Inflation Produced Magnetic Fields

Master Thesis Theoretical Physics

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Abstract

Magnetic fields are present in galaxies, clusters of galaxies and even in the intergalactic medium. Many different mechanisms have been proposed to explain the origin of these fields. One suggestion, first formulated by Turner and Widrow in 1987, is that these fields were generated during inflation. In this thesis we review this idea by studying four different models. The first model is based on standard Maxwell electromagnetism. We show that this model cannot generate fields of the required strength and magnitude. Since we want to keep the idea that the fields are generated during inflation, it is necessary to look at extensions of Maxwell electromagnetism. Three different extensions are considered in which the magnetic field couples to the gravitational field, a scalar field and a pseudo scalar field respectively. We show that each of these extended models is able to generate the observed fields. We furthermore derive that the magnetic field decays slower in a spatially open expanding universe than in a flat expanding universe. Finally we use the existence of intergalactic magnetic fields to derive a bound on the temperature during inflation.

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CHAPTER 1

Introduction

Currently more and more mysteries of the universe are getting unraveled. In the last few centuries we have discovered that the earth is not the center of the universe and that the universe is expanding. Still a lot of mysteries remain unsolved. One of these mysteries is the fact that magnetic fields are present in galaxies, clusters of galaxies and in the intergalactic medium. Up until now no one has been able to give a satisfactory answer to the question: 'Where do these fields come from?'.

To be able to find an answer to this question one must explain the two main remarkable features of the magnetic fields. The most remarkable feature is the fact that the fields exist at all in the intergalactic medium. If the magnetic fields would only exist in galaxies and clusters of galaxies, one might expect to be able to explain their origin using some astrophysical process, such as the formation of a galaxy. Since there is no electrically charged matter present in the intergalactic medium, it seems that the origin of the intergalactic magnetic fields might be a cosmological process, not an astrophysical one.

The second remarkable feature of the magnetic fields is the fact is that they have a very large coherence scale. The fields in galaxies and clusters are about the same scale as the galaxies and clusters themselves and the fields in the intergalactic medium can have a coherence scale of 1 Mpc and larger. The scale of the fields in the galaxies and clusters could be explained by an astrophysical process, since their scale is the same as the galaxies and clusters themselves. The difficulty lies again with the intergalactic fields for which it is hard to find an astrophysical explanation. Therefore if one wants to find an answer to the question where the fields come from, one needs to be able to explain both how fields could be generated in the vacuum and how they could have such a large coherence scale.

One possible answer, first formulated by Turner and Widrow in 1987 [1], is that the magnetic fields were generated during inflation, which is a period of rapid expansions in the early universe. During this period the magnetic fields would be generated from a small quantum perturbation that grows very large due to the expansion of the universe. This could first of all explain the fact that the magnetic fields are present in the intergalactic medium, since there can always be quantum perturbations in the vacuum. Secondly the rapid expansion of the universe during inflation could explain the large coherence scale of the fields. The goal of this thesis is to review this suggestion and see if there is indeed a viable mechanism that could generate magnetic fields of the required strength during inflation.

Since magnetic fields are described by Maxwell's theory of electrodynamics the most logical option would be to look at a model of Maxwell electrodynamics during inflation. However, as we will show, a model with Maxwell's theory of electrodynamics is not able to generate the observed fields, since the theory is conformal invariant. This has been known for some time [2] and therefore in the past two decades people have been searching for other models that could explain the generation of magnetic fields during inflation.

Most of the proposed models are extensions of Maxwell's theory. This thesis deals with the three most common extensions found in the literature. They consist of coupling the electromagnetic field to the gravitational field, a scalar field and a pseudo-scalar field respectively. We will review for each of these models if they are able to generated the observed magnetic fields. A positive result for one of the models is not only interesting since it would give a possible explanation for the origin of the magnetic fields, but it would also provide some evidence in favor of this extension of Maxwell theory.

In the process of dealing with the general problem of finding a theory to explain the presence of magnetic fields, this thesis also reviews some general properties of magnetic fields in an expanding universe. More specifically, we look at the influence of the curvature of spacetime on the classical evolution of a magnetic field. This is interesting because if we are able to find that the magnetic field decays slower in a universe with a particular curvature, it is possible to generate stronger magnetic fields.

Another aspect that we study is the derivation of a bound on the energy density during inflation by assuming that the magnetic fields were indeed generated during inflation. This means that if we find that this assumption is true, we automatically obtain more information about the characteristics of inflation.

The thesis is structured as follows. We start with an introduction to cosmology and inflation in chapter 2, meant for the reader that is not familiar with this subject. In chapter 3 we review the measurements that have been made of astrophysical magnetic fields and give a short argument why inflation is a good candidate for the origin of the fields. Chapters 4 and 5 contain a review of quantum field theory in curved spacetimes and electromagnetism in curved spacetimes respectively, to provide the necessary tools to evaluate the evolution of the magnetic field. In chapter 6 we use these tools to describe a formalism to derive the field strength of magnetic fields that are generated during inflation. We calculate this strength for the different models mentioned before and review whether these models are compatible with observations in chapter 7. In chapter 8 we give a classical description of the evolution of magnetic fields in a spatially curved FLRW metric and describe how this influences the strength of the field. In the last chapter we show how the energy density during inflation can be restricted from the fact that intergalactic magnetic fields exist. Finally, our conclusions are presented in chapter 10. A list of conventions can be found in Appendix A.

CHAPTER 2

Introduction to cosmology

Before we start our discussion of magnetic fields, we will give an introduction to cosmology. This chapter is meant to give a basic overview on a level that is needed to understand the rest of the thesis. In the first section we will derive how the universe evolves from the Einstein equations and discuss the different eras in the history of the universe. In the second section we will discuss the earliest information we have of our universe, the Cosmic Microwave Background. In the last section we will introduce the concept of inflation, an era of rapid expansion in the universe. We will show how this concept can solve some problems of the model given in section 2.1 and discuss some general properties.

2.1 The FRLW universe

2.1.1 FRLW metric

Observations tell us that our universe is homogeneous and isotropic on large scales, that is, the universe looks the same everywhere and in every direction. Other observations tell us that the universe is expanding. A homogeneous, isotropic, expanding universe can be described by the Friedmann-Lemaitre-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}^{(3)}dx^i dx^j = a^2(\eta)(-d\eta^2 + \gamma_{ij}^{(3)}dx^i dx^j), \tag{2.1}$$

where η is called conformal time, which is related to normal time by the relation $\dot{\eta} = a^{-1}$. From now on we will denote a derivative with respect to t with a dot and a derivative with respect to η with a prime. The quantity a is called the scale factor, it is a measure for the expansion of space. It is defined such that at present $a_0 = 1$. The coordinates x^i

are co-moving coordinates and are independent of time. The metric for the co-moving spatial part is $\gamma_{ij}^{(3)}$. When the universe is homogeneous and isotropic the spatial part must have constant curvature. This can be written in spherical coordinates as [3],

$$\gamma_{ij}^{(3)} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega, \tag{2.2}$$

where $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the two sphere. The constant K takes the values +1,0,-1 for respectively a spherical, Euclidean and hyper-spherical space.

2.1.2 Fluids in a FLRW universe

The universe is not an empty space, but filled with matter. To have a complete description of the universe we must know how the spacetime and matter interact. This interaction is described by the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{2.3}$$

We will first explain what the left hand side means and then the right hand side. The left hand side describes the curvature of spacetime. In general the curvature of spacetime is described by the Riemann tensor $R^{\rho}{}_{\sigma\mu\nu}$, defined as,

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.4}$$

where $\Gamma^{\rho}_{\mu\nu}$ are called Christoffel symbols. They are related to the metric,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right). \tag{2.5}$$

From the definition is clear that $\Gamma^{\rho}_{\mu\nu}$ is symmetric in μ and ν. Christoffel symbols also play an important role in the covariant derivative. The covariant derivative of a vector V^μ is defined as,

$$\nabla_\nu V^\mu = \partial_\nu V^\nu + \Gamma^{\mu}_{\nu\lambda} V^\lambda. \tag{2.6}$$

The quantities on the left hand side of the Einstein equations (2.3) are called the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R. They are related to the Riemann tensor as $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ and $R = R^{\mu}_{\mu} = R^{\lambda\mu}{}_{\lambda\mu}$. Because of the symmetries of the Riemann tensor, the Ricci tensor is symmetric. To evaluate the left hand side of the Einstein equations we may calculate the explicit form of the Ricci tensor and scalar for the FRLW metric (2.1). The first step is to calculate the Christoffel symbols. For example,

$$\begin{aligned}
\Gamma^{t}_{rr} &= \frac{1}{2} g^{t\lambda} \left(\partial_r g_{r\lambda} + \partial_r g_{\lambda r} - \partial_\lambda g_{rr} \right), \\
&= \frac{1}{2} \partial_t \frac{a^2}{1 - Kr^2}, \\
&= \frac{\dot{a}a}{1 - Kr^2}.
\end{aligned} \tag{2.7}$$

The other Christoffel symbols are calculated in the same manner. The result is:

$$\begin{aligned}
\Gamma^t_{\theta\theta} &= \dot{a}ar^2, \\
\Gamma^t_{\phi\phi} &= \dot{a}ar^2\sin^2\theta, \\
\Gamma^r_{tr} &= \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a}, \\
\Gamma^r_{rr} &= \frac{Kr}{1-Kr^2}, \\
\Gamma^r_{\theta\theta} &= -r(1-Kr^2), \\
\Gamma^r_{\phi\phi} &= -r(1-Kr^2)\sin^2\theta, \\
\Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}, \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^\phi_{\theta\phi} &= \cot\theta.
\end{aligned} \tag{2.8}$$

The other Christoffel symbols are zero. We can now calculate the components of the Ricci tensor. For example,

$$\begin{aligned}
R_{tt} &= R^\rho_{t\rho t}, \\
&= \partial_\rho\Gamma^\rho_{tt} - \partial_t\Gamma^\rho_{\rho t} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{tt} - \Gamma^\rho_{t\lambda}\Gamma^\lambda_{\rho t}, \\
&= 0 - 3\partial_t\frac{\dot{a}}{a} + 0 - 3\left(\frac{\dot{a}}{a}\right)^2, \\
&= -3\frac{\ddot{a}}{a}.
\end{aligned} \tag{2.9}$$

The rest of the components are calculated in the same manner and are given by,

$$R_{rr} = \frac{\ddot{a}a + 2\dot{a}^2 + 2K}{1-Kr^2}, \tag{2.10}$$

$$R_{\theta\theta} = r^2(\ddot{a}a + 2\dot{a}^2 + 2K), \tag{2.11}$$

$$R_{\phi\phi} = r^2\sin^2\theta(\ddot{a}a + 2\dot{a}^2 + 2K). \tag{2.12}$$

The other components are zero. Another way to write the spatial components is,

$$R_{ij} = \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2K}{a^2}\right] g_{ij}, \tag{2.13}$$

where g_{ij} is the spatial part of the metric. The final step is to calculate the Ricci scalar,

$$R = R^\mu_\mu = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a} + \frac{K}{a^2}\right]. \tag{2.14}$$

This is everything we need to evaluate the spacetime part of the Einstein equations.

The right hand side of the Einstein equations (2.3) describes the matter in the universe. From now on we will refer to the content of the universe as fluids, as we will later use the term matter for a non-relativistic fluid. The tensor $T_{\mu\nu}$ on the right hand side is the energy momentum tensor. Since the universe is homogeneous and isotropic we will assume that the fluids in the universe are perfect. This means that the anisotropic pressure and the energy flux are zero. A more general review of the energy momentum tensor is given in chapter 8. For a perfect fluid the energy momentum tensor $T_{\mu\nu}$ is,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{2.15}$$

where ρ is the energy density and p is the isotropic pressure of the fluid. The quantity u_μ is the four-velocity defined by,

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{2.16}$$

where τ is the proper time. A more extensive discussion of the four-velocity can be found in chapter 8. In FLRW spacetime the energy density and isotropic pressure are thus $\rho = -T^0_{\ 0}$ and $p = 1/3 T^i_{\ i}$. The energy momentum tensor is conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.17}$$

This is especially true for the zero component,

$$\begin{aligned} 0 = \nabla_\mu T^\mu_0 &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda, \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}(\rho + p). \end{aligned} \tag{2.18}$$

This is called the fluid equation.

We now know what all the quantities in the Einstein equations are, so we can start to evaluate them. Due to symmetries there are only two independent equations. The first one comes from the $\mu\nu = 00$ component,

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p). \tag{2.19}$$

The other equation comes from the $\mu\nu = ij$ component,

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2} = 8\pi G(\rho - p). \tag{2.20}$$

These are usually rewritten as,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \tag{2.21}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.22}$$

These are called the Friedmann equations. Before we will solve the Friedmann equations, it will turn out to be convenient to introduce a new parameter, the Hubble parameter.

The Hubble parameter

The Hubble parameter is defined as,

$$H = \frac{\dot{a}}{a}, \tag{2.23}$$

and is a measure for the expansion of the universe. Historically this parameter was a measure for the speed at which objects moved away from us, depending on their distance. Hubble discovered that galaxies at a distance $\vec{d} = a\vec{x}$ are moving away from us with a speed,

$$\dot{\vec{d}} = H\vec{d}. \tag{2.24}$$

It is easily checked that the Hubble parameter in definition (2.23) is the same as H in Hubble's law, since $\dot{\vec{d}} = \dot{a}\vec{x}$ and $H\vec{d} = \dot{a}\vec{x}$. The Hubble constant is the value of the Hubble parameter at present $H_0 \equiv H(a_0)$. Measurements tell us that $H_0 = 100h$ km/sec/Mpc, with $h \simeq 0.7$. Using definition (2.23) of the Hubble parameter the Friedmann equations can be rewritten as,

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.25}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.26}$$

Solutions of the Friedmann equations

To be able to solve the Friedmann equations we will make the assumption that the fluids in the universe are barotropic. That means that the energy density and the pressure are related by an equation of state, $p = \gamma\rho$. The value of γ depends on the kind of fluid. The only other thing we need to be able to solve the Friedmann equations is a relation between the scale factor a and the density ρ. We can find this relation if we solve the fluid equation (2.18), which for a barotropic fluid reads,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1 + \gamma). \tag{2.27}$$

The solution is,

$$\rho \propto a^{-3(1+\gamma)}. \tag{2.28}$$

It is now easy to solve the second Friedmann equation (2.26), the solution is,

$$a \propto t^{2/3(1+\gamma)}. \tag{2.29}$$

We can evaluate this for different kinds of fluids.

Curvature: We can view the curvature of the spacetime as a fluid. If we define,

$$\rho_K \equiv -\frac{3K}{8\pi G a^2}, \tag{2.30}$$

we can rewrite the first Friedmann equation (2.25) as,

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_K). \tag{2.31}$$

Since the curvature term is absent in the second Friedmann equation (2.26), we want the right hand site to vanish when curvature is the only fluid present in the universe. This happens when $\gamma = -1/3$. As a consequence $\rho_K \propto a^{-2}$, which is correct if we compare it to the definition of ρ_K (2.30). We also find that $a \propto t$, which means that an empty universe keeps expanding at a constant rate. When the universe is flat the first Friedmann equation becomes,

$$H^2 = \frac{8\pi G}{3}\rho. \tag{2.32}$$

In this case the density is often referred to as the critical density ρ_c. The relation also works the other way around. If,

$$\rho = \frac{3}{8\pi G}H^2 \equiv \rho_c, \tag{2.33}$$

the universe is flat. One often defines the density parameter Ω as the ratio of the density to the critical denstity,

$$\Omega \equiv \frac{\rho}{\rho_c}. \tag{2.34}$$

When the universe is flat $\Omega = 1$. With the present value of the Hubble parameter, we can calculate that the critical density is, $\rho_c(t_0) = 1.88h^2 \times 10^{26}\text{kgm}^{-3}$. Measurements indicate that $\Omega = 1.02 \pm 0.02$ [4], which implicates that the universe almost flat. In the review of the other kinds of fluids we will assume, because of this and for simplicity, that the curvature density is zero.

Matter: Matter consists of non-relativistic particles, mainly protons, neutrons and non-relativistic electrons. Since the particles are non-relativistic, they do not exert any pressure, $\gamma = 0$. As a consequence $\rho \propto a^{-3}$, which is what you would expect for matter in a volume that is expanding as $V \propto a^3$. The scale factor evolves as $a \propto t^{2/3}$. When the universe is filled with matter, it is expanding, but the expansion rate is getting smaller with time. This makes sense, since the gravitational attraction of the particles will oppose the expansion. According to measurements $\Omega_b \simeq 0.04$ [4], where the index b stands for baryons. If this was the only fluid present in the universe, it would be strongly curved, but we know that this is not the case. Measurements of the rotation of galaxies indicate that galaxies consist of more matter than we can see. This kind of matter is referred to as dark matter. The same measurements indicate that $\Omega_d = 0.23$. This is still not enough to have a flat universe, which indicates that there must be other fluids present in our universe.

Radiation: Radiation consists of relativistic particles, mainly photons. The equation of state is in this case given by $p = 1/3\rho$. As a consequence $\rho \propto a^{-4}$ and $a \propto t^{1/2}$. This shows that in an expanding universe radiation dilutes faster then matter. This makes sense, because not only the density of the photons is diluted, but also the wavelengths of the photons become stretched, which makes them lose energy. If the universe contains both radiation and matter, eventually the matter will start to dominate. This agrees with measurements we have of our universe. We know from observations that at early times there was a period of radiation domination. At present the density of radiation is negligible compared to that of matter.

Cosmological Constant: Einstein believed that the universe was static, but in the previous sections we saw that if the universe is matter or radiation dominated, it expands. To solve this problem Einstein introduced a cosmological constant Λ in the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{2.35}$$

We can again derive the Friedmann equations, the result is,

$$\begin{aligned} H^2 + \frac{K}{a^2} &= \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, && (2.36) \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. && (2.37) \end{aligned}$$

One can view the cosmological constant as an energy density ρ_Λ, which is defined as,

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}. \tag{2.38}$$

Looking at the second Friedmann equation, we can derive that the cosmological constant is a perfect fluid with $\gamma = -1$. Solving the fluid equation gives that ρ_Λ is a constant. If we solve the second Friedmann equation for a flat universe, the result is,

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho = H^2, \tag{2.39}$$

where we used the fact that for a flat universe $\rho = \rho_c$. The solution to this equation is $a = e^{Ht}$. This means that the universe is expanding exponentially, while we saw that at the same time the density stays constant. If the universe contains both matter and a cosmological constant the cosmological constant can counteract the influence of matter and thus lead to a static universe. Unfortunately this is not a stable solution. When Hubble discovered that the universe was expanding Einstein discarded the cosmological constant and called it his biggest blunder. Recent developments have renewed the interest in the cosmological constant. Observations of supernovae in distant galaxies show that the universe was expanding more slowly in the past than at present. Since all the other fluids we discussed slowed the expansion down, our description of the universe must contain the cosmological constant. This is more commonly referred to as dark energy or vacuum energy. Dark energy can also explain why our universe is almost flat. Measurements indicate that $\Omega_\Lambda \simeq 0.7$ [4]. Together with the matter density this is approximately equal to the critical density. The value of the density parameter for the dark energy indicates that it is the dominating fluid at the moment. We also saw that it was the only fluid, which density did not decrease during expansion. As a consequence the dark energy will become more dominant in time and the universe will be expanding faster and faster in the future.

2.1.3 Temperature and entropy

We would like to know the temperature of the universe at a certain time. This is equivalent to obtaining the relation between the temperature and the scale factor. The following discussion of the derivation of this relation is based on chapter 3 of the book "The Early Universe" by Kolb and Turner [5]. A more elaborate discussion can be found there.

To find the relation between the temperature and the scale factor we will assume that the expansion rate of the universe is slow enough for particles to remain in local thermal equilibrium. This means that the entropy per co-moving volume, S/a^3, remains constant. If we can find a relation between the temperature and the entropy, we automatically have a relation between the temperature and the scale factor. From the first law of thermodynamics we know that for an adiabatic expansion,

$$TdS = d(\rho V) + pdV, \tag{2.40}$$

where T is the temperature and V the volume. Together with the integrability condition, $\partial_T \partial_V S = \partial_V \partial_T S$, this gives,

$$S(T) = \frac{[\rho(T) + p(T)]V}{T} + const. \tag{2.41}$$

The entropy density is defined as,

$$s \equiv \frac{S}{V} = \frac{\rho + p}{T}, \tag{2.42}$$

up to a constant. We have assumed that the chemical potential $\mu \ll T$. One can calculate the entropy if one knows the density for a gas of weakly interacting particles. We can find an expression for ρ if we recall that average number of particles with momentum p is given by the Fermi-Dirac and Bose-Einstein distributions,

$$n(p) = \frac{1}{e^{(E-\mu)/T} \pm 1}, \tag{2.43}$$

where $E^2 = \mathbf{p}^2 + m^2$ is the energy of the particle and m is the mass of the particle. The plus sign is for fermions, the minus sign for bosons. To find the energy density we can integrate the energy density for momentum p, $E(p)n(p)$, over the momentum space. If we again assume that $\mu \ll T$, the final result is,

$$\rho(T) = \frac{g}{(2\pi)^3} \int \frac{\sqrt{\mathbf{p}^2 + m^2}}{\exp(\sqrt{\mathbf{p}^2 + m^2}/T) \pm 1} d^3p. \tag{2.44}$$

The factor g is the number of internal degrees of freedom of the particles. For a gas of relativistic particles, $\mathbf{p}^2 \gg m^2$, the density expression reduces to,

$$\text{Bosons :} \qquad \rho = \frac{\pi^2 g}{30} T^4, \tag{2.45}$$

$$\text{Fermions :} \qquad \rho = \frac{7}{8}\frac{\pi^2 g}{30} T^4. \tag{2.46}$$

Since for a relativistic gas, $p = 1/3\rho$, the entropy density is given by,

$$\text{Bosons :} \qquad s = \frac{2\pi^2 g}{45} T^3, \tag{2.47}$$

$$\text{Fermions :} \qquad s = \frac{7}{8}\frac{2\pi^2 g}{45} T^3. \tag{2.48}$$

If there are different relativistic particle species in thermal equilibrium the two equations can be combined into one,

$$s = \frac{2\pi^2}{45} g_{*S} T^3, \tag{2.49}$$

where T is the photon temperature and,

$$g_{*S} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i=\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \tag{2.50}$$

Boson and fermion species are only included when $m_i \ll T$, that is when the species are relativistic. For example, at present, $T \ll$ MeV, only the neutrinos contribute. In this case $g_{*S} = 3.91$ [5]. In the early universe $T > 300$ GeV. At this temperature all Standard Model particles are relativistic, therefore the total number of degrees of freedom is much higher, $g_{*S} = 106.75$ [5]. As can be seen from the density equation (2.44), the greatest contribution to the entropy density comes from relativistic particles, so we can assume that (2.49) is also a good approximation for a combination of relativistic and non relativistic particles. Since s is a constant as the universe expands one can derive that,

$$a \propto g_{*s}^{-1/3}(T)T^{-1}. \tag{2.51}$$

We could have suspected this relation between the scale factor and the temperature by dimension analysis.

2.2 Cosmic Microwave Background

The earliest information we have of our universe is from the Cosmic Microwave Background (CMB). It was formed during a period called 'the time of last scattering'. In the early days of the universe the temperature was very high and because of this the electrons were not bound to nuclei. As a consequence electrons and photons where in thermal equilibrium. When the universe cooled down the electrons became bound to the nuclei and photons could no longer scatter with them. Because of this the period just before the recombination of nuclei and electrons is called the time of last scattering. After this time the photons could travel more or less freely, the universe became transparent. Both before and after the time of last scattering the photons have a black-body spectrum. The only thing that changes during the evolution of the universe is that their frequency redshifts. The black-body radiation of these photons can be measured and was first discovered by Arno Penzias and Robert Wilson in 1965 [6]. The CMB is not completely isotropic, but has little temperature fluctuations around a temperature of 2.725 ± 0.002K [7]. These fluctuations are among others caused by fluctuations in the fluid configuration prior to the time of last scattering. This means that the temperature fluctuations can give us important information of the early universe and give insight in the formation of large scale structure. Because of this important role that the CMB plays in our knowledge of the universe, we will discuss some of its properties in this section. Our discussion is based on the discussion in the book of Weinberg [3] and a more extensive discussion about the history and anisotropies of the CMB can be found there. We will give an overview of the origin of the temperature fluctuations and discuss how we can relate the fluctuations that we measure to cosmological interesting quantities.

2.2.1 Dipole anisotropy

The biggest anisotropy in the CMB is the dipole anisotropy that originates from the motion of the earth. To find the magnitude of this dipole we must look at the quantity $N_\gamma(\mathbf{p})$, which is the density of photons in phase space. In other words: per unit spatial volume in a momentum space volume d^3p there are $N_\gamma(\mathbf{p})d^3p$ photons of each polarization. Thus,

$$N_\gamma(\mathbf{p}) = \frac{\text{number of photons of one polarization}}{\text{phase space volume}}. \tag{2.52}$$

The phase space volume between frequencies ν and $\nu + d\nu$ is given by $4\pi\nu^2 d\nu$. The number of photons between ν and $\nu + d\nu$ is given by the black body spectrum,

$$n_\gamma(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(\nu/k_B T) - 1}. \tag{2.53}$$

This includes both polarizations so we must divide by 2. For photons $|\mathbf{p}| = h\nu/c$, which is just ν in our units and therefore we can replace ν with $|\mathbf{p}|$. We then have,

$$N_\gamma(\mathbf{p}) = \frac{1}{\exp(|\mathbf{p}|/k_B T) - 1}. \tag{2.54}$$

Since both phase space and the number of photons are Lorentz invariant,

$$N'_\gamma(\mathbf{p}') = N_\gamma(\mathbf{p}), \tag{2.55}$$

where the prime denotes that the quantity is Lorentz transformed. Inspection of (2.54) tells us that this implies,

$$\frac{|\mathbf{p}'|}{T'} = \frac{\mathbf{p}}{T}. \tag{2.56}$$

Let $\mathbf{p}$ be the momentum of the photon in the CMB frame and $\mathbf{p}'$ the momentum that we measure on earth. If the earth is moving in the z-direction the relation between the two is given by,

$$\begin{aligned} |\mathbf{p}| &= \gamma(|\mathbf{p}'| + \beta p_z), \\ &= \gamma(1 + \beta\cos\theta)|\mathbf{p}'|, \end{aligned} \tag{2.57}$$

where θ is the angle between the z-axis and $\mathbf{p}'$. From this we can conclude that,

$$T' = \frac{T}{\gamma(1 + \beta\cos\theta)}. \tag{2.58}$$

Since β is of order 10^{-3} [3] we can expand this in β. The magnitude of the dipole is,

$$\begin{aligned} \Delta T = T' - T &= T\left(-\beta\cos\theta - \frac{1}{2}\beta^2 + \beta^2\cos\theta + ...\right), \\ &= T\left(-\beta P_1(\cos\theta) - \frac{1}{6}\beta^2 + \frac{2}{3}\beta^2 P_2(\cos\theta) + ...\right), \end{aligned} \tag{2.59}$$

where the P_l are the Legendre polynomials. Pictures of CMB are usually corrected for this dipole. Another source of anisotropy that does not originate from the early universe is that of the scattering of photons with galaxies. This is called the Sunyaev-Zel'dovich effect. We will not discuss this effect, for a discussion see [3].

2.2.2 Primairy fluctuations

Primary fluctuations are caused by anisotropies in the early universe. There are two main causes for these anisotropies. The first cause are fluctuations in the temperature of the black-body spectrum of photons, electrons and nuclei before the period of last scattering. This is for example caused by the Doppler effect of the moving photons. A second cause is the anisotropic distribution of relativistic matter prior to the period of last scattering. In areas of a higher density the gravitational attraction is stronger and the photons will redshift compared to areas with lower densities. This is known as the Sachs-Wolfe effect and shows that the temperature anisotropies are directly related to the density isotropies of the background fluid.

The temperature fluctuations that we measure are in some sense arbitrary. The CMB is the cumulative result of a lot of arbitrary fluctuations. These fluctuations could also have happened in a different way, which would have changed the CMB. Our theory of the early universe does not single out one of these processes, since the underlying physics is the same. This means that if we want to compare our theory with the experimental results we are not interested in specific fluctuations we measure, but in the average over all possible configurations of fluctuations we could have measured. In the following we will explain how we can relate the observed fluctuations to this average.

According to the ergodic theorem [3] the average over the possible configurations is the same as the average over the positions from which one can observe the CMB. We will denote this average with angular brackets $\langle\rangle$. The fact that the universe is rotational invariant will give restrictions on the averages. The easiest example is that of the average of the temperature difference in the direction $\hat{n}$:

$$\Delta T(\hat{n}) = T(\hat{n}) - T_0, \tag{2.60}$$

where T_0 is the average temperature defined as,

$$T_0 \equiv \frac{1}{4\pi}\int d^2\hat{n}T(\hat{n}). \tag{2.61}$$

Since $\langle\Delta T(\hat{n})\rangle$ is rotational invariant it cannot depend on $\hat{n}$ and,

$$\langle\Delta T(\hat{n})\rangle = 0. \tag{2.62}$$

What about the average of two temperature differences? To evaluate this expression it will turn out to be convenient to expand the temperature difference in spherical harmonics $Y_l^m(\hat{n})$ as,

$$\Delta T(\hat{n}) = \sum_{lm} a_{lm}Y_l^m(\hat{n}), \tag{2.63}$$

where a_{lm} are the coefficients in the expansion. Since the average of two temperatures $\langle \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\rangle$ must be rotational invariant it can only depend on the angle between $\hat{n}_1$ and $\hat{n}_2$ and the index l. This means that we can expand the average in Legendre polynomials P_l as,

$$\langle \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\rangle = \sum_l A_l P_l(\hat{n}_1 \cdot \hat{n}_2), \tag{2.64}$$

since these form a orthonormal basis. We also know that we can rewrite expression (2.63) into an expression for the coefficients a_{lm}:

$$a_{lm} = \int d^2\hat{n} Y_l^{m*}(\hat{n})T(\hat{n}). \tag{2.65}$$

This gives,

$$\langle a_{lm}^* a_{l'm'}\rangle = \int d^2\hat{n}_1 d^2\hat{n}_2 \langle \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\rangle Y_l^m(\hat{n}_1)Y_{l'}^{m'*}(\hat{n}_2). \tag{2.66}$$

If we insert expression (2.64) and use the addition theorem [8],

$$P_l(\hat{n}_1 \cdot \hat{n}_2) = \frac{2\pi}{2l+1}\sum_m Y_l^{m*}(\hat{n}_1)Y_l^m(\hat{n}_2), \tag{2.67}$$

and the completeness relation,

$$\sum_{lm} Y_l^{m*}(\hat{n}_1)Y_l^m(\hat{n}_2) = \delta^2(\hat{n}_1 - \hat{n}_2), \tag{2.68}$$

we can rewrite this as,

$$\begin{aligned}\langle a_{lm}^* a_{l'm'}\rangle &= \frac{4\pi}{2l+1}A_l \int d^2\hat{n}_1 d^2\hat{n}_2 Y_l^m(\hat{n}_1)Y_{l'}^{m'*}(\hat{n}_2), \\ &= \frac{4\pi}{2l+1}A_l\delta_{ll'}\delta_{mm'}, \\ &\equiv C_l\delta_{ll'}\delta_{mm'}. \end{aligned} \tag{2.69}$$

In the second line we used the orthogonality properties of the spherical harmonics.

The coefficients C_l contain all information about the average of two temperatures and are commonly used to characterize them. They can also be written as,

$$C_l = \frac{1}{4\pi}\int d^2\hat{n}_1 d^2\hat{n}_2 P_l(\hat{n}_1 \cdot \hat{n}_2)\left\langle \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\right\rangle, \tag{2.70}$$

which can be found by contracting expression (2.64) with $P_l(\hat{n}_1 \cdot \hat{n}_2)$ and integrating over the two angles. But this is still the average over all possible positions to observe the CMB. Since we can only observe from the earth we can only average over different angles, but not over different positions, so what we measure is

$$\begin{aligned} C_l^{\mathrm{obs}} &= \frac{1}{2l+1}\sum_m a_{lm}^* a_{lm}, \\ &= \frac{1}{4\pi}\int d^2\hat{n}_1 d^2\hat{n}_2 P_l(\hat{n}_1\cdot\hat{n}_2)\Delta T(\hat{n}_1)\Delta T(\hat{n}_2). \end{aligned} \tag{2.71}$$

The difference between these two quantities is called cosmic variance. To know how large this difference is we can calculate the mean square fractional difference:

$$\begin{aligned} \left\langle \left(\frac{C_l - C_l^{\mathrm{obs}}}{C_l}\right)^2 \right\rangle &= 1 - \frac{2}{(2l+1)C_l}\sum_m \langle a_{lm}^* a_{lm}\rangle + \frac{1}{(2l+1)^2 C_l^2}\sum_{mm'} \langle a_{lm}^* a_{lm} a_{lm'}^* a_{lm'}\rangle, \\ &= -1 + \frac{1}{(2l+1)^2 C_l^2}\sum_{mm'} \langle a_{lm}^* a_{lm} a_{lm'}^* a_{lm'}\rangle. \end{aligned} \tag{2.72}$$

To calculate the last term we can use the fact that the perturbations are Gaussian and thus,

$$\langle a_{lm}^* a_{lm} a_{lm'}^* a_{lm'}\rangle = \langle a_{lm}^* a_{lm}\rangle \langle a_{lm'}^* a_{lm'}\rangle + \langle a_{lm}^* a_{lm'}^*\rangle \langle a_{lm} a_{lm'}\rangle + \langle a_{lm}^* a_{lm'}\rangle \langle a_{lm'}^* a_{lm}\rangle. \tag{2.73}$$

Finally we have,

$$\left\langle \left(\frac{C_l - C_l^{\mathrm{obs}}}{C_l}\right)^2 \right\rangle = \frac{2}{2l+1}. \tag{2.74}$$

This shows that the fractional difference gets smaller when we go to larger l. As a consequence measurements at large l can tell us a lot about average temperature differences, while measurements at small l give very little information.

2.3 Inflation

The above given description of the universe has a few problems. In this section we will address these problems and see if we can find a solution for them. The solution will be to introduce a period of rapid expansion in the early universe: inflation. We will discuss general properties of inflation and see how the problems can be solved. We will also discuss the concepts of slow roll inflation and reheating.

2.3.1 Problems

There are three main problems with the description of the universe given in section 2.1. They are called the horizon problem, the flatness problem and the relic problem. We will discuss the first two problems and shortly mention the last problem.

The horizon problem

We discussed before that our universe looks homogeneous and isotropic on large scales. This is especially the case for the Cosmic Microwave Background, which would indicate that the entire CMB was once in causal contact. To investigate if this could be true we define the particle horizon d as the distance which light could have traveled from the beginning of the universe till a time t,

$$d = \int_0^t \frac{dt'}{a(t')}. \tag{2.75}$$

This relation can be derived from the fact that for light $ds^2 = 0$. Looking at the definition of the FLRW metric (2.1) one can see that this is equal to the conformal time and because of this is also proportional to $(Ha)^{-1}$. We can explicitly calculate the particle horizon using relation (2.29),

$$d \propto \frac{3(1+\gamma)}{-1+3\gamma} a^{(1+3\gamma)/2}. \tag{2.76}$$

We can now evaluate the particle horizon during the different eras of the universe. At the time of last scattering the dominant fluid in the universe was radiation. In that case $d \propto a$, which means that during this period the particle horizon expanded as much as the universe expanded. After that, the universe became matter dominated, for which $d \propto a^{1/2}$. This shows that the particle horizon expanded slower then the universe. At the moment the universe is dominated by the cosmological constant and $d \propto a^{-1}$. While the universe is expanding, the particle horizon shrinks. In the last two eras the part of the universe that is able to reach us, becomes smaller while the universe expands. Combining these results we see that parts of the sky which were not in causal contact at the time of last scattering, will never be in causal contact. We can also reverse the calculation given the part of the universe that is inside te particle horizon at present. It turns out that only a patch 1.6° [3] would have been in causal contact at the time of last scattering. This is what is known as the horizon problem.

The flatness problem

Measurements tell us that the universe is almost flat, $\Omega \simeq 1$. This is actually quite remarkable and to see why, we may explicitly write the density parameter as,

$$\begin{aligned} \Omega^{-1} &= \frac{\rho_c}{\rho}, \\ &= 1 + \frac{\rho_k}{\rho}, \\ &= 1 - \frac{3K}{8\pi G a^2 \rho}, \end{aligned} \tag{2.77}$$

where we used definition (2.30). We can make this more explicit if we use the relation between the density and the scale factor (2.28),

$$\Omega^{-1} = 1 - \frac{3K}{8\pi G\rho_0} a^{1+3\gamma}. \tag{2.78}$$

For Ω to be very close to one at present, the second term on the right hand side has to be very small. But because during radiation and matter domination γ equals 1/3 and 0 respectively, the second term grows with time. This means that a flat universe is not a stable solution. To explain the current value of Ω, the second term had to be smaller than 10^{-16} at the time of nuclei formation [9]. Although it is not impossible it seems very unlikely that the curvature was this small by accident. This is what is known as the flatness problem.

The relic problem

One would expect that during the Big Bang particles like magnetic monopoles were created. At present we do not observe these kind of particles. This discrepancy is known as the relic problem. We will not go further into this problem, a more elaborate discussion can be found in [3].

2.3.2 Properties of inflation

To solve the above problems one can introduce a period of rapid expansion in the early universe called inflation. The idea of inflation was first introduced by Guth [10], Albrecht and Steinhardt [11] and Linde [12][13] around 1981. To see that such a period can indeed solve the problems we compare expressions (2.76) and (2.78). Inspection shows that both the horizon and the flatness problem can be solved if there was a period before the time of last scattering, when,

$$1 + 3\gamma < 0 \qquad \Rightarrow \qquad \gamma < -1/3. \tag{2.79}$$

When $\gamma = -1$, this period is called the Sitter inflation, since in that case the metric becomes the de Sitter metric. For other values of γ, it is called Power-Law inflation. Comparison with the second Friedmann equation (2.26) shows that the above statement is the same as the requirement, $\ddot{a} > 0$. The universe must have had an accelerated expansion during inflation. This solves the horizon problem, because the particle horizon would have decreased prior to the time of last scattering. As a consequence the area of the present CMB could have been inside the horizon at the beginning of inflation and a thermal equilibrium could have been formed. After that the horizon would have decreased to the size of 1.6° at the time of last scattering. The flatness problem is solved in the same manner. During inflation the value of the second term in expression (2.78) decreases. If the period last long enough it could have decreased to the needed value of 10^{-16} at nuclei formation.

