Journal of Physics: Conference Series **405** (2012) 012017 doi:10.1088/1742-6596/405/1/012017

Geodesic Motions near a Five-dimensional Reissner-Nordström Anti-de Sitter black hole

Sarbari Guha[1] **and Pinaki Bhattacharya**[2]

[1] Department of Physics, St. Xavier's College (Autonomous), Kolkata 700016, India
[2] Gopal Nagar High School, Singur 712409, West Bengal, India

E-mail: `srayguha@yahoo.com`

Abstract. We have studied the geodesics of neutral particles near a non-rotating, charged five-dimensional Reissner-Nordström Anti-de Sitter black hole using the effective potential analysis and the dynamical systems analysis. The effective potential analysis is used to determine the location of the horizon and to study radial and circular trajectories. The dynamical systems method is used to determine the stability and the fixed points of the phase trajectories.

1. Introduction

With the development of string theory, the study of black holes in higher-dimensional spacetimes [1] have gained momentum, especially in the first decade of this millennium [2]. Static, spherically symmetric exterior vacuum solutions of the braneworld models were first proposed by Dadhich and others [3]. Extensive studies in higher-dimensional spacetimes over the last few decades have led many authors to investigate the geodesic motions in such spacetimes [4]. Motion of massive particles around a rotating black hole in a braneworld has been studied [5] and the effective potentials for radial null geodesics in RN-dS and Kerr-dS spacetimes were analyzed [6]. Analytic solutions of the geodesic equations in Schwarzschild-(Anti-)de Sitter spacetimes [7] and the motion of massive particles in 4-dimensions and in higher dimensions, has been analysed. Here we have investigated the radial and circular trajectories for photons and massive particles in a five-dimensional RN-AdS spacetime and have determined the fixed points of the phase trajectories. For such a non-rotating charged black hole in Anti-de Sitter spacetime, the solutions are uniquely characterized by their mass, charge and the cosmological constant [8].

2. Preliminaries

We consider a 5D spacetime with negative cosmological constant Λ. The radius of curvature $l = \sqrt{-\frac{3}{\Lambda}}$ of the spacetime provides the length scale necessary to have a horizon. The exterior metric of the black hole field is given by

$$dS^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 = -\frac{\triangle}{r^4}dt^2 + \frac{r^4}{\triangle}dr^2 + r^2 d\Omega_3^2. \tag{1}$$

The Lapse function is defined as: $f(r) = 1 - \left(\frac{2M}{r}\right)^2 + \left(\frac{q^2}{r^2}\right)^2 - \frac{\Lambda r^2}{6}$.

Journal of Physics: Conference Series **405** (2012) 012017 doi:10.1088/1742-6596/405/1/012017

For a given M, q and Λ, the ***horizon function*** $\triangle$ depends only on r. The nature of the intrinsic singularity at $r = 0$, depends on Λ and q, which we choose such that the spacetime do not have any spacelike naked singularity. The lapse function and the effective potential vanishes at the real, positive zeros of the horizon function $\triangle$, indicating the location of the horizons. The variation of the effective potential V_{eff} with specific radius r/M in the case of radial motion of massive particles, in the field of the black hole, are shown in Fig.1 and Fig. 2 for different range of values of the r/M. The location of the horizons can be easily identified from these figures.

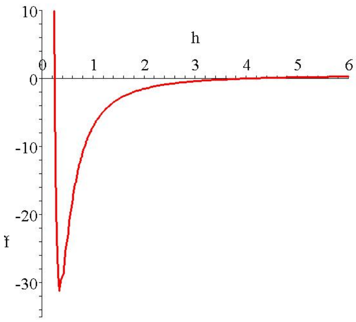

Figure 1. Plot of effective potential V_{eff} vs radial distance $h = r/M$ in the case of radial motion of massive particles with $q/M = 0.5$ and $\Lambda = -0.0005$ for $r/M = 0$ to 6.

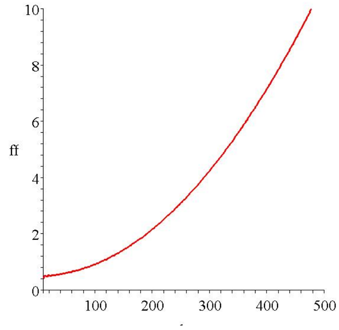

Figure 2. Plot of effective potential V_{eff} vs radial distance $h = r/M$ in the case of radial motion of massive particles with $q/M = 0.5$ and $\Lambda = -0.0005$ for $r/M = 6$ to 500.

