathcal{J}^2 + \frac{1}{\pi^2}\ln^2\mathcal{S}}},
$$

which in the case of $S \gg \sqrt{\lambda}$ becomes

$$
\tilde{C}_{123} \sim \frac{\ln S}{\sqrt{J^2 + \frac{\lambda}{\pi^2}\ln^2 S}}. \tag{3.26}
$$

There are two distinct cases now. If $J \gg \frac{\sqrt{\lambda}}{\pi}\ln S$ the folded spinning string solution goes to the BPS solution with $E = J$ and $\tilde{C}_{123}$ vanishes. This matches the expectation that the dilaton does not couple to BPS states. In the case in which $J \ll \frac{\sqrt{\lambda}}{\pi}\ln S$ the correlator approaches a constant. This is reasonable since the states cooresponding to the heavy operators are massive and hence the dilaton couples generarically to them via their mass term. It is worthwile to notice that the expression (3.26) is in agreement with the strong-coupling limit of the corresponding coupling on the gauge theory side[6].

Our second candidate for the light vertex operator was the superconformal primary scalar (3.21). Evaluating the vertex operator on the solution (3.23) yields

$$
U = \mathrm{e}^{j\nu\sigma_\mathrm{e}^0}\left[\kappa^2\left(\frac{2}{\cosh^2(\kappa\sigma_\mathrm{e}^0)} - 1\right) + \mu^2\left(\frac{2}{\cosh^2(\mu\sigma^1)} - 1\right) + \nu^2\right],
$$

with $\Delta = j$. This yields

$$
\tilde{C}_{123} = 4c_\Delta \int_{-\infty}^{\infty} \mathrm{d}\sigma_\mathrm{e}^0 \int_0^{\frac{\pi}{2}} \mathrm{d}\sigma^1 \, \frac{2\mathrm{e}^{j\nu\sigma_\mathrm{e}^0}}{\left[\cosh(\mu\sigma^1)\cosh(\kappa\sigma_\mathrm{e}^0)\right]^\Delta}\left[\frac{\kappa^2}{\cosh^2(\kappa\sigma_\mathrm{e}^0)} - \mu^2\tanh^2(\mu\sigma^1)\right],
$$

[6] Details on this can be found in [34].

which when evaluating the integrals becomes a complicated function of $\frac{\nu}{\mu}$ and j. It can be discussed in several limits. We will not repeat the whole discussion, which can be found in [34], here. There is a specific interesting limit though in which one first expands $\tilde{C}_{123}$ for small μ and fixed $l \equiv \frac{\nu}{\mu} \gg 1$ and j and afterwards lets $l \to \infty$. The resulting expression reads

$$\tilde{C}_{123} = \frac{2^{j+3}\pi c_\Delta}{j+1}\mu l \left[1 + \mathcal{O}(l^{-2})\right],$$

with the normalization[36]

$$c_\Delta = \frac{(j+1)\sqrt{j}}{2^{j+3}N}\sqrt{\lambda},$$

where N is the rank of the $\mathrm{U}(N)$ gauge group, as usual, entering the normalization via the common radius of AdS_5 and S^5. Effectively we therefore get

$$\tilde{C}_{123} = \frac{1}{N}J\sqrt{j}.$$

This result for the three-point coupling of three BMN-type operators was independently obtained in a different way and hence gives a justification for the method. In the literature the calculations have been performed for more classes of light operators and also when the heavy operators correspond to rigid circular string solutions in S^5 with three angular momenta $J_1 = J_2$ and J_3. Furthermore the method has been extended to four-point functions [37].

3.3 Stringy Correlators Using Operator Quantization

In this section we want to review how bosonic string theory correlators of the form (3.1) can be computed using operator techniques. Since the theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background comes with various complications we will do some calculations on flat spacetime. The $\mathrm{AdS}_5 \times \mathrm{S}^5$ case will be discussed in the following section. For simplicity we will restrict ourselves to the bosonic string[7]. The basic idea of these calculations in the operator technique is to use the fact that the oscillator modes α_n^μ of the world sheet fields $X^\mu(\sigma^0, \sigma^1)$ annihilate the vacuum if $n \geq 0$

$$\langle 0| \alpha_{-n}^\mu = \alpha_n^\mu |0\rangle = 0 \qquad n \geq 0.$$

Therefore the goal is to use the commutation relations of the operators to make these oscillators act on the vacuum and in this way get rid of them.

[7]Calculations involving fermions in the RNS formalism can be found in chapter 7 of [4].

Vertex operators, in general, have to be conformal primaries. Denoting their dimension by $\left(h, \bar{h}\right)$, they transform as

$$V\left(z, \bar{z}\right) \to \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} V\left(w, \bar{w}\right)$$

under the infinitesimal transformations $z \to w(z)$. Equivalently the states fulfill

$$\begin{aligned} L_0 \left|\Phi\right\rangle &= h \left|\Phi\right\rangle, & L_n \left|\Phi\right\rangle &= 0 \;\; n > 0, \\ \bar{L}_0 \left|\Phi\right\rangle &= \bar{h} \left|\Phi\right\rangle, & \bar{L}_n \left|\Phi\right\rangle &= 0 \;\; n > 0. \end{aligned}$$