Scalar field inflation

Which kind of fluid could have caused such an accelerated expansion? The simplest model for inflation is given by a scalar field ϕ, often referred to as the inflaton [14]. The action for this model is,

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]. \tag{2.80}$$

To find an explicit expression for the density and pressure of the scalar field we may derive the expression for the energy momentum tensor, using,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{2.81}$$

The result is,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{2.82}$$

Comparison with (2.15) shows that,

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.83}$$

We can insert these expressions into the fluid equation (2.18) to obtain the conservation equation,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \tag{2.84}$$

We can also rewrite the first Friedmann equation (2.25) for a flat universe,

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{2.85}$$

These two equations will govern the evolution of the scalar field.

Slow roll inflation

The requirement for an accelerated period of expansion was given by the condition $\rho + 3p < 0$. If we use the explicit expressions for the density and pressure (2.83) this is equivalent to the condition $V > \dot{\phi}^2$. To make this statement stronger we can require an almost exponential inflation, which is the same as requiring $\gamma \simeq 1$. In that case the condition is,

$$V(\phi) \gg \frac{1}{2}\dot{\phi}^2. \tag{2.86}$$

Exponential inflation occurs when the potential energy of the inflaton dominates the kinetic energy. In other words, the inflaton must be slowly rolling down the potential. That is why this kind of inflation is called slow roll inflation. Under this condition the scalar field equations (2.84) and (2.85) reduce to,

$$\begin{aligned} \dot{\phi} &\simeq -\frac{1}{3H}\frac{dV(\phi)}{d\phi}, \\ H^2 &\simeq \frac{8\pi G}{3}V(\phi), \end{aligned} \tag{2.87}$$

where we assume that during slow roll inflation we need $\ddot{\phi} \ll H\dot{\phi}$, for inflation to last a significantly long time. Another way of making sure that condition (2.86) is satisfied, is requiring that the slow roll parameters $\epsilon, \eta \ll 1$ [15], where ϵ and η are defined as,

$$\epsilon \equiv \frac{1}{16\pi G}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2 \tag{2.88}$$

$$\eta \equiv 8\pi G \frac{1}{V}\frac{d^2V}{d\phi^2} \tag{2.89}$$

2.3.3 Reheating

Inflation will end, when $\epsilon \simeq 1$. At this point the scalar field will start to oscillate around the minimum of $V(\phi)$. To find out how the density of the inflaton evolves during this period, we can may combine the expression for ρ and the conservation equation for ϕ (2.84) to obtain,

$$\left\langle \frac{d\rho}{dt} \right\rangle = -\langle 6H(\rho - V)\rangle \simeq -6H\langle \rho - V\rangle. \tag{2.90}$$

The angle brackets denote the time average. Solving for a potential of the form $V(\phi) \propto \phi^n$, the solution is [14],

$$\rho \propto a^{-6n/(n+2)}. \tag{2.91}$$

Assuming that the minimum of the potential can always be approximated by $V \propto \phi^2$, the density at the end of inflation is proportional to a^{-3}. One can also derive the time dependence of the scale factor, using the time averaged relation:

$$\langle \dot{\phi}^2 \rangle_T = n\langle V(\phi)\rangle_T. \tag{2.92}$$

We can again solve this for a potential $V(\phi) \propto \phi^n$, the result is $\phi \propto t^{2/(2-n)}$. An expression for the scale factor can then be obtained by using constraint equation (2.84), the result is [16],

$$a \propto t^{(n+2)/3n}. \tag{2.93}$$

For $n = 2$ the scale factor evolves as $t^{2/3}$.

At the end of inflation the inflaton will couple to other fields and in this way decay into other particles. The energy density of the scalar field will decrease as,

$$\rho_\phi(t) = \rho_\phi(t_I) \left(\frac{a(t_I)}{a(t)}\right)^3 e^{-\Gamma(t-t_I)}, \tag{2.94}$$

where t_I is the time of the beginning of the oscillation and Γ the decay rate [3]. The energy conservation equation for the particles the inflaton decays into is,

$$\dot{\rho}_M + 3H(\rho_M + p_M) = \Gamma \rho_\phi. \tag{2.95}$$

If we assume that the inflaton will decay in relativistic particles, $p_M = \rho_M/3$, the solution to (2.95) is,

$$\rho_M(t) = \frac{\rho_\phi(t_I)\Gamma a^3(t_I)}{a^4(t)} \int_{t_I}^{t} a(t')e^{-\Gamma(t'-t_I)}dt'. \tag{2.96}$$

In the beginning the exponential term will dominate and the energy density of the new particles will increase. At later times the a^{-4} behavior takes over and the density will fall again. One can calculate the maximum density reached in two regimes: $\Gamma \gg H(t_I)$ and $\Gamma \ll H(t_I)$.

In the limit $\Gamma \gg H(t_I)$ one can expand expression (2.96) in powers of $H(t_I)/\Gamma$ using partial integration. The result is,

$$\rho_M(t) = \rho_\phi(t_I) \left(\frac{a(t_I)}{a(t)}\right)^2 \left[1 + \frac{H(t_I)}{\Gamma} + ...\right]. \tag{2.97}$$

The inflaton decays almost instantly into the relativistic particles. Their density decays as $\rho_M \propto a^{-4}$, which we saw was characteristic for relativistic particles.

The other limit $\Gamma \ll H(t_I)$ can be calculated by setting the exponential to zero. This can be done because the density reaches its maximum before the exponential starts decaying. This also means that the universe is still dominated by the inflaton and we can use the relation $a(t) = a(t_I)(t/t_I)^{2/3}$. In that case equation (2.96) reduces to,

$$\rho_M(t) \simeq \frac{3}{5}\Gamma t_I \rho_\phi(t_I) \left(\frac{t_I}{t}\right)^{8/3} \left[\left(\frac{t}{t_I}\right)^{5/3} - 1\right], \tag{2.98}$$

The maximum density is reached when $t = (8/3)^{3/5} t_I$.

CHAPTER 3

Astrophysical magnetic fields

Measurements indicate that magnetic fields are present in galaxies, clusters of galaxies and even in the intergalactic medium. These fields can have a coherence up to a scale of 1 Mpc and the strength of the fields at the cluster scale is of the same order as the strength of magnetic fields at galactic scales. Both features are quite remarkable. Up until now it is not clear where these fields come from. One of the most promising ideas is that the fields originate from a quantum perturbation during inflation. This idea was first formulated by Turner and Widrow in 1987 [1]. In this chapter we will briefly review the measurements of magnetic fields and see how these measurements restrict the value of the magnetic field at the end of inflation. We will also give a short argument, why we believe that inflation is a good candidate to give rise to these large scale magnetic fields. The argument will become more formal in the later chapters.

3.1 Measurements of astrophysical magnetic fields

Before we will go into the results of the measurements we will first shortly review the methods that are used to measure astrophysical magnetic fields. A more extensive review of both the measuring methods and the measurements themselves can be found in a review by Widrow [17].

Measuring methods

There are four methods which are commonly used for the detection of astrophysical magnetic fields in galaxies and clusters of galaxies: synchrotron radiation, Faraday rotation, Zeeman splitting and polarization of optical starlight. The first two methods are most commonly used, since they are the easiest to observe. We will explain them in short below. After that we will explain the method that is used to detect magnetic fields in the intergalactic medium, the TeV blazars.

Synchrotron radiation: When an electron has an acceleration perpendicular to its velocity it emits electromagnetic radiation. This happens for example when electrons move in a magnetic field, since they will spiral around the field lines. When the electrons have a relativistic speed, the radiation is called synchrotron radiation. The emitted radiation can give information about the strength of the magnetic field perpendicular to the radiation.

Faraday rotation: When electromagnetic waves travel trough a magnetic field in a medium, the polarization gets rotated. This was discovered by Michael Faraday in 1845, hence the name Faraday rotation. Astrophysical electromagnetic waves travel mostly trough a medium of free electrons. In that case linearly polarized electromagnetic waves get rotated by an angle,

$$\phi = \frac{e^3\lambda^2}{2\pi m_e^2}\int_0^{l_s} n_e(l)B_{\parallel}(l)dl, \tag{3.1}$$

where λ is the wavelength of the electromagnetic wave, m_e the mass of the electron, n_e the density of electrons along the path l, $B_{\parallel}$ the magnetic field parallel to the path and l_s the position of the source. This is often rewritten as,

$$\phi = RM\lambda^2, \tag{3.2}$$

where RM is called the rotation measure. The value of the rotation measure can tell us something about the strength of the magnetic field parallel to the electromagnetic wave.

TeV blazars: Blazars are compact stellar like objects which can emit gamma rays with an energy of order TeV. When these gamma rays travel through the interstellar medium they interact with photons from the diffuse extragalactic background light. This is radiation from objects such as active galactic nuclei and remnants of star formation. Through the interaction an electron-positron pair is created. These pairs can scatter in turn with photons from the CMB, which gives these photons an energy of order GeV. This process, through which the photon gains energy, is called Inverse Compton scattering. If there is no intergalactic magnetic field present these photons travel in a straight line and seem to originate from the same point as the blazar. If a magnetic field is present the path of the electron-positron pairs will be bended. Therefore the GeV photons will seem to originate from points around the blazar. By measuring these deflections a lower bound on the intergalactic magnetic field can be found. Since this method of measuring intergalactic magnetic field is quite recent the method is still under discussion. For example in [18] they argue that Inverse Compton scattering is not the main process through which the electron-positron pair loses energy and therefore the results are unreliable.

Measurements

Using the methods described above a lot of different measurements have been made of magnetic fields in galaxies. It turns out that most of the galaxies have magnetic fields with an average strength of $B \sim 10^{-6} - 10^{-5}$ G [17] and references therein. These fields are coherent on the scale of the galaxies themselves. One has also observed these fields in galaxies at a large redshifts, which indicates that they also existed just after galaxy formation. There are also measurements of magnetic fields in clusters of galaxies. These fields are coherent up to the scale of the clusters and have a strength which is only a little smaller than the galactic magnetic fields, $B \sim 10^{-7} - 10^{-6}$ G [17] and references therein.

Recently a lower bound has been found for magnetic fields in the intergalactic medium of $B \geq 10^{-16} - 10^{-15}$ G, for a coherence scale of the field of 1 Mpc or larger [19][20]. For these measurements the method of TeV blazars was used under the assumption that the strength of the emission of the gamma rays has been constant on long time scales. Since it is not guaranteed that the emission was indeed constant others have assumed that the emission has only been constant during the time of measuring, which was only a few years, when this assumption could be checked. They found a lower bound of $B \geq 10^{-18} - 10^{-17}$ G [21] [22][23].

3.2 Intergalactic magnetic fields

The discovery of magnetic fields in the intergalactic medium is quite remarkable since there seems to be no explanation of how these fields were generated. The reason is that there is no matter and especially no electrically charged matter in the intergalactic medium. The origin of the fields in galaxies and clusters is also still under debate, but one could think of an astrophysical process, such as the birth of a galaxy, as the generator of the magnetic fields. Since no matter is present in the intergalactic medium there is also no astrophysical process that could account for the presence of magnetic fields. This is the main reason that people are looking for a more exotic origin, such as the idea that these fields were generated during inflation. The main reason for this assumption is that it could explain how the fields were generated in the vacuum. We explain in short how this is possible in section 3.5 and the technical details can be found in chapters 6 and 7.

Another thing that is remarkable about the intergalactic magnetic fields is their large coherence scale of 1 Mpc and larger. These large scales are also found in galaxies and clusters of galaxies, but since they are the same size as the galaxies and clusters themselves, it is again possible to find an astrophysical explanation for these scales. The absence of such an explanation in the case of intergalactic magnetic fields points again in the direction of a more exotic origin. In section 3.5 we explain how the assumption that the fields were generated during inflation is also able to explain this feature of the intergalactic fields. For the above reasons the discovery of the intergalactic fields is very important. It provides us with the main reason to look at inflation as the origin of magnetic fields and discards astrophysical processes as the generator of these fields.

3.3 Amplification of the magnetic field

It is possible that the galactic magnetic fields have been amplified in the history of the universe. One of the main mechanism to amplify the magnetic field is called the dynamo mechanism. It arises when conducting matter moves trough a magnetic field. The motion of the matter through the magnetic field can induce a current in the matter. This induced current can in turn amplify the original magnetic field. When there is a magnetic field present in a galaxy, the motion of this galaxy trough the field can amplify this magnetic field. This is what is called the galactic dynamo mechanism. An extensive review of this mechanism is given by Widrow [17] and we refer the reader there for more details. What is important for our purposes is that the dynamo mechanism could have amplified a seed magnetic field. How much depends on the specifics of the dynamo mechanism, which are still in debate. Different values can be found in [17][24] and references therein. We will take one of the upper limits. The advantage of taking this limit is that, if a mechanism does not generate fields of this strength, it will not be able to generate strong enough fields for all different theories of the dynamo mechanism. Since we will be evaluating the strength of the magnetic field at present, without including the influence of the dynamo mechanism, we need to know how much weaker the field would have been if the dynamo mechanism was not present. For example if the dynamo mechanism has amplified the magnetic field strength with a factor of 10^{10} and we ignore this process, our calculated field strength will be a a factor 10^{-10} to small. If we assume that the present magnetic field has a strength of $B = 10^{-6}$ G, and our calculations find a field strength of $B = 10^{-16}$, our calculation is correct with observations, since the dynamo mechanism would have amplified the field to the observed strength. The limit we will use, is that a magnetic field strength at present of $B^0 \sim 10^{-33}$ G on a scale $\lambda = 10$ kpc, could still have been amplified by the dynamo mechanism to the observed values, as discussed in [24].

There is also another mechanism that could cause amplification of magnetic fields. Since the magnetic flux is conserved, the magnetic field is amplified when a protogalaxy collapses into a galactic disc in the early ages of the universe [17]. We will again take an upper limit, when dealing with the specifics and assume that without any amplification the needed seed magnetic field at present would be of order $B^0 \sim 10^{-14}$ G on scales of 1 Mpc, as discussed in [24]. We will use both these limits to evaluate if inflation can indeed generate large scale magnetic fields.

It is important to notice that these amplification mechanisms are only able to amplify the magnetic fields in galaxies and clusters of galaxies. Since there is no matter in the intergalactic medium the intergalactic magnetic fields can not be amplified by the above mechanisms. Therefore the measurements of intergalactic magnetic fields usualy provides us with the strongest lower bound. Any viable theory needs to be able to produce a present magnetic field of at least $B^0 \sim 10^{-18}$ at the scale of 1 Mpc.

3.4 CMB constraints

Magnetic fields create anisotropic pressures as will be explained in chapter 8. This will influence the gravitational field, which will also become anisotropic. The gravitational anisotropy has an influence on the Cosmic Microwave Background as explained in section 2.2.2. This means that if the magnetic fields were generated during from inflation they would leave an imprint on the CMB. Barrow, Ferreira and Silk analyzed 4 years of data from the Cosmic Background Explorer (COBE) and found an upper limit of $B^0 < 10^{-9}$ G [25]. This is again the strength of the present magnetic field if no amplification mechanisms would have been present.

3.5 Inflation as the origin of astrophysical magnetic fields

In 1987 Turner and Widrow [1] suggested that the observed magnetic fields could have been generated during inflation. One reason for this suggestion was that both the amplifying mechanisms that we have discussed in section 3.3 only work with a seed magnetic field and are not able to create a magnetic field out of nothing. There are different theories about how this seed magnetic field was generated. The fact that we have observed magnetic fields in galaxies at large redshift indicates that they have been generated in the early universe. On the other hand, the fact that magnetic fields are present in the intergalactic medium, where no electrically charged matter is present, suggest that the magnetic fields could not have been generated by an astrophysical process. For this reason it was suggested that the origin of the fields is a quantum perturbation during inflation, since in this way the field could have been created in the vacuum. This could also explain the fact that the magnetic fields are coherent on such large scales. We saw in chapter 2.3, that during inflation the universe had an accelerated expansion and therefore the particle horizon became smaller with time. Therefore, during inflation a quantum perturbation can grow very fast and finally become larger than the horizon. After inflation the horizon will become larger again, as we have seen, and the 'perturbation' will cross back in the horizon.

To make this description more formal we will first introduce the notion of the Hubble radius. This is the distance beyond which particles are moving away from us faster than the speed of light. Since Hubble's law was defined as $v = Hd$, the Hubble radius is H^{-1}, if we put c to one. The Hubble radius is not equivalent to the particle horizon, although in papers they are often interchanged [26]. If we make the assumption that the universe was matter dominated during most of its history, then at present the particle horizon is two times the size of the Hubble radius [15]. Therefore if one wants to know if a perturbation is inside the particle horizon at present, it is a good approximation to look if it is inside the Hubble radius. From now on we will often refer to the Hubble radius as the horizon.

To find out when perturbations cross in- and outside the horizon we need to know how the Hubble radius evolves during time. We know from the Friedmann equations that,

$$H^{-1} \propto a^{3(1+\gamma)/2}. \tag{3.3}$$

During inflation, when $\gamma < -1/3$, the Hubble radius expands less than the universe itself. This means a co-moving scale λ can be within the horizon at early times and move outside the horizon during inflation. After that the universe is radiation and matter dominated and the Hubble radius is respectively proportional to a^2 and $a^{3/2}$. Therefore the horizon increases and the scale λ moves back into the horizon. This is schematically depicted in figure 3.1. One can see that larger scales exit the horizon at earlier times and cross back in the horizon later than smaller scales.

We can calculate when the co-moving length λ crosses outside the Hubble radius, that is when $a\lambda = H^{-1}$. This happens N(λ) e-folds before the end of inflation, where the number of e-folds is defined as,

$$N(\lambda) = \ln\left(\frac{a_{end}}{a_1}\right). \tag{3.4}$$

Here a_1 is the scale factor at the time of horizon crossing and a_{end} the scale factor at the end of inflation. Since during reheating $a \propto \rho^{1/3}$, we can rewrite this as,

$$N(\lambda) = \ln\left[\frac{M^2}{m_{\rm pl}}\lambda a_{\rm rad}\left(\frac{\rho_{\rm end}}{\rho_{\rm rad}}\right)^{1/3}\right], \tag{3.5}$$

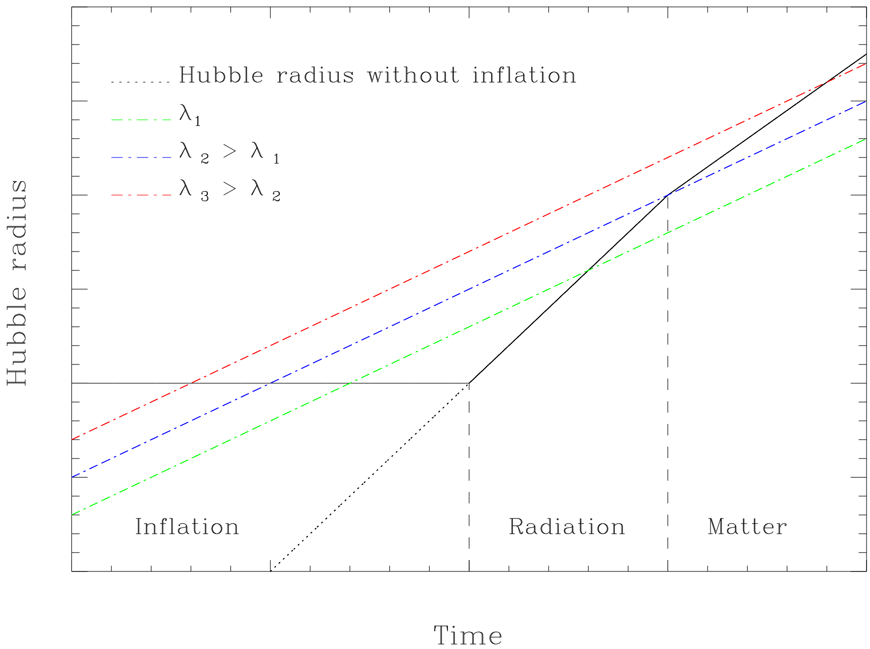

Figure 3.1 – Schematic evolution of the horizon (solid line) during de Sitter inflation ($\gamma = -1$), the radiation- and matter dominated era. The red line corresponds to the largest scale λ_3, the green line to the smallest scale λ_1. During de Sitter inflation the horizon stays constant and because of the expansion of the universe a co-moving scale λ can cross outside the Hubble radius. After inflation the horizon expands faster than the universe and the scales cross back inside the horizon. Larger scales exit the horizon before smaller scales and cross back in the horizon at later times. The dotted line indicates the evolution of the horizon in absence of an inflationary period. Figure from [27].

where M is the temperature at the end of inflation. We used the fact that $a_1 = H_1^{-1}\lambda^{-1}$ and that during inflation, $H^2 \simeq \rho/m_{\rm pl} \simeq M^4/m_{\rm pl}^2$ [1], since we assume that the universe is flat and we saw in section 2.1.3 that $\rho \simeq T^4$. The subscript rad indicates the value of the quantity at the end of reheating. Using $\rho_{\rm end}/\rho_{\rm rad} \simeq M^4/T_{\rm rad}^4$ and $a_{\rm rad} \simeq a_0 T_0 T_{\rm rad}^{-1}$ we find,

$$\begin{aligned} N(\lambda) &= \ln\left[\lambda \frac{T_0}{m_{\rm pl}} M^{2/3} T_{\rm rad}^{1/3}\right], \\ &= 45 + \ln\lambda_{\rm Mpc} + \frac{2}{3}\ln(M_{14}) + \frac{1}{3}\ln(T_{10}), \end{aligned} \tag{3.6}$$

where $\lambda_{\rm Mpc} \equiv \lambda/{\rm Mpc}$, $M = M_{14}10^{14}$GeV and $T_{\rm rad} = T_{10}10^{10}$ GeV. The current value of the Hubble parameter is $H_0 = 100h\text{kms}^{-1}\text{Mpc}^{-1}$. This leads to a present Hubble radius of $H_0^{-1} \simeq 3000$ Mpc, where we put c to one and used $h \simeq 1$. This indicates that perturbations, which cross back into the horizon at present, left the horizon,

$$N(\lambda) = 53 + \frac{2}{3}\ln(M_{14}) + \frac{1}{3}\ln(T_{10}), \tag{3.7}$$

e-folds before the end of inflation. One can calculate that at this time the density was given by [1],

$$\frac{\rho_{\text{tot}}}{m_{\text{pl}}^4} = (1.6 \times 10^{26})^s \lambda_{\text{Mpc}}^s \left(\frac{M}{m_{\text{pl}}}\right)^{4+2s/3} \left(\frac{T_{\text{rad}}}{m_{\text{pl}}}\right)^{s/3}, \tag{3.8}$$

where $s = -6(1+\gamma)/(1+3\gamma)$.

The above given description holds for a general perturbation. In the following chapters we will show explicitly how the astrophysical magnetic fields arise from quantum perturbations during inflation and what their strength is. To do this we need to know how quantum field theory in curved spacetimes works, since the FLRW metric is not normal Minkowski spacetime. This will be the subject of the next chapter.

CHAPTER 4

Quantum field theory in curved spacetime

Quantum field theory in a curved spacetime is not the same as quantum field theory in Minkowski spacetime. One of the main differences is that the concept of a particle is no longer well defined, as we will show. Since the FRLW metric is not the Minkowski metric, we have to keep this in mind when we want to evaluate the magnetic fields. It will turn out that due to symmetries of the FLRW metric it is still possible to come to a definition of the particle concept. In this chapter we will discuss some general properties of quantum field theory in curved spacetime, by taking the example of a scalar field. We will use the obtained results in chapter 5 to get a description of the magnetic field in an FLRW metric. This chapter is build op as follows: first we will recap some results from quantum field theory in Minkowski space. Secondly we will discuss the notion of conformal symmetry. In section 4.3 we will give properties of a scalar field in a general curved spacetime. Last we will discuss the notion of a particle and see how particles can be created in a 2d FLRW universe. The discussions in this chapter are based on the book by Birrell and Davies [28]. A more extensive review of quantum field theory in curved spacetime can be found there.

4.1 Scalar field in Minkowski space

A scalar field $\phi(x)$ in Minkowski space satisfies the Klein-Gordon equation,

$$(\Box - m^2)\phi = 0, \tag{4.1}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\eta_{\mu\nu}$ is the metric for Minkowski spacetime and m is the mass of the scalar particle. A solution of the Klein-Gordon equation are the plane waves,

$$u_k(x) \propto e^{ik^\mu x_\mu} = e^{i\mathbf{kx}-i\omega t}, \tag{4.2}$$

where $\omega^2 = \mathbf{k}^2 + m^2$. Because ω depends only on $\mathbf{k}$ up to a sign, different solutions can be characterized by $\mathbf{k}$ and the sign of ω. The most general solution to the Klein-Gordon equation can be found by constructing a complete, orthonormal set of modes in which we can expand ϕ. To have an idea of what orthonormal in this case means, we may use the inner product,

$$(\phi_1, \phi_2) = -i\int_{\Sigma_t}(\phi_1\partial_t\phi_2^* - \phi_2\partial_t\phi_1^*)d^{n-1}x, \tag{4.3}$$

where Σ_t is a constant-time hypersurface. We can calculate the inner product of two plane waves,

$$\begin{aligned}(u_k, u_k') &\propto -i\int_{\Sigma_t}(e^{i\mathbf{kx}-i\omega t}\partial_t e^{-i\mathbf{k}'\mathbf{x}-i\omega' t} - e^{i\mathbf{k}'\mathbf{x}-i\omega' t}\partial_t e^{-i\mathbf{kx}-i\omega t})d^{n-1}x,\\ &= (\omega+\omega')e^{i(\omega-\omega')t}(2\pi)^{n-1}\delta^{(n-1)}(\mathbf{k}-\mathbf{k}'). \end{aligned}\tag{4.4}$$

This becomes zero if the two plane waves are different, which is the same as stating that they have different $\mathbf{k}$. If we want the plane waves to form an orthonormal set of modes we require,

$$(u_k, u_k') = \delta^{(n-1)}(\mathbf{k}-\mathbf{k}'). \tag{4.5}$$

Comparison with equation (4.4) tells us that,

$$u_k = \frac{1}{\sqrt{2\omega(2\pi)^{n-1}}}e^{i\mathbf{kx}-i\omega t}. \tag{4.6}$$

Since the sign of ω is not determined by $\mathbf{k}$, we can require that ω is positive. The negative solutions can then be obtained by taking the complex conjugate of u_k. From now on we will use the following definition. A mode is a positive-frequency mode if,

$$\partial_t u_k = -i\omega u_k, \qquad \omega > 0, \tag{4.7}$$

and is a negative-frequency mode if,

$$\partial_t u_k = i\omega u_k, \qquad \omega > 0. \tag{4.8}$$

In our case the u_k are the positive frequency modes and the u_k^* are the negative-frequency modes. Notice that the positive- and negative-frequency are defined using the operator ∂_t, which is a timelike Killing vector in Minkowski space.

Quantization of the scalar field is done by imposing the equal time commutation relations,

$$\begin{aligned}
&[\phi(t,\mathbf{x}),\pi(t,\mathbf{x}')] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}'),\\
&\big[\phi(t,\mathbf{x}),\phi(t,\mathbf{x}')\big] = 0,\\
&\big[\pi(t,\mathbf{x}),\pi(t,\mathbf{x}')\big] = 0,
\end{aligned} \tag{4.9}$$

where π is the conjugate variable of ϕ, which is defined as,

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}. \tag{4.10}$$

In the case of a scalar field in Minkowski space the Lagrangian is given by,

$$\mathcal{L}(x) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{4.11}$$

It is easy to see that variation of the action with respect to ϕ gives back the Klein-Gordon equation (4.1). In this case the conjugate variable is given by $\pi = \dot{\phi}$.

We saw that the plane waves (4.2) and their complex conjugates form an orthonormal set of modes, so we can expand ϕ as,

$$\phi(x) = \sum_k [a_k u_k(x) + a_k^\dagger u_k^*(x)]. \tag{4.12}$$

Notice that the positive-frequency modes are the coefficients of the annihilation operator and the negative-frequency modes are the coefficients of the creation operator. We can use the above given commutation relations (4.9) to compute the commutation relations for the creation and annihilation operators $a_k^\dagger$ and a_k:

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \qquad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \tag{4.13}$$

The vacuum state $|0\rangle$ is defined as,

$$a_k|0\rangle = 0, \quad \forall k. \tag{4.14}$$

A one-particle state can be created by acting with $a_k^\dagger$ on the vacuum. The same way multiple particle states can be constructed. The number of particles in a particular state can be calculated with the number operator $N_k = a_k^\dagger a_k$. This obeys,

$$N_{k_i}|n_1, n_2, ..., n_j\rangle = n_i|n_1, n_2, ..., n_j\rangle, \tag{4.15}$$

where n_i is the number of particles in the i state.

4.2 Conformal transformations

A conformal transformation is a transformation which leaves the metric invariant up to a scale factor,

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{4.16}$$

One can calculate how other quantities transform under conformal transformations. It will turn out to be very convenient to know the transformation of,

$$\left[\Box - \frac{1}{4}\frac{(n-2)}{(n-1)}R\right]\phi \to \left[\bar{\Box} - \frac{1}{4}\frac{(n-2)}{(n-1)}\bar{R}\right]\bar{\phi}, \tag{4.17}$$

where we conveniently define,

$$\bar{\phi}(x) \equiv \Omega^{(2-n)/2}(x)\phi(x). \tag{4.18}$$

To calculate this transformation we must start with one of the most basic quantities, the Christoffel symbol. This changes under a conformal transformation as,

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} \to \bar{\Gamma}^\rho_{\mu\nu} &= \frac{1}{2}\bar{g}^{\rho\sigma}\left[\partial_\mu \bar{g}_{\nu\sigma} + \partial_\nu \bar{g}_{\sigma\mu} - \partial_\sigma \bar{g}_{\mu\nu}\right], \\
&= \Gamma^\rho_{\mu\nu} + \frac{1}{2}\Omega^{-2} g^{\rho\sigma}\left[g_{\nu\sigma}\partial_\mu\Omega^2 + g_{\sigma\mu}\partial_\nu\Omega^2 - g_{\mu\nu}\partial_\sigma\Omega^2\right], \\
&= \Gamma^\rho_{\mu\nu} + \Omega^{-1}\left[\delta^\rho_\nu\partial_\mu\Omega + \delta^\rho_\mu\partial_\nu\Omega - g_{\mu\nu}g^{\rho\sigma}\partial_\sigma\Omega\right].
\end{aligned} \tag{4.19}$$

Now we know how the Christoffel symbol changes we can use this to calculate what happens to the Ricci scalar R:

$$\begin{aligned}
R \to \bar{R} &= \bar{g}^{\mu\alpha}\left[\partial_\lambda \bar{\Gamma}^\lambda_{\mu\alpha} - \partial_\mu \bar{\Gamma}^\lambda_{\lambda\alpha} + \bar{\Gamma}^\lambda_{\lambda\beta}\bar{\Gamma}^\beta_{\mu\alpha} - \bar{\Gamma}^\lambda_{\mu\beta}\bar{\Gamma}^\beta_{\lambda\alpha}\right], \\
&= \Omega^{-2}R - 2(n-1)\Omega^{-3}g^{\mu\nu}\partial_\mu\partial_\nu\Omega - (n-1)(n-4)\Omega^{-4}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega,
\end{aligned} \tag{4.20}$$

where n is the dimension of spacetime. The other transformation we need to know is that of,

$$\begin{aligned}
\Box \to \bar{\Box} &= \frac{1}{\sqrt{-\bar{g}}}\partial_\mu\left[\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\right], \\
&= \Omega^{-2}\Box + \Omega^{-3}(n-2)g^{\mu\nu}(\partial_\mu\Omega)\partial_\nu,
\end{aligned} \tag{4.21}$$

where we used that $\bar{g} = \Omega^{2n}g$. Now we have all the ingredients to calculate the transformation (4.17).

$$\begin{aligned}
\left[\bar{\Box} - \frac{1}{4}\frac{(n-2)}{(n-1)}\bar{R}\right]\bar{\phi} =& \big[\Omega^{-2}\Box + \Omega^{-3}(n-2)g^{\mu\nu}(\partial_\mu\Omega)\partial_\nu \\
&- \frac{1}{4}\frac{(n-2)}{(n-1)}\Omega^{-2}R + \frac{1}{2}(n-2)\Omega^{-3}g^{\mu\nu}\partial_\mu\partial_\nu\Omega \\
&+ \frac{1}{4}(n-2)(n-4)\Omega^{-4}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega\big]\bar{\phi}.
\end{aligned} \tag{4.22}$$

If you calculate this the first term on the right hand side will generate $\Omega^{-(n+2)/2}\Box\phi$ and three other terms which cancel exactly against the second, fourth and fifth term on the right hand side. So the final result is,

$$\left[\bar{\Box} - \frac{1}{4}\frac{(n-2)}{(n-1)}\bar{R}\right]\bar{\phi} = \Omega^{-(n+2)/2}\left[\Box - \frac{1}{4}\frac{(n-2)}{(n-1)}R\right]\phi \tag{4.23}$$

4.3 Scalar field in curved spacetime

We can generalize section 4.1 to curved spacetime characterized by the metric $g_{\mu\nu}$. The Lagrangian is in this case given by,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2\right]. \tag{4.24}$$

In the case of Minkowski space, where $R = 0$ and $g^{\mu\nu} = \eta^{\mu\nu}$ this reduces to the Lagrangian (4.11). The resulting equation of motion is,

$$\left[\Box - m^2 - \xi R\right]\phi = 0. \tag{4.25}$$

Again note that this reduces to the Klein-Gordon equation if $R = 0$ and $g^{\mu\nu} = \eta^{\mu\nu}$. There are two cases of special interest. The first is $\xi = 0$, this is called the minimally coupled case. The second is called the conformally coupled case,

$$\xi = \frac{1}{4}\frac{n-2}{n-1} \equiv \xi(n). \tag{4.26}$$

The reason for this name becomes clear if we insert this into the equation of motion (4.25):

$$\left[\Box - m^2 - \frac{1}{4}\frac{n-2}{n-1}R\right]\phi = 0. \tag{4.27}$$

Comparison with equation (4.23) shows that the equations of motions are invariant under conformal transformations if $m = 0$, since we can always multiply by $\Omega^{(n+2)/2}$. We would like to find solutions u_i to the equation of motion (4.25) similar to the plane wave solutions in Minkowski space, such that we can again expand ϕ as,

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)]. \tag{4.28}$$

If we then quantize the system we will get the same commutation relations for $a_i^\dagger$ and a_i as in Minkowski spacetime. How can we find these solutions? Recall that in Minkowski spacetime the plane waves were a natural solution since they where eigenfunctions of the timelike Killing vector ∂_t. This allowed us to define positive- and negative-frequency modes, which were the coefficients of the creation and annihilation operators. The problem in general curved spacetime is that there will not always be timelike Killing vectors. As a consequence we cannot naturally define positive- and negative-frequency modes. The result is that there is no 'natural' choice for the modes u_i. We could also choose a different set of modes $\bar{u}_i$. In that case ϕ can be expanded as,

$$\phi(x) = \sum_i [\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)]. \tag{4.29}$$

Because both u_i and $\bar{u}_i$ form a complete set of modes we can write one as an expansion of the other,

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*), \tag{4.30}$$

$$u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*). \tag{4.31}$$

These are called Bogoliubov transformations and the matrices α and β are Bogoliubov coefficients. They satisfy the normalization conditions,

$$\sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij}, \tag{4.32}$$

$$\sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0. \tag{4.33}$$

Because both expansions of ϕ should be equal we can derive using the Bogoliubov transformations that,

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger), \tag{4.34}$$

$$\bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger). \tag{4.35}$$

This shows that an annihilation operator in one mode expansion is a combination of an annihilation and a creation operator in another mode expansion if $\beta_{ji} \neq 0$. The above story has a very important consequence if we look at the vacuum state. We have two vacuum states $|0\rangle$ and $|\bar{0}\rangle$ which are defined by,

$$a_i|0\rangle = 0, \quad \forall i, \tag{4.36}$$

$$\bar{a}_j|\bar{0}\rangle = 0, \quad \forall j. \tag{4.37}$$

One can use (4.34) to see what happens if we act with a_i on $|\bar{0}\rangle$.

$$a_i|\bar{0}\rangle = \sum_j \beta_{ji}^* \bar{a}_j^\dagger |\bar{0}\rangle \neq 0. \tag{4.38}$$

So what is the vacuum state in one mode expansion is a many particle state in another mode expansion. More specifically, if one defines the number operator as $N_i = a_i^\dagger a_i$, then the number of particles in the u_i mode in the vacuum state $|\bar{0}\rangle$ are,

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2. \tag{4.39}$$

This means that the vacuum states are only equal if $\beta_{ji} = 0$. In the same way the number of particles in many particle states will depend on the mode expansion that is used.

4.4 Particles

How can we define the vacuum that lies closest to the physical vacuum? In what state would a detector not detect particles? It turns out that even in Minkowski space there is not one answer to this question. An accelerated observer in Minkowski space will observe particles that an inertial observer will not. This is called the Unruh effect, a detailed discussion is given in the book by Carroll [29].

To understand better what is going on in curved spacetime we will first have a look at inertial observers in Minkowski spacetime. The isometry group of Minkowski spacetime is the Pointcaré group. This consist of translations, rotations and boosts. We would like to know what happens with the plane waves if we perform a Poincaré transformation. The most non-trivial transformation is a boost, so lets examine what happens if we boost by a velocity $\mathbf{v} = d\mathbf{x}/dt$. The coordinates $x^{\mu'}$ are then given by,

$$t' = \gamma t - \gamma \mathbf{v}\cdot\mathbf{x}, \tag{4.40}$$
$$\mathbf{x}' = \gamma\mathbf{x} - \gamma\mathbf{v}t, \tag{4.41}$$

where $\gamma = (1-v^2)^{-1/2}$. The inverse transformations are given by,

$$t = \gamma t' + \gamma\mathbf{v}\cdot\mathbf{x}', \tag{4.42}$$
$$\mathbf{x} = \gamma\mathbf{x}' + \gamma\mathbf{v}t'. \tag{4.43}$$

Now we can calculate the time derivative of the plane wave in the boosted frame

$$\begin{aligned}\partial_t' u_k &= \frac{\partial x^\mu}{\partial t'}\partial_\mu u_k, \\ &= -i\gamma(\omega - \mathbf{v}\cdot\mathbf{k})u_k, \\ &= -i\omega' u_k. \end{aligned} \tag{4.44}$$

This is the same as in the non boosted frame, except that the frequency changed to the frequency in the boosted frame. If a plane wave is a positive-frequency mode in the non boosted frame it will stay a positive-frequency mode in the boosted frame. The same is true for the negative-frequency modes. In the previous section we saw that this has the consequence that both the vacuum and the number operator stays the same. Thus in Minkowski spacetime all inertial observers see the same vacuum and the same number of particles. This is why it is natural to define the vacuum state as in (4.14), it is the agreed vacuum for all inertial observers.