3. Five-dimensional Geodesics

Due to **spherical symmetry**, we analyze the motion of neutral particles on the **equatorial hyperplane**, $\theta, \phi = \pi/2$. We have the following equations:

$$\frac{d^2t}{d\lambda^2} + \frac{B(r)}{A(r)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0, \qquad \frac{d^2\psi}{d\lambda^2} + \frac{1}{r}\frac{dr}{d\lambda}\frac{d\psi}{d\lambda} = 0, \tag{2}$$

$$\frac{d^2r}{d\lambda^2} + A(r)B(r)\left(\frac{dt}{d\lambda}\right)^2 - \frac{B(r)}{A(r)}\left(\frac{dr}{d\lambda}\right)^2 + rA(r)\left(\frac{d\psi}{d\lambda}\right)^2 = 0, \tag{3}$$

where $A(r) = -f(r)$ and $B(r) = \frac{1}{r}\left(-\left(\frac{2M}{r}\right)^2 + 2\left(\frac{q^2}{r^2}\right)^2 + \frac{\Lambda r^2}{6}\right)$.

4. Effective Potential Analysis

The **Lagrangian** for particle motion is: $Ł = -\left(1 - \left(\frac{2M}{r}\right)^2 + \left(\frac{q^2}{r^2}\right)^2 - \frac{\Lambda r^2}{6}\right)\dot{t}^2$
$+\left(\frac{\dot{r}^2}{1-\left(\frac{2M}{r}\right)^2+\left(\frac{q^2}{r^2}\right)^2-\frac{\Lambda r^2}{6}} + r^2(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2 + sin^2\theta sin^2\phi\dot{\psi}^2)\right)$.

There are two conserved quantities: **Energy**, $E = g_{tt}\frac{dt}{d\lambda} = -f(r)\frac{dt}{d\lambda} = A(r)\frac{dt}{d\lambda}$, and **momentum** $L = r^2\frac{d\psi}{d\lambda}$ conjugate to ψ.

On the equatorial hyperplane we have, $\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{\triangle}{r^4}\left(\epsilon + \frac{L^2}{r^2}\right) = E^2 + A(r)\left(\epsilon + \frac{L^2}{r^2}\right)$

so that $\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 = E_{eff} - V_{eff}(r)$, where $E_{eff} = \frac{1}{2}E^2$ and
$V_{eff}(r) = \frac{\triangle}{2r^4}\left(\epsilon + \frac{L^2}{r^2}\right) = \frac{1}{2}\left(1 - \left(\frac{2M}{r}\right)^2 + \left(\frac{q^2}{r^2}\right)^2 - \frac{\Lambda r^2}{6}\right)\left(\epsilon + \frac{L^2}{r^2}\right)$.

Radial motion: Here $L = 0$. For bound states of massive particles, we have $\frac{4M^2}{r^3} > \frac{2q^4}{r^5} + \frac{\Lambda r}{6}$.

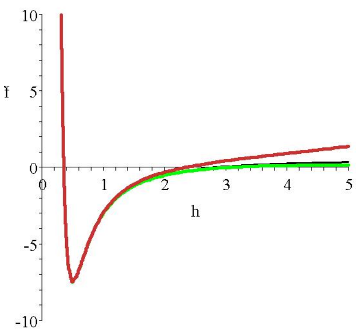

Figure 3. V_{eff} vs radial distance $h = r/M$ in the case of radial motion of massive particles with $q/M = 1/\sqrt{2}$ and $\Lambda = -0.5, -0.0005$ (orange and black).

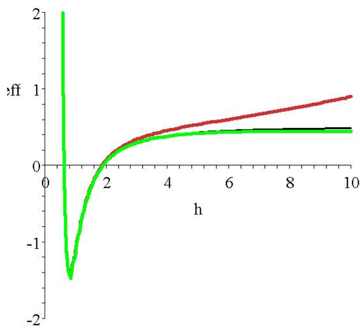

Figure 4. V_{eff} vs radial distance for circular motion of massive particles with $q/M = 1.1$ and $\Lambda = -0.05, -0.0005$ (orange and black).

Circular motion: We now introduce the variable change $u = r^{-1}$. For equilibrium circular orbits,we have
$L^2 = \frac{\epsilon[6q^4u^6 - 12M^2u^4 + \Lambda/2]}{3u^4[-3q^4u^4 + 8M^2u^2 - 1]}$ and $E^2 = \frac{\epsilon[6q^4u^6 - 24M^2u^4 + 6u^2 - \Lambda][3q^4u^6 - 12M^2u^4 + 3u^2 - \Lambda]}{18u^4[3q^4u^4 - 8M^2u^2 + 1]}$.

For photons, we have $u^2 = \frac{4M^2 \pm \sqrt{16M^4 - 3q^4}}{3q^4}$. Therefore, circular orbits occur for only two values of u in the case of photons.

4.1. Stability of the orbits

The stable circular orbits occur for those values of r which are located at the local minimum of the potential. Here we obtain

$$V_{eff}(r) = \frac{1}{2}\left(1 - \left(\frac{2M}{r}\right)^2 + \left(\frac{q^2}{r^2}\right)^2 - \frac{\Lambda r^2}{6}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{4}$$

For $4M^2L^2 = q^4$ and $4M^2 \neq L^2$, we have $r^2_{ISCOM} = \frac{\sqrt{30}Lq^2}{(4M^2 - L^2) + \sqrt{(4M^2 - L^2)^2 + 30L^2q^4\Lambda}}$, giving only two real values of r for **massive particles**, if $(4M^2 - L^2)^2 > |30L^2q^4\Lambda|$.