Since vertex operators correspond to physical states, which fulfill the conditions

$$\begin{aligned} \left(L_0 - 1\right) \left|\Phi\right\rangle_{\text{phys}} &= 0, & L_n \left|\Phi\right\rangle_{\text{phys}} &= 0 \;\; n > 0, \\ \left(\bar{L}_0 - 1\right) \left|\Phi\right\rangle_{\text{phys}} &= 0, & \bar{L}_n \left|\Phi\right\rangle_{\text{phys}} &= 0 \;\; n > 0, \end{aligned}$$

the operators are restricted to have scaling dimension $h = \bar{h} = 1$. There are various ways to check that this is fulfilled, one of which is the condition

$$\left[L_m, V(k, z, \bar{z})\right] = \left(z^{m+1} \frac{\mathrm{d}}{\mathrm{d}z} + m z^m\right) V(k, z, \bar{z}).$$

There is a more convenient way though which we will use in the following. Under the infinitesimal transformations $\delta z = \epsilon(z)$, $\delta \bar{z} = \bar{\epsilon}(\bar{z})$ the vertex operators transform as

$$\begin{aligned} \delta_\epsilon V(w, \bar{w}) &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \; \epsilon(z) \left[T(z), V(w, \bar{w})\right], \\ \delta_{\bar{\epsilon}} V(w, \bar{w}) &= \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \; \epsilon(z) \left[\bar{T}(\bar{z}), V(w, \bar{w})\right] \end{aligned}$$

respectively. Where, as a consequence of time-ordering, the integrals are evaluated along a closed z or $\bar{z}$ loop encircling the point w or $\bar{w}$ respectively. Complex analysis now tells us that, in order to calculate the integrals we only need to know the singular terms in $(z - w)$. For conformal primary fields of dimensions $(h, \bar{h})$ the singular terms are poles and the short-distance singularities in the *operator product expansion* (OPE) with the energy-momentum tensor are given by

$$\begin{aligned} T(z) V(w, \bar{w}) &= \frac{h}{\left(z - w\right)^2} V(w, \bar{w}) + \frac{1}{z - w} \partial V(w, \bar{w}) + \ldots, \\ \bar{T}(\bar{z}) V(w, \bar{w}) &= \frac{\bar{h}}{\left(\bar{z} - \bar{w}\right)^2} V(w, \bar{w}) + \frac{1}{\bar{z} - \bar{w}} \partial V(w, \bar{w}) + \ldots, \end{aligned}$$

where the ... stand for non-sigular terms. Therefore, in order to check the conformal dimension of a given vertex operators, we have to calculate its OPE with the energy-momentum tensor and check whether it has the above form with $h = \bar{h} = 1$.

The vertex operators of flat target space bosonic open string theory have the generic form

$$V_\Lambda(k,z) =: W_\Lambda(z) V_0(k,z) : ,$$

where

$$V_0(k,z) = \mathrm{e}^{\mathrm{i}\,k\cdot X(z)},$$

which, using the oscillator expansion

$$X^\mu(z) = x^\mu - \mathrm{i}\,p^\mu \ln z + \mathrm{i}\sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu z^{-n},$$

can be recast in the form

$$\begin{aligned}
V_0 &= Z_0 W_0, \\
Z_0 &= \exp\left(\mathrm{i}\,k\cdot x + k\cdot p \ln z\right), \\
W_0 &= \exp\left(k\cdot \sum_{n=1}^{\infty} \frac{1}{n}\alpha_{-n} z^n\right) \exp\left(-k\cdot \sum_{n=1}^{\infty} \frac{1}{n}\alpha_n z^{-n}\right).
\end{aligned}$$

Closed strings, now, possess a left- and a right-moving sector. As is well known they can be described by a direct product of open string states each carrying half of the momentum k of the corresponding state. Therefore the vertex operators factorize according to

$$V(k,\sigma^0,\sigma^1) = V_\mathrm{L}\left(k/2, \sigma^0+\sigma^1\right) V_\mathrm{R}\left(k/2, \sigma^0-\sigma^1\right).$$

While the position of the vertex operator on the world sheet of closed strings is fixed to be at one of the end points $\sigma^1 = 0, \pi$, on a closed string the vertex operator can be at any σ^1 position. Therefore, the vertex operators have to be integrated over σ^1 and take the form

$$V^\mathrm{closed}(k,\sigma^0) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma^1 \; V_\mathrm{L}\left(k/2, \sigma^0+\sigma^1\right) V_\mathrm{R}\left(k/2, \sigma^0-\sigma^1\right).$$

Now we can use the fact that σ^1 translations on the worldsheet are generated by $L_0 - \bar{L}_0$ to rewrite the closed string vertex operator in the form

$$V^\mathrm{closed}(k,\sigma^0) = \int_0^\pi \frac{\mathrm{d}\sigma^1}{\pi}\, \mathrm{e}^{-2\mathrm{i}\sigma^1\left(L_0-\bar{L}_0\right)} V_\mathrm{L}\left(k/2,\sigma^0\right) V_\mathrm{R}\left(k/2,\sigma^0\right) \mathrm{e}^{2\mathrm{i}\sigma^1\left(L_0-\bar{L}_0\right)}. \quad (3.27)$$

In the same way the closed string states factorize according to

$$|\Phi\rangle_{\text{closed}} = |\Phi\rangle_{\text{L}} \otimes |\Phi\rangle_{\text{R}} \,.$$