In general curved spacetime does not have Pointcaré symmetry or any other symmetry. It is then no longer the case that inertial observers measure the same vacuum or the same number of particles. How do we know what a particle detector measures? A detector moves along a trajectory characterized by a proper time τ. If it is possible to find modes that obey,

$$\frac{D}{d\tau}u_i = \frac{dx^\mu}{d\tau}\nabla_\mu u_i = -i\omega u_i, \tag{4.45}$$

we can define these as the positive frequency modes. A more general discussion about particle detectors in curved spacetime is given in Birrell and Davies [28].

Although it is not possible to give a natural definition of the vacuum in curved spacetime, there is one way in which we can make a natural choice. A lot of spacetimes are asymptotically Minkowskian in the past or in the future. A natural choice for a vacuum is the state where all inertial observers see no particles in the past or in the future. The time in the past is usually referred to as the in-region and the time in the future as the out-region. In some cases the in- and out-vacuum are not the same. If all inertial observers do not see particles in the in-state, they might see particles in the out state. One can say that in this case the particles were created by the gravitational field. To make this idea more clear we will look at an example in the next section.

4.5 Particle creation in a 2d FLRW universe

The example we will look at is the 2d FLRW universe. We choose a 2d universe, because this makes the calculation simpler since the conformally coupled case is equal to the minimally coupled case. A 2d FLRW universe is described by the metric in conformal time,

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2). \tag{4.46}$$

One can immediately see that this is conformal to Minkowski spacetime. The metric will become Minkowskian in the in- and out-region if $a(\eta)$ becomes a constant when

$\eta \to \pm\infty$. One way to realize this is to write $a^2(\eta)$ as,

$$a^2(\eta) = A + B \tanh \rho\eta, \tag{4.47}$$

where A, B and ρ are constants. Then

$$a^2(\eta) \to A \pm B, \qquad \eta \to \pm\infty. \tag{4.48}$$

How can we define a good basis to expand our scalar field in? A first thing to notice is that a only depends on the conformal time, so ∂_i is a Killing vector of spacetime. This leads us to the assumption that,

$$u_k(\eta, x) = \frac{1}{\sqrt{2\pi}} \chi_k(\eta) e^{i\mathbf{kx}}. \tag{4.49}$$

We will try to find a solution in the conformally coupled case, which in two dimensions is equal to $\xi(n) = 0$. The first step to solve the equation of motion (4.25) is to notice that,

$$\Box = a^{-2}(\eta)(-\partial_\eta^2 + \partial_x^2). \tag{4.50}$$

Acting with this on u_k gives us an equation for $\chi(\eta)$,

$$\partial_\eta^2 \chi_k(\eta) + (k^2 + a^2(\eta)m^2)\chi_k(\eta) = 0. \tag{4.51}$$

This equation is not easy to solve, but it can be done in terms of hypergeometric functions. The first hypergeometric function is a function ${}_2F_1(a, b, c, z)$, which solves the hypergeometrical differential equation,

$$z(1-z)y''(z) + [c - (a+b+1)z]y'(z) - aby(z) = 0. \tag{4.52}$$

We will give the solution of equation (4.51) and then show that it is correct. There is more than one solution, but we will give the solutions that behaves like positive-frequency modes in the in and out region. The solution of the former is given by,

$$\chi_k^{\text{in}}(\eta) = \frac{1}{\sqrt{2\omega_{\text{in}}}} \exp\left[-i\omega_+\eta - \frac{i\omega_-}{\rho}\ln(2\cosh\rho\eta)\right] \quad {}_2F_1(\eta), \tag{4.53}$$

where

$$\omega_{\text{in}} = [k^2 + m^2(A - B)]^{1/2}, \tag{4.54}$$

$$\omega_{\text{out}} = [k^2 + m^2(A + B)]^{1/2}, \tag{4.55}$$

$$\omega_\pm = \frac{1}{2}(\omega_{\text{out}} \pm \omega_{\text{in}}). \tag{4.56}$$

To see that this is indeed a solution of (4.51) we must calculate the derivatives of (4.53) and insert them into the equation. We then get a differential equation for ${}_2F_1(\eta)$,

$$\begin{aligned}{}_2F_1''(\eta) + 2{}_2F_1'(\eta)[-i\omega_+ - i\omega_- \tanh\rho] +_2 F_1(\eta)[-i\omega_-\rho(1-\tanh^2\rho\eta)\\ + (-i\omega_+ i\omega_- \tanh\rho\eta)^2 + k^2 + Am^2 + Bm^2\tanh\rho\eta] = 0.\end{aligned} \tag{4.57}$$

We can make a change of coordinates,

$$z \equiv \frac{1}{2}(1+\tanh\rho\eta). \tag{4.58}$$

Then after some lines of calculation the differential equation becomes,

$$\begin{aligned}z(z-1){}_2F_1''(z) + [-2z - 2i\omega_-\rho^{-1}z + 1 + i\rho(\omega_- - \omega_+)]{}_2F_1'(z)\\ +[-i\omega_-\rho^{-1} + \omega_-^2\rho^{-2}]{}_2F_1(z) = 0.\end{aligned} \tag{4.59}$$

Comparison with equation (4.52) shows that the solution to this equation is indeed given by ${}_2F_1(1+i\omega_-\rho^{-1}, i\omega_-\rho^{-1}, 1-i\omega_{\text{in}}\rho^{-1}, 1/2(1+\tanh\rho\eta))$. We can now calculate what happens to the solution in the in-region, if $\eta \to -\infty$. For negative large arguments ${}_2F_1(1/2(1+\tanh\rho\eta)) \to 1$ and $\ln[2\cosh\rho\eta] \to -\rho\eta$. So,

$$u_k^{\text{in}} \to (4\pi\omega_{\text{in}})^{-1/2} e^{i\mathbf{kx} - i\omega_{\text{in}}\eta}, \tag{4.60}$$

in the in-region. It is clear that this is indeed a positive-frequency mode. The solution that behaves like a positive-frequency mode in the out-region is given by,

$$\begin{aligned}\chi_k^{\text{out}}(\eta) = \frac{1}{\sqrt{2\omega_{\text{out}}}} \exp\left[-i\omega_+\eta - \frac{i\omega_-}{\rho}\ln(2\cosh\rho\eta)\right] \quad \times\\ {}_2F_1(1+i\omega_-\rho^{-1}, i\omega_-\rho^{-1}, 1+i\omega_{\text{out}}\rho^{-1}, 1/2(1-\tanh\rho\eta)).\end{aligned} \tag{4.61}$$

It is possible to check that this is indeed a solution in the same way as we did for the previous case. Except for the pre-factor it looks similar to the expression we had for the vacuum in the in-region. The hypergeometric function has the limit ${}_2F_1(1/2(1-\tanh\rho\eta)) \to 1$ for $\eta \to \infty$, just as for the in-region. The difference in the exponent will come from the fact that $\ln[2\cosh\rho\eta] \to \rho\eta$ instead of $-\rho\eta$, when $\eta \to \infty$. In the out-region this becomes,

$$u_k^{\text{out}} \to (4\pi\omega_{\text{out}})^{-1/2} e^{i\mathbf{kx} - i\omega_{\text{out}}\eta}. \tag{4.62}$$

The two solutions are not equal, so the vacuum state in the in-region may contain particles in the out-region. To know how many particles the out-region contains we must now how to write u_k^{in} in terms of u_k^{out}.

$$u_k^{\text{in}} = \sum_{k'} (\alpha_{kk'} u_{k'}^{\text{out}} + \beta_{kk'} u_{k'}^{\text{out}*}), \tag{4.63}$$

$$\chi_k^{\text{in}} e^{i\mathbf{kx}} = \sum_{k'} (\alpha_{kk'} \chi_{k'}^{\text{out}} e^{i\mathbf{k'x}} + \beta_{kk'} \chi_{k'}^{\text{out}*} e^{-i\mathbf{k'x}}). \tag{4.64}$$

Multiplying both sides by $e^{-i\mathbf{kx}}$ and integrating both sides with respect to $\mathbf{x}$ gives,

$$\chi_k^{\text{in}} = \sum_{k'} (\delta_{kk'} \alpha_{kk'} \chi_{k'}^{\text{out}} + \delta_{-kk'} \beta_{kk'} \chi_{k'}^{\text{out}*}), \tag{4.65}$$

$$= \alpha_k \chi_k^{\text{out}} + \beta_k \chi_k^{\text{out}*}, \tag{4.66}$$

where the relation with the Bogoliubov coefficients is given by,

$$\alpha_{kk'} = \alpha_k \delta_{kk'}, \qquad \beta_{kk'} = \beta_k \delta_{-kk'}. \tag{4.67}$$

The coefficient $\chi_k^{\text{out}*}$ does not have a minus sign because the value of χ_k can only be determined by k up to a sign. We can now calculate the number of particles in the out region, $\sum_{k'} |\beta_{k'k}|^2 = |\beta_k|^2$. The first step is to solve equation (4.66),

$$\frac{1}{\sqrt{2\omega_{\text{in}}}} e^x \ {}_2F_1(\text{in}) = \frac{\alpha_k}{\sqrt{2\omega_{\text{out}}}} e^x \ {}_2F_1(\text{out}) + \frac{\beta_k}{\sqrt{2\omega_{\text{out}}}} e^{-x} \ {}_2F_1^*(\text{out}), \tag{4.68}$$

where we have defined,

$$\begin{aligned} e^x &\equiv \exp\left[-i\omega_+\eta - \frac{i\omega_-}{\rho} \ln(2\cosh\rho\eta)\right], \\ {}_2F_1(\text{in}) &\equiv \ {}_2F_1(1 + i\omega_-\rho^{-1}, i\omega_-\rho^{-1}, 1 - i\omega_{\text{in}}\rho^{-1}, 1/2(1 + \tanh\rho\eta)), \\ {}_2F_1(\text{out}) &\equiv \ {}_2F_1(1 + i\omega_-\rho^{-1}, i\omega_-\rho^{-1}, 1 + i\omega_{\text{out}}\rho^{-1}, 1/2(1 - \tanh\rho\eta)). \end{aligned} \tag{4.69}$$

We can use linear transformation properties of the hypergeometric functions, which can be found for example in Abramowitz and Stegun [30],

$$\begin{aligned} {}_2F_1(a, b, c, z) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \ {}_2F_1(a, b, a+b-c+1, 1-z) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \ {}_2F_1(c-a, c-b, c-a-b+1, 1-z), \\ {}_2F_1(a, b, c, z) =& (1-z)^{c-a-b} \ {}_2F_1(c-a, c-b, c, z), \end{aligned} \tag{4.70}$$

to rewrite,

$$\begin{aligned} {}_2F_1(\text{in}) =& \frac{\Gamma(1 - i\omega_{\text{in}}\rho^{-1})\Gamma(-i\omega_{\text{out}}\rho^{-1})}{\Gamma(-i\omega_+\rho^{-1})\Gamma(1 - i\omega_+\rho^{-1})} \ {}_2F_1(\text{out}) \\ &+ \frac{(1/2(1 + \tanh\rho\eta))^{i\omega_{\text{in}}\rho^{-1}}}{(1/2(1 - \tanh\rho\eta))^{i\omega_{\text{out}}\rho^{-1}}} \frac{\Gamma(1 - i\omega_{\text{in}}\rho^{-1})\Gamma(i\omega_{\text{out}}\rho^{-1})}{\Gamma(i\omega_-\rho^{-1})\Gamma(1 + i\omega_-\rho^{-1})} \ {}_2F_1^*(\text{out}), \\ =& \frac{\Gamma(1 - i\omega_{\text{in}}\rho^{-1})\Gamma(-i\omega_{\text{out}}\rho^{-1})}{\Gamma(-i\omega_+\rho^{-1})\Gamma(1 - i\omega_+\rho^{-1})} \ {}_2F_1(\text{out}) \\ &+ e^{-2x} \frac{\Gamma(1 - i\omega_{\text{in}}\rho^{-1})\Gamma(i\omega_{\text{out}}\rho^{-1})}{\Gamma(i\omega_-\rho^{-1})\Gamma(1 + i\omega_-\rho^{-1})} \ {}_2F_1^*(\text{out}). \end{aligned} \tag{4.71}$$

Equation (4.68) can therefore be rewritten as,

$$\begin{aligned}\frac{\Gamma(1-i\omega_{\text{in}}\rho^{-1})\Gamma(-i\omega_{\text{out}}\rho^{-1})}{\Gamma(-i\omega_{+}\rho^{-1})\Gamma(1-i\omega_{+}\rho^{-1})}\ {}_2F_1(\text{out})+e^{-2x}\frac{\Gamma(1-i\omega_{\text{in}}\rho^{-1})\Gamma(i\omega_{\text{out}}\rho^{-1})}{\Gamma(i\omega_{-}\rho^{-1})\Gamma(1+i\omega_{-}\rho^{-1})}\ {}_2F_1^*(\text{out})\\ =\left(\frac{\omega_{\text{in}}}{\omega_{\text{out}}}\right)^{1/2}\left[\alpha_k\ {}_2F_1(\text{out})+\beta_k e^{-2x}\ {}_2F_1^*(\text{out})\right].\end{aligned} \tag{4.72}$$

Comparison of both sides shows that,

$$\alpha_k=\left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2}\frac{\Gamma(1-i\omega_{\text{in}}\rho^{-1})\Gamma(-i\omega_{\text{out}}\rho^{-1})}{\Gamma(-i\omega_{+}\rho^{-1})\Gamma(1-i\omega_{+}\rho^{-1})}, \tag{4.73}$$

$$\beta_k=\left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2}\frac{\Gamma(1-i\omega_{\text{in}}\rho^{-1})\Gamma(i\omega_{\text{out}}\rho^{-1})}{\Gamma(i\omega_{-}\rho^{-1})\Gamma(1+i\omega_{-}\rho^{-1})}. \tag{4.74}$$

From this result we can calculate the number of particles in the out-region, using the properties of the Gamma function [30],

$$|\Gamma(ix)|^2=\frac{\pi}{x\sinh(\pi x)}, \tag{4.75}$$

$$|\Gamma(1+ix)|^2=\frac{\pi x}{\sinh(\pi x)}. \tag{4.76}$$

The result is,

$$|\beta_k|^2=\frac{\sinh^2(\pi\omega_{-}\rho^{-1})}{\sinh(\pi\omega_{\text{in}}\rho^{-1})\sinh(\pi\omega_{\text{out}}\rho^{-1})}. \tag{4.77}$$

This is only zero if $m=0$, since for this value of m, $\omega_{-}=0$. Notice that in this case the equation of motion is conformally invariant. It turns out that this is a general feature: if a spacetime is conformal to Minkowski spacetime and if the field equation is conformally invariant there are no particles created by the gravitational field.

CHAPTER 5

Electromagnetism

In the previous chapter we reviewed the properties of a scalar field in curved spacetime. In this chapter we will do the same for the magnetic field. We will begin by a recap of electromagnetism in Minkowski spacetime. In the second section we will generalize this to a general curved spacetime using the concepts of chapter 4. We then make this description more specific and describe electromagnetism in a FLRW spacetime.

5.1 Electromagnetism in Minkowski spacetime

5.1.1 Maxwell equations

The full description of electromagnetism was first given by Maxwell at the end of the 19th century. He discovered that the electric and magnetic field obeyed the following equations,

$$\begin{aligned} \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{J}, \\ \nabla \cdot \mathbf{E} &= \rho, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned} \tag{5.1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field, $\mathbf{J}$ is the current and ρ the charge density. It turns out to be possible to write these equation is a covariant way. To do this we may define the field strength tensor $F_{\mu\nu}$ as,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{5.2}$$

The electric and magnetic field are then given by,

$$E^i = F^{0i}, \tag{5.3}$$

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{\mu\nu}, \tag{5.4}$$

where ϵ^{ijk} is the Levi-Citiva symbol. For a covariant description we must also define the current four-vector as,

$$j^\mu = (\rho, J^x, J^y, J^z). \tag{5.5}$$

The first two Maxwell equations can then be written as,

$$\partial_\mu F^{\mu\nu} = j^\mu. \tag{5.6}$$

The last two Maxwell equations are given by the Bianchi identity:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{5.7}$$

5.1.2 Vector field

The Maxwell equations can also be described by a vector field A^μ, such that the electric and magnetic field are given by,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{5.8}$$

$$\mathbf{E} = -\nabla A_0 - \partial_t \mathbf{A}. \tag{5.9}$$

As a consequence the field tensor can be written as,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{5.10}$$

With this definition the Bianchi identity which described the last two Maxwell equations becomes trivial. The first two Maxwell equations can be found by varying the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu, \tag{5.11}$$

with respect to A_μ. If we look more closely to definition (5.10) it turns out that the equations do not change if we perform the transformation,

$$A^\mu \to A^{\mu'} = A^\mu + \partial_\mu \lambda, \tag{5.12}$$

where λ is a scalar function. The fact that we can choose any λ is called gauge freedom and the transformations (5.12) are called gauge transformations. A common gauge choice is called the Lorentz gauge which is defined by,

$$\partial_\mu A^\mu = 0. \tag{5.13}$$

Another common gauge choice is the Coulomb gauge defined by

$$A_0 = \partial_i A^i = 0. \tag{5.14}$$

In the next chapters we will usually use the Coulomb gauge.

5.1.3 Quantization

Since we have found a field description of electromagnetism the next step is to quantize it. We would like to do the quantization in a way that the usual commutation relation is realized,

$$[A_i(t,\mathbf{x}), \pi_j(t,\mathbf{y})] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}),$$

where $\pi_j = \partial\mathcal{L}/\partial\dot{A}_i$. We said before that we will mostly work in the Coulomb gauge, but in that case the above expression cannot be right, since

$$[\nabla \cdot \mathbf{A}(t,\mathbf{x}), \pi_j(t,\mathbf{y})] = i\partial_i\delta^3(\mathbf{x}-\mathbf{y}) \neq 0.$$

To solve this problem we can write the delta function in its exponential form,

$$\delta^3(\mathbf{x}-\mathbf{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})},$$

and replace δ_{ij} with a rank 2 tensor Δ_{ij}. If we then take the divergence the result is,

$$[\nabla \cdot \mathbf{A}(t,\mathbf{x}), \pi_j(t,\mathbf{y})] = i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} k^i \Delta_{ij}.$$

This should be zero, so we can derive the condition,

$$\Delta_{ij} = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}.$$

The right commutation relation is thus,

$$[A_i(t,\mathbf{x}), \pi_j(t,\mathbf{y})] = i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \Delta_{ij}. \tag{5.15}$$

Using this commutation relation we can expand the electromagnetic potential just as we did for the scalar field in the previous chapter. In Minkowski spacetime the vacuum Maxwell equations in the Coulomb gauge become,

$$\Box A^i = 0. \tag{5.16}$$

A natural solution is of the form $e^{\pm ikx}$ times a vector. Thus we can expand the electromagnetic field as,

$$A_i(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k}} \sum_{\lambda=1}^{2} \epsilon_{i\lambda}(k)[a_\lambda(k)e^{ikx} + a_\lambda^\dagger(k)e^{-ikx}]. \tag{5.17}$$

The ϵ_λ are called polarization vectors. In the Coulomb gauge $k^i\epsilon_{i\lambda} = 0$, which means that the ϵ_λ are transverse to the direction of propagation. Because the polarization vectors act as a basis they satisfy the completeness relation,

$$\sum_{\lambda=1}^{2} \epsilon_{i\lambda}(\mathbf{k})\epsilon_{j\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2} = \Delta_{ij}. \tag{5.18}$$

If we insert the expansion into the commutation relation (5.15) we recover the usual commutation relations for the creation and annihilation operators,

$$\begin{aligned}
&[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'},\\
&[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0,\\
&[a_\lambda^\dagger(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] = 0.
\end{aligned} \tag{5.19}$$

5.2 Electromagnetism in a flat FLRW spacetime

To see how the magnetic field in a FLRW universe behaves we must first find out which of the above expressions holds for general spacetime and which are specific to Minkowski spacetime.

5.2.1 General spacetime

The Lagrangian

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu,\\
&= -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} + j^\mu A_\mu,
\end{aligned} \tag{5.20}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is still valid in general spacetime. By varying the action with respect to A_μ we can derive the equations of motions for the electric and magnetic field,

$$-\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}F_{\nu\beta}] = j^\alpha. \tag{5.21}$$

These are equivalent to the first two Maxwell equations in Minkowski spacetime. Since the expression for $F_{\mu\nu}$ in terms of the vector field stays the same, the Bianchi identity still describes the last two Maxwell equations. The expressions (5.3) and (5.4) are not valid in general spacetime. A covariant definition for the electric and magnetic field is given by [31],

$$E_\mu = u^\nu F_{\mu\nu}, \tag{5.22}$$

$$B_\mu = \frac{1}{2}\epsilon_{\mu\nu\kappa}F^{\nu\kappa}, \tag{5.23}$$

where the tensor $\epsilon_{\mu\nu\kappa}$ is given by,

$$\epsilon_{\mu\nu\kappa} = u^\lambda \eta_{\lambda\mu\nu\kappa}. \tag{5.24}$$

In these equations u^μ is the four-velocity vector and $\eta_{\mu\nu\kappa\lambda}$ is the Levi-Civita tensor, which is defined as,

$$\eta_{\mu\nu\kappa\lambda} = \sqrt{-g}\tilde{\eta}_{\mu\nu\kappa\lambda}, \tag{5.25}$$

where $\tilde{\eta}_{\mu\nu\kappa\lambda}$ is the Levi-Civita symbol in four dimensions. The definitions of the Lorentz and Coulomb gauge where already described in a covariant way, so they are still valid. What about quantization? In general spacetime we still want to satisfy the commutation relation,

$$[A_i(t,\mathbf{x}), \pi_j(t,\mathbf{y})] = i\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\Delta_{ij}. \tag{5.26}$$

The expansion of the vector field is not as straightforward as in Minkowski spacetime. In the last chapter we saw that the equation of motion of the scalar field in curved spacetime had different solutions which led to fundamentally different expansions. The same is true for a vector field in curved spacetime.

5.2.2 Flat FLRW spacetime

Since we found the general description of the electric and magnetic field we can look at their expressions in FLRW spacetime. Recall that the flat FLRW metric in conformal time was given by,

$$ds^2 = a^2(\eta)(-d\eta^2 + d\mathbf{x}^2). \tag{5.27}$$

The determinant of the FLRW metric is $g = -a^8(\eta)$. The equations of motions then become,

$$-a^{-4}\partial_\mu[a^4 g^{\mu\nu}g^{\alpha\beta}F_{\nu\beta}] = j^\alpha, \tag{5.28}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{5.29}$$

The Coulomb gauge in conformal time is given by,

$$A_0(\eta, \mathbf{x}) = 0, \tag{5.30}$$

$$\partial_j A^j(\eta, \mathbf{x}) = 0. \tag{5.31}$$

Since we will work in the Coulomb gauge most of the time, we will give the expressions for the electric and magnetic field in a FLRW universe in this gauge. A co-moving observer in conformal time is given by $u^\mu = (a^{-1}, \mathbf{0})$, since the definition $u_\mu u^\mu = -1$ must hold. The electric and magnetic field for a co-moving observer in the Coulomb gauge is then given by,

$$E_i = -a^{-1}\partial_\eta A_i, \tag{5.32}$$

$$B_i = a^{-1}\epsilon_{ijk}\partial_j A_k. \tag{5.33}$$

We can also write the the equation of motion for the vector field in the Coulomb gauge,

$$A_i'' - a^2\partial_j\partial^j A_i = a^2 j_i. \tag{5.34}$$

In the vacuum this becomes,

$$A_i'' - a^2\partial_j\partial^j A_i = 0. \tag{5.35}$$

Since the scale factor only depends on η a solution to this equation will be of the form $A(\eta)e^{i\mathbf{kx}}$ times a vector. So it would be natural to expand the vector field as,

$$A_i(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k}} \sum_{\lambda=1}^{2} \epsilon_{i\lambda}(\mathbf{k})[a_\lambda(\mathbf{k})A_k(\eta)e^{i\mathbf{kx}} + a_\lambda^\dagger(\mathbf{k})A_k^*(\eta)e^{-i\mathbf{kx}}], \tag{5.36}$$

where ϵ_λ are the same polarization vectors as before. If we insert this into the commutation relation (5.26) we can only find the usual commutation relations if we require,

$$A_k(\eta)A_k^{*'}(\eta) - A_k'(\eta)A_k^*(\eta) = 2ik. \tag{5.37}$$

If we insert the expansion into the equation of motion for the vacuum we get an equation of motion for $A_k(\eta)$ given by,

$$A_k''(\eta) + k^2 A_k(\eta) = 0. \tag{5.38}$$

The general solution is,

$$A_k(\eta) = c_{k1}e^{ik\eta} + c_{k2}e^{-ik\eta}. \tag{5.39}$$

5.2.3 High conductivity

In the last section we saw what the characteristics are of the vector field in a vacuum. We would also like to know how the vector field behaves when the conductivity of the surrounding matter becomes high. To see what happens we can covariantly define the the current four-vector as [31],

$$j_\mu = \rho_e u_\mu + \sigma_c E_\mu, \tag{5.40}$$

where ρ_e is the charge density, u_μ is the four velocity and σ_c the conductivity. In the Coulomb gauge for a co-moving observer, this becomes,

$$j_0 = a^{-1}\rho_e, \tag{5.41}$$
$$j_i = -a^{-1}\sigma_c A_i'. \tag{5.42}$$

The equation of motion for the vector field (5.34) thus becomes,

$$A_i''(\eta, x) + a^{-1}\sigma_c A_i'(\eta, x) - a^{-2}\partial^j\partial_j A_i(\eta, k) = 0. \tag{5.43}$$

We can solve this exactly in the large scale and high conductivity limit. The large scale limit makes sure we can neglect the derivatives of A_i. And in the high conductivity limit $\sigma_c \gg H$. The solution is then,

$$A_i(\eta, \mathbf{x}) = \frac{a c_1(\mathbf{x})}{\sigma_c} e^{-\sigma_c \eta / a} + c_2(\mathbf{x}). \tag{5.44}$$

The electric and magnetic field are then,

$$E_i(\eta, \mathbf{x}) = 0, \tag{5.45}$$
$$B_i(\eta, \mathbf{x}) = a^{-1}\left(\nabla \times c_2(\mathbf{x})\right), \tag{5.46}$$

where we used that $\sigma_c \gg H$, so the exponential goes to zero. If the conductivity becomes high the electric field is zero and the magnetic field decays as $\mathbf{B} \propto a^{-1}$.

5.2.4 Co-moving fields

Since we will be interested in describing magnetic fields at co-moving length scales λ is convenient to have a co-moving description of the electric and magnetic field. This is quite simple,

$$E_i^{\text{co-moving}} = a^{-1} E_i^{\text{physical}}, \tag{5.47}$$
$$B_i^{\text{co-moving}} = a^{-1} B_i^{\text{physical}}. \tag{5.48}$$

CHAPTER 6

Inflation produced magnetic fields

How large are the magnetic fields that were generated during inflation? The answer to this question depends on a lot of different things. First of all it depends on the specific model that is used for describing electromagnetism. We will look at different models in chapter 7, but here we will give a short insight into why we use different models. Beside the model that is used, the strength of the magnetic field also depends on the evolution of the field during different era's. We will give an overview of what happens to the magnetic field from inflation till now.
Another important thing that we must think about, before we can answer the main question, is: what do we actually measure? The answer to this consists of three parts. First we must take into account that we cannot measure the magnetic field at exactly one point, but always measure a little area. Secondly, what we measure is a vacuum expectation value, and not the field itself, therefore we must find an explicit expression for this. Lastly, since we saw in chapter 4 that the vacuum is not uniquely defined, we must think about which vacuum we use and how this relates to what we observe. The main part of the chapter will consist of addressing these three problems and by doing this we will arrive at an expression for the present magnetic fields that were generated during inflation. We will use this expression in chapter 7 to evaluate different models.
As a final step we must make sure that the assumption we made, namely that the electromagnetic field does not influence the background, is correct. In the last section we will develop a method to check that this is indeed the case.

6.1 Model

Why do we have different models and do we not use normal Maxwell theory described by the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{6.1}$$

in the vacuum? We will explicitly show in chapter 7 that this theory does not produce strong enough magnetic fields. The reason is that this Lagrangian gives rise to conformal invariant equations of motions. In chapter 4 we saw that if the equations of motions and the background where both conformally invariant, there were no particles produced by the gravitational field. To produce interesting magnetic fields we must break the conformal invariance by adding another term to the Lagrangian that is not conformal invariant. There are different ways to do this and we will look at the most general models in chapter 7. These models are constrained by the fact that we know that after inflation we have the usual Maxwell electrodynamics. So all the models must reduce to the Maxwell Lagrangian at the end of inflation.

6.2 Evolution

The evolution of the magnetic field is quite straightforward, but to simplify our calculation we will make some assumptions.

Inflation: The magnetic field is created during inflation and during this period the fields will cross outside the Hubble radius. While inflation lasts the magnetic field is described by the equation of motion that can be derived from the Lagrangian, that differs for different models. During inflation the density of particles is very low. That is why we can use the vacuum equations of motion, i.e. $j_\mu = 0$. Also this means that the conductivity is very low. Quantization is done exactly as described in section 5.2.2. The only thing that is different with respect to the Maxwell model is the expression for $A_k(\eta)$ and condition (5.37).

Reheating: After inflation the universe reheats. To make things easier we will assume instantaneous reheating. This is the same as taking the limit $\Gamma \gg H$ as we did in chapter 2.3. As we said in the previous section we will assume that at this point the Lagrangian reduces to the Maxwell Lagrangian. Last we will assume that at this point the conductivity becomes very high.

Radiation and matter domination During radiation and matter dominated eras the conductivity will stay high. In the last chapter we saw that if the conductivity is high

$B \propto a^{-2}$ in co-moving coordinates, so

$$B_\lambda^0 = a_{\rm rad}^2 B_\lambda(\eta_{\rm rad}), \tag{6.2}$$

where the subscript rad means at the beginning of the radiation era and the superscript 0 stands for the present field. We used the fact that $a_0 = 1$.

6.3 Average magnetic field

Now we know how the magnetic field evolves, we must address the question, what it is we measure, when we measure large scale magnetic fields. First we will address the fact that our measuring devices are not sensitive enough to measure the magnetic field at one point, since the fields are far away. In the last chapter we saw that a co-moving magnetic field in a FLRW metric is given by,

$$\mathbf{B} = \frac{1}{a^2}\nabla \times \mathbf{A}(\eta, \mathbf{x}). \tag{6.3}$$

To account for the fact that we do not measure one point, but a little area in space we define the average magnetic field on a co-moving scale λ as [32][33],

$$\mathbf{B}_\lambda(\eta, \mathbf{x}) = \frac{1}{a^2}\int d^3y W_\lambda(|\mathbf{x}-\mathbf{y}|)\nabla \times \mathbf{A}(\eta, \mathbf{y}), \tag{6.4}$$

where W_λ is a Gaussian window function given by,

$$W_\lambda(|\mathbf{x}|) = (2\pi\lambda^2)^{-3/2} e^{-|\mathbf{x}|^2/(2\lambda^2)}. \tag{6.5}$$

Here λ is a measure for the area in space we average over.

6.4 Vacuum expectation value during inflation

If we measure a magnetic field we do not measure the field itself, but the vacuum expectation value of the field. The vacuum expectation value of the co-moving average magnetic field is defined as,

$$B_\lambda^2(\eta) = \left\langle 0 || \mathbf{B}_\lambda(\eta, \mathbf{x})|^2 | 0 \right\rangle. \tag{6.6}$$

In this section we will explicitly calculate this using the definition of the average magnetic field (6.4) and the expansion (5.36). This means we want to calculate,

$$B_\lambda^2(\eta) = \frac{1}{a^4}\langle 0 | \int d^3y d^3y' W_\lambda(|\mathbf{x}-\mathbf{y}|) W_\lambda^*(|\mathbf{x}-\mathbf{y}'|)(\nabla \times \mathbf{A}(\eta, \mathbf{y}))(\nabla \times \mathbf{A}(\eta, \mathbf{y}'))^* | 0 \rangle. \tag{6.7}$$

The first step is to notice that,

$$
\begin{aligned}
(\nabla\times\mathbf{A}(\eta,\mathbf{y}))_i &= \partial_j A_z - \partial_z A_j,\\
&= \int \frac{d^3k}{(2\pi)^2\sqrt{2k}} \sum_{\lambda=1,2} \big[\epsilon_{z\lambda}(\mathbf{k})\big(ik_j a_\lambda(\mathbf{k})A_k(\eta)e^{i\mathbf{ky}}\\
&\quad - ik_j a^\dagger_\lambda(\mathbf{k})A^*_k(\eta)e^{-i\mathbf{ky}}\big) + \epsilon_{j\lambda}(\mathbf{k})\big(-ik_z a_\lambda(\mathbf{k})A_k(\eta)e^{i\mathbf{ky}}\\
&\quad + ik_z a^\dagger_\lambda(\mathbf{k})A^*_k(\eta)e^{-i\mathbf{ky}}\big)\big].
\end{aligned}
$$

Then using that $a_\lambda|0\rangle = \langle 0|a^\dagger_\lambda = 0$ and the commutation relations for a_λ one can derive,

$$
\begin{aligned}
&\big\langle 0|(\nabla\times\mathbf{A}(\eta,\mathbf{y}))_i(\nabla\times\mathbf{A}(\eta,\mathbf{y}'))^*_i|0\big\rangle = \int \frac{d^3kd^3k'}{(2\pi)^3 2k}\sum_{\lambda,\lambda'}\delta_{\lambda,\lambda'}\delta(\mathbf{k}-\mathbf{k}')\times\\
&\big[A_kA^*_{k'}k_jk'_j\epsilon_{z\lambda}(\mathbf{k})\epsilon_{z\lambda'}(\mathbf{k}')e^{i(\mathbf{yk}-\mathbf{y}'\mathbf{k}')} - A_kA^*_{k'}k_jk'_z\epsilon_{z\lambda}(\mathbf{k})\epsilon_{j\lambda'}(\mathbf{k}')e^{i(\mathbf{yk}-\mathbf{y}'\mathbf{k}')}\\
&-A_kA^*_{k'}k_zk'_j\epsilon_{j\lambda}(\mathbf{k})\epsilon_{z\lambda}(\mathbf{k}')e^{i(\mathbf{yk}-\mathbf{y}'\mathbf{k}')} + A_kA^*_{k'}k_zk'_z\epsilon_{j\lambda}(k)\epsilon_{j\lambda}(\mathbf{k}')e^{i(\mathbf{yk}-\mathbf{y}'\mathbf{k}')}\big].
\end{aligned}
$$

Performing the integral over k and k' and summing over λ' this reduces to,

$$
\begin{aligned}
&\big\langle 0|(\nabla\times\mathbf{A}(\eta,\mathbf{y}))_i(\nabla\times\mathbf{A}(\eta,\mathbf{y}'))^*_i|0\big\rangle = \frac{1}{2k}\sum_\lambda |A_k|^2\delta(\mathbf{y}-\mathbf{y}')\times\\
&\big[k_j^2\epsilon_{z\lambda}(\mathbf{k})\epsilon_{z\lambda}(\mathbf{k}) - k_jk_z\epsilon_{z\lambda}(\mathbf{k})\epsilon_{j\lambda}(\mathbf{k}) - k_zk_j\epsilon_{j\lambda}(\mathbf{k})\epsilon_{z\lambda}(\mathbf{k}) + k_z^2\epsilon_{j\lambda}(\mathbf{k})\epsilon_{j\lambda}(\mathbf{k})\big].
\end{aligned}
\tag{6.8}
$$

Using the completeness relation (5.18) this reduces to,

$$
\begin{aligned}
&\big\langle 0|(\nabla\times\mathbf{A}(\eta,\mathbf{y}))_i(\nabla\times\mathbf{A}(\eta,\mathbf{y}'))^*_i|0\big\rangle = \frac{1}{2k}|A_k|^2\delta(\mathbf{y}-\mathbf{y}')\times\\
&\quad\big[k_j^2\left(1-\frac{k_z^2}{k^2}\right) + k_jk_z\left(\frac{k_zk_j}{k^2}\right) + k_zk_j\left(\frac{k_zk_j}{k^2}\right) + k_z^2\left(1-\frac{k_j^2}{k^2}\right)\big],\\
&\quad = \frac{1}{2k}|A_k|^2\delta(\mathbf{y}-\mathbf{y}')\left(k_z^2+k_j^2\right).
\end{aligned}
$$

The final result is,

$$
\big\langle 0|(\nabla\times A(\eta,y))(\nabla\times A(\eta,y'))^*|0\big\rangle = k|A_k|^2\delta(\mathbf{y}-\mathbf{y}'). \tag{6.9}
$$

With this result we can calculate the vacuum expectation value,

$$
\begin{aligned}
B^2_\lambda(\eta) &= \frac{1}{a^4}\langle 0|\int d^3yd^3y' W_\lambda(|\mathbf{x}-\mathbf{y}|)W^*_\lambda(|\mathbf{x}-\mathbf{y}'|)(\nabla\times\mathbf{A}(\eta,\mathbf{y}))(\nabla\times\mathbf{A}(\eta,\mathbf{y}'))^*|0\rangle,\\
&= \frac{1}{a^4}\int d^3yd^3y' W_\lambda(|\mathbf{x}-\mathbf{y}|)W_\lambda(|\mathbf{x}-\mathbf{y}'|)\times k|A_k|^2\delta(\mathbf{y}-\mathbf{y}'),\\
&= \frac{1}{a^4}\int d^3y W^2_\lambda(|\mathbf{x}-\mathbf{y}|)k|A_k|^2,
\end{aligned}
\tag{6.10}
$$

where we have used that W_λ is real. The final step is to Fourier expand the window function,

$$W_\lambda(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} W_\lambda(k) e^{i\mathbf{kx}},$$

where $W_\lambda(k) = e^{-\lambda^2 k^2/2}$. Then

$$\begin{aligned}
B_\lambda^2(\eta) &= \frac{1}{a^4} \int \frac{d^3y d^3k d^3k'}{(2\pi)^6} W_\lambda(k) W_\lambda(k') e^{i|\mathbf{x}-\mathbf{y}|(\mathbf{k}-\mathbf{k}')} k|A_k|^2, \\
&= \frac{1}{a^4} \int \frac{d^3k d^3k'}{(2\pi)^3} W_\lambda(k) W_\lambda(k') e^{i\mathbf{x}(\mathbf{k}-\mathbf{k}')} \delta(\mathbf{k}-\mathbf{k}') k|A_k|^2, \\
&= \frac{1}{a^4} \int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |A_k|^2. \qquad (6.11)
\end{aligned}$$

So our final result is,

$$B_\lambda^2(\eta) = \frac{1}{a^4} \int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |A_k|^2. \qquad (6.12)$$

6.5 Vacuum expectation value after reheating

In the previous section we used the vacuum defined as,

$$a_\lambda(\mathbf{k})|0\rangle = 0. \qquad (6.13)$$

The annihilation operators where defined by the expansion of the vector field and had as coefficients the positive frequency modes $\sum_{\lambda=1}^2 \epsilon_{i\lambda}(\mathbf{k}) A_k(\eta) e^{i\mathbf{kx}}$. These coefficients were defined by the model during inflation. What we want to know is, how many particles we see now, after inflation. All the different models reduce at the beginning of the radiation era to standard Maxwell theory with the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \qquad (6.14)$$

After inflation the positive-frequency mode is therefore given by,

$$u_k^{\rm rad} = \sum_{\lambda=1}^{2} \epsilon_{i\lambda}(\mathbf{k}) e^{-ik\eta + i\mathbf{kx}}. \qquad (6.15)$$

Recall that we calculated the A_k part from the differential equation,

$$A_k''(\eta) + k^2 A_k'' = 0, \qquad (6.16)$$

so the value of k in the exponential can only be calculated up to a sign. We saw in chapter 4 that we can use this fact to require $k > 0$ and define the positive-frequency mode as,

$$\partial_\eta u_k^{\rm rad} = -iku_k^{\rm rad}. \qquad (6.17)$$

In general the expression for the positive-frequency mode after inflation will be different than the expression we found during inflation. In chapter 4 we saw that we could express the positive frequency mode in the in-region in terms of the modes in the out-region as,

$$u_k^{\text{in}} = \sum_{k'} (\alpha_{kk'} u_{k'}^{\text{out}} + \beta_{kk'} u_{k'}^{\text{out}*}). \tag{6.18}$$

Since the spatial part of the modes is the same during and after inflation we can do the same trick as in the example of the 2d FLRW universe and write,

$$\begin{aligned} A_k^{\text{inflation}}(\eta) &= \alpha_k A_k^{\text{rad}}(\eta) + \beta_k A_k^{\text{rad}*}(\eta), \\ &= \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}, \end{aligned} \tag{6.19}$$

where the relation with the Bogoliubov coefficients is given by,

$$\alpha_{kk'} = \alpha_k \delta_{kk'}, \qquad \beta_{kk'} = \beta_k \delta_{-kk'}. \tag{6.20}$$

Just as in the example of chapter 4 the coefficient $A_k^{\text{rad}*}(\eta)$ has a positive subscript because we required that $k > 0$. The number of particles in the out-region is given by $|\beta_k|^2$. Therefore the average vacuum expectation value of the magnetic field at the beginning of the radiation era is,

$$B_\lambda^2(\eta) = \frac{1}{a^4} \int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |\beta_k(\eta)|^2. \tag{6.21}$$

Finally the magnetic field we measure at present is given by,

$$B_\lambda^0 = \left[\int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |\beta_k(\eta_{\text{rad}})|^2 \right]^{1/2}, \tag{6.22}$$

where we used expression (6.2).