The minimum radius of stable circular orbits of **photons** is obtained as:

$$r_{mc} > \frac{\sqrt{15}q^2}{\sqrt{12M^2 - \sqrt{144M^4 - 15q^4}}}.$$

5. Dynamical Systems Analysis

Let us define **three new variables** [9]: $U = \frac{dt}{d\lambda}$, $V = \frac{dr}{d\lambda}$ and $W = \frac{d\psi}{d\lambda}$.

Thus the geodesics equations are:

$\frac{dU}{d\lambda} + \frac{B(r)}{A(r)}UV = 0$, $\frac{dW}{d\lambda} + \frac{1}{r}VW = 0$ and $\frac{dV}{d\lambda} + A(r)B(r)U^2 - \frac{B(r)}{A(r)}V^2 + rA(r)W^2 = 0$, which are related through $A(r)U^2 - \frac{1}{A(r)}V^2 + r^2W^2 = -\epsilon$.

Real non-linear dynamical system:

We now assume $\frac{dU}{d\lambda} = H(U,V,r)$, $\frac{dr}{d\lambda} = V$ and $\frac{dV}{d\lambda} = J(U,V,r)$, where $H(U,V,r) = -\frac{B(r)}{A(r)}UV$, $J(U,V,r) = \frac{\epsilon A(r)}{r} - \frac{A(r)}{r}(rB(r) - A(r))U^2 - \frac{1}{rA(r)}(A(r) - rB(r))V^2$.

Fixed Points: For a black hole with a given M and q, (0,0) is the unique fixed point on the (U-V) plane for null geodesics. The timelike geodesics possess definite fixed points.

The **phase evolution** of the system on the (U-V) phase plane is determined from the condition $\frac{dV}{dU} = \frac{J}{H}$, provided $H \neq 0$ (except at the point (0,0) for null geodesics). Simplifying, we get $f_1U^2 - f_2V^2 - f_3 = 0$, where f_1, f_2 and f_3 are functions of r. For null geodesics, we get $f_3 = 0$.

Thus the geodesics of massive particles near a black hole in a RN-AdS$_5$ possesses definite fixed points and the orbits are either elliptic (periodic bound) or hyperbolic (escape orbits). Moreover, for a black hole of a given charge and mass, the null geodesics possesses a unique fixed point ($U_0 = 0$, $V_0 = 0$, r_0). The trajectories are linear, with their slopes changing according to the values of the parameters f_1 and f_2. Hence the null geodesics are terminating orbits.

6. Conclusions

We have determined the location of the black hole horizons from the plot of effective potential. The nature of the trajectories depend upon the particle energies and their angular momenta, as well as on Λ and q. The radius of the innermost stable circular orbit of massive particles is totally defined in terms of their angular momenta and M and q of the black hole. Photons trace out circular trajectories for only two distinct values of r. The geodesics of massive particles near a black hole in a RN-AdS$_5$ are either periodic bound or escape orbits and these have definite fixed points. The null geodesics have a unique fixed point and these are terminating orbits.

[1] Emparan R and Reall H S 2008 *Liv. Rev. Rel.* 2008-6 (http://www.livingreviews.org/lrr-2008-6)
[2] Gibbons G W, Lu H, Page D N and Pope C N 2004 *Phys. Rev. Lett.* **93** 171102; Sen A 2005 *J. High Energy Phys.* 07(2005)073; Frolov V and Stojkovic D 2003 *Phys. Rev. D* **68** 064011
[3] Dadhich N K, Maartens R, Papodopoulos P and Rezania V 2000 *Phys. Lett. B* **487** 1
[4] Page D N, Kubiznak D, Vasudevan M and Krtous P 2007 *Phys. Rev. Lett.* **98** 061102; Cardoso V, Cavaglia M and Gualtieri L 2006 *Phys. Rev. Lett.* **96** 071301; Konoplya R A and Zhidenko A 2008 *Phys. Rev. D* **78** 104017; Cruz N, Olivares M and Villanueva J R 2005 *Class. Quant. Grav.* **22** 1167
[5] Abdujabbarov A and Ahmedov B 2010 *Phys. Rev. D* **81** 044022
[6] Stuchlik Z and Calvani M 1991 *Gen. Relativ. Grav.* **23** 507
[7] Hackmann E and Lämmerzahl C 2008 *Phys. Rev. Lett.* **100** 171101
[8] Gibbons G W, Ida D and Shiromizu T 2002 *Prog. Theor. Phys. Suppl.* **148** 284
[9] Guha S and Chakraborty S 2010 *Gen. Relativ. Grav.* **42** 1739
[10] Wainwright J and Ellis G F R 1997 *Dynamical Systems in Cosmology* (Cambridge: Cambridge University Press)
[11] Dahia F, Romero C, da Silva L F P and Tavakol R 2007 *J. Math. Phys.* **48** 072501, Dahia F, Romero C, da Silva L F P and Tavakol R 2008 *Gen. Rel. Grav.* **40** 1341