The form of (3.27) has an important implication for the calculation of three-point functions. Since physical states are constrained to fulfill the level matching condition they are annihilated by $L_0 - \bar{L}_0$ and therefore the σ^1 integral is trivial. Using the factorization properties of the closed string vertex operators and states as well as the fact that the left- and right-moving oscillators commute a three-point function of closed string states can be written as

$$\left(\langle\Phi_1|_{\text{L}}\, V_{2,\text{L}}\left(k/2,\sigma^0\right)|\Phi_3\rangle_{\text{L}}\right)\cdot\left(\langle\Phi_1|_{\text{R}}\, V_{2,\text{R}}\left(k/2,\sigma^0\right)|\Phi_3\rangle_{\text{R}}\right).$$

Before we start calculating such an amplitude we will go back and look at the conformal dimension condition for vertex operators. The energy momentum in flat target space bosonic string theory is given by

$$T(z) = -2 : \partial X \cdot \partial X : \,, \qquad \bar{T}(\bar{z}) = -2 : \bar{\partial} X \cdot \bar{\partial} X :$$

and the OPE of the bosonic field $X^\mu(z)$ with itself is

$$X^\mu(z) X^\nu(w) = -\frac{1}{4}\ln\left(z-w\right) + \ldots .$$

With this information at hand it is straightforward to compute the conformal dimension of the tachyon vertex operator

$$T(z) : \mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : = \frac{k^2}{8(z-w)^2} : \mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : + \frac{1}{z-w} : k\cdot\partial X(w)\mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : + \ldots,$$
$$\bar{T}(\bar{z}) : \mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : = \frac{k^2}{8(\bar{z}-\bar{w})^2} : \mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : + \frac{1}{\bar{z}-\bar{w}} : k\cdot\bar{\partial} X(\bar{w})\mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : + \ldots .$$

Hence the conformal dimension of this operator is $(k^2/8, k^2/8)$ and in order to make it a consistent vertex operator we have to demand that $k^2 = 8$. Similarly one can compute the OPE of the graviton operator and the stress tensor

$$-2f_{\mu\nu} : \partial X\cdot\partial X : \underbrace{: \partial X^\mu(w)\bar{\partial}X^\nu(\bar{w})\mathrm{e}^{ik\cdot X(w,\bar{w})} :}_{\equiv V^{\mu\nu}_{\text{grav}}} = f_{\mu\nu}\left(-\frac{\mathrm{i}}{4}k^\mu\frac{\bar{\partial}X^\nu(\bar{\omega})}{(z-w)^3} + \frac{1+k^2/8}{(z-w)^2}V^{\mu\nu}_{\text{grav}}\right.$$
$$\left. + \,\mathrm{i}\frac{k\cdot\partial X(w)}{z-w}f_{\mu\nu}V^{\mu\nu}_{\text{grav}} + \ldots\right),$$

which gives the conditions

$$k^\mu f_{\mu\nu} = 0 \qquad \text{and} \qquad k^2 = 0\,,$$

that have to be fulfilled in order for $f_{\mu\nu}V^{\mu\nu}_{\text{grav}}$ to be a good vertex operator. Analogously one can calculate the OPE of the vertex operator with the anti-holomorphic part of the stress tensor $\bar{T}(\bar{z})$ which doesn't impose additional constraints. Having done these exercises we can now look at a candidate for a vertex operator corresponding to a large spin S, closed, bosonic string state on flat target space. In analogy to the graviton operator we will assume

$$V_S = f_{\mu_1\bar{\mu}_1\ldots\mu_{S/2}\bar{\mu}_{S/2}} : \partial X^{\mu_1}\bar{\partial}X^{\bar{\mu}_1}\ldots\partial X^{\mu_{S/2}}\bar{\partial}X^{\bar{\mu}_{S/2}}\,\mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} : . \tag{3.28}$$

The calculation of the conformal dimension is analogous to the graviton vertex. There is an additional contribution of order four in $(z-w)^{-1}$, though, which comes from the contraction of the two ∂X terms in the stress tensor with two of such terms in the spin-S vertex operator. It is straightforwardly calculated to be given by

$$-\frac{1}{8\,(z-w)^4}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{\infty} f_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\bar{\mu}_{S/2}}\eta^{\mu_i\mu_j} : \partial X^{\mu_1}\ldots\widehat{\partial X^{\mu_i}}\ldots\widehat{\partial X^{\mu_j}}\ldots\bar{\partial}X^{\bar{\mu}_{S/2}}\,\mathrm{e}^{\mathrm{i}k\cdot X} :\right),$$

where $\widehat{\partial X^{\mu_i}}$ means that this term is omitted in the product. Demading that the vertex operator is a conformal primary therefore imposes the condition

$$f_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\bar{\mu}_{S/2}}\eta^{\mu_i\mu_j} = 0 \qquad \forall\, i,j \in \{1,\ldots,S/2\}\,,\ i\neq j.$$

The order three divergent term in the OPE looks similar to the one of the graviton

$$-\frac{\mathrm{i}}{8\,(z-w)^3}\left(\sum_{i=1}^{\infty} f_{\mu_1\ldots\mu_i\ldots\bar{\mu}_{S/2}}k^{\mu_i} : \partial X^{\mu_1}\ldots\widehat{\partial X^{\mu_i}}\ldots\bar{\partial}X^{\bar{\mu}_{S/2}}\,\mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} :\right)$$

and the corresponding condition ensuring the right transformation behaviour of the operator is

$$f_{\mu_1\ldots\mu_i\ldots\bar{\mu}_{S/2}}k^{\mu_i} = 0 \qquad \forall\, i \in \{1,\ldots,S/2\}\,.$$