6.6 Backreaction

In the above explained procedure we have assumed that the electromagnetic field is a perturbation compared to the inflaton field. If this would not be the case, the electromagnetic field would influence the spacetime and we could no longer work with the FLRW metric (2.1). To show that we indeed could have made this assumption, we must check that the energy density of the electromagnetic field is smaller than the total energy density during inflation. The easiest way to check this is to calculate the energy momentum tensor by varying the action, S,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}. \tag{6.23}$$

The density is then $\rho = -T^0_0$. At the wavelengths we are interested in this should be smaller then the total energy density,

$$\rho_{\text{tot}} = \frac{3}{8\pi} H_{\text{inf}}^2 m_{\text{pl}}^2. \tag{6.24}$$

We are not interested in the density above, but in the vacuum expectation value of the co-moving average electromagnetic field. This can be found by replacing the electric and magnetic field by their average vacuum expectation value. The expression for the vacuum expectation value of the co-moving average electric field can be found in exactly the same manner as we did for the magnetic field in sections 6.3 and 6.4. The result is,

$$E^2_\lambda(\eta) = \frac{1}{a^4} \int \frac{dk}{k} W^2_\lambda(k) \frac{k^2}{2\pi^2} |A'_k|^2, \tag{6.25}$$

where $W_\lambda(k) = e^{-\lambda^2 k^2/2}$.

CHAPTER 7

Models

We are now at a point that we can evaluate different models using the method described in chapter 6. In this chapter we will review four different models. First we will look at Maxwell electrodynamics and see why this model cannot generate large scale magnetic fields. We will then review three other models, where we break the conformal invariance of the Lagrangian. In the second model the electromagnetic field is coupled to the gravitational field. The third model describes a coupling of the electromagnetic field to a scalar field, for example the inflaton field. In the last model the electromagnetic field is coupled to a pseudoscalar field, for example an axion. We will show that for certain specifics of these three models, strong enough magnetic fields can be generated.

7.1 Maxwell electrodynamics

In chapter 5 we saw that the Lagrangian for electromagnetism was given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{7.1}$$

From this we could derive the equation of motion,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}F_{\nu\beta}\right] = 0. \tag{7.2}$$

If we perform a conformal transformation, $g^{\mu\nu} \to \bar{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$, and, $g \to \bar{g} = \Omega^8 g$. The transformed equation of motion is,

$$\begin{aligned}
\frac{1}{\sqrt{-\bar{g}}} \partial_\mu \left[\sqrt{-\bar{g}} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} F_{\nu\beta} \right] &= 0, \\
\frac{1}{\Omega^4 \sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\nu\beta} \right] &= 0, \\
\frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\nu\beta} \right] &= 0.
\end{aligned} \tag{7.3}$$

The equation of motion is invariant under conformal transformations. We will show that this indeed leads to a vanishing magnetic field as discussed in chapter 4. In chapter 5 we calculated that in a flat FLRW spacetime the equations of motions are,

$$A_i'' - a^2 \partial_j \partial^j A_i = 0. \tag{7.4}$$

where the prime denoted the derivative with respect to η. We expanded the vector field as (5.36) and got an equation of motion for $A_k(\eta)$,

$$A_k''(\eta) + k^2 A_k(\eta) = 0, \tag{7.5}$$

which had the positive-frequency solution,

$$A_k(\eta) = c e^{-ik\eta}. \tag{7.6}$$

To obtain the value of the constant we must use condition (5.37), which gives the constraint,

$$|c|^2 = 1. \tag{7.7}$$

The next step to obtain the present value of the magnetic field is to obtain the Bogoliubov coefficients (6.19), that is we must solve,

$$e^{-ik\eta} = \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}. \tag{7.8}$$

Obviously $\beta_k = 0$. Since the present magnetic field was given by,

$$B_\lambda^0 = \left[\int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |\beta_k(\eta_{\rm rad})|^2 \right]^{1/2}, \tag{7.9}$$

it will be zero. This explicitly shows that a conformal invariant theory leads to vanishing magnetic fields. The only way to obtain non-vanishing fields is to break this conformal invariance. The rest of this chapter gives a review of different models that break conformal invariance by a coupling to the gravitational field, the inflaton field or an axion field. In the next chapter we will break the conformal invariance of the background spacetime and review how this influences the evolution of the magnetic field.

7.2 The R^nF^2 model, coupling to gravity

One way to break conformal invariance is to couple the electromagnetic field to the gravitational field through the coupling R^nF^2, where R is the Ricci scalar and n an integer. This was first suggested by Turner and Widrow [1] for the case $n = 1$. The general case was later investigated by Mazzitelli and Spedalieri [34]. We will follow the discussion given by Campanelli, Cea, Fogli and Tedesco [35].

The model

The Lagrangian of this model is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{c(n)}{4}R^n F_{\mu\nu}F^{\mu\nu}, \tag{7.10}$$

where c is a constant dependent on n. Dimensional analysis tells us that $c(n)R^n$ must be dimensionless. Since R has units m^2 we can define $c(n) \equiv m^{-2n}$. What this mass is, depends on the model that is used. For example Mazzitelli and Spedalieri use in their article [34] the electron mass, $m_e = 0.51$ MeV. We will show below that in that case the produced magnetic fields are smaller than the dynamo limit, therefore this is not a good model. To see if the R^nF^2 model can generate strong enough magnetic fields at all, we will also review the model for the largest possible value of m. This value can be found as follows. We assume that during inflation the second term is dominant, which means that $R \gg m^2$. To know how R evolves we use the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{m_{\mathrm{pl}}^2}T_{\mu\nu}. \tag{7.11}$$

If we contract this with $g^{\mu\nu}$ and assume that the universe is filled with a perfect fluid this can be rewritten as,

$$\begin{aligned} -R &= \frac{8\pi}{m_{\mathrm{pl}}^2}\left[(1+\gamma)\rho u^\mu u_\mu + 4\gamma\rho\right], \\ R &= \frac{8\pi}{m_{\mathrm{pl}}^2}(1-3\gamma)\rho. \end{aligned} \tag{7.12}$$

Since the density decreases with time, R will also decrease. If we want the coupling term to be dominant during the entire inflation period we must impose that $R_{\mathrm{end}} \geq m^2$. Therefore the maximum value of $m = R_{\mathrm{end}}^{1/2}$.

Equation of motion

To obtain the equations of motion we must vary the second term in the action with respect to A_μ. The result is,

$$a^{-4}\partial_\mu \left[a^4 \frac{R^n}{m^{2n}} g^{\mu\nu} g^{\alpha\beta} F_{\nu\beta} \right] = 0. \tag{7.13}$$

In the Coulomb gauge this becomes,

$$A_i'' + \frac{(R^n)'}{R^n} A_i' - a^2 \partial_j \partial^j A_i = 0, \tag{7.14}$$

where the prime denotes the derivative with respect to the conformal time. We can use expansion (5.36) to rewrite this in terms of $A_k(\eta)$. To find the normal commutation relations (5.19) we must replace condition (5.37) with

$$A_k(\eta) A_k^{*\prime}(\eta) - A_k'(\eta) A_k^*(\eta) = 2ik \frac{m^{2n}}{R^n}. \tag{7.15}$$

The equation of motion for $A_k(\eta)$ is,

$$A_k''(\eta) + \frac{(R^n(\eta))'}{R^n(\eta)} A_k'(\eta) + k^2 A_k(\eta) = 0. \tag{7.16}$$

To find the solution we must know the explicit η dependence of R^n. If we use expression (7.12) we find that,

$$\frac{(R^n(\eta))'}{R^n(\eta)} = \frac{(\rho^n(\eta))'}{\rho^n(\eta)} = n \frac{\rho'(\eta)}{\rho(\eta)}. \tag{7.17}$$

One can derive that $\rho \propto \eta^s$, where $s = -6(1+\gamma)(1+3\gamma)$, so finally,

$$\frac{(R^n(\eta))'}{R^n(\eta)} = \frac{ns}{\eta}, \tag{7.18}$$

and the equation of motion is,

$$A_k''(\eta) + \frac{ns}{\eta} A_k'(\eta) + k^2 A_k(\eta) = 0. \tag{7.19}$$

Solution

The difficulty in solving the equation of motion lies in the fact that the coefficient of the second term depends on η. The rest of the equation is similar to the Bessel equation,

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0. \tag{7.20}$$

This equation is solved by Bessel functions $J_\nu(x)$ and linear combinations of them. We will make the assumption that the solution of (7.19) is of the form,

$$A_k = c\eta^x H_\nu^{(1)}(-k\eta), \tag{7.21}$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind, which is a linear combination of Bessel functions. We choose the first Hankel function since for large arguments [30],

$$H_\nu^{(1)}(-k\eta) \to C(\eta)e^{-ik\eta}, \tag{7.22}$$

which resembles the positive-frequency modes in Maxwell electromagnetism. If we insert our assumption into the equation of motion and multiply it with η^{-x+2} we get,

$$\begin{aligned}(-k\eta)^2\frac{\partial^2}{\partial(-k\eta)^2}H_\nu^{(1)}(-k\eta) + (2x+ns)(-k\eta)\frac{\partial}{\partial(-k\eta)}H_\nu^{(1)}(-k\eta)\\ + \left[x(x-1) + nsx + (-k\eta)^2\right]H_\nu^{(1)}(-k\eta) = 0.\end{aligned} \tag{7.23}$$

Comparison with the Bessel equation tells us that $x = (1-ns)/2$. This gives,

$$\begin{aligned}(-k\eta)^2\frac{\partial^2}{\partial(-k\eta)^2}H_\nu^{(1)}(-k\eta) + (-k\eta)\frac{\partial}{\partial(-k\eta)}H_\nu^{(1)}(-k\eta)\\ + \left[-\frac{1}{4} + \frac{ns}{2} - \frac{(ns)^2}{4} + (-k\eta^2)\right]H_\nu^{(1)}(-k\eta) = 0.\end{aligned} \tag{7.24}$$

This is indeed the Bessel equation with

$$\nu = \frac{ns-1}{2}. \tag{7.25}$$

The solution to the equation of motion is therefore,

$$A_k = c\eta^{(1-ns)/2}H_\nu^{(1)}(-k\eta), \tag{7.26}$$

where ν is given above. The constant c can be found using condition (7.15). To be able to calculate the left hand side we need some properties of Hankel functions. The complex conjugate of the first Hankel function is the second Hankel function. The derivative of both is,

$$\begin{aligned}H_\nu^{'}(-k\eta) &= -k\frac{\partial}{\partial(-k\eta)}H_\nu(-k\eta)\\ &= -\frac{k}{2}\left[H_{\nu-1}(-k\eta) - H_{\nu+1}(-k\eta)\right].\end{aligned} \tag{7.27}$$

We can now write condition (7.15) as,

$$\begin{aligned}\frac{|c|^2}{2}k\eta^{(1-ns)}\big[H_\nu^{(1)}(-k\eta)H_{\nu-1}^{(2)}(-k\eta) - H_\nu^{(1)}(-k\eta)H_{\nu+1}^{(2)}(-k\eta)\\ - H_\nu^{(2)}(-k\eta)H_{\nu-1}^{(1)}(-k\eta) + H_\nu^{(2)}(-k\eta)H_{\nu+1}^{(1)}(-k\eta)\big] = 2ki\frac{m^{2n}}{R^n}.\end{aligned} \tag{7.28}$$

Notice that the terms with derivatives of $\eta^{(1-ns)/2}$ drop out. We can combine the first and third term and the second and fourth term and rewrite them using Wronskian identities [30], such that,

$$\frac{4i|c|^2}{\pi\eta^{ns}} = 2ki\frac{m^{2n}}{R^n}. \tag{7.29}$$

Finally the constant becomes,

$$c = \sqrt{\frac{\pi k}{2}}\frac{m^n}{R^{n/2}}\eta^{ns/2}, \tag{7.30}$$

up to a phase, which we will take to be zero. We can make this assumption without lose of generality since we will be interested in the absolute value of A_k. The solution to the equation of motion is finally given by,

$$A_k = \sqrt{\frac{\pi}{2}}\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2}H_\nu^{(1)}(-k\eta). \tag{7.31}$$

Matching after inlation

At the end of inflation the wavelength of the magnetic field is greater then the Hubble radius $a\lambda \gg H^{-1}$. Using $k = 1/\lambda$ and $a \propto \eta^{2/(1+3\gamma)}$ this is equal to $|k\eta| \ll (a'/a)\eta = 2/|(1+3\gamma)|$. We can therefore expand the Hankel function for small arguments. For example for $\nu > 0$ [30],

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \tag{7.32}$$

$$\rightarrow \frac{1}{\Gamma(\nu+1)}2^{-\nu}x^\nu - i\frac{\Gamma(\nu)}{\pi}2^\nu x^{-\nu}, \tag{7.33}$$

$$\simeq -i\frac{\Gamma(\nu)}{\pi}2^\nu x^{-\nu}, \tag{7.34}$$

where J_ν is the Bessel function of the first kind and Y_ν is the Bessel function of the second kind and $\Gamma(\nu)$ the Gamma function. In the same manner one can find that the small argument expansion for $\nu = 0$ and $\nu < 0$ are respectively given by $H_\nu^{(1)}(x) \rightarrow 2i\pi^{-1}\ln x$ and $H_\nu^{(1)}(x) \rightarrow i\pi^{-1}2^{-\nu}\Gamma(-\nu)e^{-i\pi\nu}x^\nu$.

First we will evaluate the case $\nu > 0$. The small argument expansion of A_k is given by,

$$A_k(\eta) = -i\sqrt{\frac{1}{2\pi}}2^\nu\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2-\nu}. \tag{7.35}$$

We want to calculate the Bogoliubov coefficients as we explained in chapter 6. This means we want to solve,

$$-i\sqrt{\frac{1}{2\pi}}2^\nu\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2-\nu} = \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}, \tag{7.36}$$

and the first derivative of this equation,

$$ik\left(\frac{1}{2}-\nu\right)\sqrt{\frac{1}{2\pi}}2^{\nu}\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{-1/2-\nu}$$
$$+\frac{in}{2}\sqrt{\frac{1}{2\pi}}2^{\nu}\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2-1}R'(-k\eta)^{-1/2-\nu}=-ik\alpha_k e^{-ik\eta}+ik\beta_k e^{ik\eta}. \tag{7.37}$$

This can be done by rewriting the first equation into an expression for $\alpha_k e^{-ik\eta}$ and insert it into the second equation. The result is an expression for β_k,

$$\beta_k=-\frac{1}{2}\sqrt{\frac{1}{2\pi}}2^{\nu}\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2-\nu}\left[-\left(\frac{1}{2}-\nu\right)(-k\eta)^{-1}-\frac{n}{2k}\frac{R'}{R}+i\right]e^{-ik\eta},$$
$$=-\frac{1}{2}\sqrt{\frac{1}{2\pi}}2^{\nu}\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2-\nu}\left[-2\nu(-k\eta)^{-1}+i\right]e^{-ik\eta}. \tag{7.38}$$

Since the argument $(-k\eta)$ is small, we only keep the first term. Then,

$$|\beta_k|^2=\frac{2^{2\nu}\Gamma^2(\nu+1)}{2\pi}\left(\frac{R}{m^2}\right)^{-n}|k\eta|^{-1-2\nu}. \tag{7.39}$$

We used the fact that $\nu\Gamma(\nu)=\Gamma(\nu+1)$.

When $\nu=0$ the small argument expansion of A_k is given by,

$$A_k=i\sqrt{\frac{2}{\pi}}\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2}\ln(-k\eta). \tag{7.40}$$

The Bogoliubov coefficients are calculated in the same way as for the previous case. Keeping only the leading order term the result is,

$$|\beta_k|^2=\frac{1}{2\pi}\left(\frac{R}{m^2}\right)^{-n}|k\eta|^{-1}. \tag{7.41}$$

Comparison with the previous case shows that this is the same as expression (7.39) for $\nu=0$.

The last case is $\nu<0$. The A_k small argument expansion is given by,

$$A_k(\eta)=i\sqrt{\frac{1}{2\pi}}e^{i\pi\nu}2^{-\nu}\Gamma(-\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2+\nu}. \tag{7.42}$$

We will neglect the phase $e^{i\pi\nu}$ since $|\beta_k|^2$ will not depend on it. Again the calculation of the Bogoliubov coefficients is done in the exact same way. The result is,

$$|\beta_k|^2=\frac{2^{-2\nu}\Gamma^2(-\nu)}{4\pi}\left(\frac{R}{m^2}\right)^{-n}|k\eta|^{1+2\nu}. \tag{7.43}$$

Since we were looking at small arguments this will lead to vanishing magnetic fields. That is why in the rest of this section we will restrict ourselves to the case $\nu\geq 0$ or $ns\geq 1$. During inflation s is always positive since $-1<\gamma<-1/3$, so n must be positive as well.

The present magnetic field

To calculate the value of the magnetic field at present we must use expression (6.22) as explained in chapter 6.

$$\begin{aligned} B_\lambda^0 &= \left[\int \frac{dk}{k} W_\lambda^2(k) \frac{k^4}{2\pi^2} |\beta_k(\eta_{\rm rad})|^2\right]^{1/2}, \\ &= \frac{2^\nu \Gamma(\nu+1)}{2\pi} \left(\frac{R_{\rm rad}}{m^2}\right)^{-n/2} |\eta_{\rm rad}|^{-1/2-\nu} \left[\frac{1}{\pi}\int_0^\infty dk e^{-\lambda^2 k^2} k^{2-2\nu}\right]^{1/2}, \\ &= \frac{2^\nu}{(2\pi)^{3/2}} \Gamma(\nu+1)[\Gamma(3/2-\nu)]^{1/2} \left(\frac{R_{\rm rad}}{m^2}\right)^{-n/2} |\eta_{\rm rad}|^{-1/2-\nu} \lambda^{\nu-3/2}. \end{aligned} \tag{7.44}$$

The unknowns in this expression are $n, \gamma, R_{\rm rad}, m$ and $\eta_{\rm rad}$. The values of n and γ are contained in ν. We had already restricted ν to $\nu \geq 0$. We can further restrict it by noticing that for $\nu \geq 3/2$ the integral in expression (7.44) has an infrared divergence. To avoid this we impose $\nu < 3/2$. The value of $R_{\rm rad}$ can be found with the help of expression (7.12). We can rewrite this in terms of the temperature at the end of inflation, since for relativistic particles $\rho \sim T^4$, as explained in section 2.1.3. The result is,

$$R_{\rm rad} = \frac{8\pi}{m_{\rm pl}^2}(1-3\gamma)M^4, \tag{7.45}$$

where M is the temperature at the end of inflation. The value of m is unknown except for the limit, $R_{\rm rad} \simeq R_{\rm end} \geq m^2$. In their article Mazzitelli and Spedalieri [34] have $m = m_e$, where, $m_e = 0.51$MeV, is the electron mass. Campanelli, Cea, Fogli and Tedesco [35] argued that this value of m will lead to too small fields and instead use $m = R_{\rm rad}^{1/2}$, the largest possible value of m. We will evaluate expression (7.44) for both of these values of m. The value of $\eta_{\rm rad}$ is a bit more involved. In section 2.1.3 we found a relation between the scale factor and the temperature. To find an expression for $\eta_{\rm rad}$ in temperature, we may calculate,

$$\begin{aligned} \eta &= \frac{2}{(1+3\gamma)} \frac{a}{a'}, \\ &= \frac{2}{(1+3\gamma) a H}. \end{aligned} \tag{7.46}$$

We can rewrite this using the relation,

$$a \propto g_{*S}^{-1/3}(T) T^{-1}, \tag{7.47}$$

which we found in section 2.1.3 and the fact that we can rewrite the Hubble parameter as,

$$
\begin{aligned}
H &= \sqrt{\frac{8\pi}{3}} \frac{\sqrt{\rho_{\text{tot}}}}{m_{\text{pl}}}, \\
&= \sqrt{\frac{8\pi}{3}} \frac{M^2}{m_{\text{pl}}}.
\end{aligned} \tag{7.48}
$$

The result is,

$$
\eta_{\text{rad}} = \sqrt{\frac{3}{2\pi}} \frac{1}{1+3\gamma} \left(\frac{g_{*S}(T_{\text{rad}})}{g_{*S}(T_0)} \right)^{1/3} \frac{m_{\text{pl}}}{MT_0}. \tag{7.49}
$$

The values of the unknown quantities are [5],

$$
T_0 = 2.35 \times 10^{-13}\text{GeV}, \tag{7.50}
$$

$$
g_{*S}(T_{\text{rad}}) = 106.75, \tag{7.51}
$$

$$
g_{*S}(T_0) = 3.91. \tag{7.52}
$$

We explained the values of g_{*S} in section 2.1.3. We can also calculate that in Gaussian units, that is when $\epsilon_0 = (4\pi)^{-1}$, where ϵ_0 is the vacuum permittivity,

$$
1G = 6.9 \times 10^{-20}\text{GeV}^2, \tag{7.53}
$$

$$
1\text{Mpc} = 1.56 \times 10^{38}\text{GeV}^{-1}. \tag{7.54}
$$

Finally,

$$
\eta_{\text{rad}} \simeq 2.3 \times 10^3 \frac{1}{1+3\gamma} \left(\frac{M}{m_{\text{pl}}} \right)^{-1} \sqrt{G}. \tag{7.55}
$$

The larger the temperature M is, the larger the magnetic fields will be. The exact value of M is not known, but graviton production leads to the constraint that $M/m_{\text{pl}} < 10^{-2}$ [1]. The value of M depends on the value of s, as was shown in equation (3.8). Since we assumed that $M = T_{\text{rad}}$, this equation reduces to,

$$
\frac{\rho_{\text{tot}}}{m_{\text{pl}}^4} = (1.6 \times 10^{26})^s \lambda_{\text{Mpc}}^s \left(\frac{M}{m_{\text{pl}}} \right)^{4+s}. \tag{7.56}
$$

If we use the above mentioned graviton constraint, which in terms of the density reads, $\rho_{\text{tot}} = 10^{-8} m_{\text{pl}}^4$, at the present horizon scale, we can find the maximum value of s,

$$
s_{\max} = -\frac{8 + 4\log_{10}(M/m_{\text{pl}})}{29.8 + \log_{10}(M/m_{\text{pl}})}. \tag{7.57}
$$

When we evaluate the strength of the magnetic field for different values of M, we must keep in mind that this also changes the maximum value of s, which in turn changes the value of ν. Since $0 \leq \nu < 3/2$, the possible values of M are restricted. We know that $\lambda/|\eta_{\rm rad}| \gg 1$, and therefore the magnetic field gets larger for a larger ν. As a consequence the field is maximal for $s = s_{\rm max}$. Campanelli, Cea, Fogli and Tedesco [33] evaluated the strength of the magnetic field in the case, $s = s_{\rm max}$, for $m = m_e$ and $m = R_{\rm rad}^{1/2}$ at a scale of 10 kpc. The result is shown in figure 7.1. The values of M are restricted for different n, such that $0 < \nu < 1.49$. The upper panel corresponds to the case $m = m_e$. One can see that the present magnetic field lies below the dynamo limit for all values of n and M. Therefore we can conclude that for this value of m the $R^n F^2$ model is not a suitable model, as we had already anticipated. The lower panel corresponds to the case that $m = R_{\rm rad}^{1/2}$, which was the largest possible value for m. The figure shows that the magnetic field is stronger than the dynamo mechanism limit when $n \geq 2$, for a certain lower limit on M, which differs for different n. When the limit for intergalactic magnetic fields is satisfied is not clear from this figure. As one can see for each power of n the strongest magnetic fields is generated for the lowest value of M, which corresponds to $\nu = 1.49$. We have plotted the strength of the magnetic field for this value of ν for different powers of n in figure 7.2. One can see that the intergalactic magnetic field limit is satisfied for $n \geq 6$. This corresponds to a temperature of $M \geq 10^{-6} m_{\rm pl}$. Therefore the $R^n F^2$ model is a possible model to explain the observed magnetic fields, when the power of the Ricci scalar is $n \geq 6$ and the temperature during inflation $M \geq 10^{-6} m_{\rm pl}$.

Backreaction

To check that our model is consistent we must check that at the wavelengths we are interested in, the energy density of the electromagnetic field is smaller than the total energy density, as explained in section 6.6. The energy momentum tensor $T_{\mu\nu}$ is defined as,

$$\delta S = -\int d^4x \sqrt{-g} \frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu}. \tag{7.58}$$

In the current model the left hand side is equal to,

$$\begin{aligned} \delta S = &\int d^4x \sqrt{-g} \frac{1}{8} \left(\frac{R}{m^2}\right)^n g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \delta g^{\mu\nu} - \int d^4x \sqrt{-g} \frac{1}{2} \left(\frac{R}{m^2}\right)^n F_{\mu\beta} F_\nu^\beta \delta g^{\mu\nu} \\ &- \int d^4x \sqrt{-g} \frac{n}{4} \left(\frac{R}{m^2}\right)^n F_{\alpha\beta} F^{\alpha\beta} \frac{R_{\mu\nu}}{R} \delta g^{\mu\nu} - \int d^4x \frac{n}{4} \sqrt{-g} \frac{R^{n-1}}{m^{2n}} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu} \delta R_{\mu\nu}. \end{aligned} \tag{7.59}$$

We will first evaluate the last term on the right hand side. We can use the identities,

$$\delta R_{\mu\nu} = \nabla_\rho (\delta \Gamma^\rho_{\mu\nu}) - \nabla_\nu (\delta \Gamma^\rho_{\rho\mu}), \tag{7.60}$$

$$\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left[\nabla_\nu \delta g_{\sigma\mu} + \nabla_\mu \delta g_{\nu\sigma} - \nabla_\sigma \delta g_{\mu\nu}\right], \tag{7.61}$$

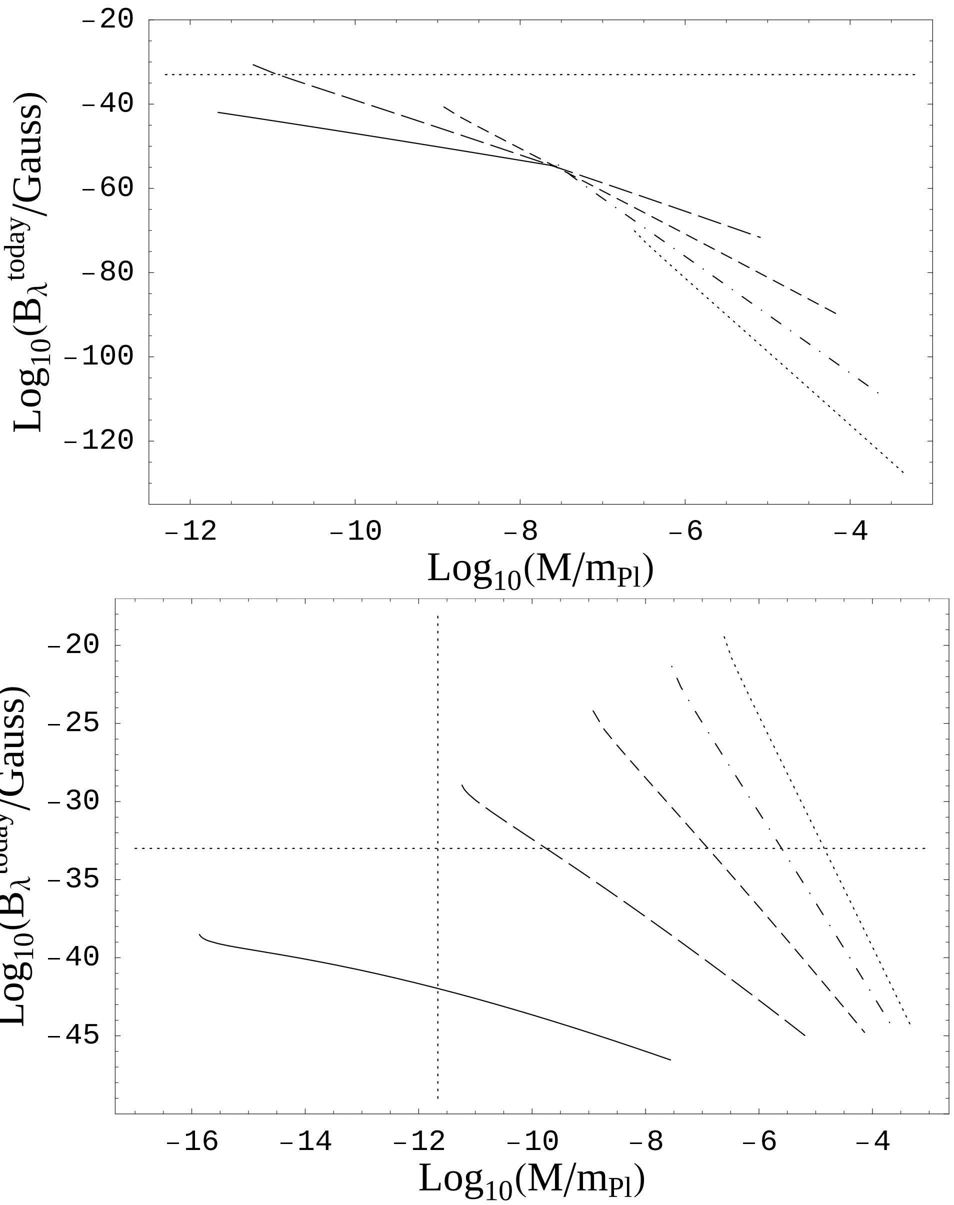

Figure 7.1 – The present magnetic field as a function of M at the scale $\lambda = 10$ kpc. The upper plot is the case $m = m_e$ and in the lower plot $m = R_{\rm rad}^{1/2}$. The different lines are the different values of n: $n = 1$ (continuous line), $n = 2$ (long dashed line), $n = 3$ (dashed line), $n = 4$ (dot-dashed line) and $n = 5$ (dotted line). The horizontal dotted line is the dynamo limit $B^0 > 10^{-33}$ G at scales $\lambda = 10$ kpc. The plots are from [35].