Finally the last interesting term in the OPE is given by

$$-\frac{\mathrm{i}}{8\,(z-w)^2}\left(k^2 V_S + \sum_{i=1}^{\infty} f_{\mu_1\ldots\mu_i\ldots\bar{\mu}_{S/2}} : \partial X^{\mu_i}\partial X^{\mu_1}\ldots\widehat{\partial X^{\mu_i}}\ldots\bar{\partial}X^{\bar{\mu}_{S/2}}\,\mathrm{e}^{\mathrm{i}k\cdot X(w,\bar{w})} :\right),$$

such that we can guarantee that V_S has scaling dimension $h=1$ when

$$f_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\bar{\mu}_{S/2}} = f_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\bar{\mu}_{S/2}} \qquad \forall\, i,j \in \{1,\ldots,S/2\}\,,\ i\neq j$$

and $k^2 = 0$. Analogous conditions can be obtained for the $\bar{\mu}$ indices of f by the OPE of the anti-holomorphic part of the stress-tensor $\bar{T}(\bar{z})$ with V_S.
To summarize, the conditions we have to impose on V_S are the following

$$\begin{aligned}
f_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\bar{\mu}_{S/2}}\eta^{\mu_i\mu_j} &= f_{\mu_1\ldots\bar{\mu}_i\ldots\bar{\mu}_j\ldots\bar{\mu}_{S/2}}\eta^{\bar{\mu}_i\bar{\mu}_j} = 0 && \forall\, i,j \in \{1,\ldots,S/2\}\,,\ i \neq j,\\
f_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\bar{\mu}_{S/2}} &= f_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\bar{\mu}_{S/2}} && \forall\, i,j \in \{1,\ldots,S/2\}\,,\ i \neq j,\\
f_{\mu_1\ldots\bar{\mu}_i\ldots\bar{\mu}_j\ldots\bar{\mu}_{S/2}} &= f_{\mu_1\ldots\bar{\mu}_j\ldots\bar{\mu}_i\ldots\bar{\mu}_{S/2}} && \forall\, i,j \in \{1,\ldots,S/2\}\,,\ i \neq j,\\
f_{\mu_1\ldots\mu_i\ldots\bar{\mu}_{S/2}}k^{\mu_i} &= f_{\mu_1\ldots\bar{\mu}_i\ldots\bar{\mu}_{S/2}}k^{\bar{\mu}_i} = 0 && \forall\, i \in \{1,\ldots,S/2\}\,, \qquad (3.29)\\
k^2 &= 0.
\end{aligned}$$

Knowing that (3.28) is a reasonable vertex operator we can now start calculating correlation functions. We will, in analogy to the previous section investigate a three-point function of two large spin closed string states and the dilaton. In order to simplify the corresponding expression let us first examine the state corresponding to the vertex operator (3.28). According to the standard operator-state correspondence it is given by[8]

$$\begin{aligned}
|2S,2k\rangle_{\mathrm{L}} &= \lim_{y\to 0} y^{-1} :\partial X^{\mu_1}(y)\ldots\partial X^{\mu_S}(y)\mathrm{e}^{\mathrm{i}k\cdot X(y)}: |0\rangle\\
&= \lim_{y\to 0} y^{-1} : \prod_{i=1}^{S}\left(\sum_{n\in\mathbb{Z}} \alpha_n^{\mu_i} y^{-n-1}\right) y\, \mathrm{e}^{\mathrm{i}k\cdot x} y^{k\cdot p}\mathrm{e}^{k\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}y^n}\mathrm{e}^{-k\cdot\sum_{n>0}\frac{1}{n}\alpha_n y^{-n}} : |0\rangle\\
&= \lim_{y\to 0} : \sum_{n_1,\ldots,n_S\in\mathbb{Z}} \alpha_{n_1}^{\mu_1}\ldots\alpha_{n_S}^{\mu_S}\, y^{-\left(\sum_{i=1}^{S} n_i\right)-S}\mathrm{e}^{k\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}y^n} : |k\rangle\,,
\end{aligned}$$

where we have used the definition $|k\rangle \equiv \mathrm{e}^{\mathrm{i}k\cdot x}y^{k\cdot p}\,|0\rangle$. Now, if any of the n_i is positive the corresponding α_{n_i} gets shifted to the right by normal ordering and annihilates $|k\rangle$. Therefore oly the terms in which n_i are negative or zero survive.

$$|2S,2k\rangle_{\mathrm{L}} = \lim_{y\to 0} \sum_{n_1,\ldots,n_S\geq 0} \alpha_{-n_1}^{\mu_1}\ldots\alpha_{-n_S}^{\mu_S}\, y^{\left(\sum_{i=1}^{S} n_i\right)-S}\mathrm{e}^{k\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}y^n}\,|k\rangle\,.$$

In the above expressios we suppressed the "polarization tensor" $f_{\mu_1\ldots\mu_S}$ that the terms are contracted with. As we have seen in (3.29) this tensor has the property $k^{\mu_i}f_{\mu_1\ldots\mu_i\ldots\mu_S} = 0$ for any i. This property becomes important now because whenever one of the $n_i = 0$, the resulting expression will vanish since $\alpha_0^{\mu_i}\,|k\rangle = k^{\mu_i}\,|k\rangle$. Therefore we get

$$\begin{aligned}
|2S,2k\rangle_{\mathrm{L}} &= \lim_{y\to 0} \sum_{n_1,\ldots,n_S> 0} \alpha_{-n_1}^{\mu_1}\ldots\alpha_{-n_S}^{\mu_S}\, y^{\left(\sum_{i=1}^{S} n_i\right)-S}\mathrm{e}^{k\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}y^n}\,|k\rangle\\
&= \alpha_{-1}^{\mu_1}\ldots\alpha_{-1}^{\mu_S}\,|k\rangle\,. \qquad (3.30)
\end{aligned}$$

[8]We focus on the left-moving sector here. The calculation for the right-moving sector is analogous.