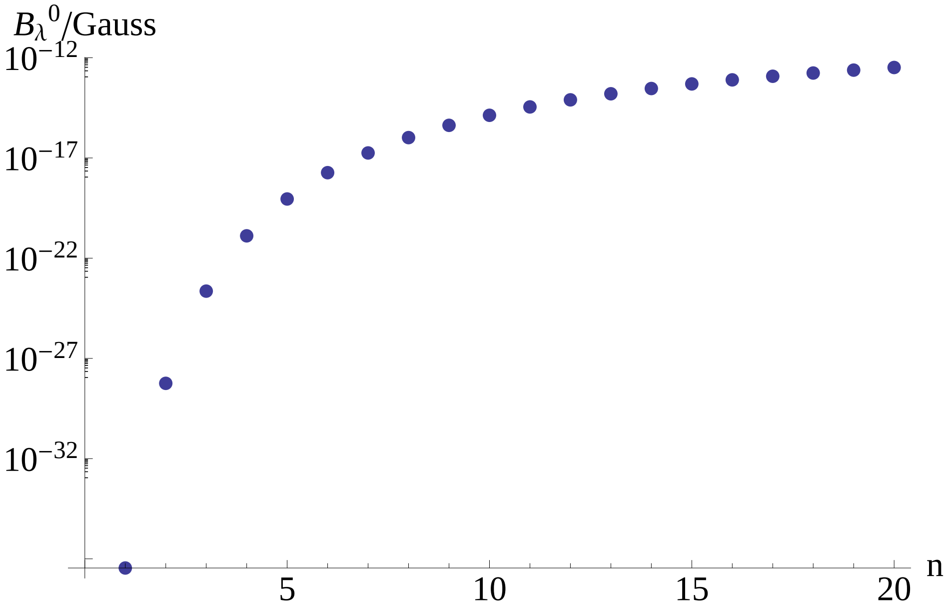

Figure 7.2 – The present magnetic field as a function of n at the scale $\lambda = 1$ Mpc.

and partial integration, to calculate that,

$$\begin{aligned}
-\int d^4x \frac{n}{4}\sqrt{-g}\frac{R^{n-1}}{m^{2n}} F_{\alpha\beta}F^{\alpha\beta}g^{\mu\nu}\delta R_{\mu\nu} \\
&= -\int d^4x \frac{1}{2}\delta g^{\mu\nu}\left[g_{\mu\nu}\nabla^2 - \nabla_\nu\nabla_\mu\right]\left[\frac{n}{2}\sqrt{-g}\frac{R^{n-1}}{m^{2n}}F_{\alpha\beta}F^{\alpha\beta}\right], \\
&= -\int d^4x \frac{1}{2}\delta g^{\mu\nu}\left[g_{\mu\nu}\partial^\rho\partial_\rho - g_{\mu\nu}g^{\rho\sigma}\Gamma^\lambda_{\sigma\rho}\partial_\lambda - \partial_\nu\partial_\mu + \Gamma^\lambda_{\mu\nu}\partial_\lambda\right] \\
&\qquad \times \left[\frac{n}{2}\sqrt{-g}\frac{R^{n-1}}{m^{2n}}F_{\alpha\beta}F^{\alpha\beta}\right].
\end{aligned} \tag{7.62}$$

We therefore find that the energy momentum tensor is given by,

$$\begin{aligned}
T_{\mu\nu} = &-\frac{1}{4}\left(\frac{R}{m^2}\right)^n g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \left(\frac{R}{m^2}\right)^n F_{\mu\beta}F^\beta_\nu + \frac{n}{2}\left(\frac{R}{m^2}\right)^n F_{\alpha\beta}F^{\alpha\beta}\frac{R_{\mu\nu}}{R} \\
&+ \frac{1}{\sqrt{-g}}\left[g_{\mu\nu}\partial^\rho\partial_\rho - g_{\mu\nu}g^{\rho\sigma}\Gamma^\lambda_{\sigma\rho}\partial_\lambda - \partial_\nu\partial_\mu + \Gamma^\lambda_{\mu\nu}\partial_\lambda\right]\left[\frac{n}{2}\sqrt{-g}\frac{R^{n-1}}{m^{2n}}F_{\alpha\beta}F^{\alpha\beta}\right].
\end{aligned} \tag{7.63}$$

The energy density of the electromagnetic field in the Coulomb gauge is then,

$$
\begin{aligned}
\rho &= -T^0_0, \\
&= \frac{1}{2}\left(\frac{R}{m^2}\right)^n (B^2 - E^2) + \left(\frac{R}{m^2}\right)^n E^2 - n\left(\frac{R}{m^2}\right)^n (B^2 - E^2)\left(1 - \frac{a'^2}{a''a}\right) \\
&\quad - a^{-4}\left[\partial^j\partial_j - g^{\rho\sigma}\Gamma^\lambda_{\sigma\rho}\partial_\lambda - a^{-2}\Gamma^\lambda_{00}\partial_\lambda\right]\left[na^4\frac{R^{n-1}}{m^{2n}}(B^2 - E^2)\right], \\
&= \frac{1}{2}\left(\frac{R}{m^2}\right)^n (B^2 + E^2) - n\left(\frac{R}{m^2}\right)^n (B^2 - E^2)\left(1 - \frac{a'^2}{a''a}\right) \\
&\quad + \frac{n}{2}\frac{a'}{a^7}\partial_\eta\left[\frac{a^7}{a''}\left(\frac{R}{m^2}\right)^n (B^2 - E^2)\right], \\
&= \frac{1}{2}\left(\frac{R}{m^2}\right)^n (B^2 + E^2) + \left[-n + \frac{n}{2}(9 - 3n)\frac{a'^2}{a''a} + \frac{n}{2}(n-1)\frac{a'a'''}{a''^2}\right]\left(\frac{R}{m^2}\right)^n (B^2 - E^2) \\
&\quad + \frac{n}{2}\frac{a'}{a''}\left(\frac{R}{m^2}\right)^n \partial_\eta (B^2 - E^2). \qquad (7.64)
\end{aligned}
$$

We can get rid of the scale factor derivatives by using the fact that, $a \propto \eta^{2/(1+3\gamma)}$. We then find,

$$
\begin{aligned}
\rho &= \frac{1}{2}\left(\frac{R}{m^2}\right)^n (B^2 + E^2) + n\frac{5+3\gamma}{1-3\gamma}\left(\frac{R}{m^2}\right)^n (B^2 - E^2) \\
&\quad + \frac{n}{2}\frac{1+3\gamma}{1-3\gamma}\eta\left(\frac{R}{m^2}\right)^n \partial_\eta (B^2 - E^2). \qquad (7.65)
\end{aligned}
$$

Therefore the vacuum expectation value of the co-moving average electromagnetic density on a scale λ is,

$$
\begin{aligned}
\rho_\lambda &= \left(\frac{R}{m^2}\right)^n \left[\left(\frac{1}{2} + n\frac{5+3\gamma}{1-3\gamma}\right)B^2_\lambda + \left(\frac{1}{2} - n\frac{5+3\gamma}{1-3\gamma}\right)E^2_\lambda\right] \\
&\quad + \frac{n}{2}\frac{1+3\gamma}{1-3\gamma}\eta\left(\frac{R}{m^2}\right)^n \partial_\eta (B^2_\lambda - E^2_\lambda). \qquad (7.66)
\end{aligned}
$$

We already found the expression for A_k during inflation:

$$
A_k(\eta) = -i\sqrt{\frac{1}{2\pi}}2^\nu\Gamma(\nu)\left(\frac{R}{m^2}\right)^{-n/2}(-k\eta)^{1/2-\nu}, \qquad (7.67)
$$

which was the small argument expansion for $\nu > 0$. The vacuum expectation values of the average co-moving electric and magnetic fields during inflation can be calculated by filling in (6.12) and (6.25):

$$
B^2_\lambda = \frac{4^\nu[\Gamma(\nu)]^2\Gamma(5/2-\nu)}{a^4 8\pi^3}\left(\frac{R}{m^2}\right)^{-n}\lambda^{-5+2\nu}\eta^{1-2\nu}, \qquad (7.68)
$$

$$
E^2_\lambda = \frac{4^\nu[\Gamma(\nu)]^2\Gamma(3/2-\nu)(1/2-\nu)^2}{a^4 8\pi^3}\left(\frac{R}{m^2}\right)^{-n}\lambda^{-3+2\nu}\eta^{-1-2\nu}. \qquad (7.69)
$$

It follows that,

$$\begin{aligned}
\rho_\lambda \simeq\ & \frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}\left(\frac{1}{2}+n\frac{5+3\gamma}{1-3\gamma}\right)\Gamma\left(\frac{5}{2}-\nu\right)\lambda^{-5+2\nu}\eta^{1-2\nu} \\
&+\frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}\left(\frac{1}{2}-n\frac{5+3\gamma}{1-3\gamma}\right)\left(\frac{1}{2}-\nu\right)^2\Gamma\left(\frac{3}{2}-\nu\right)\lambda^{-3+2\nu}\eta^{-1-2\nu} \\
&+\frac{n}{2}\frac{1+3\gamma}{1-3\gamma}\eta\frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}\left[-4\frac{a'}{a}-n\frac{R'}{R}\right] \\
&\times\left[\Gamma\left(\frac{5}{2}-\nu\right)\lambda^{-5+2\nu}\eta^{1-2\nu}-\left(\frac{1}{2}-\nu\right)^2\Gamma\left(\frac{3}{2}-\nu\right)\lambda^{-3+2\nu}\eta^{-1-2\nu}\right] \\
&+\frac{n}{2}\frac{1+3\gamma}{1-3\gamma}\frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}(1-2\nu)\Gamma\left(\frac{5}{2}-\nu\right)\lambda^{-5+2\nu}\eta^{1-2\nu} \\
&-\frac{n}{2}\frac{1+3\gamma}{1-3\gamma}\frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}(-1-2\nu)\left(\frac{1}{2}-\nu\right)^2\Gamma\left(\frac{3}{2}-\nu\right)\lambda^{-3+2\nu}\eta^{-1-2\nu}.
\end{aligned} \tag{7.70}$$

We can get rid of the R' and a' terms in the same way as before. Therefore we can simplify the above expression to,

$$\begin{aligned}
\rho_\lambda \simeq\ & \frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}\left[\frac{1}{2}+n\frac{1+3\gamma}{1-3\gamma}+3n^2\frac{1-\gamma}{1-3\gamma}+\frac{n}{2}\frac{1+3\gamma}{1-3\gamma}(1-2\nu)\right] \\
&\qquad\times\Gamma\left(\frac{5}{2}-\nu\right)\lambda^{-5+2\nu}\eta^{1-2\nu} \\
&+\frac{4^\nu[\Gamma(\nu)]^2}{a^4 8\pi^3}\left[\frac{1}{2}-n\frac{1+3\gamma}{1-3\gamma}-3n^2\frac{1-\gamma}{1-3\gamma}+\frac{n}{2}\frac{1+3\gamma}{1-3\gamma}(1+2\nu)\right] \\
&\qquad\times\left(\frac{1}{2}-\nu\right)^2\Gamma\left(\frac{3}{2}-\nu\right)\lambda^{-3+2\nu}\eta^{-1-2\nu}.
\end{aligned} \tag{7.71}$$

The terms in the square brackets are all of order 1, and therefore not important for our purposes. We will neglect them in the following. We can rewrite $a = 2[(1+3\gamma)\eta H]^{-1}$. Then,

$$\begin{aligned}
\frac{\rho_\lambda}{\rho_{\rm tot}} \simeq & \frac{4^\nu[\Gamma(\nu)]^2}{3\pi^2}\frac{(1+3\gamma)^4}{16}\left(\frac{H}{m_{\rm pl}}\right)^2\times \\
&\left[\Gamma\left(\frac{5}{2}-\nu\right)\left(\frac{\eta}{\lambda}\right)^{5-2\nu}+\Gamma\left(\frac{3}{2}-\nu\right)\left(\frac{1}{2}-\nu\right)^2\left(\frac{\eta}{\lambda}\right)^{3-2\nu}\right].
\end{aligned} \tag{7.72}$$

We argued before that $\nu < 3/2$, so the numerical factor in front will not be larger then $\sim 10^2$. From graviton production we have the constraint that $H/m_{\rm pl} < 10^{-4}$ [1] and since we are in the large scale limit $\eta/\lambda \ll 1$. If we use these requirements,

$$\frac{\rho_\lambda}{\rho_{\rm tot}} \ll 1. \tag{7.73}$$

This shows that we have correctly neglected the backreaction of the electromagnetic field.

7.3 The $I(\phi)F^2$ model, coupling to a scalar field

Another way to break conformal invariance is to couple the electromagnetic field to the inflaton field through the coupling $I(\phi)F^2$, where ϕ is the inflaton field. This was first done by Ratra [36][37] for the model $I(\phi) = e^{c\phi}$, were c is an arbitrary constant. We will show later that this is the general form of the coupling when considering Power-Law inflation. He found that for certain values of c a magnetic field could be generated with a strength of 10^{-9} G, which is well above both the dynamo limit and the protogalaxy collapsing limit. The general case was later investigated by Martin and Yokoyama [38] and others. We will evaluate this general case, following the paper of Campanelli, Cea, Fogli and Tedesco [35].

The model

The electromagnetic part of the Lagrangian is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}I(\phi)F_{\mu\nu}F^{\mu\nu}. \tag{7.74}$$

The term $I(\phi)$ must be dimensionless. We assume that during inflation the second term is dominant and that at the end of inflation the first term will become dominant. That is during inflation $I(\phi) \geq 1$.

Equation of motion

The model is similar to the R^nF^2 model. Instead of recalculating the equation of motion we can replace $(R/m^2)^n$ with $I(\phi)$ in the later model. Comparison with expression (7.16) shows that the equation of motion for A_k is given by,

$$A_k''(\eta) + \frac{I(\phi)'}{I(\phi)}A_k'(\eta) + k^2A_k(\eta) = 0. \tag{7.75}$$

Condition (7.15) becomes,

$$A_k(\eta)A_k^{*'}(\eta) - A_k'(\eta)A_k^*(\eta) = \frac{2ik}{I(\phi)}. \tag{7.76}$$

To find the solution we must know the explicit η dependence of $I(\phi)$. Martin and Yokoyama [38] argue that it is reasonable to assume that $I(\eta) \propto a^\alpha$. The reason is that the magnetic fields that we measure on a galactic scale and cluster scale are almost of the same strength. This would suggest that a correct model should have an almost scale invariant power spectrum for the magnetic field. In their paper Martin and Yokoyama show that for a certain value $\alpha = \alpha_{\rm si}$ the power spectrum is scale invariant. That means that $I(\phi) \propto a^{\alpha_{\rm si}}$ is a reasonable model and the assumption $I(\phi) \propto a^{\alpha}$ is the generalization of this. Since we know that,

$$a(\eta) \propto \eta^{2/(1+3\gamma)}, \tag{7.77}$$

the η dependence of $I(\phi)$ is given by,

$$I(\phi) \propto \eta^{2\alpha/(1+3\gamma)} \equiv \eta^{b}. \tag{7.78}$$

We can use this to write the equation of motion as,

$$A_k''(\eta) + \frac{b}{\eta} A_k'(\eta) + k^2 A_k(\eta) = 0. \tag{7.79}$$

This is again similar to the $R^n F^2$ model and we can use those results if we replace ns by b.

Example of a particle physics model

The would like to know which particle physics models have a coupling of the form we suggested, $I(\phi) \propto \eta^{b}$. We will show that the coupling is compatible with Power-Law inflation. This is the simplest case, since an exact form of the potential is known,

$$V(\phi) \propto \exp\left[-\sqrt{2p}\phi\right]. \tag{7.80}$$

If we use that $a \propto \eta^{b/\alpha}$, we can solve the Friedmann equation (2.85). The result is,

$$\phi(\eta) = \frac{1}{a}\sqrt{2(\alpha b + b^2)} \ln|\eta|. \tag{7.81}$$

Therefore, if $I(\phi) \propto \eta^{b}$, we must have,

$$I(\phi) \propto \exp\left[\alpha\sqrt{\frac{b}{2(\alpha+b)}}\phi\right]. \tag{7.82}$$

We can express α in terms of b and rewrite it as,

$$I(\phi) \propto \exp\left[\frac{(1+3\gamma)}{2\sqrt{3(1+\gamma)}}\, b\phi\right]. \tag{7.83}$$

This is is equivalent to the coupling proposed by Ratra [36][37].

Solution

The solution to the equation of motion is equivalent to (7.26), namely,

$$A_k = c\eta^{(1-b)/2} H_\nu^{(1)}(-k\eta), \tag{7.84}$$

where $\nu = (b-1)/2$. The constant can be determined with condition (7.76) using the Hankel function identities just as in the previous section. The final result is,

$$A_k = \sqrt{\frac{\pi}{2}} I^{-1/2}(\phi)(-k\eta)^{1/2} H_\nu^{(1)}(-k\eta). \tag{7.85}$$

Calculating the Bogoliubov coefficients is easy, since we can again use the results from the previous section. Only $\nu \geq 0$ will give significant magnetic fields, namely,

$$B_\lambda^0 = \frac{2^\nu}{(2\pi)^{3/2}} \Gamma(\nu+1)[\Gamma(3/2-\nu)]^{1/2} I_{\rm rad}^{-1/2}(\phi)|\eta_{\rm rad}|^{-1/2-\nu}\lambda^{\nu-3/2}. \tag{7.86}$$

The unknowns in expression (7.86) are $b, \gamma, I_{\rm rad}(\phi)$ and $\eta_{\rm rad}$. To avoid infrared divergence we have the condition that $\nu < 3/2$. Together with the condition $\nu \geq 0$ we can put a limit on b, namely $1 \leq b < 4$. As before $\eta_{\rm rad}$ is given by expression (7.55). The value of $I_{\rm rad}$ is limited by $I_{\rm rad} \geq 1$ as we saw before. We will take the upper limit for both $I_{\rm rad} = 1$ and $\eta_{\rm rad}$, since it will lead to an upper limit of the magnetic field. Taking the upper limit for $\eta_{\rm rad}$ corresponds to taking the maximum value for $M = 10^{-2} m_{\rm pl}$. In this case, $\eta_{\rm rad} \simeq 10^5$. We have plotted the result in figure 7.3. As one can see the dynamo limit is satisfied for $b > 2$, the intergalactic field limit for $b > 3.4$ and the protogalaxy collapsing limit for $b > 3.7$. For all values of b the strength of the magnetic field lies below the CMB limit and is therefore in agreement with CMB observations. We can therefore conclude that the $I(\phi)F^2$ model is a possible explanation for the observed magnetic fields if $3.4 < b < 4$. We can also look the relation between the strength of the magnetic field and the temperature during inflation. Since the strongest fields are obtained for $b = 3.99$ we have evaluated expression (7.86) for this value. The result can be seen in figure 7.4. It shows that the intergalactic field limit is satisfied for $M > 10^{-6} m_{\rm pl}$. Therefore the temperature during inflation must have been at least this high for the $I(\phi)F^2$ model to work.

In the derivation of this result we assumed that $I(\phi) \propto a^\alpha$, since Martin and Yokoyama [38] argued that in this case a scale invariant spectrum could be obtained, which is in agreement with observations. If we look at expression (7.86) we see that the present magnetic field is scale invariant if $\nu = 3/2$, that is when $\alpha = 2(1+3\gamma)$. However at this value the solution blows up due to the gamma function, which came from integrating the window function, and therefore the expression becomes meaningless. We could have avoided this problem if we would not have taken the average of the field on a scale λ, but

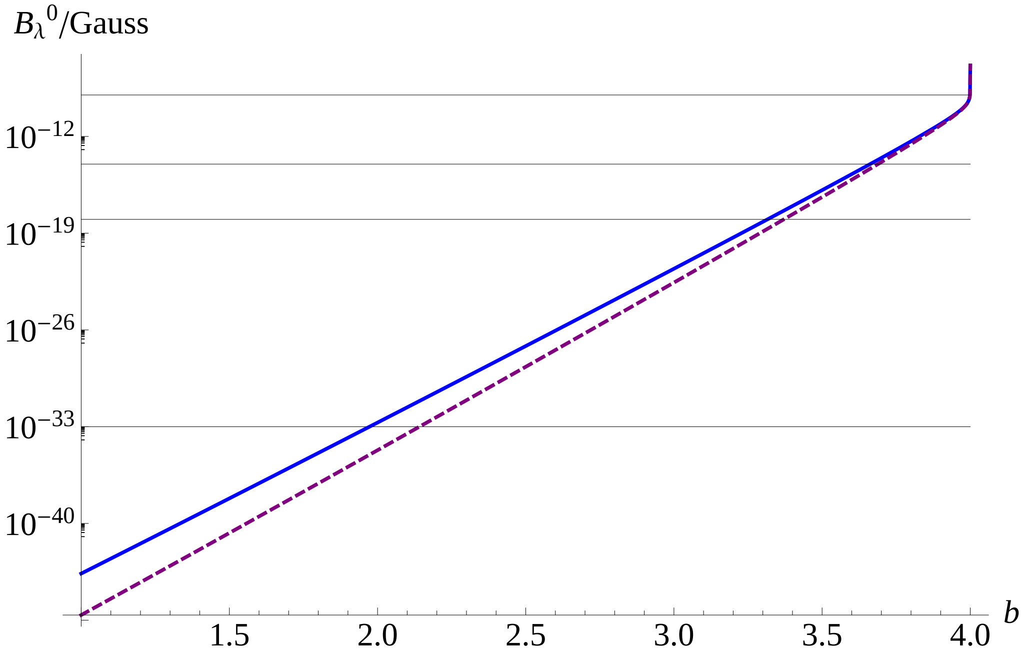

Figure 7.3 – The present magnetic field as a function of b. The solid blue line corresponds to the the scale $\lambda = 10$ kpc and the dashed purple line to $\lambda = 1$ Mpc. The thin black lines correspond to the limits from the CMB ($B < 10^{-9}$), protogalaxy collapsing ($B > 10^{-14}$ for $\lambda = 1$ Mpc), intergalactic magnetic fields ($B > 10^{-19}$ for $\lambda = 1$ Mpc) and the dynamo mechanism ($B > 10^{-33}$ for $\lambda = 10$ kpc).

just used the delta-function $\delta(k - 1/\lambda)$. This is what Martin and Yokoyama do in their paper [38] and therefore they are able to find a scale invariant spectrum. The downside of this approach is that one does no longer take into account the fact that observations measure the strength of the entire field and not the strength at specific points in space. The questions is if we can still justify the assumption $I(\phi) \propto a^\alpha$ in our framework. The fact that a coupling of this form is compatible with Power-Law inflation suggests that it is.

Backreaction

As a last step we may calculate the backreaction. We can again use the similarity with the $R^n F^2$ model. Since we assume that $I(\phi)$ does not depend on $g^{\mu\nu}$, the density is equivalent to the first term in expression (7.64) if we replace $(R/m^2)^n$ with $I(\phi)$,

$$\rho = \frac{1}{2} I(\phi) \left(B^2 + E^2\right) . \tag{7.87}$$

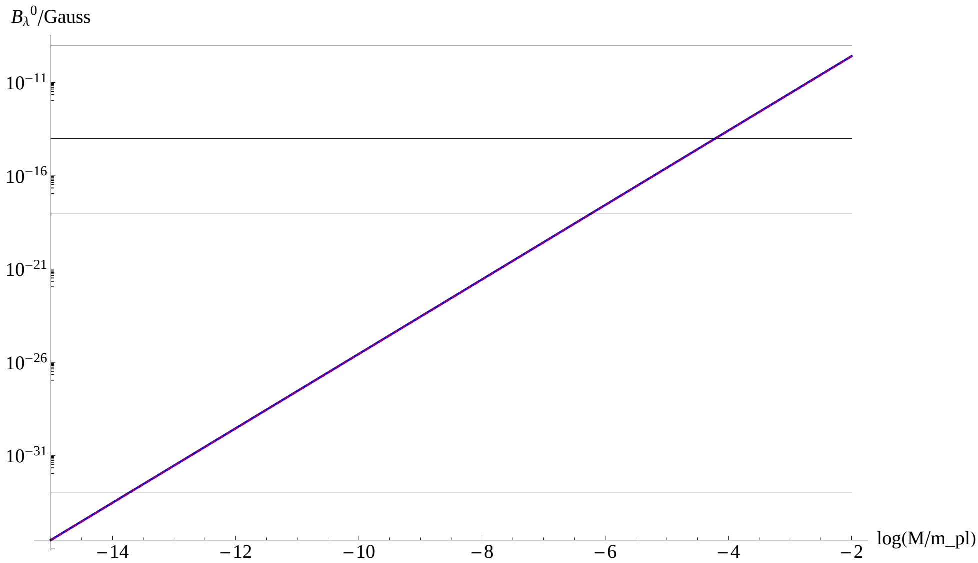

Figure 7.4 – The present magnetic field as a function of the temperature during inflation M. The solid blue line corresponds to the the scale $\lambda = 10$ kpc and the dashed purple line to $\lambda = 1$ Mpc. The thin black lines correspond to the limits from the CMB ($B < 10^{-9}$), protogalaxy collapsing ($B > 10^{-14}$ for $\lambda = 1$ Mpc), intergalactic magnetic fields ($B > 10^{-19}$ for $\lambda = 1$ Mpc) and the dynamo mechanism ($B > 10^{-33}$ for $\lambda = 10$ kpc).

We can calculate the ratio $\rho_\lambda/\rho_{\text{tot}}$ in the same manner as for the R^nF^2 model. Here we can also use the similarity of the expression we found for $A_k(\eta)$ during inflation with that of the R^nF^2 model. The final result will be,

$$\frac{\rho_\lambda}{\rho_{\text{tot}}} \simeq \frac{4^\nu[\Gamma(\nu)]^2}{6\pi^2}\frac{(1+3\gamma)^4}{16}\left(\frac{H}{m_{\text{pl}}}\right)^2 \times \left[\Gamma\left(\frac{5}{2}-\nu\right)\left(\frac{\eta}{\lambda}\right)^{5-2\nu} + \Gamma\left(\frac{3}{2}-\nu\right)\left(\frac{1}{2}-\nu\right)^2\left(\frac{\eta}{\lambda}\right)^{3-2\nu}\right]. \tag{7.88}$$

From the same arguments as in the R^nF^2 model it follows that the backreaction is negligible. Therefore our assumption that the electromagnetic field does not influence the background was correct.

7.4 The $I(\phi)F\tilde{F}$ model, coupling to a pseudo-scalar field

The last model we will review is a model with a Chern-Simons term $I(\phi)F\tilde{F}$, where $\tilde{F}_{\mu\nu} \propto \eta_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and I depends on a pseudo-scalar field ϕ. This term can cause the magnetic field to have a non-zero helicity, as we will explain in this section. The model was first introduced by Turner and Widrow [1]. They suggested that the electromagnetic field could couple to an axion field, trough the coupling $I = g_\alpha \phi$, where g_α is the coupling constant. However they did not perform a full analysis of the model, this was done later by Garretson, Field and Carrol in [39] and by Field and Carrol in [40]. Unfortuanately the model did not generate large enough magnetic fields. This led people to investigate models with other forms of I. In this section we will derive the equation of motion for a general I and then review the model where the Fourier transform of I is of the form $I_{\mathbf{k}} \propto (-k\eta)^b$. Our discussion is based on the paper by Campanelli [33] and we will closely follow his arguments.

Equations of motion

The Lagrangian for this model is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}I(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{7.89}$$

where $\tilde{F}^{\mu\nu} = \eta_{\mu\nu\rho\sigma}F^{\rho\sigma}/(2\sqrt{-g})$ and I is a function of a pseudo-scalar field ϕ. The action is then,

$$S = \int d^4x \left[-\sqrt{-g}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}I(\eta)\eta^{\mu\nu\rho\sigma}F_{\mu\nu}\tilde{F}_{\rho\sigma} \right]. \tag{7.90}$$

Dimensional analysis tells us that I must be dimensionless. To obtain the equations of motion we vary the action with respect to A_μ. This leads to,

$$\partial_\rho[\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}] + \frac{1}{2}\partial_\rho\left[I\eta^{\mu\nu\rho\sigma}F_{\mu\nu}\right] = 0. \tag{7.91}$$

For $\sigma = i$, this reduces in the Coulomb gauge to the equation of motion,

$$A_i'' - a^2\partial_j\partial^j A_i + I'\epsilon_{ijl}\partial_j A_l - \epsilon_{ijl}(\partial_j I)A_l' = 0. \tag{7.92}$$

We use expansion (5.36) to rewrite this as,

$$\begin{aligned} &\sum_\lambda \frac{1}{\sqrt{2k}}\epsilon_{i\lambda}(\mathbf{k})\left(A_k'' + k^2 A_k\right)a_\lambda(\mathbf{k}) + i\int \frac{d^3q}{(2\pi)^3\sqrt{2q}} \\ &\quad \times \sum_{\lambda'}\left(I'_{\mathbf{k}-\mathbf{q}}\epsilon_{ijl}q_j\epsilon_{l\lambda'}(\mathbf{q})A_k - I_{\mathbf{k}-\mathbf{q}}\epsilon_{ijl}(k_j - q_j)\epsilon_{l\lambda'}(\mathbf{q})A_k'\right)a_\lambda(\mathbf{q}) + \text{h.c.} = 0, \end{aligned} \tag{7.93}$$

where $I_{\mathbf{k}}$ is the Fourier transform of I. To simplify the equation of motion we may assume that I is only nonzero at small k. This is a reasonable assumption, since it means that I is only non-zero at large wavelengths. If I would be peaked at a large k, the third and fourth term in equation (7.92) are negligible compared to the second term at the large scales we are interested in. The equation of motion will then just reduce to Maxwell electrodynamics, which led to vanishing magnetic fields. For this reason we will consider $I_{\mathbf{k}}$ to be of the form,

$$I_{\mathbf{k}} = (2\pi)^3 \delta(\mathbf{k}) I_k. \tag{7.94}$$

The equation of motion depends on $I_{\mathbf{k}-\mathbf{q}} = (2\pi)^3\delta(\mathbf{k}-\mathbf{q})I_{k-q}$. We want to consider this function for small k. Because of the delta function we know that q also must be small. Therefore we can expand I_{k-q} for small q, the result is,

$$I_{\mathbf{k}-\mathbf{q}} = (2\pi)^3\delta(\mathbf{k}-\mathbf{q})\left[I_k - \mathbf{q}\nabla_{\mathbf{k}} I_k + \mathcal{O}(\mathbf{q}^2)\right]. \tag{7.95}$$

The equation of motion will then reduce, to leading order, to,

$$\epsilon_{i\lambda}(\mathbf{k})\left(A_k'' + k^2 A_k\right) + iI_k'\epsilon_{ijl}k_j\epsilon_{l\lambda}(\mathbf{k})A_k \simeq 0. \tag{7.96}$$

The last term disappeared because of the delta function. If we choose the momentum to lie along the x^3-axis, the equations of motions for $i = 1, 2$ are,

$$\begin{aligned} \epsilon_{1\lambda}\left(A_k'' + k^2 A_k\right) - i\epsilon_{2\lambda}I_k' k A_k = 0, \\ \epsilon_{2\lambda}\left(A_k'' + k^2 A_k\right) + i\epsilon_{1\lambda}I_k' k A_k = 0, \end{aligned} \tag{7.97}$$

where $k = |\mathbf{k}|$. We can define circular polarization vectors as $\epsilon_\pm = \epsilon_{1\lambda} \pm i\epsilon_{2\lambda}$. It is possible to expand A_i in these polarization vectors as,

$$A_i(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k}} \sum_{\alpha=\pm}^{2} \epsilon_{i\alpha}(\mathbf{k})[a_\alpha(\mathbf{k})A_{k\alpha}(\eta)e^{i\mathbf{kx}} + a_\alpha^\dagger(\mathbf{k})A_{k\alpha}^*(\eta)e^{-i\mathbf{kx}}]. \tag{7.98}$$

If we look at expressions (7.97) it is easy to see that in terms of the circular expansion the equation of motion is,

$$A_{k\pm}'' + k^2 A_{k\pm} \mp I_k' k A_{k\pm} = 0. \tag{7.99}$$

The model

We will assume that I_k is of the form,

$$I_k = c(-k\eta)^b, \tag{7.100}$$

where c is a positive constant and b an arbitrary (real) constant. The reason we take this model is, that it is compatible with different pseudo-scalar models. For example the model,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi), \tag{7.101}$$

with,

$$V(\phi) = m^2\phi^2, \qquad \text{or,} \tag{7.102}$$
$$V(\phi) = \xi R\phi^2, \tag{7.103}$$

where m is the mass of the pseudo-scalar field, ξ a constant and R the Ricci scalar. The first case corresponds to a non-coupled massive pseudo-scalar field and in the second case the pseudo-scalar field is non-minimally coupled to gravity. For these models one can solve the conservation equation for the pseudo-scalar field in momentum space, which in the case of a field that has a space dependence takes the form,

$$\phi_k'' + 2\frac{a'}{a}\phi_k' + a^2\frac{dV_k}{d\phi_k} + k^2\phi_k = 0. \tag{7.104}$$

If we assume that we have de Sitter inflation $a = -1/H\eta$ and that H is a constant during de Sitter inflation, the conservation equation is given by,

$$\phi_k'' + \frac{2}{\eta}\phi_k' + \left(\frac{x}{H^2\eta^2} + k^2\right)\phi_k = 0, \tag{7.105}$$

where x depends on which model we use, either $x = m^2$ or $x = \xi R$. If we have de Sitter inflation we can write the second case as $x = 12H^2$. The equation can be solved using Hankel functions,

$$\phi_k \propto (-k\eta)^{3/2}H_\nu^{(1)}(-k\eta), \tag{7.106}$$

where,

$$\nu = \sqrt{\frac{9}{4} - \frac{x}{H^2}}. \tag{7.107}$$

Since we have assumed that the coupling function is only non-zero for large scales, we can expand the solution for small arguments. Using expansion (7.34) we find,

$$\phi_k(\eta) \propto (-k\eta)^{3/2-\nu}. \tag{7.108}$$

Therefore if we take $I \propto \phi$, one can derive that I_k is of the form (7.100), with $b = 3/2 - \nu$.

Solution

With definition (7.100) of I_k the equation of motion becomes,

$$A_{k\pm}'' + k^2A_{k\pm} \pm k^2bc(-k\eta)^{b-1}A_{k\pm} = 0. \tag{7.109}$$

The equation of motion can only be solved exactly for $b = \pm 1$. The case $b = 1$ is equivalent to Maxwell electrodynamics, which led to vanishing magnetic fields. When $b = -1$ The equation of motion reduces to,

$$A''_{k\pm} + k^2 A_{k\pm} \mp k^2 c(-k\eta)^{-2} A_{k\pm} = 0. \tag{7.110}$$

We can solve this in the same manner as for the $R^n F^2$ model by substituting,

$$A_{k\pm} = c_k \eta^x H_\nu^{(1)}(-k\eta). \tag{7.111}$$

The value of c_k and x can be found in the same way as for the previous two models. The result is,

$$A_{k\pm} = \sqrt{\frac{\pi}{2}}(-k\eta)^{1/2} H_{\nu_\pm}^{(1)}(-k\eta), \tag{7.112}$$

where,

$$\nu_\pm = \sqrt{\frac{1}{4} \pm c}. \tag{7.113}$$

For other values of b the model is not exactly solvable. To simplify the model, we may use the fact that we are interested in large scale fields. We saw that for scales much larger than the horizon, $|k\eta| \ll 1$. If we take $b < 0$ the second term is negligible compared to the other terms in this limit. The equation of motion then simplifies to,

$$A''_{k\pm} \pm k^2 bc(-k\eta)^{b-1} A_{k\pm} = 0. \tag{7.114}$$

The solution to this equation is,

$$A_{k\pm} = c_1(-k\eta)^{1/2} H_{1/(1+b)}^{(1)}(z_\pm), \tag{7.115}$$

where c_1 is a constant and,

$$z_\pm = \frac{1\sqrt{\pm bc}}{1+b}(-k\eta)^{(1+b)/2}. \tag{7.116}$$

One can check that this is correct by inserting the result into equation (7.114) and see that one indeed recovers the Bessel equation (7.20). When $b < -1$, $|z_\pm| \gg 1$, since we were working in the limit $|k\eta| \ll 1$. In this case we can use the large argument expansion of the Hankel function [30], to obtain,

$$A_{k\pm} = c'_1(-k\eta)^{(1-b)/2} e^{iz_\pm}, \tag{7.117}$$

where c'_1 is a constant. For the positive solution the exponent is just an oscillating term, since z_+ is real, and therefore not interesting. The solution is proportional to $A_{k+} \propto (-k\eta)^{(1-b)/2}$, but since $b < -1$ and $|k\eta| \ll 1$, this uninterestingly small. On the other hand z_- is complex, so $A_{k-} \propto (-k\eta)^{(1-b)/2} e^{|z_-|}$. This term will diverge for small k and is therefore not physical. For this reason will require that $b > -1$.

When $-1 < b < 0$, we have $|z_\pm| \ll 1$ and we can use the small argument expansion for the Hankel function [30],

$$A_{k\pm} = c_1' + c_2'(-k\eta). \tag{7.118}$$

To find the value of the constants c_1' and c_2' we can use a trick. For small scales $|k\eta| \gg 1$, the third term in the equation of motion (7.109) is negligible and we recover Maxwell electrodynamics. We already saw that the solution in this case was $A_k \propto e^{-ik\eta}$. We can determine the constants by matching these two solutions and their derivatives at the horizon, $|k\eta| = 1$. The result is,

$$A_{k\pm} = e^{-i}\left[1 + i(1 - k\eta)\right]. \tag{7.119}$$

Matching after inflation

Since we are calculating the positive and negative solution separately we cannot use expression (6.6). Instead we want to find an expression for,

$$B^2_{\lambda\pm}(\eta) = \left\langle 0 \right| |\mathbf{B}_{\lambda\pm}(\eta, \mathbf{x})|^2 \left| 0 \right\rangle. \tag{7.120}$$

If we look at the derivation in section 6.4, we see that the only difference is, that there is no longer a sum over $\lambda = \pm$. Instead of expression (6.8), we have,

$$\begin{aligned}&\left\langle 0 \right| (\nabla \times \mathbf{A}(\eta, \mathbf{y}))_i (\nabla \times \mathbf{A}(\eta, \mathbf{y}'))^*_i \left| 0 \right\rangle = \frac{1}{2k} |A_k|^2 \delta(\mathbf{y} - \mathbf{y}') \times \\ &\left[k_j^2 \epsilon_{z\pm}(\mathbf{k})\epsilon_{z\pm}(\mathbf{k}) - k_j k_z \epsilon_{z\pm}(\mathbf{k})\epsilon_{j\pm}(\mathbf{k}) - k_z k_j \epsilon_{j\pm}(\mathbf{k})\epsilon_{z\pm}(\mathbf{k}) + k_z^2 \epsilon_{j\pm}(\mathbf{k})\epsilon_{j\pm}(\mathbf{k})\right].\end{aligned} \tag{7.121}$$

This is the same as,

$$\left\langle 0 \right| (\nabla \times \mathbf{A}(\eta, \mathbf{y}))_i (\nabla \times \mathbf{A}(\eta, \mathbf{y}'))^*_i \left| 0 \right\rangle = \frac{1}{2k} |A_k|^2 \delta(\mathbf{y} - \mathbf{y}') \left[(k_j \epsilon_{z\pm} - k_z \epsilon_{j\pm})^2\right]. \tag{7.122}$$

All the components together are,

$$\begin{aligned}\left\langle 0 \right| (\nabla \times \mathbf{A}(\eta, \mathbf{y}))(\nabla \times \mathbf{A}(\eta, \mathbf{y}'))^* \left| 0 \right\rangle &= \frac{1}{2k} |A_k|^2 \delta(\mathbf{y} - \mathbf{y}')(\mathbf{k} \times \epsilon_\pm)^2, \\ &= \frac{k}{2} |A_k|^2 \delta(\mathbf{y} - \mathbf{y}').\end{aligned} \tag{7.123}$$

Comparison with expression (6.9) shows that,

$$\begin{aligned}B^0_{\lambda\pm} &= \frac{1}{\sqrt{2}} B^0_\lambda, \\ &= \left[\int \frac{dk}{k} W^2_\lambda(k) \frac{k^4}{4\pi^2} |\beta_k(\eta_{\rm rad})|^2\right]^{1/2}.\end{aligned} \tag{7.124}$$

We can now find the present value of the magnetic field by solving $A_{k\pm} = \alpha_{k\pm}e^{-ik\eta} + \beta_{k\pm}e^{ik\eta}$ and the first derivative, for $\beta_{k\pm}$, as explained in chapter 6. In the case that $-1 < b < 0$, the result is,

$$\beta_{k\pm} = \frac{ie^{-i}}{2}\left(1 - k\eta\right)e^{-ik\eta}. \tag{7.125}$$

In the large scale limit the second term is negligible to the first term, so approximately,

$$|\beta_{k\pm}|^2 = \frac{1}{4}. \tag{7.126}$$

We may do the same thing for the case $b = -1$. We will first restrict the negative solution to $c < 1/4$, since for these values ν_- is real and non-zero. We can use the small argument expansion of the Hankel function to rewrite solution (7.112) as,

$$A_{k\pm} = i(2\pi)^{-1/2}2^{\nu_\pm}\Gamma(\nu_\pm)(-k\eta)^{-\nu_\pm+1/2}, \tag{7.127}$$

up to a phase, for $\nu_\pm > 0$. If we solve for $\beta_{k\pm}$ the result is,

$$\beta_{k\pm} = \frac{2^{\nu_\pm}\Gamma(\nu_\pm)}{\sqrt{8\pi}}(-k\eta)^{-\nu_\pm-1/2}\left[-ik\eta + \left(\frac{1}{2} - \nu_\pm\right)\right]. \tag{7.128}$$

The first term is negligible compared to the second, which leads to,

$$|\beta_{k\pm}|^2 = \frac{2^{2\nu_\pm}\Gamma^2(\nu_\pm)}{8\pi}\left(\frac{1}{2} - \nu_\pm\right)^2|k\eta|^{-2\nu_\pm-1}. \tag{7.129}$$

We can also evaluate the negative solution for ν_- is zero or complex, that is for $c \geq 1/4$. We can expand the Hankel function for these values and solve for $\beta_{k\pm}$, the results are [33],

$$\nu_- = 0: \qquad |\beta_{k\pm}|^2 = \frac{\ln^2|k\eta|}{8\pi|k\eta|}, \tag{7.130}$$

$$\nu_- = i\tilde{\nu}_-: \qquad |\beta_{k\pm}|^2 = \frac{a_{\nu_-} + b_{\nu_-}\cos[2\tilde{\nu}_-\ln(|k\eta|/2)]}{4\pi|k\eta|} + \frac{c_{\nu_-}\sin[2\tilde{\nu}_-\ln(|k\eta|/2)]}{4\pi|k\eta|}, \tag{7.131}$$

where $\tilde{\nu}$ is the imaginary part of ν and,

$$\begin{aligned} a_{\nu_-} &= \frac{\pi}{\tilde{\nu}_-}\coth(\pi\tilde{\nu}_-), \\ b_{\nu_-} &= (1 - 4\tilde{\nu}_-^2)\Re[\Gamma^2(\nu_-)] + 4\tilde{\nu}_-\Im[\Gamma^2(\nu_-)], \\ c_{\nu_-} &= (1 - 4\tilde{\nu}_-^2)\Im[\Gamma^2(\nu_-)] - 4\tilde{\nu}_-\Re[\Gamma^2(\nu_-)]. \end{aligned} \tag{7.132}$$

The present magnetic field

We can insert the results (7.126), (7.129), (7.130) and (7.131) into expression (7.124) to obtain,

$$-1 < b < 0 : \quad B^0_{\lambda\pm} = \frac{1}{4\pi\sqrt{2}}\lambda^{-2}, \tag{7.133}$$

$$b = -1, \nu_\pm > 0 : \quad B^0_{\lambda\pm} = \frac{2^{\nu_\pm}\Gamma(\nu_\pm)\left[\Gamma(3/2-\nu_\pm)\right]^{1/2}|1/2-\nu_\pm|}{(4\pi)^{3/2}}|\eta_{\rm rad}|^{-\nu_\pm-1/2}\lambda^{\nu_\pm-3/2}, \tag{7.134}$$

$$b = -1, \nu_- = 0 : \quad B^0_{\lambda\pm} = \frac{1}{2(4\pi)^5/4}\ln\left(\frac{\lambda}{|\eta_{\rm rad}|}\right)|\eta_{\rm rad}|^{-1/2}\lambda^{-3/2}, \tag{7.135}$$

$$b = -1, \nu_- = i\tilde{\nu}_- : \quad B^0_{\lambda\pm} = \frac{(a_{\nu_-} + b_{\nu_-}/\sqrt{2} + c_{\nu_-}/\sqrt{2})^{1/2}}{\sqrt{2}(4\pi)^5/4}|\eta_{\rm rad}|^{-1/2}\lambda^{-3/2}. \tag{7.136}$$

When $-1 < b < 0$, the magnetic field at scales $\lambda = 10$ kpc is $B^0_{\lambda\pm} \simeq 10^{-54}$, which is a lot lower then the dynamo limit, $B > 10^{-33}$. therefore we can conclude that this model does not generate large enough magnetic fields.