This results is in accordance with the naive expectation for an open spin S state one has from operator quantization of flat space bosonic string theory. Using this result we can write down the holomorphic (i.e. left-moving) part of the three-point correlator of two such states and a dilaton. It reads

$$-\frac{\mathrm{i}}{2}\langle k|\,\alpha_1^{\mu_1}\ldots\alpha_1^{\mu_S} : \left(\sum_{n\in\mathbb{Z}}\alpha_n^{\mu}z^{-n-1}\right)\mathrm{e}^{\mathrm{i}\tilde{k}\cdot x}z^{\tilde{k}\cdot p} \\ \times\,\mathrm{e}^{\tilde{k}\cdot\sum_{n>0}\frac{1}{n}\alpha_{-n}z^{n}}\mathrm{e}^{-\tilde{k}\cdot\sum_{n>0}\frac{1}{n}\alpha_{n}z^{-n}} : \alpha_{-1}^{\rho_1}\ldots\alpha_{-1}^{\rho_S}\,|l\rangle\,,$$

where $2k$ and $2l$ are the momenta of the in- and outgoing states, $2\tilde{k}$ is the momentum of the dilaton and we have omitted the polarization tensors $f_{\mu_1\ldots\mu_S}$ and $f_{\rho_1\ldots\rho_S}$ as well as the $\delta_{\mu\nu}$ factor from the dilaton vertex operator that multiply the expression. The calculation for general S is straightforward but quite complicated and does not provide any physical insight. Therefore we will not discuss it here. To obtain the result one has to work out the normal ordering first and then use commutators to get rid of the oscillators. To give the general idea let us consider $S = 2$ here, which corresponds to the graviton.

The commutation relations of the operators enable us to get rid of most of the terms such that the correlator is given by

$$-\frac{\mathrm{i}}{2}\langle k|\,\alpha_1^{\mu_1}\ldots\alpha_1^{\mu_S} : \left(\alpha_1^{\mu}z^{-2}+(l+\tilde{k})^{\mu}z^{-1}+\alpha_{-1}^{\mu}\right)\mathrm{e}^{\tilde{k}\cdot\alpha_{-1}z}\mathrm{e}^{-\tilde{k}\cdot\alpha_1 z^{-1}} : \alpha_{-1}^{\rho_1}\ldots\alpha_{-1}^{\rho_S}\,|l+\tilde{k}\rangle\,,$$

where we have used that $\alpha_0^{\mu}\,|k\rangle = k^{\mu}$. Now making normal ordering explicit and using the diffeomorphism invariance on the worldsheet to set $z = 1$ we get

$$\begin{aligned}&-\frac{\mathrm{i}}{2}\langle k|\,\alpha_1^{\mu_1}\alpha_1^{\mu_2}\left[\alpha_{-1}^{\mu}\mathrm{e}^{\tilde{k}\cdot\alpha_{-1}}\mathrm{e}^{-\tilde{k}\cdot\alpha_1}+\mathrm{e}^{\tilde{k}\cdot\alpha_{-1}}\mathrm{e}^{-\tilde{k}\cdot\alpha_1}\alpha_1^{\mu}\right]\alpha_{-1}^{\rho_1}\alpha_{-1}^{\rho_2}\,|l+\tilde{k}\rangle\\&-\frac{\mathrm{i}}{2}\langle k|\,\alpha_1^{\mu_1}\alpha_1^{\mu_2}\left[(l+\tilde{k})^{\mu}\mathrm{e}^{\tilde{k}\cdot\alpha_{-1}}\mathrm{e}^{-\tilde{k}\cdot\alpha_1}\right]\alpha_{-1}^{\rho_1}\alpha_{-1}^{\rho_2}\,|l+\tilde{k}\rangle\,.\end{aligned}$$

Now we can use the relations

$$\begin{aligned}\left[\alpha_1^{\mu},\alpha_{-1}^{\rho_1}\ldots\alpha_{-1}^{\rho_N}\right] &= \sum_{k=1}^{N}\left(\eta^{\mu\rho_k}\,\alpha_{-1}^{\rho_1}\ldots\widehat{\alpha}_{-1}^{\rho_k}\ldots\alpha_{-1}^{\rho_N}\right),\\ \langle p|\,\alpha_1^{\mu_1}\ldots\alpha_1^{\mu_N}\mathrm{e}^{c\,k\cdot\alpha_{-1}} &= \langle p|\sum_{\sigma\in S^N}\sum_{k=0}^{N}\binom{N}{k}\frac{c^k}{N!}\left(\prod_{j=1}^{N-k}\alpha_1^{\mu_{\sigma(j)}}\right)\left(\prod_{j=N-k+1}^{N}k^{\mu_{\sigma(j)}}\right),\\ \langle l|\,\alpha_1^{\mu_1}\ldots\alpha_1^{\mu_N}\alpha_{-1}^{\rho_1}\ldots\alpha_{-1}^{\rho_M}\,|k\rangle &= \delta_{N,M}\delta(k-l)\sum_{\sigma\in S^N}\prod_{i=1}^{N}\eta^{\mu_i\rho_{\sigma(i)}}.\end{aligned}$$