To evaluate the case $b = -1$, we can use the value of $\eta_{\rm rad}$ in equation (7.55). We take the maximum value for $M = 10^{-2}m_{\rm pl}$, to obtain an upper limit for the magnetic field. In this case, $\eta_{\rm rad} \simeq 10^5$. In figure 7.5 we have plot the strength of the positive solution for different values of c. Just as in the R^nF^2 model, we must have $c < 2$, since a larger value would lead to divergent fields at large scales. The solid blue line corresponds to the case $\lambda = 10$ kpc and the dashed purple line to $\lambda = 1$ Mpc. One can see that for $c > 0.1$ the generated fields are stronger then the dynamo constraint and for $c > 1.2$ the intergalactic field limit is satisfied, which is the stronger limit. When $c > 1.7$ the fields are strong enough to be generated without the dynamo mechanism. For all values of c the produced field lies below the CMB constraint and are therefore compatible with CMB observations. The negative solution is smaller then the positive solution for all values of ν since it is proportional to $|\eta_{\rm rad}|^{-1/2}\lambda^{-3/2}$. The positive solution was minimal for $\nu \simeq 0$. At that value of ν the strength of the magnetic field was lower then the dynamo limit and therefore the same will be true for the negative solution. As before we have also plotted the relation between the temperature and the strength of the field for $c = 1.99$, which corresponds to the strongest fields. The result can be found in figure 7.6. Again we find that the intergalactic field limit is satisfied if the temperature during inflation $M > 10^{-6}m_{\rm pl}$. This is the minimal value needed for this model to work.

We can compare these results with the particle physics models we suggested in (7.102) and (7.103). We saw that if we take the coupling $I(\phi) \propto \phi$, then for de Sitter inflation

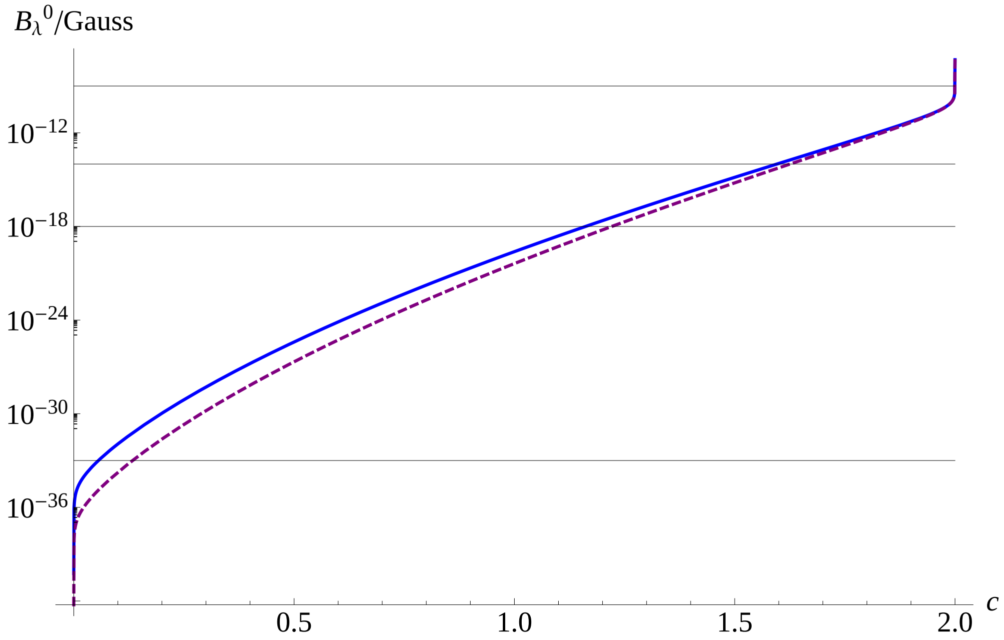

Figure 7.5 – The positive present magnetic field solution for $b = -1$ as a function of c. The solid blue line corresponds to the the scale $\lambda = 10$ kpc and the dashed purple line to $\lambda = 1$ Mpc. The thin black lines correspond to the limits from the CMB ($B < 10^{-9}$), protogalaxy collapsing ($B > 10^{-14}$ for $\lambda = 1$ Mpc), the intergalactic field limit ($B > 10^{-18}$ for $\lambda = 1$ Mpc) and the dynamo mechanism ($B > 10^{-33}$ for $\lambda = 10$ kpc).

$b = 3/2 - \nu$, where,

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \tag{7.137}$$

$$\nu = \sqrt{\frac{9}{4} - 12\xi}, \tag{7.138}$$

where the first value is the case that ϕ is a non-coupled massive pseudo-scalar field and the second value that the pseudo-scalar field is non-minimally coupled to gravity. The case that leads to interesting magnetic fields, $b = -1$, corresponds to, $m^2 = -4H^2$ and $\xi = 1/3$ respectively.

Helicity

The circular polarization vectors are also called the helical polarization vectors. Helicity is a measure of a the polarization with respect the direction of motion of the photon.

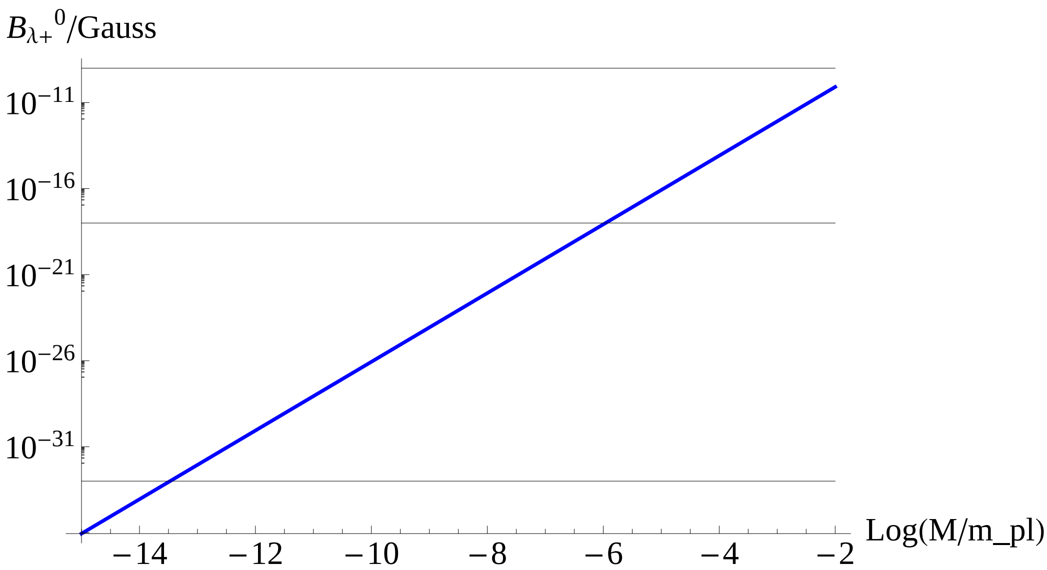

Figure 7.6 – The positive present magnetic field solution for $b = -1$ as a function of the temperature during inflation M. The solid blue line corresponds to the the scale $\lambda = 10$ kpc and the dashed purple line to $\lambda = 1$ Mpc. The thin black lines correspond to the limits from the CMB ($B < 10^{-9}$), the intergalactic field limit ($B > 10^{-18}$ for $\lambda = 1$ Mpc) and the dynamo mechanism ($B > 10^{-33}$ for $\lambda = 10$ kpc).

The helicity of a magnetic field in a volume V is classically defined as,

$$H = \frac{1}{V}\int_V \mathbf{A}\cdot\mathbf{B}d^3x. \tag{7.139}$$

It describes the twisting of the field lines. The average vacuum value on a co-moving scale λ of the helicity is defined, just as for the magnetic field, as,

$$H_\lambda(\eta) = \langle 0|\int d^3y d^3z W_\lambda(|\mathbf{x}-\mathbf{y}|)W_\lambda(|\mathbf{x}-\mathbf{z}|)\mathbf{A}(\eta,\mathbf{y})\cdot\mathbf{B}(\eta,\mathbf{z})|0\rangle. \tag{7.140}$$

One can calculate this explicitly, just as we did for B_λ^2 in chapter 6. The result is,

$$H_\lambda(\eta) = \frac{1}{a^2}\int_0^\infty \frac{dk}{k}W_\lambda^2(k)\frac{k^3}{4\pi^2}\left[|A_{k+}|^2 - |A_{k-}|^2\right]. \tag{7.141}$$

The helicity is proportional to the difference between the positive polarized and negative polarized photons, as one would expect. The helicity is zero if the positive and negative helicity solution are equal, as was the case for $-1 < b < 0$. One can calculate the helicity in the case $b = -1$. We already saw that the negative helicity solution was much smaller then the positive solution. The difference is proportional to $(\lambda/\eta_{\mathrm{rad}})^\nu$, which for $\lambda = 10$kpc is proportional to $10^{21\nu}$. This difference is so large that we can neglect the negative solution and,

$$H^0_\lambda(\eta_{\rm rad}) \simeq \int_0^\infty \frac{dk}{k} W^2_\lambda(k) \frac{k^3}{4\pi^2} |A_{k+}|^2. \tag{7.142}$$

Comparison with expression (7.124) shows that,

$$H^0_\lambda(\eta_{\rm rad}) \simeq (B^0_{\lambda+})^2 \lambda. \tag{7.143}$$

Unfortunately up until now the measurements of helicity of magnetic fields in the CMB and galaxies are not precise enough to detect this strength of helicity [33]. If this would become possible in the future, it may give us a reason to accept or refute the $IF\tilde{F}$ model.

Backreaction

Lastly we must check that our assumptions are still correct, as explained in section 6.6. That is we must check that the energy density of the electromagnetic field is smaller then the total energy density. The energy momentum tensor is,

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\partial S}{g^{\mu\nu}}, \\ &= -\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F_{\mu\beta} F^\beta_\nu, \end{aligned} \tag{7.144}$$

which is the same as for the Maxwell theory, since in the case we evaluated $I(\phi)$ does not depend on $g^{\mu\nu}$. The energy density of the electromagnetic field in the Coulomb gauge is then,

$$\begin{aligned} \rho &= -T^0_0, \\ &= \frac{1}{2}\left(B^2 - E^2\right) + E^2, \\ &= \frac{1}{2}\left(B^2 + E^2\right). \end{aligned} \tag{7.145}$$

We can now deduce that the vacuum expectation value of the co-moving average electromagnetic density on a scale λ is,

$$\rho_{\lambda\pm} \simeq \frac{1}{2} \sum_\pm \left(B^2_{\lambda\pm} + E^2_{\lambda\pm}\right). \tag{7.146}$$

The expression for $E_{\lambda\pm}$ can be found just as we did for $B_{\lambda\pm}$ above. The result is,

$$\begin{aligned} E^2_{\lambda\pm} &= \frac{1}{2} E^2_\lambda, \\ &= \frac{1}{a^4} \int \frac{dk}{k} W^2_\lambda(k) \frac{k^2}{4\pi^2} |A'_{k\pm}|^2. \end{aligned} \tag{7.147}$$

We will evaluate the density for both $-1 < b < 0$ and $b = -1$.

For $-1 < b < 0$, we found that in the large scale limit, during inflation,

$$A_{k\pm} = e^{-i}\left[1 + i(1 - k\eta)\right]. \tag{7.148}$$

Using this, we can calculate the vacuum expectation values of the average co-moving electric and magnetic fields during inflation,

$$B^2_{\lambda\pm} \simeq \frac{1}{a^4 4\pi^2}\lambda^{-6}\eta^2, \tag{7.149}$$
$$E^2_{\lambda\pm} \simeq \frac{1}{a^4 8\pi^2}\lambda^{-4}, \tag{7.150}$$

where we only kept the leading terms. It follows that,

$$\rho_\lambda \simeq \frac{1}{a^4 4\pi^2}\left(\lambda^{-6}\eta^2 + \frac{1}{2}\lambda^{-4}\right). \tag{7.151}$$

We can rewrite $a = 2[(1+3\gamma)\eta H]^{-1}$. Then,

$$\frac{\rho_\lambda}{\rho_{\rm tot}} \simeq \frac{(1+3\gamma)^4}{64\pi^2}\left(\frac{H}{m_{\rm pl}}\right)^2\left[\left(\frac{\eta}{\lambda}\right)^6 + \frac{1}{2}\left(\frac{\eta}{\lambda}\right)^4\right]. \tag{7.152}$$

We have the constraint that $H/m_{\rm pl} < 10^{-4}$ from graviton production [1] and since we are in the large scale limit, $\eta/\lambda \ll 1$. If we use these requirements,

$$\frac{\rho_\lambda}{\rho_{\rm tot}} \ll 1. \tag{7.153}$$

This shows that we have correctly neglected the backreaction of the electromagnetic field, when $-1 < b < 0$.

In the case that $b = -1$ we found the solution,

$$A_{k\pm} = \sqrt{\frac{\pi}{2}}(-k\eta)^{1/2}H^{(1)}_{\nu_\pm}(-k\eta). \tag{7.154}$$

We saw that only the positive solution led to interesting fields, so we will only evaluate the positive solution and assume that the negative solution is negligible. We can use the small argument expansion, to find,

$$B^2_\lambda = \frac{4^{\nu_+}[\Gamma(\nu_+)]^2\Gamma(5/2-\nu_+)}{a^4 4\pi^3}\lambda^{-5+2\nu_+}\eta^{1-2\nu_+}, \tag{7.155}$$
$$E^2_\lambda = \frac{4^{\nu_+}[\Gamma(\nu_+)]^2\Gamma(3/2-\nu_+)(1/2-\nu_+)^2}{a^4 4\pi^3}\lambda^{-3+2\nu_+}\eta^{-1-2\nu_+}. \tag{7.156}$$

Following the same steps as above, we finally arrive at,

$$\frac{\rho_\lambda}{\rho_{\rm tot}} \simeq \frac{4^{\nu_+}[\Gamma(\nu_+)]^2}{3\pi^2}\frac{(1+3\gamma)^4}{16}\left(\frac{H}{m_{\rm pl}}\right)^2 \times \left[\Gamma\left(\frac{5}{2}-\nu_+\right)\left(\frac{\eta}{\lambda}\right)^{5-2\nu_+} + \Gamma\left(\frac{3}{2}-\nu_+\right)\left(\frac{1}{2}-\nu_+\right)^2\left(\frac{\eta}{\lambda}\right)^{3-2\nu_+}\right]. \quad (7.157)$$

For the same reasons as before this is vanishingly small and we have correctly neglected the backreaction of the electromagnetic field.

CHAPTER 8

Evolution of magnetic fields in a spatially curved FLRW spacetime

In the previous chapter we saw how magnetic fields arise if we break the conformal invariance of the field equations. There is another way to break conformal invariance. We can look at a background that is no longer conformal to Minkowski spacetime. In this chapter we will qualitatively evaluate how this influences the strength of the magnetic field. For simplicity we will keep the discussion classical.
Before we can start the discussion we must introduce a new description for spacetime and fluids. This description is called the $1 + 3$ covariant description. We will go through the basics of this description in section 8.1. In the next two sections we will apply this description to the electromagnetic field and the gravitational field. At that point we have all our ingredients to derive the wave equation for the magnetic field in an arbitrary curved spacetime. The derivation is done explicitly in section 8.4. The result is evaluated for the case of a constant spatially curved FLRW spacetime in section 8.5. We compare the result of a flat universe with a spatially open or closed universe and show that in the case of an open universe the magnetic field decays slower. For simplicity we will set $8\pi G = 1$. In this chapter we mainly follow the discussion of Tsagas [31]. Another, more extensive review can be found in [41].

8.1 1+3 covariant description

The description of spacetime we used before contained the metric $g_{\mu\nu}$, which is coordinate dependent. Since all the physical quantities do not depend on the coordinate system that is used, we would like to have a description that is coordinate independent. A description that has this property is the $1+3$ covariant description. In this chapter we will often refer to this description just as covariant. The main idea is that it separates a timelike direction along the four-velocity, which will be defined below, and spacelike direction orthogonal to the four-velocity. In this section we will give the basic ingredients for such a description and show how kinematical quantities and the energy momentum tensor can be expressed. This section is based on [42], a more elaborate discussion and applications can be found there.

Four-velocity

Consider a particle that moves through spacetime. The path of the particle can be parametrized by the proper time τ and is called a worldline. The vector tangent to the path of the particle is called the four-velocity and defined by,

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{8.1}$$

Since the proper time is defined as $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$, one can derive that,

$$u_\mu u^\mu = -1. \tag{8.2}$$

If the metric is known one can derive the explicit expression for the four-velocity of a co-moving observer. For example for Minkowski spacetime the four-velocity of a co-moving observer has the form,

$$u^\mu = (1, \mathbf{0}). \tag{8.3}$$

In cosmology we have to deal with fluids instead of single particles, therefore one considers the worldlines that describe the average motion of the fluid at each point [42].

Projection tensors

To be able to project along and orthogonal to the four-velocity we define projection tensors as,

$$\begin{aligned} U^\mu_\nu &= -u^\mu u_\nu, \\ h_{\mu\nu} &= g_{\mu\nu} + u_\mu u_\nu. \end{aligned} \tag{8.4}$$

The projection tensor U^μ_ν projects along the four-velocity and $h_{\mu\nu}$ projects orthogonal to the four-velocity into what is called the observers instantaneous rest space. We will show this explicitly by letting the operators act on a vector V^μ and taking the inner-product with the four-velocity,

$$\begin{aligned}(U_{\mu\nu}V^\mu)u^\nu &= -u_\mu u_\nu V^\mu u^\nu, \\ &= u_\mu V^\mu,\end{aligned} \tag{8.5}$$

and,

$$\begin{aligned}(h_{\mu\nu}V^\mu)u^\nu &= g_{\mu\nu}V^\mu u^\nu + u_\mu u_\nu V^\mu u^\nu, \\ &= V^\mu u_\mu - V^\mu u_\mu, \\ &= 0.\end{aligned} \tag{8.6}$$

This is the behavior we expect from projection tensors. We can also calculate that the the tensors satisfy the relations below, as one would expect from projection tensors,

$$\begin{aligned}U^\mu_\nu U^\nu_\sigma &= U^\mu_\sigma, \qquad U^\mu_\mu = 1, \qquad U_{\mu\nu}u^\nu = u_\mu, \\ h^\mu_\nu h^\nu_\sigma &= h^\mu_\sigma, \qquad h^\mu_\mu = 3, \qquad h_{\mu\nu}u^\nu = 0.\end{aligned} \tag{8.7}$$

The orthogonal projection tensor $h_{\mu\nu}$ can been viewed as the metric of the spatial sections orthogonal to u^μ since [43],

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(u_\mu dx^\mu)^2 + h_{\mu\nu}dx^\mu dx^\nu. \tag{8.8}$$

From now on we will denote the orthogonal projection of vectors and the orthogonally projected symmetric trace free part of tensors with angle brackets,

$$\begin{aligned}v^{\langle\mu\rangle} &= h^\mu_\nu v^\nu, \\ T^{\langle\mu\nu\rangle} &= \left[h^{(\mu}_\lambda h^{\nu)}_\sigma - \frac{1}{3}h^{\mu\nu}h_{\lambda\sigma}\right]T^{\lambda\sigma}.\end{aligned} \tag{8.9}$$

Volume element

There is a volume element for the observers instantaneous rest space given by,

$$\epsilon_{\mu\nu\kappa} = u^\lambda \eta_{\lambda\mu\nu\kappa}, \tag{8.10}$$

where $\eta_{\mu\nu\kappa\lambda}$ is the Levi-Civita tensor. Recall that it is defined as,

$$\eta_{\mu\nu\kappa\lambda} = \sqrt{-g}\,\tilde{\eta}_{\mu\nu\kappa\lambda}, \tag{8.11}$$

where $\tilde{\eta}_{\mu\nu\kappa\lambda}$ is the Levi-Civita symbol in four dimensions. Since $\eta_{\mu\nu\kappa\lambda}$ is antisymmetric under exchange of indices, the same is true for $\epsilon_{\mu\nu\kappa}$. Looking at the definition of $\epsilon_{\mu\nu\kappa}$ it is easy to see that the identity $\epsilon_{\mu\nu\kappa}u^\mu = 0$ holds.

Derivatives

We also define two derivatives. The first one is a covariant derivative in the timelike direction, along the wordline,

$$\tilde{\dot{T}}^{\nu_1\nu_2\ldots}_{\mu_1\mu_2\ldots} = u^\lambda \nabla_\lambda T^{\nu_1\nu_2\ldots}_{\mu_1\mu_2\ldots}, \tag{8.12}$$

where T can be any tensor. The second is fully orthogonal projected derivative, which operates in the observers rest space,

$$\tilde{\nabla}_\rho T^{\nu_1\nu_2\ldots}_{\mu_1\mu_2\ldots} = h^{\lambda_1}_{\mu_1} h^{\lambda_2}_{\mu_2} \ldots h^{\nu_1}_{\sigma_1} h^{\nu_2}_{\sigma_2} \ldots h^{\kappa}_{\rho} \nabla_\kappa T^{\sigma_1\sigma_2\ldots}_{\lambda_1\lambda_2\ldots}. \tag{8.13}$$

Kinematical quantities

To obtain a covariant description for $\nabla_\mu u_\nu$ we may split it into its irreducible parts,

$$\begin{aligned}
\nabla_\mu u_\nu &= -u_\mu \tilde{\dot{u}}_\nu + \tilde{\nabla}_\mu u_\nu \\
&= -u_\mu \tilde{\dot{u}}_\nu + \frac{1}{3}\Theta h_{\nu\mu} + \sigma_{\nu\mu} + \omega_{\nu\mu}, \\
&= -u_\mu \tilde{\dot{u}}_\nu + \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu} - \omega_{\mu\nu}.
\end{aligned} \tag{8.14}$$

The first line can be checked using the definitions of the derivatives (8.12) and (8.13). The second line is equivalent to the decomposition into the trace, the symmetric trace free part and the anti-symmetric part. In this case $\Theta = \tilde{\nabla}_\mu u^\mu$ is the trace. This quantity can be interpreted as the rate of volume expansion of the fluid. This means that we can connect it to the Hubble parameter, which was the rate of coordinate expansion, by $H = \Theta/3$. The quantity $\sigma_{\mu\nu} = \tilde{\nabla}_{\langle\nu} u_{\mu\rangle}$ is the trace free symmetric part, which, because of this symmetry, obeys the relations $\sigma_{\mu\nu} = \sigma_{(\mu\nu)}$, $\sigma_{\mu\nu} u^\nu = 0$ and $\sigma^\mu_\mu = 0$. This quantity is called the rate of shear tensor, which can be interpreted as the rate of distortion of the fluid. The fourth quantity $\omega_{\mu\nu} = \tilde{\nabla}_{[\nu} u_{\mu]}$ is the anti-symmetric part and is called the vorticity tensor, which obeys the relations $\omega_{\mu\nu} = \omega_{[\mu\nu]}$, $\omega_{\mu\nu} u^\nu = 0$. It can be interpreted as the rotation of the fluid with respect to a non rotating frame. A schematical interpretation of the decomposition into irreducible parts is given in figure 8.1. It will turn out to be usefull to define $\omega^\mu = 1/2\eta^{\mu\nu\lambda}\omega_{\nu\lambda}$ called the vorticity vector. This vector has the properties $\omega_\mu u^\mu = \omega_{\mu\nu}\omega^\nu = 0$. The quantity $\tilde{\dot{u}}^\mu$ can be interpreted as the relativistic acceleration due to forces other than gravity or inertia. If the fluid is in free fall this quantity is zero.

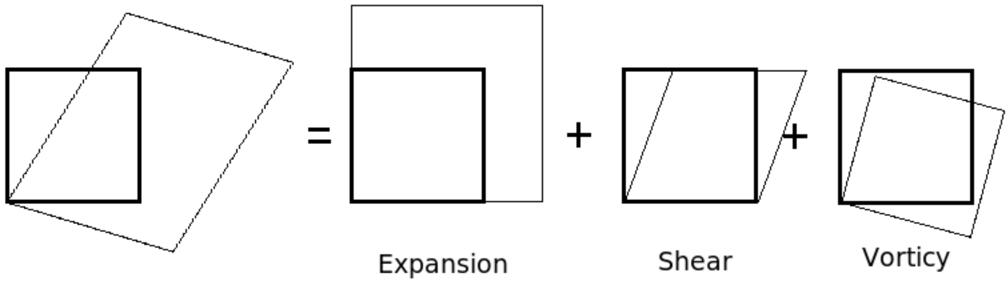

Figure 8.1 – A schematic interpretation of the decomposition into an expansion rate, rate of shear tensor and a vorticity tensor.

The energy momentum tensor

The energy momentum tensor $T_{\mu\nu}$ can be decomposed with respect to u^μ into its irreducible parts as,

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu}, \tag{8.15}$$

where $\rho = T_{\mu\nu} u^\mu u^\nu$ is the energy density and $p = \frac{1}{3} T_{\mu\nu} h^{\mu\nu}$ is the isotropic pressure of the fluid. The quantity $\pi_{\mu\nu} = T_{\lambda\kappa} h^\lambda_{\langle\mu} h^\kappa_{\nu\rangle}$ is the trace free anisotropic pressure, also referred to as stress. It is symmetric so it has the properties $\pi_{\mu\nu} = \pi_{(\mu\nu)}$ and $\pi^\mu_\mu = 0$. The quantity $q^\mu = -T_{\nu\lambda} u^\nu h^{\lambda\mu}$ is the momentum density or the energy-flux relative to u^μ. From the definitions of $\pi_{\mu\nu}$ and q^μ it is easy to see that they obey the relations $q_\mu u^\mu = \pi_{\mu\nu} u^\nu = 0$.

A perfect fluid is defined as,

$$\pi_{\mu\nu} = q_\mu = 0 \qquad \Rightarrow \qquad T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}. \tag{8.16}$$

If we go to the rest frame of a co-moving observer in a Minkowski spacetime,

$$T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p). \tag{8.17}$$

This shows that we can indeed identify ρ with the energy density of the fluid and p with the isotropic pressure. A perfect fluid is called barotropic if ρ and p are related by an equation of state $p = p(\rho)$.

8.2 Electromagnetic fields

The electromagnetic field can be thought of as an imperfect fluid. To show this we may split the field strength tensor $F_{\mu\nu}$ into an electric and magnetic part relative to u^μ

as,

$$F_{\mu\nu} = 2u_{[\mu}E_{\nu]} + \epsilon_{\mu\nu\lambda}B^{\lambda}, \tag{8.18}$$

where $E_\mu = F_{\mu\nu}u^\nu$ is the electric field and $B_\mu = \epsilon_{\mu\nu\lambda}F^{\nu\lambda}/2$ is the magnetic field seen by the observer moving along the worldline. One can check that this is indeed the case by going to the rest space of the observer in Minkowski spacetime, where $u^\mu = (1, \mathbf{0})$. The equations will then reduce to the normal definitions of the electric and magnetic field. From the covariant definitions it follows that $E_\mu u^\mu = B_\mu u^\mu = 0$, so both fields are indeed vectors in the observers instantaneous rest space.

The energy momentum tensor of the electromagnetic field is obtained by varying the usual Maxwell action,

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \\ &= -F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}F_{\lambda\kappa}F^{\lambda\kappa}g_{\mu\nu}. \end{aligned} \tag{8.19}$$

Using (8.18) this can be written after some lines of calculations as,

$$T_{\mu\nu} = \frac{1}{2}(E^2 + B^2)u_\mu u_\nu + \frac{1}{6}(E^2 + B^2)h_{\mu\nu} + 2\mathcal{Q}_{(\mu}u_{\nu)} + \mathcal{P}_{\mu\nu}, \tag{8.20}$$

where,

$$\mathcal{Q}_\mu = \epsilon_{\mu\nu\lambda}E^\nu B^\lambda, \tag{8.21}$$

$$\mathcal{P}_{\mu\nu} = \mathcal{P}_{\langle\mu\nu\rangle} = \frac{1}{3}(E^2 + B^2)h_{\mu\nu} - E_\mu E_\nu - B_\mu B_\nu. \tag{8.22}$$

The quantity $\mathcal{Q}_\mu$ is called the electromagnetic Poynting vector. Comparison with equation (8.15) shows that the electromagnetic field can indeed be described as an imperfect fluid with energy density $(E^2 + B^2)/2$, isotropic pressure $(E^2 + B^2)/6$, anisotropic pressure $\mathcal{P}_{\mu\nu}$ and energy-flux vector $\mathcal{Q}_\mu$.

We would like to put the Maxwell equations into a covariant form. Recall that they are given by,

$$\nabla_{[\mu}F_{\nu\lambda]} = 0, \tag{8.23}$$

$$\nabla^\nu F_{\mu\nu} = j_\mu. \tag{8.24}$$

Just as we did for the field strength tensor we can split j_μ into irreducible parts relative to u^μ,

$$j_\mu = \rho_e u_\mu + \mathcal{J}_\mu, \tag{8.25}$$

where $\rho_e = -j_\mu u^\mu$ is the charge density and $\mathcal{J}_\mu = h_\mu^{\ \nu} j_\nu$ is the orthogonally projected current, which implies that $\mathcal{J}_\mu u^\mu = 0$. There is also another interpretation we can assign to $\mathcal{J}_\mu$. We know that the relation between the conductivity, the current and the electric field is given by Ohm's law, which takes the covariant form [31][41],

$$j_\mu = \rho_e u_\mu + \sigma_c E_\mu, \tag{8.26}$$

where σ_c is the conductivity. If we act on this with the orthogonal projection tensor the result is,

$$\mathcal{J}_\mu = \sigma_c E_\mu. \tag{8.27}$$

When the conductivity is zero, the current in the observers instantaneous rest space will be zero. On the other hand, if the conductivity is infinite, a finite current implies that the electric field will go to zero.

We can separate both Maxwell equations into a component along the four-velocity and orthogonal to the four-velocity. These components can be found by acting with the projection tensors on the equations. This results in to two propagation equations [31][41],

$$\tilde{\dot{E}}_{\langle\mu\rangle} = \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\lambda}\omega^\lambda - \frac{2}{3}\Theta h_{\mu\nu}\right) E^\nu + \epsilon_{\mu\nu\lambda}\tilde{\dot{u}}^\nu B^\lambda + \mathrm{curl} B_\mu - \mathcal{J}_\mu, \tag{8.28}$$

$$\tilde{\dot{B}}_{\langle\mu\rangle} = \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\lambda}\omega^\lambda - \frac{2}{3}\Theta h_{\mu\nu}\right) B^\nu - \epsilon_{\mu\nu\lambda}\tilde{\dot{u}}^\nu E^\lambda - \mathrm{curl} E_\mu, \tag{8.29}$$

and two constraints,

$$\tilde{\nabla}^\mu E_\mu + 2\omega^\mu B_\mu = \rho_e, \tag{8.30}$$

$$\tilde{\nabla}^\mu B_\mu - 2\omega^\mu E_\mu = 0. \tag{8.31}$$

Here curl $v_\mu \equiv \epsilon_{\mu\nu\lambda}\tilde{\nabla}^\nu v^\lambda$ for any orthogonally projected vector v_μ. Notice that these four equations resemble the non-covariant Maxwell equations (5.1) plus terms which originate from the observers motion. There is also another conservation law, which comes from the fact that j_μ is a conserved current,

$$\nabla^\mu j_\mu = 0. \tag{8.32}$$

That this is indeed the case can easily be derived by acting with ∇^μ on the second Maxwell equation (8.24). The field strength tensor is anti-symmetric and both covariant derivatives are symmetric, so the left hand side becomes zero. We can write this conservation law in a covariant way using the decomposition of j_μ (8.25):

$$\begin{aligned} 0 = \nabla^\mu j_\mu &= \nabla^\mu(\rho_e u_\mu) + \nabla^\mu \mathcal{J}_\mu, \\ &= \tilde{\dot{\rho}}_e + \rho_e \nabla^\mu u_\mu + \nabla^\mu \mathcal{J}_\mu. \end{aligned} \tag{8.33}$$

The second term on the right hand side can be rewritten using decomposition (8.14). We can also rewrite the third term by noticing that

$$\begin{aligned}\left(\tilde{\nabla}^\mu + \tilde{\dot{u}}^\mu\right)\mathcal{J}_\mu &= h^\mu_\lambda h^\sigma_\mu \nabla^\lambda \mathcal{J}_\sigma + u^\lambda \left(\nabla_\lambda u^\mu\right)\mathcal{J}_\mu \\ &= \left(h^{\lambda\mu} - u^\lambda u^\mu\right)\nabla_\lambda \mathcal{J}_\mu \\ &= \nabla^\mu \mathcal{J}_\mu. \end{aligned} \tag{8.34}$$

In the second line we used the chain rule and the fact that $u^\mu \mathcal{J}_\mu = 0$. The final result is,

$$\tilde{\dot{\rho}}_e = -\Theta\rho_e - \tilde{\nabla}^\mu \mathcal{J}_\mu - \tilde{\dot{u}}^\mu \mathcal{J}_\mu. \tag{8.35}$$

8.3 Gravitational field

Information about the curvature of spacetime is contained in the Riemann tensor $R_{\sigma\mu\lambda\nu}$. We can can decompose it, just as we did for the other tensors. The usual decomposition is done into the Ricci tensor $R_{\mu\nu} = R^\lambda_{\ \mu\lambda\nu}$, the Ricci scalar $R = R^\mu_\mu$ and the Weyl tensor $C_{\sigma\mu\lambda\nu}$. The first two contain the traces of the Riemann tensor and the latter is the trace free part. The exact decomposition takes the form,

$$R_{\sigma\mu\lambda\nu} = C_{\sigma\mu\lambda\nu} + \frac{1}{2}\left(g_{\sigma\lambda}R_{\mu\nu} + g_{\mu\nu}R_{\sigma\lambda} - g_{\mu\lambda}R_{\sigma\nu} - g_{\sigma\nu}R_{\mu\lambda}\right) - \frac{1}{6}R\left(g_{\sigma\lambda}g_{\mu\nu} - g_{\sigma\nu}g_{\mu\lambda}\right). \tag{8.36}$$

Weyl Tensor

First we will look at the covariant description of the Weyl tensor. To get a covariant expression we must decompose the Weyl tensor further into its irreducible parts as,

$$C_{\sigma\mu\lambda\nu} = (g_{\sigma\mu\alpha\beta}g_{\lambda\nu\gamma\delta} - \eta_{\sigma\mu\alpha\beta}\eta_{\lambda\nu\gamma\delta})u^\alpha u^\gamma E^{\beta\delta} - (\eta_{\sigma\mu\alpha\beta}g_{\lambda\nu\gamma\delta} + g_{\sigma\mu\alpha\beta}\eta_{\lambda\nu\gamma\delta})u^\alpha u^\gamma H^{\beta\delta}, \tag{8.37}$$

where $g_{\sigma\mu\alpha\beta} = g_{\sigma\alpha}g_{\mu\beta} - g_{\sigma\beta}g_{\mu\alpha}$. The quantity $E_{\mu\nu}$ is called the electric Weyl component and is given by,

$$E_{\mu\nu} = C_{\mu\nu\rho\sigma}u^\rho u^\sigma. \tag{8.38}$$

The quantity $H_{\mu\nu}$ is the magnetic Weyl component and is given by,

$$H_{\mu\nu} = \frac{1}{2}\epsilon_\mu^{\ \sigma\rho}C_{\sigma\rho\nu\lambda}u^\lambda. \tag{8.39}$$

Because all the information about the contractions of the Riemann tensor is contained in the Ricci tensor and Ricci scalar, the Weyl tensor is trace free. This means that also the electric and magnetic Weyl components are trace free. Further we can deduce from the definitions of the later that $E_{\mu\nu}u^\nu = H_{\mu\nu}u^\nu = 0$.

Einstein equations

The next step is to obtain a covariant description of the Ricci tensor and Ricci scalar. We know from general relativity that the curvature of spacetime and matter influence each other. The exact relations between these two quantities are given by the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}. \tag{8.40}$$

Since we are interested in the evolution of magnetic fields in curved spacetime, we want the energy momentum tensor to contain both the fluid that is dominant in the universe and the electric and magnetic fields. Combining the expressions for the energy momentum tensor of a general fluid (8.15) and of the electromagnetic field (8.20) we have,

$$T_{\mu\nu} = \left(\rho + \frac{1}{2}B^2 + \frac{1}{2}E^2\right) u_\mu u_\nu + \left(p + \frac{1}{6}B^2 + \frac{1}{6}E^2\right) h_{\mu\nu} + 2\left(q_{(\mu} + \mathcal{Q}_{(\mu}\right) u_{\nu)} + \pi_{\mu\nu} + \mathcal{P}_{\mu\nu}. \tag{8.41}$$

To obtain an expression for the relation between the Ricci tensor and the properties of the fluids it will be convenient to rewrite the Einstein equations as,

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}, \tag{8.42}$$

where $T = T^\mu_\mu$ is the trace of the energy momentum tensor. One can check that this expression is correct by taking the trace of the original expression (8.40), which gives $R = -T$, and insert it into the equation above. The result is the original equation. The trace of (8.41) is given by,

$$T = -\rho + 3p. \tag{8.43}$$

We used the fact that $u_\mu u^\mu = -1$ and $h^\mu_\mu = 3$. Also we used that,

$$\begin{aligned} q_\mu u^\mu &= -u^\mu h^\nu_\mu T_{\nu\lambda} u^\lambda = 0, \\ \mathcal{Q}_\mu u^\mu &= u^\sigma u^\mu \eta_{\sigma\mu\nu\lambda} E^\nu B^\lambda = 0. \end{aligned} \tag{8.44}$$

We can now derive $1+3$ covariant equations from the Einstein equations by contracting with two times u^μ, once u^μ and once h^μ_ν or two times h^μ_ν. The first one is given by,

$$\begin{aligned} R_{\mu\nu} u^\mu u^\nu &= T_{\mu\nu} u^\mu u^\nu - \frac{1}{2} T g_{\mu\nu} u^\mu u^\nu, \\ &= \left(\rho + \frac{1}{2}B^2 + \frac{1}{2}E^2\right) + \frac{1}{2}\left(-\rho + 3p\right), \\ &= \frac{1}{2}\left(\rho + 3p + B^2 + E^2\right). \end{aligned} \tag{8.45}$$

In the second line we used the fact that $h_{\mu\nu}u^\mu = q_\mu u^\mu = \mathcal{Q}_\mu u^\mu = \pi_{\mu\nu}u^\mu = \mathcal{P}_{\mu\nu}u^\mu = 0$. The second relation is given by,

$$\begin{aligned} h^\nu_\mu R_{\nu\lambda}u^\lambda &= h^\nu_\mu T_{\nu\lambda}u^\lambda - \frac{1}{2}Th^\nu_\mu g_{\nu\lambda}u^\lambda, \\ &= 2h^\nu_\mu \left(q_{(\nu} + \mathcal{Q}_{(\nu)}\right) u_{\lambda)}u^\lambda - \frac{1}{2}Th_{\mu\lambda}u^\lambda, \\ &= -h^\nu_\mu \left(q_\nu + \mathcal{Q}_\nu\right), \\ &= -\left(q_\mu + \mathcal{Q}_\mu\right). \end{aligned} \tag{8.46}$$

In the last line we used the fact that $q_\mu u^\mu = \mathcal{Q}_\mu u^\mu = 0$. The last relation is,

$$\begin{aligned} h^\lambda_\mu h^\sigma_\nu R_{\lambda\sigma} &= h^\lambda_\mu h^\sigma_\nu T_{\lambda\sigma} - \frac{1}{2}Th^\lambda_\mu h^\sigma_\nu g_{\lambda\sigma}, \\ &= \left(p + \frac{1}{6}B^2 + \frac{1}{6}E^2\right) h_{\mu\nu} + h^\lambda_\mu h^\sigma_\nu \left(\pi_{\lambda\sigma} + \mathcal{P}_{\lambda\sigma}\right) - \frac{1}{2}Th_{\mu\nu}, \\ &= \left(\frac{1}{2}\rho - \frac{1}{2}p + \frac{1}{6}B^2 + \frac{1}{6}E^2\right) h_{\mu\nu} + \pi_{\mu\nu} + \mathcal{P}_{\mu\nu}. \end{aligned} \tag{8.47}$$

In the last line we used the explicit expressions for $\pi_{\lambda\sigma}$ and $\mathcal{P}_{\lambda\sigma}$ and the fact that $E_\mu u^\mu = B_\mu u^\mu = 0$.

Ricci identies

For any vector v^μ there exist the Ricci identity,

$$[\nabla_\mu, \nabla_\mu]\, v_\lambda = R_{\mu\nu\lambda\sigma}v^\sigma. \tag{8.48}$$

This identity also holds for the four-velocity,

$$[\nabla_\mu, \nabla_\mu]\, u_\lambda = R_{\mu\nu\lambda\sigma}u^\sigma. \tag{8.49}$$

From this identity a number of propagation and constraint equations can be derived. To find them one may project the identity in to the observers instantaneous rest space and parallel to the four-velocity. To evaluate the left hand side one can use the identity (8.14). The right hand side can be evaluated using the decomposition of the Riemann tensor (8.36), the decomposition of the Weyl tensor (8.37) and the identities (8.47), (8.45) and (8.43). Finally one can split the resulting equations into the trace, the symmetric trace free part and the antisymmetric part. The final results are three propagation equations,

$$\dot{\tilde{\Theta}} - \tilde{\nabla}_\mu \dot{\tilde{u}}^\mu = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\rho + 3p + E^2 + B^2) - 2(\sigma^2 + \omega^2) + \dot{\tilde{u}}_\mu \dot{\tilde{u}}^\mu, \tag{8.50}$$

$$\dot{\tilde{\omega}}_{<\mu>} + \frac{1}{2}\,\mathrm{curl}\dot{\tilde{u}}_\mu = -\frac{2}{3}\Theta\omega_\mu + \sigma_{\mu\nu}\omega^\nu, \tag{8.51}$$

$$\dot{\tilde{\sigma}}_{<\mu\nu>} - \tilde{\nabla}_{<\mu}\dot{\tilde{u}}_{\nu>} = -\frac{2}{3}\Theta\sigma_{\mu\nu} - \sigma_{\lambda<\mu}\sigma^\lambda_{\ \nu>} - \omega_{<\mu}\omega_{\nu>} - \dot{\tilde{u}}_{<\mu}\dot{\tilde{u}}_{\nu>} - E_{\mu\nu} + \frac{1}{2}\left(\pi_{\mu\nu} + \mathcal{P}_{\mu\nu}\right), \tag{8.52}$$

where $\sigma^2 = 1/2\sigma^{\mu\nu}\sigma_{\mu\nu}$ and $\omega^2 = 1/2\omega^{\mu\nu}\omega_{\mu\nu} = \omega^\mu\omega_\mu$. The other part of the result are three constraint equations,

$$0 = \frac{2}{3}\tilde{\nabla}_\mu\Theta - \tilde{\nabla}^\nu\sigma_{\mu\nu} + \mathrm{curl}\omega_\mu + 2\epsilon_{\mu\nu\sigma}\dot{\tilde{u}}^\nu\omega^\sigma - q_\mu - \mathcal{Q}_\mu, \tag{8.53}$$

$$0 = \tilde{\nabla}^\mu\omega_\mu - \dot{\tilde{u}}^\mu\omega_\mu, \tag{8.54}$$

$$0 = H_{\mu\nu} - \mathrm{curl}\sigma_{\mu\nu} - \tilde{\nabla}_{<\mu}\omega_{\nu>} - 2\dot{\tilde{u}}_{<\mu}\omega_{\nu>}. \tag{8.55}$$

It will turn out to be convenient to have a Ricci identity for the orthogonal projected derivatives of the electric and magnetic field. The identity for the electric field is [31],

$$\left[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\right] E_\sigma = -2\epsilon_{\mu\nu\lambda}\omega^\lambda \dot{\tilde{E}}_{<\sigma>} + \mathcal{R}_{\lambda\sigma\nu\mu}E^\lambda. \tag{8.56}$$

The same equation holds for the magnetic field. $\mathcal{R}_{\lambda\sigma\nu\mu}$ is the orthogonal projected Riemann tensor defined by,

$$\mathcal{R}_{\lambda\sigma\nu\mu} = h^\alpha_\lambda h^\beta_\sigma h^\gamma_\nu h^\delta_\mu R_{\alpha\beta\gamma\delta} - v_{\lambda\nu}v_{\sigma\mu} + v_{\lambda\mu}v_{\sigma\nu}, \tag{8.57}$$

where $v_{\mu\nu} = \tilde{\nabla}_\nu u_\mu$.

8.4 Wave equation of the magnetic field in curved spacetime

We now have all the ingredients to derive how the magnetic field evolves in a general spacetime. This evolution is described by the wave equation for the magnetic field, therefore we want an equation of the form $\ddot{\tilde{B}}_{<\mu>} - \tilde{\nabla}^2 B_\mu = (...)$. We will derive the wave equation for the case that both the background matter and the electromagnetic field have a perfect fluid form, ($q_\mu = \pi_{\mu\nu} = \mathcal{Q}_\mu = \mathcal{P}_{\mu\nu} = 0$). We will also assume that the background fluid is barotropic with the explicit relation $p = \gamma\rho$. To find the wave equation we must take the time derivative of equation (8.29) and project it orthogonal to u_μ. If we do this we are calculating,

$$h_\mu^{\ \nu}u^\lambda\nabla_\lambda h_\nu^{\ \sigma}u^\rho\nabla_\rho B_\sigma = \ddot{\tilde{B}}_{<\mu>} + \dot{\tilde{u}}_\mu u_\nu \dot{\tilde{B}}^\nu. \tag{8.58}$$

Under these operations the right hand side of equation (8.29) will become,

$$h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \left[\left(\sigma_{\sigma\nu} + \epsilon_{\sigma\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\sigma\nu} \right) B^\nu \right] - h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \epsilon_{\sigma\nu\rho} \tilde{\dot{u}}^\nu E^\rho - h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \epsilon_{\sigma\nu\rho} \tilde{\nabla}^\nu E^\rho. \tag{8.59}$$

We will evaluate this term by term. First we will let the derivative act on the terms in the brackets,

$$\begin{aligned} h_\mu^{\ \sigma} \tilde{\dot{\sigma}}_{\sigma\nu} B^\nu &= \tilde{\dot{\sigma}}_{<\mu\nu>} B^\nu, \\ &= -\frac{2}{3}\Theta\sigma_{\mu\nu}B^\nu - \sigma_{\rho<\mu}\sigma^\rho_{\ \nu>}B^\nu - \omega_{<\mu}\omega_{\nu>}B^\nu + \tilde{\nabla}_{<\mu}\tilde{\dot{u}}_{\nu>}B^\nu \\ &\quad + \tilde{\dot{u}}_{<\mu}\tilde{\dot{u}}_{\nu>}B^\nu - E_{\mu\nu}B^\nu. \end{aligned} \tag{8.60}$$

In the second line we used propagation equation (8.52) and $h_{\mu\nu}B^\nu = B_\mu$. We can rewrite,

$$\tilde{\nabla}_{<\mu}\tilde{\dot{u}}_{\nu>}B^\nu = \frac{1}{2}\left(\tilde{\nabla}_\mu\tilde{\dot{u}}_\nu + \tilde{\nabla}_\nu\tilde{\dot{u}}_\mu\right)B^\nu - \frac{1}{3}B_\mu\tilde{\nabla}^\nu\tilde{\dot{u}}_\nu. \tag{8.61}$$

The term with the time derivative of $\epsilon_{\sigma\nu\rho}$ is zero. The next term in the brackets is,

$$\begin{aligned} h_\mu^{\ \sigma}\epsilon_{\sigma\nu\rho}\tilde{\dot{\omega}}^\rho B^\nu &= \epsilon_{\mu\nu\rho}\tilde{\dot{\omega}}^{<\rho>}B^\nu, \\ &= -\frac{2}{3}\Theta\epsilon_{\mu\nu\rho}\omega^\rho B^\nu - \frac{1}{2}\epsilon_{\mu\nu\rho}B^\nu\,\mathrm{curl}\tilde{\dot{u}}^\rho + \epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\omega_\sigma B^\nu. \end{aligned} \tag{8.62}$$

In the second line we used propagation equation (8.51). The second term in this expression can be rewritten using the identity,

$$\begin{aligned} \epsilon_{\mu\nu\rho}\epsilon^{\rho\sigma\lambda} &= u^\tau u_\kappa \eta_{\rho\tau\mu\nu}\eta^{\rho\kappa\sigma\lambda}, \\ &= u^k u_\kappa(-1)3!\ \delta_k^{[\kappa}\delta_\mu^\sigma\delta_\nu^{\lambda]}, \\ &= \delta_\mu^\sigma\delta_\nu^\lambda + u^\sigma u_\mu\delta_\lambda^\nu - u^\sigma u_\nu\delta_\mu^\lambda + u^\lambda u_\nu\delta_\mu^\sigma - u^\lambda u_\mu\delta_\nu^\sigma - \delta_\mu^\lambda\delta_\nu^\sigma, \end{aligned} \tag{8.63}$$

as,

$$-\frac{1}{2}\epsilon_{\mu\nu\rho}B^\nu\,\mathrm{curl}\tilde{\dot{u}}^\rho = \frac{1}{2}\left(\tilde{\nabla}_\nu\tilde{\dot{u}}_\mu - \tilde{\nabla}_\mu\tilde{\dot{u}}_\nu\right)B^\nu. \tag{8.64}$$

The last term in the brackets is given by,

$$\begin{aligned} -\frac{2}{3}h_\mu^{\ \sigma}\tilde{\dot{\Theta}}h_{\sigma\nu}B^\nu &= -\frac{2}{3}\tilde{\dot{\Theta}}B_\mu, \\ &= \frac{2}{9}\Theta^2 B_\mu + \frac{1}{3}\left[\rho(1+3\gamma) + E^2 + B^2\right]B_\mu + \frac{4}{3}(\sigma^2 - \omega^2)B_\mu \\ &\quad - \frac{2}{3}\tilde{\nabla}^\nu\tilde{\dot{u}}_\nu B_\mu - \frac{2}{3}\tilde{\dot{u}}_\nu\tilde{\dot{u}}^\nu B_\mu. \end{aligned} \tag{8.65}$$

To obtain this we used propagation equation (8.50). To complete the calculation of the first term of expression (8.59) we may let the derivative act on B^ν. This will give the term,

$$h_\mu^{\ \sigma}\left(\sigma_{\sigma\nu} + \epsilon_{\sigma\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\sigma\nu}\right)\tilde{\dot{B}}^\nu = \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\mu\nu}\right)\tilde{\dot{B}}^\nu. \tag{8.66}$$

Combining the results we arrive at an expression for the first term:

$$
\begin{aligned}
&h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \left[\left(\sigma_{\sigma\nu} + \epsilon_{\sigma\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\sigma\nu} \right) B^\nu \right] \\
&\quad = \frac{1}{3}\left[\rho(1+3\gamma) + E^2 + B^2\right] B_\mu + \left(\sigma_{\mu\nu} + \epsilon_{\mu\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\mu\nu} \right) \dot{\tilde{B}}^\nu \\
&\qquad - \frac{2}{3}\left(\Theta\sigma_{\mu\nu} + \epsilon_{\mu\nu\rho}\omega^\rho - \frac{1}{3}\Theta^2 h_{\mu\nu} \right) B^\nu + \frac{4}{3}(\sigma^2 - \omega^2) B_\mu - \sigma_{\rho<\mu}\sigma^\rho_{\ \nu>} B^\nu - \omega_{<\mu}\omega_{\nu>} B^\nu \\
&\qquad + \epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\omega_\sigma B^\nu - E_{\mu\nu} B^\nu + B^\nu \tilde{\nabla}_\nu \dot{\tilde{u}}_\mu - \tilde{\nabla}^\nu \dot{\tilde{u}}_\nu B_\mu + \dot{\tilde{u}}_{<\mu}\dot{\tilde{u}}_{\nu>} B^\nu - \frac{2}{3}\dot{\tilde{u}}_\nu \dot{\tilde{u}}^\nu B_\mu.
\end{aligned} \tag{8.67}
$$

The second term in expression (8.59) is,

$$
- h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \epsilon_{\sigma\nu\rho} \dot{\tilde{u}}^\nu E^\rho = -\epsilon_{\mu\nu\rho} \ddot{\tilde{u}}^\nu E^\rho - h_\mu^\sigma \epsilon_{\sigma\nu\rho} \dot{\tilde{u}}^\nu \dot{\tilde{E}}^\rho. \tag{8.68}
$$

The term with the time derivative of $\epsilon_{\mu\nu\rho}$ is zero. The second term on the right hand side can be rewritten using the propagation equation for the electric field (8.28),

$$
\begin{aligned}
-h_\mu^{\ \sigma} \epsilon_{\sigma\nu\rho} \dot{\tilde{u}}^\nu \dot{\tilde{E}}^\rho &= -\epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \dot{\tilde{E}}^{<\rho>}, \\
&= -\epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \left(\sigma^{\rho\sigma} + \epsilon^{\rho\sigma\lambda}\omega_\lambda - \frac{2}{3}\Theta h^{\rho\sigma} \right) E_\sigma - \epsilon_{\mu\nu\rho}\epsilon^{\rho\sigma\lambda} \dot{\tilde{u}}^\nu \dot{\tilde{u}}_\sigma B_\lambda \\
&\quad - \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \mathrm{curl} B^\rho + \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \mathcal{J}^\rho.
\end{aligned} \tag{8.69}
$$

The second term in the brackets can be rewritten using identity (8.63) and the fact that $u_\mu E^\mu = u_\mu \omega^\mu = \dot{\tilde{u}}^\mu u_\mu = 0$,

$$
- \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \epsilon^{\rho\sigma\lambda} \omega_\lambda E_\sigma = \omega_\mu \dot{\tilde{u}}^\nu E_\nu - \dot{\tilde{u}}^\nu \omega_\nu E_\mu. \tag{8.70}
$$

In the same way we can rewrite the second term,

$$
\begin{aligned}
-\epsilon_{\mu\nu\rho}\epsilon^{\rho\sigma\lambda} \dot{\tilde{u}}^\nu \dot{\tilde{u}}_\sigma B_\lambda &= -\dot{\tilde{u}}_\mu \dot{\tilde{u}}^\nu B_\nu + \dot{\tilde{u}}_\nu \dot{\tilde{u}}^\nu B_\mu, \\
&= -\dot{\tilde{u}}_{<\mu} \dot{\tilde{u}}_{\nu>} B^\nu + \frac{2}{3} \dot{\tilde{u}}_\nu \dot{\tilde{u}}^\nu B_\mu,
\end{aligned} \tag{8.71}
$$

where we used that $u_\mu B^\mu = \dot{\tilde{u}}^\mu u_\mu = 0$. We can also rewrite the third term in the same manner,

$$
- \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \mathrm{curl} B^\rho = \dot{\tilde{u}}^\nu \tilde{\nabla}_\nu B_\mu - \dot{\tilde{u}}^\nu \tilde{\nabla}_\mu B_\nu. \tag{8.72}
$$

Finally the total second term of expression (8.59) is,

$$
\begin{aligned}
-h_\mu^{\ \sigma} u^\lambda \nabla_\lambda \epsilon_{\sigma\nu\rho} \dot{\tilde{u}}^\nu E^\rho = &- \epsilon_{\mu\nu\rho} \ddot{\tilde{u}}^\nu E^\rho - \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \left(\sigma^{\rho\sigma} - \frac{2}{3}\Theta h^{\rho\sigma} \right) E_\sigma + \dot{\tilde{u}}^\nu \tilde{\nabla}_\nu B_\mu - \dot{\tilde{u}}^\nu \tilde{\nabla}_\mu B_\nu \\
&+ \epsilon_{\mu\nu\rho} \dot{\tilde{u}}^\nu \mathcal{J}^\rho + \omega_\mu \dot{\tilde{u}}^\nu E_\nu - \dot{\tilde{u}}^\nu \omega_\nu E_\mu - \dot{\tilde{u}}_{<\mu} \dot{\tilde{u}}_{\nu>} B^\nu + \frac{2}{3} \dot{\tilde{u}}_\nu \dot{\tilde{u}}^\nu B_\mu.
\end{aligned} \tag{8.73}
$$

The third term of expression (8.59) is a bit more complicated. It is given by,

$$\begin{aligned}
-h^{\sigma}_{\mu}u^{\lambda}\nabla_{\lambda}\epsilon_{\sigma\nu\rho}\tilde{\nabla}^{\nu}E^{\rho} &= -\epsilon_{\mu\nu\rho}u^{\lambda}\nabla_{\lambda}h^{\nu}_{\ \kappa}h^{\rho}_{\ \tau}\nabla^{\kappa}E^{\tau}, \\
&= -\epsilon_{\mu\nu\rho}u^{\lambda}(\nabla_{\lambda}h^{\nu}_{\ \kappa})\nabla^{\kappa}E^{\rho} - \epsilon_{\mu\nu\rho}u^{\lambda}(\nabla_{\lambda}h^{\rho}_{\ \tau})\nabla^{\nu}E^{\tau} \\
&\quad - \epsilon_{\mu\nu\rho}u^{\lambda}\nabla_{\lambda}\nabla^{\nu}E^{\rho}, \\
&= -\epsilon_{\mu\nu\rho}\tilde{\dot{u}}^{\nu}\dot{E}^{\rho} + \epsilon_{\mu\nu\rho}\tilde{\dot{u}}^{\rho}E^{\sigma}(\nabla^{\nu}u_{\sigma}) - \epsilon_{\mu\nu\rho}u^{\lambda}\left[\nabla_{\lambda},\nabla^{\nu}\right]E^{\rho} \\
&\quad - \epsilon_{\mu\nu\rho}u^{\lambda}\nabla^{\nu}\nabla_{\lambda}E^{\rho}. \qquad (8.74)
\end{aligned}$$

We will again evaluate this expression term by term. The first term is exactly the same as expression (8.69). The second term can be found with the help of (8.14),

$$\epsilon_{\mu\nu\rho}\tilde{\dot{u}}^{\rho}E^{\sigma}(\nabla^{\nu}u_{\sigma}) = \epsilon_{\mu\nu\rho}E_{\sigma}\tilde{\dot{u}}^{\rho}\sigma^{\nu\sigma} + \tilde{\dot{u}}^{\nu}E_{\mu}\omega_{\nu} - \tilde{\dot{u}}^{\nu}E_{\nu}\omega_{\mu} + \frac{1}{3}\epsilon_{\mu\nu\rho}\Theta E_{\nu}\tilde{\dot{u}}^{\rho}. \qquad (8.75)$$

We used the fact that $\omega^{\nu\sigma} = \epsilon^{\lambda\nu\sigma}\omega_{\lambda}$. The third term can be found using the Ricci identity (8.48),

$$-\epsilon_{\mu\nu\rho}u^{\lambda}\left[\nabla_{\lambda},\nabla^{\nu}\right]E^{\rho} = -\epsilon_{\mu\nu\rho}u^{\lambda}R_{\sigma\ \ \lambda}^{\ \rho\nu}E^{\sigma}. \qquad (8.76)$$

Since the Riemann tensor is coordinate dependent we want to rewrite it in terms of covariant quantities. This can be done using the relations (8.36) and (8.37). This gives,

$$\begin{aligned}
-\epsilon_{\mu\nu\rho}u^{\lambda}R_{\sigma\ \ \lambda}^{\ \rho\nu}E^{\sigma} &= -\epsilon_{\mu\nu\rho}u_{\lambda}E_{\sigma}C^{\sigma\rho\nu\lambda} - \frac{1}{2}\epsilon_{\mu\nu\rho}u_{\lambda}E^{\nu}R^{\rho\lambda}, \\
&= -\epsilon_{\mu\nu\rho}\epsilon^{\sigma\rho\tau}E_{\sigma}H_{\tau}^{\ \nu} + 0, \\
&= H_{\mu\nu}E^{\nu}. \qquad (8.77)
\end{aligned}$$

In the second line we used the relation (8.46) and the fact that $\mathcal{Q}_{\mu} = 0$, since we were looking at perfect fluids. The last term of expression (8.74) is,

$$\begin{aligned}
-\epsilon_{\mu\nu\rho}u^{\lambda}\nabla^{\nu}\nabla_{\lambda}E^{\rho} &= -\epsilon_{\mu\nu\rho}\nabla^{\nu}\tilde{\dot{E}}^{\rho} + \epsilon_{\mu\nu\rho}(\nabla^{\nu}u^{\lambda})\nabla_{\lambda}E^{\rho}, \\
&= -\epsilon_{\mu\nu\rho}\nabla^{\nu}\tilde{\dot{E}}^{<\rho>} + \epsilon_{\mu\nu\rho}u_{\sigma}\tilde{\dot{E}}^{\sigma}\nabla^{\nu}u^{\rho} + \epsilon_{\mu\nu\rho}(\nabla^{\nu}u^{\lambda})\nabla_{\lambda}E^{\rho}. \qquad (8.78)
\end{aligned}$$

The second of these terms can be evaluated using identity (8.14), the fact that $\omega^{\nu\rho} = \epsilon^{\mu\nu\rho}\omega_{\mu}$ and identity (8.63),

$$\begin{aligned}
\epsilon_{\mu\nu\rho}u_{\sigma}\tilde{\dot{E}}^{\sigma}\nabla^{\nu}u^{\rho} &= \epsilon_{\mu\nu\rho}u_{\sigma}\tilde{\dot{E}}^{\sigma}\omega^{\rho\nu}, \\
&= -2u_{\nu}\tilde{\dot{E}}^{\nu}\omega_{\mu}, \\
&= 2E^{\nu}\tilde{\dot{u}}_{\nu}\omega_{\mu}. \qquad (8.79)
\end{aligned}$$

The last term in expression (8.78) can be calculated using the same identities,

$$\begin{aligned}
\epsilon_{\mu\nu\rho}(\nabla^{\nu}u^{\lambda})\nabla_{\lambda}E^{\rho} &= \epsilon_{\mu\nu\rho}\nabla_{\lambda}E^{\rho}\left(\sigma^{\nu\lambda} - \omega^{\nu\lambda} + \frac{1}{3}\Theta h^{\nu\lambda}\right), \\
&= \epsilon_{\mu\nu\rho}\sigma^{\nu\lambda}\tilde{\nabla}_{\lambda}E^{\rho} - \omega_{\mu}\tilde{\nabla}_{\nu}E^{\nu} + \omega_{\nu}\tilde{\nabla}_{\mu}E^{\nu} + \frac{1}{3}\Theta\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}E^{\rho}. \qquad (8.80)
\end{aligned}$$

The first term of expression (8.78) can be calculated with the propagation equation (8.28),

$$-\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\dot{\tilde{E}}^{<\rho>} = -\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\left(\sigma^{\rho\sigma} + \epsilon^{\rho\sigma\lambda}\omega_{\lambda} - \frac{2}{3}\Theta h^{\rho\sigma}\right)E_{\sigma} - \epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\epsilon^{\rho\sigma\lambda}\dot{\tilde{u}}_{\sigma}B_{\lambda}$$
$$- \epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\mathrm{curl}B^{\rho} + \epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\mathcal{J}^{\rho}. \tag{8.81}$$

The term in the brackets can be written out as,

$$-\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\left(\sigma^{\rho\sigma} + \epsilon^{\rho\sigma\lambda}\omega_{\lambda} - \frac{2}{3}\Theta h^{\rho\sigma}\right)E_{\sigma}$$
$$= -\epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\tilde{\nabla}^{\nu}E_{\sigma} - \omega_{\nu}\tilde{\nabla}^{\nu}E_{\mu} + \omega_{\mu}\tilde{\nabla}^{\nu}E_{\nu} + \frac{2}{3}\epsilon_{\mu\nu\rho}\Theta\tilde{\nabla}^{\nu}E^{\rho} - \epsilon_{\mu\nu\rho}E_{\sigma}\tilde{\nabla}^{\nu}\sigma^{\rho\sigma}$$
$$- E_{\mu}\tilde{\nabla}^{\nu}\omega_{\nu} + E_{\nu}\tilde{\nabla}^{\nu}\omega_{\mu} + \frac{2}{3}\epsilon_{\mu\nu\rho}E^{\rho}\tilde{\nabla}^{\nu}\Theta. \tag{8.82}$$

The second term is,

$$-\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\epsilon^{\rho\sigma\lambda}\dot{\tilde{u}}_{\sigma}B_{\lambda} = \tilde{\nabla}^{\nu}\left(\dot{\tilde{u}}_{\nu}B_{\mu}\right) - \tilde{\nabla}^{\nu}\left(\dot{\tilde{u}}_{\mu}B_{\nu}\right),$$
$$= B_{\mu}\tilde{\nabla}^{\nu}\dot{\tilde{u}}_{\nu} - B_{\nu}\tilde{\nabla}^{\nu}\dot{\tilde{u}}_{\mu} + \dot{\tilde{u}}_{\nu}\tilde{\nabla}^{\nu}B_{\mu} - \dot{\tilde{u}}_{\mu}\tilde{\nabla}^{\nu}B_{\nu}. \tag{8.83}$$

The third term is,

$$-\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\mathrm{curl}B^{\rho} = -\tilde{\nabla}^{\nu}\tilde{\nabla}_{\mu}B_{\nu} + \tilde{\nabla}^{2}B_{\mu},$$
$$= -\left[\tilde{\nabla}^{\nu}, \tilde{\nabla}_{\mu}\right]B_{\nu} - \tilde{\nabla}_{\mu}\tilde{\nabla}^{\nu}B_{\nu} + \tilde{\nabla}^{2}B_{\mu},$$
$$= -2\epsilon_{\mu\nu\rho}\omega^{\rho}\dot{\tilde{B}}^{\nu} - \mathcal{R}_{\mu\nu}B^{\nu} - 2E_{\nu}\tilde{\nabla}_{\mu}\omega^{\nu} - 2\omega^{\nu}\tilde{\nabla}_{\mu}E_{\nu} + \tilde{\nabla}^{2}B_{\mu}. \tag{8.84}$$

In the last line we used identities (8.56) and (8.31). Finally we can combine these results to get the the total third term of expression (8.59). The result is,

$$-h_{\mu}^{\ \sigma}u^{\lambda}\nabla_{\lambda}\epsilon_{\sigma\nu\rho}\tilde{\nabla}^{\nu}E^{\rho} = \epsilon_{\mu\nu\rho}\dot{\tilde{u}}^{\nu}\mathcal{J}^{\rho} - \dot{\tilde{u}}_{<\mu}\dot{\tilde{u}}_{\nu>}B^{\nu} + \frac{2}{3}\dot{\tilde{u}}_{\nu}\dot{\tilde{u}}^{\nu}B_{\mu} - 2\epsilon_{\mu\nu\rho}\omega^{\rho}\dot{\tilde{B}}^{\nu} - \mathcal{R}_{\mu\nu}B^{\nu}$$
$$+ 2\epsilon_{\mu\nu\rho}E_{\sigma}\dot{\tilde{u}}^{\rho}\sigma^{\nu\sigma} - \frac{1}{3}\epsilon_{\mu\nu\rho}\Theta E_{\nu}\dot{\tilde{u}}^{\rho} + H_{\mu\nu}E^{\nu} + \epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}\mathcal{J}^{\rho}$$
$$+ 2E^{\nu}\dot{\tilde{u}}_{\nu}\omega_{\mu} + \epsilon_{\mu\nu\rho}\sigma^{\nu\lambda}\tilde{\nabla}_{\lambda}E^{\rho} - \epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\tilde{\nabla}^{\nu}E_{\sigma} + \Theta\epsilon_{\mu\nu\rho}\tilde{\nabla}^{\nu}E^{\rho}$$
$$- \epsilon_{\mu\nu\rho}E_{\sigma}\tilde{\nabla}^{\nu}\sigma^{\rho\sigma} + \frac{2}{3}\epsilon_{\mu\nu\rho}E^{\rho}\tilde{\nabla}^{\nu}\Theta + \tilde{\nabla}^{2}B_{\mu}$$
$$+ B_{\mu}\tilde{\nabla}^{\nu}\dot{\tilde{u}}_{\nu} - B_{\nu}\tilde{\nabla}^{\nu}\dot{\tilde{u}}_{\mu} + 2\dot{\tilde{u}}_{\nu}\tilde{\nabla}^{\nu}B_{\mu} - \dot{\tilde{u}}_{\mu}\tilde{\nabla}^{\nu}B_{\nu} - \dot{\tilde{u}}^{\nu}\tilde{\nabla}_{\mu}B_{\nu}$$
$$- E_{\mu}\tilde{\nabla}^{\nu}\omega_{\nu} + E_{\nu}\tilde{\nabla}^{\nu}\omega_{\mu} - 2E_{\nu}\tilde{\nabla}_{\mu}\omega^{\nu} - \omega^{\nu}\tilde{\nabla}_{\mu}E_{\nu} - \omega_{\nu}\tilde{\nabla}^{\nu}E_{\mu}. \tag{8.85}$$

The last step is to combine the results of the three terms (8.67), (8.73) and (8.85). The final result is,

$$
\begin{aligned}
\tilde{\dot{B}}_{<\mu>} - \tilde{\nabla}^2 B_\mu =& \frac{1}{3}\left[\rho(1+3\gamma) + E^2 + B^2\right] B_\mu + \left(\sigma_{\mu\nu} - \epsilon_{\mu\nu\rho}\omega^\rho - \frac{2}{3}\Theta h_{\mu\nu}\right)\tilde{\dot{B}}^\nu \\
& - \frac{2}{3}\left(\Theta\sigma_{\mu\nu} + \epsilon_{\mu\nu\rho}\omega^\rho - \frac{1}{3}\Theta^2 h_{\mu\nu}\right) B^\nu + \frac{4}{3}(\sigma^2 - \omega^2) B_\mu - \sigma_{\rho<\mu}\sigma^\rho_{\ \nu>} B^\nu \\
& - \omega_{<\mu}\omega_{\nu>} B^\nu + \epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\omega_\sigma B^\nu - E_{\mu\nu}B^\nu + \epsilon_{\mu\nu\rho}\tilde{\nabla}^\nu \mathcal{J}^\rho + 2\epsilon_{\mu\nu\rho}\tilde{\dot{u}}^\nu \mathcal{J}^\rho \\
& + \tilde{\dot{u}}_\nu \tilde{\dot{u}}^\nu B_\mu + 3\tilde{\dot{u}}_\nu \tilde{\nabla}^\nu B_\mu - \tilde{\dot{u}}_\mu \tilde{\nabla}^\nu B_\nu - 2\tilde{\dot{u}}^\nu \tilde{\nabla}_\mu B_\nu - \mathcal{R}_{\mu\nu}B^\nu + H_{\mu\nu}E^\nu \\
& + 3\epsilon_{\mu\nu\rho}E_\sigma \tilde{\dot{u}}^\rho \sigma^{\nu\sigma} - \epsilon_{\mu\nu\rho}\Theta E_\nu \tilde{\dot{u}}^\rho + 3E^\nu \tilde{\dot{u}}_\nu \omega_\mu - \tilde{\dot{u}}^\nu \omega_\nu B_\mu - \epsilon_{\mu\nu\rho}\tilde{\ddot{u}}^\nu E^\rho \\
& + \epsilon_{\mu\nu\rho}\sigma^{\nu\lambda}\tilde{\nabla}_\lambda E^\rho - \epsilon_{\mu\nu\rho}\sigma^{\rho\sigma}\tilde{\nabla}^\nu E_\sigma + \Theta\epsilon_{\mu\nu\rho}\tilde{\nabla}^\nu E^\rho - \epsilon_{\mu\nu\rho}E_\sigma \tilde{\nabla}^\nu \sigma^{\rho\sigma} \\
& + \frac{2}{3}\epsilon_{\mu\nu\rho}E^\rho \tilde{\nabla}^\nu \Theta - E_\mu \tilde{\nabla}^\nu \omega_\nu + E_\nu \tilde{\nabla}^\nu \omega_\mu - 2E_\nu \tilde{\nabla}_\mu \omega^\nu - \omega^\nu \tilde{\nabla}_\mu E_\nu \\
& - \omega_\nu \tilde{\nabla}^\nu E_\mu.
\end{aligned} \tag{8.86}
$$

8.5 Evolution of the magnetic field in a curved FLRW spacetime

In the previous section we found the wave equation for the magnetic field in a general curved spacetime. We are interested in cosmological magnetic fields, therefore we want to consider a FLRW spacetime. We already saw in the previous chapter that, if we consider a spacetime that is conformal to Minkowski spacetime, we will not obtain interesting magnetic fields. On the other had we want our model to be compatible with the observations of the curvature of our universe. Taking both arguments into account we will look at an isotropic and homogeneous FLRW spacetime with curved spatial sections. When the model is isotropic the universe looks the same in every direction. Recall that the dynamics of the background fluid where governed by the quantities Θ, $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$ and $\tilde{\dot{u}}$. If the fluid is isotropic $\sigma_{\mu\nu} = \omega_{\mu\nu} = \tilde{\dot{u}} = 0$. When the background spacetime is isotropic the Weyl tensor vanishes, so $E_{\mu\nu} = H_{\mu\nu} = 0$. If the model is homogeneous the universe looks the same everywhere. This means that the properties of the background fluid are the same everywhere, so $\tilde{\nabla}\rho = 0$. We do not need to specify the other quantities since our model had a perfect barotropic fluid. The fact that the model is homogeneous also means that the expansion of the universe must be the same everywhere, $\tilde{\nabla}\Theta = 0$. Since the conductivity during inflation is negligible expression (8.27) tells us that $\mathcal{J}_\mu = 0$. If we put these requirements into the wave equation of the magnetic field (8.86) the result is,

$$\tilde{\ddot{B}}_\mu - \tilde{\nabla}^2 B_\mu = \frac{1}{3}\left[\rho(1+3\gamma) + E^2 + B^2\right] B_\mu - \frac{2}{3}\Theta\tilde{\dot{B}}_\mu + \frac{2}{9}\Theta^2 B_\mu - \mathcal{R}_{\mu\nu}B^\nu + \Theta\epsilon_{\mu\nu\lambda}\tilde{\nabla}^\nu E^\lambda. \tag{8.87}$$

In the first term on the left hand side we have used the fact that $\tilde{\dot{u}}_\mu = 0$ and therefore the angle brackets have vanished. The last term can be rewritten using the propagation equation for the magnetic field (8.29), which in this model reduces to,

$$\tilde{\dot{B}}_\mu = -\frac{2}{3}\Theta B_\mu - \epsilon_{\mu\nu\lambda}\tilde{\nabla}^\nu E^\lambda. \tag{8.88}$$

If we insert this into expression (8.87) the result is,

$$\tilde{\ddot{B}}_\mu - \tilde{\nabla}^2 B_\mu = \frac{1}{3}\left[\rho(1+3\gamma) + E^2 + B^2\right] B_\mu - \frac{5}{3}\Theta\tilde{\dot{B}}_\mu - \frac{4}{9}\Theta^2 B_\mu - \mathcal{R}_{\mu\nu}B^\nu. \tag{8.89}$$

The first term on the right hand side can be rewritten using the propagation equation for Θ (8.50), which in this model reduces to,

$$\tilde{\dot{\Theta}} = -\frac{1}{3}\Theta^2 - \frac{1}{2}\left[\rho(1+3\gamma) + E^2 + B^2\right]. \tag{8.90}$$