to get

$$A^{\mu\mu_1\mu_2\rho_1\rho_2} = -\frac{\mathrm{i}}{2}\left(\frac{1}{2}\tilde{k}^{\mu_1}\tilde{k}^{\rho_1}\tilde{k}^{\rho_2}\eta^{\mu_2\mu} - \eta^{\mu_1\rho_2}\eta^{\mu_2\mu}\tilde{k}^{\rho_1} - \frac{1}{2}\tilde{k}^{\mu_1}\tilde{k}^{\mu_2}\tilde{k}^{\rho_1}\eta^{\mu\rho_2} + \tilde{k}^{\mu_2}\eta^{\mu\rho_1}\eta^{\mu_1\rho_2}\right.$$
$$\left. + \frac{1}{4}k^{\mu}\tilde{k}^{\mu_1}\tilde{k}^{\mu_2}\tilde{k}^{\rho_1}\tilde{k}^{\rho_2} + \frac{1}{4}k^{\mu}\eta^{\mu_2\rho_1}\eta^{\mu_1\rho_2} - k^{\mu}\eta^{\mu_1\rho_2}\tilde{k}^{\mu_2}\tilde{k}^{\rho_1} + \begin{pmatrix}\mu_1 \leftrightarrow \mu_2 \\ \rho_1 \leftrightarrow \rho_2\end{pmatrix}\right),$$

where we have suppressed the momentum conservation factor $\delta\left(l + \tilde{k} - k\right)$. Since the right moving sector looks very similar, the three-point correlator of two spin-4 states and a dilaton is finally given by

$$f_{\mu_1\bar{\mu}_1\mu_2\bar{\mu}_2} f_{\rho_1\bar{\rho}_1\rho_2\bar{\rho}_2}\delta_{\mu\nu}A^{\mu\mu_1\mu_2\rho_1\rho_2}A^{\nu\bar{\mu}_1\bar{\mu}_2\bar{\rho}_1\bar{\rho}_2}.$$

3.4 $\mathrm{AdS}_5 \times \mathrm{S}^5$ Correlators in the Operator Formalism

In principle it would be a good check of the results obtained in [34] to recalculate the correlation functions using the perturbative operator quantization of [26] and the methods reviewed in the last section. Extending this quantization procedure to the fermionic sector using the light-cone action of section 2.4.5 one could furthermore include fermionic contributions to the correlators and calculate corrections to higher orders in α'. However, performing such a calculation is much harder, even in the bosonic case, in the context of a string on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. The first complication arises from the fact that the quantization relies crucially on the light-cone gauge. As we saw in section 2.4.5 it is then not consistent to impose the conformal gauge for the worldsheet metric - a gauge choice that simplified the calculations in the previous section a lot and made it possible to use tools from complex analysis in the calculation of OPEs. The second complication comes from the complicated equations of motion for the fields. In the mode expansion of section 2.5 the time (σ^0) dependence of the oscillators is not explicit. This time dependence is needed however to perform *general* calculations in the operator formalism since the vertex operator insertions on the worldsheet have to be separated. There are several ways to calculate the explicit form for $\alpha(\sigma^0)$, one of which uses the equations of motion for the fields but turns out to be very complicated. An alternative approach uses the worldsheet Hamiltonian $\mathcal{H}$ as a time-evolution operator and Heisenberg's equation

$$\dot{\mathcal{O}} \equiv \frac{\partial}{\partial\sigma^0}\mathcal{O} = [\mathcal{H}, \mathcal{O}]\,.$$

The light-cone worldsheet hamiltonian is, in terms of the oscillators, given by[26]

$$\begin{aligned} H_{\mathrm{ws}} =& gM^2 + \sqrt{g}\left(\vec{p}^2 + \vec{q}^2 + \left(\vec{y}^2 + \vec{z}^2\right)\left(p_+^{(0)}\right)^2 + \frac{\vec{z}^2 - \vec{y}^2}{2} M^2 \right. \\ & \left. -\vec{z}^2 \sum_{n\neq 0} \vec{\alpha}_n \cdot \vec{\tilde{\alpha}} + \vec{y}^2 \sum_{n\neq 0} \vec{\beta}_n \cdot \vec{\tilde{\beta}} + \frac{\vec{z}^2 + \vec{y}^2}{\left(p_+^{(0)}\right)^2} \sum_{n\in\mathbb{Z}} L_n \tilde{L}_n \right) + \mathcal{O}\left(g^{\frac{1}{4}}\right) . \end{aligned}$$

Then one can expand Heisenberg's equation for α_n^i in g and, to leading order, get

$$\dot{\alpha}_n^i = -\mathrm{i}gn\alpha_n^i,$$

which is solved by

$$\dot{\alpha}_n^i = \mathrm{e}^{-\mathrm{i}ng\sigma^0}.$$

The time dependence, hence, looks similar to the flat space case as expected. An importance difference is apparent though. Namely a factor of g turns up in the exponent. This is another evidence for the fact that we cannot impose conformal gauge for the worldsheet metric anymore. More explicitly the worldsheet metric will depend on g. The next-to-leading order differential equation for α_n^i takes the following form

$$\begin{aligned} \dot{\alpha}_n^i = -\mathrm{i}gn\alpha_n^i + \frac{\sqrt{g}}{2}&\left(-\mathrm{i}n\left(\vec{z}^2 - \vec{y}^2\right)\alpha_n^i + \mathrm{i}\vec{z}^2 n\tilde{\alpha}_{-n}^i \right. \\ & \left. -\frac{\mathrm{i}}{2}\sum_{\substack{r,s\in\mathbb{Z}\\ r\neq s, r\neq 0}} \frac{\vec{z}^2 - \vec{y}^2}{\left(p_+^{(0)}\right)^2} n\alpha_{r+m}^i \left(\vec{\tilde{\alpha}}_{r-s}\cdot\vec{\tilde{\alpha}}_s + \vec{\tilde{\beta}}_{r-s}\cdot\vec{\tilde{\beta}}_s\right)\right), \end{aligned}$$

which is a coupled non-linear partial differential equation. Finding a solution, hence, is a quite non-trivial problem, which we were not able to solve. Still it is possible to calculate *special* correlation functions in which all operator insertions are at the same world-sheet time σ^0 but at different σ^1.

The third complication comes from the vertex operators themselves. As we discussed in section 3.2.2 no systematic way to construct vertex operators is known in the case of non-trivial backgrounds. Still, as we have seen, it is possible to write down some special vertex operators and perform calculations. In order to calculate amplitudes using the quantization procedure we have to rewrite the vertex operators in the coordinate system used in section 2.5, which gives quite complicated expressions. The spin-J vertex operator (3.14), for instance, is, in this coordinate system, given by

$$V_J = C \left(\frac{1 + \vec{Z}^2/4}{1 - \vec{Z}^2/4} \right)^{-E} \mathrm{e}^{-\mathrm{i}Et} \left(\partial \left(\frac{Z_1 + \mathrm{i}Z_2}{1 - \vec{Z}^2/4} \right) \bar{\partial} \left(\frac{Z_1 + \mathrm{i}Z_2}{1 - \vec{Z}^2/4} \right) \right)^{J/2} ,$$

which has to be expanded in g and afterwards the oscillators have to be plugged in. The resulting expression is quite lengthy and therefore we will not present it here.

Chapter 4

Conclusions

In this thesis we have discussed the Maldacena conjecture connecting type IIB superstring theory on an $\mathrm{AdS}_5 \times \mathrm{S}^5$ background to $\mathcal{N} = 4$ SU(N) super-Yang-Mills theory on the boundary of AdS_5. Also the theories on both sides of the duality have been examined. We presented the Wess–Zumino–Witten-like Green–Schwarz coset action for the superstring and discussed an approach to perturbatively quantize it in the light-cone gauge. In the second part of the thesis we presented *spinning folded string* solutions to the classical equations of motion of the bosonic string on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and showed how these solutions can be used to test the AdS/CFT correspondence. Furthermore we discussed a semi-classical path integral method which enabled us to calculate three-point functions of these non-BPS states. In this context we also examined the structure of vertex operators and their construction both in flat space and on the curved $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. We showed how to calculate correlation functions in the operator quantization and as an example calculated a three-point function on flat space. The last section of the thesis deals with the computation of correlators using operator techniques in the AdS/CFT context and the various complications that arise.

We saw that no general way to construct vertex operators corresponding to given states on curved space-time backgrounds is known. Still one can guess expressions that have the correct flat-space limit and carry the right quantum numbers. Such an expression at hand, one can calculate the correlation function of two such operators and check whether it yields the correct dependence of the energy on the other quantum numbers. In the AdS/CFT context all vertex operators should depend on boundary points, since the corresponding fields are sources for CFT operators on the boundary. This makes vertex operators in general, and especially for the spinning folded strings, being quite complicated expressions. In order to calculate correlation functions using the quantization scheme of section 2.5 it would be good to know the (worldsheet-)time-dependence of the oscillators. We calculated the corresponding differential equations and to leading order got a time-dependence similar to the one in flat space. The difference was given by the fact that the worldsheet-metric, in this framework, has to depend on the 't Hooft parameter λ. To next-to-leading order the differential equations for the operators could not be solved.

Still, one can calculate correlation functions of operators inserted on the worldsheet at the same time σ^0 but with different positions σ^1 on the closed strings, which is work in progress. After having performed these calculations we plan to extend the computation to higher orders in α' and incorporate fermions. Another approach that we want to follow in the future is to try to construct expressions for vertex operators in the context of a field theory on the worldsheet, in which the metric explicitly depends on λ. This will hopefully enable us to obtain expressions order-by-order in the 't Hooft parameter and therefore simplify calculations and give a consistency check for the methods developed in [26].

Bibliography

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231-252. [arXiv:hep-th/9711200].

[2] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253-291. [arXiv:hep-th/9802150].

[3] M. B. Green and J. H. Schwarz, "Covariant Description of Superstrings," Phys. Lett. B **136** (1984) 367.

[4] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," *Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)*

[5] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," *Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)*

[6] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: A modern introduction," *Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p*

[7] C. Vafa, "Evidence for F theory," Nucl. Phys. **B469** (1996) 403-418. [hep-th/9602022].

[8] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498** (1997) 467 [arXiv:hep-th/9612115].

[9] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," [arXiv:hep-th/0201253].

[10] D. Bailin, A. Love, "Supersymmetric gauge field theory and string theory," Bristol, UK: IOP (1994) 322 p. (Graduate student series in physics).

[11] H. P. Nilles, "Supersymmetry, Supergravity and Particle Physics," Phys. Rept. **110** (1984) 1.

[12] J. A. Minahan, ‘Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills,” [arXiv:1012.3983 (hep-th)].

[13] M. T. Grisaru, M. Rocek, W. Siegel, “Zero Three Loop beta Function in N=4 Superyang-Mills Theory,” Phys. Rev. Lett. **45** (1980) 1063-1066.

[14] S. Mandelstam, “Light Cone Superspace and the Ultraviolet Finiteness of the N=4 Model,” Nucl. Phys. **B213** (1983) 149-168.

[15] L. Brink, O. Lindgren, B. E. W. Nilsson, “The Ultraviolet Finiteness of the N=4 Yang-Mills Theory,” Phys. Lett. **B123** (1983) 323.

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. **323** (2000) 183-386. [arXiv:hep-th/9905111].

[17] J. L. Petersen, “Introduction to the Maldacena conjecture on AdS / CFT,” Int. J. Mod. Phys. **A14** (1999) 3597-3672. [arXiv:hep-th/9902131].

[18] J. Polchinski, “Dirichlet Branes and Ramond-Ramond charges,” Phys. Rev. Lett. **75** (1995) 4724 [arXiv:hep-th/9510017].

[19] S. Förste, “Strings, branes and extra dimensions,” Fortsch. Phys. **50** (2002) 221-403. [hep-th/0110055].

[20] I. R. Klebanov, “World volume approach to absorption by nondilatonic branes,” Nucl. Phys. **B496** (1997) 231-242. [arXiv:hep-th/9702076].

[21] S. S. Gubser, I. R. Klebanov, A. A. Tseytlin, “String theory and classical absorption by three-branes,” Nucl. Phys. **B499** (1997) 217-240. [arXiv:hep-th/9703040].

[22] G. Arutyunov and S. Frolov, “Foundations of the $AdS_5 \times S^5$ Superstring. Part I,” J. Phys. A **42** (2009) 254003 [arXiv:0901.4937 (hep-th)].

[23] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in AdS(5) x S**5 background,” Nucl. Phys. B **533** (1998) 109 [arXiv:hep-th/9805028].

[24] M. Nakahara, “Geometry, topology and physics,” Boca Raton, USA: Taylor & Francis (2003) 573 p.

[25] S. Frolov, J. Plefka and M. Zamaklar, “The AdS(5) x S**5 superstring in light-cone gauge and its Bethe equations,” J. Phys. A **39** (2006) 13037 [arXiv:hep-th/0603008].

[26] F. Passerini, J. Plefka, G. W. Semenoff, D. Young, “On the Spectrum of the $AdS_5 \times S^5$ String at large λ,” JHEP **1103** (2011) 046. [arXiv:1012.4471 (hep-th)].

[27] D. E. Berenstein, J. M. Maldacena, H. S. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” JHEP **0204** (2002) 013. [arXiv:hep-th/0202021].

[28] S. Frolov, A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S**5,” JHEP **0206** (2002) 007. [arXiv:hep-th/0204226].

[29] S. Frolov and A. A. Tseytlin, “Multispin string solutions in AdS(5) x S**5,” Nucl. Phys. B **668** (2003) 77 [arXiv:hep-th/0304255].

[30] A. A. Tseytlin, “On semiclassical approximation and spinning string vertex operators in AdS(5) x S**5,” Nucl. Phys. **B664** (2003) 247-275. [arXiv:hep-th/0304139].

[31] A. A. Tseytlin, “Spinning strings and AdS / CFT duality,” In *Shifman, M. (ed.) et al.: From fields to strings, vol. 2* 1648-1707. [arXiv:hep-th/0311139].

[32] E. I. Buchbinder, “Energy-Spin Trajectories in $AdSS$ from Semiclassical Vertex Operators,” JHEP **1004** (2010) 107. [arXiv:1002.1716 (hep-th)].

[33] E. I. Buchbinder, A. A. Tseytlin, “On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT,” JHEP **1008** (2010) 057. [arXiv:1005.4516 (hep-th)].

[34] R. Roiban, A. A. Tseytlin, “On semiclassical computation of 3-point functions of closed string vertex operators in $AdS_5 x S^5$,” Phys. Rev. **D82** (2010) 106011. [arXiv:1008.4921 (hep-th)].

[35] D. E. Berenstein, C. Corrado, W. Fischler, J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” Phys. Rev. **D59** (1999) 105023 [arXiv:hep-th/9809188].

[36] K. Zarembo, “Holographic three-point functions of semiclassical states,” [arXiv:1008.1059 (hep-th)].

[37] E. I. Buchbinder, A. A. Tseytlin, “Semiclassical four-point functions in $AdS_5 \times S^5$,” JHEP **1102** (2011) 072. [arXiv:1012.3740 (hep-th)].

I hereby certify that the work presented here was accomplished by myself and without the use of illegitimate means or support, and that no sources and tools were used other than those cited.

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