The wave equation becomes,

$$\tilde{\ddot{B}}_\mu - \tilde{\nabla}^2 B_\mu = -\frac{5}{3}\Theta\tilde{\dot{B}}_\mu - \frac{2}{3}\Theta^2 B_\mu - \frac{2}{3}\tilde{\dot{\Theta}} B_\mu - \mathcal{R}_{\mu\nu}B^\nu. \tag{8.91}$$

The next step is to obtain an expression for $\mathcal{R}_{\mu\nu}$. From definition (8.57) we know that,

$$\begin{aligned}
\mathcal{R}^\lambda{}_{\sigma\lambda\mu} &= h^\lambda_\alpha h^\beta_\sigma h^\gamma_\lambda h^\delta_\mu R^\alpha{}_{\beta\gamma\delta} - v^\lambda{}_\lambda v_{\sigma\mu} + v^\lambda{}_\mu v_{\sigma\lambda}, \\
&= h^\gamma_\alpha h^\beta_\sigma h^\delta_\mu R^\alpha{}_{\beta\gamma\delta} + \frac{2}{9}\Theta^2 h_{\sigma\mu}, \\
&= h^\beta_\sigma h^\delta_\mu R_{\beta\delta} + u^\gamma u_\alpha h^\beta_\sigma h^\delta_\mu R^\alpha{}_{\beta\gamma\delta} + \frac{2}{9}\Theta^2 h_{\sigma\mu}.
\end{aligned} \tag{8.92}$$

In the second line we used the explicit expression for $v_{\mu\nu} = \tilde{\nabla}_\nu u_\mu$ and identity (8.14) which in this model reduces to,

$$\tilde{\nabla}_\mu u_\nu = \frac{1}{3}\Theta h_{\mu\nu}. \tag{8.93}$$

The second term can be evaluated using identity (8.36), where one may recall that the Weyl tensor is zero,

$$u^\gamma u_\alpha h^\beta_\sigma h^\delta_\mu R^\alpha{}_{\beta\gamma\delta} = -\frac{1}{2}h^\beta_\sigma h^\delta_\mu R_{\beta\delta} + \frac{1}{2}u^\alpha u^\gamma h_{\sigma\mu}R_{\alpha\gamma} + \frac{1}{6}R h_{\sigma\mu}. \tag{8.94}$$

The terms on the right hand side and the first term of equation (8.92) can be evaluated using expressions (8.47) and (8.45),

$$\mathcal{R}_{\mu\nu} = \frac{1}{3}\left(E^2 + B^2 + 2\rho + \frac{2}{3}\Theta^2\right) h_{\mu\nu}. \tag{8.95}$$

We can contract the indices to obtain an expression for $\mathcal{R}$:

$$\mathcal{R} = E^2 + B^2 + 2\rho + \frac{2}{3}\Theta^2. \tag{8.96}$$

This shows that we can write $\mathcal{R}_{\mu\nu} = 1/3\mathcal{R}h_{\mu\nu}$. We can also explicitly calculate $\mathcal{R}$ with definition (8.57),

$$\begin{aligned}\mathcal{R}^{\lambda\mu}{}_{\lambda\mu} &= h^{\lambda}_{\alpha} h^{\mu}_{\beta} h^{\gamma}_{\lambda} h^{\delta}_{\mu} R^{\alpha\beta}{}_{\gamma\delta} - v^{\lambda}{}_{\lambda} v^{\mu}{}_{\mu} + v^{\lambda}{}_{\mu} v^{\mu}{}_{\lambda}, \\ &= h^{\gamma}_{\alpha} h^{\delta}_{\beta} R^{\alpha\beta}{}_{\gamma\delta} - \Theta^2 + \frac{1}{3}\Theta^2. \end{aligned} \tag{8.97}$$

The first term can be evaluated using identity (8.36), just as we did in the case of $\mathcal{R}_{\mu\nu}$,

$$\begin{aligned} h^{\gamma}_{\alpha} h^{\delta}_{\beta} R^{\alpha\beta}{}_{\gamma\delta} &= 2h^{\alpha\beta} R_{\alpha\beta} - R, \\ &= R + 2u^{\alpha} u^{\beta} R_{\mu\nu} \\ &= R + \rho\,(1 + 3\gamma) + E^2 + B^2. \end{aligned} \tag{8.98}$$

In the last line we used identity (8.45). The right hand side of expression (8.98) can be rewritten using the propagation equation for Θ (8.90). Finally $\mathcal{R}$ is given by,

$$\mathcal{R} = R - \frac{4}{3}\Theta^2 - 2\dot{\tilde{\Theta}}. \tag{8.99}$$

If we combine this result with the fact that $\mathcal{R}_{\mu\nu} = 1/3\mathcal{R}h_{\mu\nu}$, the wave equation for the magnetic field becomes,

$$\ddot{\tilde{B}}_{\mu} - \tilde{\nabla}^2 B_{\mu} = -\frac{5}{3}\Theta \dot{\tilde{B}}_{\mu} - \frac{2}{9}\Theta^2 B_{\mu} - \frac{1}{3} R B_{\mu}. \tag{8.100}$$

At this point we would like to make the connection with the previous chapters. We want to describe the field seen by a co-moving observer, $u^{\mu} = (1, \mathbf{0})$. The covariant time derivative reduces to $\dot{\tilde{V}}^{\mu} = u^{\nu}\nabla_{\nu} V^{\mu} = \partial_t V^{\mu}$. The orthogonal projected derivative is a little bit more involved. We are only interested in $\tilde{\nabla}^2 B_{\mu}$. For a co-moving observer this reduces to,

$$\begin{aligned} \tilde{\nabla}^{\nu}\tilde{\nabla}_{\nu} B_{\mu} &= h^{\sigma}_{\rho} h^{\lambda}_{\mu} \nabla^{\rho} h^{\alpha}_{\sigma} h^{\beta}_{\lambda} \nabla_{\alpha} B_{\beta} \\ &= h^{\alpha}_{\rho} h^{\beta}_{\mu} \nabla^{\rho} \nabla_{\alpha} B_{\beta} \\ &= h^{\beta}_{\mu} \nabla^{j} \nabla_{j} B_{\beta}. \end{aligned} \tag{8.101}$$

In the second line we used the fact that for a co-moving observer the derivatives of $h_{\mu\nu}$ are zero. If $\mu = 0$ this expression is zero, if $\mu = i$,

$$\tilde{\nabla}^{\nu}\tilde{\nabla}_{\nu}B_i = \nabla^j\nabla_j B_i. \tag{8.102}$$

We can now write down the wave equation for B_i for a co-moving observer,

$$\ddot{B}_i - \nabla^j\nabla_j B_i = -\frac{5}{3}\Theta\dot{B}_i - \frac{2}{9}\Theta^2 B_i - \frac{1}{3}RB_i. \tag{8.103}$$

In a curved FLRW spacetime R is given by,

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right]. \tag{8.104}$$

Recall also that $\Theta = 3H = 3\dot{a}/a$. In terms of the scale factor a the wave equation is,

$$\ddot{B}_i - \nabla^j\nabla_j B_i = -5\frac{\dot{a}}{a}\dot{B}_i - 4\left(\frac{\dot{a}}{a}\right)^2 B_i - 2\frac{\ddot{a}}{a}B_i - \frac{2K}{a^2}B_i. \tag{8.105}$$

We want to change to conformal time, such that $\partial_t = a^{-1}\partial_\eta$. In these variables the wave equation is,

$$B_i'' - a^2\nabla^j\nabla_j B_i = -4\frac{a'}{a}B_i' - 2\left(\frac{a'}{a}\right)^2 B_i - 2\frac{a''}{a}B_i - 2KB_i, \tag{8.106}$$

where we multiplied the entire equation with a^2. Following the article by Tsagas [31] we decompose B_i in vector spherical harmonics Q_i^n. They form a orthonormal basis, just as the spherical harmonics in the case of scalars. The decomposition is defined as,

$$B_i = \sum_n B_n(\eta)Q_i^n. \tag{8.107}$$

The vector spherical harmonics Q_i^n are eigenfunctions of the Laplace-Beltrami operator, which is the generalization of the Laplace operator in curved spacetime, ∇^2, just as the spherical harmonics. The co-moving eigenvalues of the n-th harmonic component is n, so in the case of a FLRW metric $\nabla^2 Q_i^n = -(n/a)^2 Q_i^n$. For the case $K = 1$, the eigenvalues take the form $n^2 = \nu(\nu+1)$, with ν a positive integer. In the other cases $K = -1$ and $K = 0$, we have respectively $n^2 = \nu^2 + 1$ and $n^2 = \nu^2$ [31], where ν can have any real continuous value. Under this decomposition the wave equation of the n-th component takes the form,

$$B_n'' + n^2 B_n = -4\frac{a'}{a}B_n' - 2\left(\frac{a'}{a}\right)^2 B_n - 2\frac{a''}{a}B_n - 2KB_n. \tag{8.108}$$

This simplifies if we define,

$$B_n \equiv a^{-2}\mathcal{B}_n(\eta), \tag{8.109}$$

to,

$$\mathcal{B}_n'' + (n^2 + 2K)\mathcal{B}_n = 0. \tag{8.110}$$

In chapter 7, we saw that if spacetime was flat, $K = 0$, the magnetic fields vanished. Since the discussion in this chapter is classical, we can only describe the evolution of an already existing magnetic field. To find the effect of a constantly curved universe, we must compare it to the flat case. When $K = 0$ the wave equation reduces to,

$$\mathcal{B}_n'' + n^2\mathcal{B}_n = 0. \tag{8.111}$$

Since we had the restriction that $n \geq 0$, the total magnetic field evolves as,

$$B_n = \frac{1}{a^2}\left[c_1 e^{in\eta} + c_2 e^{-in\eta}\right]. \tag{8.112}$$

The exponentials are just oscillating terms, so on average the magnetic fields decays proportional to a^{-2}.

When the universe is closed, $K = 1$, the wave equation takes the form,

$$\mathcal{B}_n'' + (\nu(\nu+1) + 2)\mathcal{B}_n = 0. \tag{8.113}$$

Since the minimum value of $\nu = 1$, the factor in front of the second term is positive and the solution is,

$$\mathcal{B}_n = c_1 \exp[i(\nu(\nu+1)+2)^{1/2}\eta] + c_2 \exp[-i(\nu(\nu+1)+2)^{1/2}\eta]. \tag{8.114}$$

This is also just an oscillatory term and the magnetic field will decay as $B \propto a^{-2}$. On average the fields will be as small as in the case $K = 0$ and therefore not interesting.

For a spatially open universe, $K = -1$, the wave equation is,

$$\mathcal{B}_n'' + (\nu^2 - 1)\mathcal{B}_n = 0. \tag{8.115}$$

When $\nu^2 > 1$ the solutions are again oscillatory and not interesting. This can be explained by the fact that this limit corresponds to short wavelengths. Curvature becomes less important on short length scales and therefore has less influence on the fields. When evaluating the case $\nu^2 < 1$ it is convenient to define $k^2 = 1 - \nu^2$, following [44][45]. The restriction $-1 < \nu^2 < 1$ translates to $0 < k < 2$. Recall that k is related to the physical wavelength by $\lambda_{\text{phys}} = a/k$. The solution of the wave equation is,

$$\mathcal{B}_k = c_1 e^{k\eta} + c_2 e^{-k\eta}. \tag{8.116}$$

To evaluate this further we must find a relation between the conformal time and the scale factor in a open universe. When $K = -1$ the first Friedmann equation (2.25) is,

$$a'^2 - a^2 = ca^{1-3\gamma}, \tag{8.117}$$

where c is a constant. The solution to this equation is,

$$a(\eta) = \pm\sqrt{c}\left[\left(e^{\eta(1+3\gamma)/2} - e^{-\eta(1+3\gamma)/2}\right)\right]^{2/(1+3\gamma)}. \tag{8.118}$$

This can be rewritten as,

$$a(\eta) = a_0 \left[\frac{1 - e^{-\eta(1+3\gamma)}}{1 - e^{-\eta_0(1+3\gamma)}}\right]^{2/(1+3\gamma)} e^{\eta-\eta_0}. \tag{8.119}$$

One can derive that $(1 + 3\gamma)\eta > 0$ [46]. For this reason, when $|\eta| \gg 0$ the scale factor reduces to,

$$a(\eta) \propto e^{\eta}. \tag{8.120}$$

We can substitute this result into expression (8.116) and redefine the constants such that,

$$B_k = c_1 e^k a^{-1} + c_3 e^k a^{-3}. \tag{8.121}$$

This result is true for all values of γ as long as $|\eta| \gg 0$. In a spatially open universe the magnetic field will decay slower then in a flat universe, since the decay is proportional to a^{-1} and a^{-2} respectively. As a consequence the strength of the magnetic field at the end of inflation we found in chapter 7 could be larger than we calculated.

CHAPTER 9

Constraining the energy density during inflation

The presence of magnetic fields in the intergalactic medium can constrain the energy density during inflation, if we assume that the fields are generated during inflation. In this chapter we will review two recent papers that derive a limit for the energy density. In the first section we will review the paper by Fujita and Mukohyama [47] that claims to be able to derive an upper bound on the energy density. In the second section we review the paper by Suyama and Yokoyama [48] and see how they derive a lower limit on the energy density.

9.1 Upper limit

First we will review the paper by Fujita and Mukohyama [47]. In this paper they claim to have found a way to derive an upper limit for the energy density during inflation. However it turns out that their derivation is not valid using our assumptions as described in chapter 6. In this section we will first present their argument and then show why it is not valid in our case.

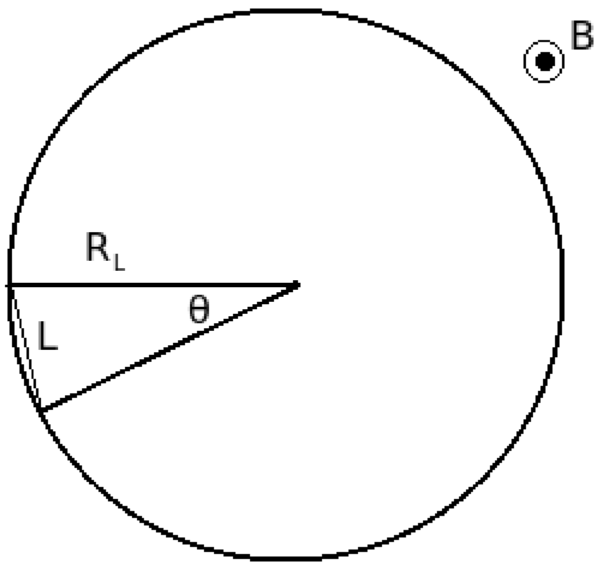

Figure 9.1 – The bending angle θ of a charged particle in a homogeneous magnetic field B. L is the distance traveled by the particle and R_L is called the Larmor radius.

9.1.1 Derivation of the upper limit

Constraining the Power spectrum

The first step Fujita and Mukohyama take to derive an upper limit is to re-express the limit on the magnetic fields, given by observations, into a limit on the power spectrum. The limits found by observations are all expressed in terms of the coherence scale of the magnetic field. These limits were derived from the fact that the path of the electron-positron pair, created from the interaction between the gamma-rays and photons from the diffuse extragalactic background light, is bended in the presence of a magnetic field (see section 3.1). To derive an expression in terms of the power spectrum they consider how large the bending angle θ is in the presence of a homogeneous magnetic field with an effective strength B_{eff}^2. The following discussion is in the classical limit $v \ll c$.

If we assume that the angle θ is small it can be approximated by $\theta = L/R_L$, see figure 9.1, where L is the distance traveled by the particle and R_L is called the Larmor radius. This last quantity can be found by equating the Lorentz force with the centripetal force. One then finds,

$$B_{\perp} = \frac{mv}{eL}\,\theta, \tag{9.1}$$

where v is the speed of the particle, m the mass and e the charge. The subscript $\perp$ denotes the perpendicular component with respect to the plane of motion. If we take into account that we need the vacuum expectation value of the magnetic field and the fact that the variance of the magnetic field in three dimensions is three halves times the variance of the magnetic field perpendicular the plane of motion, we find,

$$B_{\text{eff}}^2 = \frac{3}{2}\left(\frac{mv}{eL}\right)^2 \langle \boldsymbol{\theta}^2 \rangle. \tag{9.2}$$

We can rewrite this equation by recalling that the bending angle $\boldsymbol{\theta}$ is also given by,

$$\boldsymbol{\theta} = \frac{\mathbf{v}(t_1) - \mathbf{v}(t_2)}{v}, \tag{9.3}$$

where $\mathbf{v}(t)$ is the velocity of the particle at time t. Using the expression for the Lorentz force, we can write this as,

$$\begin{aligned}\boldsymbol{\theta} &= \frac{1}{v}\int_{t_1}^{t_2} dt\ \dot{\mathbf{v}}(t), \\ &= \frac{e}{mv}\int_{t_1}^{t_2} dt\ \mathbf{v}(t) \times \mathbf{B}(t), \\ &= \frac{e}{mv}\int_0^L d\mathbf{x} \times \mathbf{B}(x).\end{aligned} \tag{9.4}$$

Since θ is small and we can approximate $\mathbf{x}(t) = x_1\hat{\mathbf{e}}_1$. We then find,

$$\begin{aligned}B_{\text{eff}}^2 &= \frac{3}{2L^2}\int_0^L dx_1 dx_1' \langle (\hat{\mathbf{e}}_1 \times \mathbf{B}(x_1\hat{\mathbf{e}}_1))(\hat{\mathbf{e}}_1 \times \mathbf{B}(x_1'\hat{\mathbf{e}}_1))\rangle, \\ &= \frac{3}{2L^2}\int_0^L dx_1 dx_1' (\delta_{ij} - \delta_{i1}\delta_{j1})\langle B_i(x_1\hat{\mathbf{e}}_1)B_j(x_1'\hat{\mathbf{e}}_1)\rangle, \\ &= \frac{3}{2L^2}\int_0^L dx_1 dx_1' \langle B_2(x_1\hat{\mathbf{e}}_1)B_2(x_1'\hat{\mathbf{e}}_1) + B_3(x_1\hat{\mathbf{e}}_1)B_3(x_1'\hat{\mathbf{e}}_1)\rangle.\end{aligned} \tag{9.5}$$

In section 6.4 we computed the vacuum expectation value of one component of the magnetic field. We can use these results to find,

$$\begin{aligned}B_{\text{eff}}^2(\eta) = \frac{3}{2L^2}\frac{1}{a^4}\int_0^L dx_1 dx_1' \int \frac{d^3y d^3k d^3k'}{(2\pi)^6} \\ \times W_\lambda(k)W_\lambda(k')|A_k(\eta)|^2 \frac{k_1^2 + k^2}{2k} e^{ik_1x_1 + ik_1'x_1'} e^{-i\mathbf{y}(\mathbf{k}+\mathbf{k}')}.\end{aligned} \tag{9.6}$$

Performing the integral over y and k' this is equal to,

$$B_{\text{eff}}^2(\eta) = \frac{3}{2L^2}\frac{1}{a^4}\int_0^L dx_1 dx_1' \int \frac{d^3k}{(2\pi)^3} W_\lambda^2(k)|A_k(\eta)|^2 \frac{k_1^2 + k^2}{2k} e^{ik_1(x_1 - x_1')}. \tag{9.7}$$

We can also perform the integral over x_1 and x_1', then,

$$B_{\text{eff}}^2(\eta) = \frac{3}{2L^2}\frac{1}{a^4}\int\frac{d^3k}{(2\pi)^3}W_\lambda^2(k)|A_k(\eta)|^2\frac{k_1^2+k^2}{2k}\left(\frac{e^{ik_1L}-1}{ik_1}\right)\left(\frac{e^{-ik_1L}-1}{-ik_1}\right),$$
$$= \frac{3}{2L^2}\frac{1}{a^4}\int\frac{d^3k}{(2\pi)^3}W_\lambda^2(k)|A_k(\eta)|^2\frac{k_1^2+k^2}{kk_1^2}\left[1-\cos(k_1L)\right]. \tag{9.8}$$

Finally performing the integral over the angular parts of k one finds [47],

$$B_{\text{eff}}^2(\eta) = \frac{1}{a^4}\int\frac{dk}{k}W_\lambda^2(k)F(kL)\frac{k^4}{2\pi^2}|A_k(\eta)|^2, \tag{9.9}$$

where,

$$F(z) = \frac{3}{2}\frac{1}{z^2}\left[\cos(z)-\frac{\sin(z)}{z}+z\int_0^z dz'\frac{\sin(z')}{z'}\right]. \tag{9.10}$$

Although the function $F(z)$ looks complicated, it has the useful property that for $z \geq 0$,

$$0 \leq zF(z) \leq \alpha \simeq 2.48. \tag{9.11}$$

Comparison with expression (6.12) shows us that expression (9.9) is equal to the power spectrum during inflation times the function $F(kL)$. To intuitively understand where this function comes from we can use the expansions,

$$F(z) \simeq 1+\mathcal{O}(z^2), \qquad (z \ll 1),$$
$$F(z) \simeq \frac{3\pi}{4z}+\mathcal{O}(z^{-2}), \qquad (z \gg 1). \tag{9.12}$$

We can then write,

$$B_{\text{eff}}^2 \simeq \int_0^{1/L}\frac{dk}{k}\mathcal{P}_B(\eta,k)+\int_{1/L}^{\infty}\frac{dk}{k}\frac{1}{kL}\mathcal{P}_B(\eta,k), \tag{9.13}$$

where,

$$\mathcal{P}_B(\eta,k) = W_\lambda^2(k)\frac{k^4}{2\pi^2}|A_k(\eta)|^2. \tag{9.14}$$

If the path of the particle is smaller than the coherence length $(1/k)$ of the magnetic field, that is if $k < 1/L$, the particle just feels like it is traveling through a homogeneous magnetic field. Therefore the first term on the right hand side in expression (9.13) is equal to expression (6.12). If the path is longer than the coherence length, that is if $k > 1/L$, the particle is traveling trough $N = kL$ different homogeneous fields. Each of these fields can be in a different random direction and therefore the bending angle is on average $\sqrt{N}$ times smaller than if the particle would travel through only one field. For this reason the second term in expression (9.13) has an additional factor of $1/N = 1/kL$. Fujita and Mukohyama finally re-express the limit on the intergalactic magnetic fields found by observations as,

$$\left(B_{\text{eff}}^0\right)^2 = \int\frac{dk}{k}W_\lambda^2(k)F(kL)\frac{k^4}{2\pi^2}|A_k(\eta_{\text{rad}})|^2 \geq 10^{-36}. \tag{9.15}$$

Constraining the energy density

Using the found expression for the magnetic field, Fujita and Mukohyama, derive an upper limit for the energy density during in inflation, by looking at the equality,

$$|A_k(\eta_{\rm end})|^2 - |A_k(\eta_i)|^2 = \int_{\eta_i}^{\eta_{\rm end}} d\eta\, 2|A_k(\eta)||A_k(\eta)|', \tag{9.16}$$

which is true for any function A_k. In this case the subscript i denotes the beginning of inflation and end the end of inflation as usual. Using the inequality $2xy \leq x^2 + y^2$ for real numbers, they rewrite the equality as,

$$\begin{aligned} |A_k(\eta_{\rm end})|^2 - |A_k(\eta_i)|^2 &\leq \int_{\eta_i}^{\eta_{\rm end}} \frac{d\eta}{k} 2k|A_k(\eta)||A_k'(\eta)|, \\ &\leq \int_{\eta_i}^{\eta_{\rm end}} \frac{d\eta}{k} \left(k^2|A_k(\eta)| + |A_k'(\eta)|\right). \end{aligned} \tag{9.17}$$

As a next step they multiply both sides with $W_\lambda^2(k)F(kL)k^3/2\pi^2$ and integrate over k. The result is,

$$a_{\rm end}^4 B_{\rm eff}^2(\eta_{\rm end}) - a_i^4 B_{\rm eff}^2(\eta_i) \leq \int_{\eta_i}^{\eta_{\rm end}} d\eta \int \frac{dk}{k} W_\lambda^2(k) F(kL) \frac{k^3}{2\pi^2} \left(k^2|A_k(\eta)| + |A_k'(\eta)|\right). \tag{9.18}$$

They then use inequality (9.11) to rewrite the right hand side as,

$$a_{\rm end}^4 B_{\rm eff}^2(\eta_{\rm end}) - a_i^4 B_{\rm eff}^2(\eta_i) \leq \frac{\alpha}{L} \int_{\eta_i}^{\eta_{\rm end}} d\eta\; a^4(\eta) \left(B_\lambda^2(\eta) + E_\lambda^2(\eta)\right), \tag{9.19}$$

where they used definitions (6.12) and (6.25). Notice that they could only express the terms on the right hand side in the electric and magnetic field, because of the properties of $F(kL) \leq \alpha/k$. In chapter 7 we saw that for every model we used $B_\lambda^2(\eta) + E_\lambda^2(\eta) \simeq \rho_{\rm em} < \rho_{\rm inf}$, where $\rho_{\rm em}$ is the energy density of the electromagnetic field and $\rho_{\rm inf}$ the energy density during inflation, which we assume to be a constant. This, together with the fact that $a_{\rm end}^4 B_{\rm eff}^2(\eta_{\rm end}) \gg a_i^4 B_{\rm eff}^2(\eta_i)$, is used to rewrite the inequality as,

$$a_{\rm end}^4 B_{\rm eff}^2(\eta_{\rm end}) < \frac{\alpha}{L} \rho_{\rm inf} \int_{\eta_i}^{\eta_{\rm end}} d\eta\; a^4(\eta). \tag{9.20}$$

They then integrate the last term using the fact that $H = a'/a^2$ is constant during inflation and that $a_{\rm end} > a_i$. The result is,

$$\begin{aligned} B_{\rm eff}^2(\eta_{\rm now}) &< \frac{\alpha}{3LH_{\rm inf}} \rho_{\rm inf} a_{\rm end}^3, \\ &= \sqrt{\frac{1}{24\pi}} \frac{\alpha m_{\rm pl}}{L} \rho_{\rm inf}^{1/2} a_{\rm end}^3. \end{aligned} \tag{9.21}$$

They rewrite $a_{\rm end}$ using the relation $a \propto g_{*S}^{-1/3}(T)T^{-1}$ and the fact that $T_{\rm end} = \rho_{\rm inf}^{1/4}$. They then find that,

$$\rho_{\rm inf}^{1/4} < \sqrt{\frac{1}{24\pi}\frac{\alpha m_{\rm pl}}{L}\frac{g_{*S}(T_0)}{g_{*S}(T_{\rm end})}T_0^3\left(B_{\rm eff}^0\right)^{-2}}, \\ \simeq 1.8\times 10^{-4} m_{\rm pl}\left(\frac{B_{\rm eff}^0}{10^{-18}{\rm G}}\right)^{-2}. \tag{9.22}$$

Where they took a coherence length of $L = 1$ Mpc. So they finally find,

$$\rho_{\rm inf} < 10^{-15} m_{\rm pl}^4\left(\frac{B_{\rm eff}^0}{10^{-18}{\rm G}}\right)^{-8}. \tag{9.23}$$

If this would be true, we have a maximum temperature during inflation of $M < 10^{-4}m_{\rm pl}$. This is not a problem for the models in chapter 7, since we derived that for all the models we must have $M > 10^{-6}m_{\rm pl}$. It does restrict our models further, since we used before the upper limit of $M < 10^{-2}m_{\rm pl}$.

9.1.2 Why the derivation is wrong

We will now show why the argument of Fujita and Mukohyama does not hold in our framework. Not only during but also after inflation Fujita and Mukohyama use the following definition of the magnetic field,

$$B_{\rm eff}^2(\eta) = \frac{a^4(\eta_{\rm end})}{a^4(\eta)}\int\frac{dk}{k}W_\lambda^2(k)F(kL)\frac{k^4}{2\pi^2}|A_k(\eta_{\rm end})|^2. \tag{9.24}$$

In our framework after inflation the magnetic field is given by,

$$B_{\rm eff}^2(\eta) = \frac{a^4(\eta_{\rm end})}{a^4(\eta)}\int\frac{dk}{k}W_\lambda^2(k)F(kL)\frac{k^4}{2\pi^2}|\beta_k(\eta_{\rm end})|^2. \tag{9.25}$$

Comparing the expressions for $\beta_k(\eta)$ and $A_k(\eta)$ in all three models of chapter 7 we find that for all the models,

$$|A_k(\eta)|^2 \simeq (\eta k)^2|\beta_k(\eta)|^2, \tag{9.26}$$

and therefore,

$$\left(B_{\rm eff}^0\right)_{\rm FM}^2 \simeq \left(\frac{\eta_{\rm rad}}{\lambda}\right)^2\left(B_{\rm eff}^0\right)_{\rm ACT}^2, \tag{9.27}$$

where FM denotes the value found by Fujita and Mukohyama and ACT the actual value in our framework. The limit (9.23) in our framework is therefore,

$$\rho_{\rm inf} < 10^{-15} m_{\rm pl}^4\left(\frac{\lambda}{\eta_{\rm rad}}\right)^8\left(\frac{B_{\rm eff}^0}{10^{-18}{\rm G}}\right)^{-8}. \tag{9.28}$$

For a coherence scale of $\lambda = 1$ Mpc, $\lambda/|\eta_{\rm rad}| \gg 1$ and so the limit will be considerably higher then found by Fujita and Mukohyama. We can calculate it explicitly using the value of $\eta_{\rm rad}$ (7.55) and find,

$$\rho_{\rm inf} < 10^{63} m_{\rm pl}^4 \left(\frac{B_{\rm eff}^0}{10^{-18}{\rm G}} \right)^{-8/3} . \tag{9.29}$$

This will lead to a temperature limit of $M < 10^{16} m_{\rm pl}$, which is a much higher limit than the limit from graviton production and therefore not very useful.

9.2 Lower limit

In this section we will explain the recent paper of Suyama and Yokoyama [48], where they derive a lower limit on the energy density during inflation. They derive this limit from the fact that the amplitude of the metric perturbation can be observed in the CMB. Since, as explained in chapter 8, a magnetic field perturbs the metric, the presence of a magnetic field during inflation would be party responsible for this amplitude. Observations indicate that the amplitude has a value of $\sim 10^{-5}$ [49] and to derive a lower limit on the energy density Suyama and Yokoyama assume that this is solely caused by the magnetic field. Before we will show how this limit is derived we will first give a short overview of how one can derive the perturbed Einstein equations, which tell us how a fluid can influence metric perturbations.

9.2.1 Perturbed Einstein equations

Perturbations in a fluid and of the metric are related to each other through the Einstein equations. In this section we will give a short introduction to perturbations of both the metric and the energy momentum tensor and then derive the perturbed Einstein equations and conservation equations. These are all the tools we need to calculate the amplitude of the metric perturbation due to a magnetic field.

Metric perturbations

If we assume that the metric perturbations are small, we can treat them as first order perturbations. We can then write the metric as,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{9.30}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed FLRW metric and $h_{\mu\nu}$ is a small perturbation. In the following all the unperturbed quantities will be denoted with a bar. The inverse metric perturbation is defined as,

$$h^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}. \tag{9.31}$$

such that,

$$h^{00} = -h_{00}, \qquad h^{i0} = a^{-2}h_{i0}, \qquad h^{ij} = -a^{-4}h_{ij}, \tag{9.32}$$

if we use normal time coordinates. Because of the symmetries of the FLRW metric it is always possible to write the metric perturbation in the form [3],

$$h_{00} = -2A, \tag{9.33}$$
$$h_{i0} = a^2\left[\partial_i F + G_i\right], \tag{9.34}$$
$$h_{ij} = a^2\left[2\psi\delta_{ij} + 2\partial_i\partial_j K + 2\partial_j C_i + 2\partial_i C_j + D_{ij}\right], \tag{9.35}$$

with the additional constraints,

$$\partial_i C_i = \partial_i G_i = \partial_i D_{ij} = D_{ii} = 0. \tag{9.36}$$

It turns out that the Einstein equations can be decoupled in a scalar, vector and tensor part. Therefore we can evaluate these parts separately. We will focus on the scalar part and therefore have,

$$h_{00} = -2A, \tag{9.37}$$
$$h_{i0} = a^2\partial_i F, \tag{9.38}$$
$$h_{ij} = 2a^2\left[\psi\delta_{ij} + \partial_i\partial_j K\right]. \tag{9.39}$$

Energy momentum tensor perturbations

As explained in chapter 8 the energy momentum tensor is given by,

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}. \tag{9.40}$$

Due to the symmetries of the universe the unperturbed energy momentum tensor is a perfect fluid,

$$\bar{T}_{\mu\nu} = \bar{p}\bar{g}_{\mu\nu} + (\bar{\rho} + \bar{p})\bar{u}_\mu\bar{u}_\nu. \tag{9.41}$$

The energy-flux and anisotropic pressure are therefore considered to be perturbations. Since we are interested in the scalar perturbations we only have to consider the scalar part of the anisotropic stress,

$$\pi^{i\,S}_{j} \equiv \left(\frac{1}{3}\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\Pi. \tag{9.42}$$

Since we assume that the anisotropy is only created by the magnetic field, this scalar part of the anisotropic stress is solely due to the magnetic field. To be able to describe the perturbations of the energy momentum tensor we need to know how the perturbations of the four-velocity behave. The four-velocity satisfies the condition $u^\mu u_\mu = -1$. In a FLRW spacetime using normal time coordinates $\bar{u}_i = 0$ and $\bar{u}_0 = -1$. Using this we can calculate that,

$$\begin{aligned} g^{\mu\nu} u_\mu u_\nu &= -1, \\ (\bar{g}^{\mu\nu} + h^{\mu\nu})(\bar{u}_\mu \bar{u}_\nu + 2\bar{u}_\mu \delta u_\nu) &= -1, \\ h^{\mu\nu} \bar{u}_\mu \bar{u}_\nu + 2\bar{g}^{\mu\nu} \bar{u}_\mu \delta u_\nu) &= 0, \\ \delta u_0 &= \frac{1}{2} h_{00}. \end{aligned} \tag{9.43}$$

In the same way one can calculate that $\delta u^0 = h_{00}/2$ and $\delta u_i \equiv a^2 \partial_i v$ is an independent variable. The perturbed energy momentum tensor is given by,

$$T_{\mu\nu} = (\bar{p} + \delta p)(\bar{g}_{\mu\nu} + h_{\mu\nu}) + (\bar{\rho} + \bar{p} + \delta\rho + \delta p)(\bar{u}_\mu \bar{u}_\nu + 2\bar{u}_\mu \delta u_\nu) + \pi^S_{\mu\nu}, \tag{9.44}$$

such that the first order perturbation is,

$$\delta T_{\mu\nu} = \bar{p} h_{\mu\nu} + 2(\bar{\rho} + \bar{p})\bar{u}_\mu \delta u_\nu + \delta\rho \bar{u}_\mu \bar{u}_\nu + \delta p(\bar{g}_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu) + \pi^S_{\mu\nu}. \tag{9.45}$$

It is then easy to compute that,

$$\delta T_{00} = \delta\rho - \bar{\rho} h_{00}, \tag{9.46}$$
$$\delta T_{0i} = -(\bar{\rho} + \bar{p}) a^2 \partial_i v + \bar{p} h_{0i} \tag{9.47}$$
$$\delta T_{ij} = \bar{p} h_{ij} + \delta p a^2 \delta_{ij} + \pi^S_{ij}, \tag{9.48}$$

or

$$\delta T^0_{\ 0} = -\delta\rho, \tag{9.49}$$
$$\delta T^0_{\ i} = (\bar{\rho} + \bar{p}) a^2 \partial_i v, \tag{9.50}$$
$$\delta T^i_{\ 0} = -(\bar{\rho} + \bar{p}) \partial_i (v - h^{i0}), \tag{9.51}$$
$$\delta T^i_{\ j} = \delta p \delta^i_j + \pi^{i\ S}_{\ j}, \tag{9.52}$$

and

$$\delta T = -\delta\rho + 3\delta p. \tag{9.53}$$

Einstein equations

With these ingredients we can compute how the perturbations in the fluids influence the metric trough the Einstein equations. The Einstein equations can be written in the form,

$$R_{\mu\nu} = 8\pi G \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]. \tag{9.54}$$

We want to calculate the perturbations given by,

$$\delta R^{\mu}_{\nu} = 8\pi G \left[\delta T^{\mu}_{\nu} - \frac{1}{2} \bar{g}^{\mu}_{\nu} \delta T \right]. \tag{9.55}$$

The first step is to notice that $\delta R^{\mu}_{\nu} = \delta(g^{\mu\lambda} R_{\lambda\nu}) = h^{\mu\lambda} \bar{R}_{\lambda\nu} + \bar{g}^{\mu\lambda} \delta R_{\lambda\nu}$. To be able to calculate $\delta R_{\lambda\nu}$ we first have to calculate the perturbation of the Christoffel symbols, which is given by,

$$\delta\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} \bar{g}^{\mu\nu} \left[-2h_{\rho\sigma} \bar{\Gamma}^{\sigma}_{\nu\lambda} + \partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} \right]. \tag{9.56}$$

It is easy to calculate that for a flat FLRW metric,

$$\bar{\Gamma}^{i}_{j0} = \bar{\Gamma}^{i}_{0j} = \frac{\dot{a}}{a} \delta_{ij} = H \delta_{ij}, \tag{9.57}$$
$$\bar{\Gamma}^{0}_{ij} = \dot{a} a \delta_{ij} = a^2 H \delta_{ij}, \tag{9.58}$$

and therefore the perturbations of the Christoffel symbols are,

$$\delta\Gamma^{i}_{jk} = \frac{1}{2a^2} \left[-2a^2 H h_{i0} \delta_{jk} + \partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk} \right], \tag{9.59}$$
$$\delta\Gamma^{i}_{j0} = \frac{1}{2a^2} \left[-2H h_{ij} + \dot{h}_{ij} + \partial_j h_{i0} - \partial_i h_{j0} \right], \tag{9.60}$$
$$\delta\Gamma^{0}_{ij} = \frac{1}{2} \left[2a^2 H \delta_{ij} h_{00} - \partial_j h_{i0} - \partial_i h_{j0} + \dot{h}_{ij} \right], \tag{9.61}$$
$$\delta\Gamma^{i}_{00} = \frac{1}{2a^2} \left[2\dot{h}_{i0} - \partial_i h_{00} \right], \tag{9.62}$$
$$\delta\Gamma^{0}_{i0} = H h_{i0} - \frac{1}{2} \partial_i h_{00}, \tag{9.63}$$
$$\delta\Gamma^{0}_{00} = -\frac{1}{2} \dot{h}_{00}. \tag{9.64}$$

The perturbation of $R_{\mu\nu}$ is given by,

$$\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \partial_\nu \delta\Gamma^{\lambda}_{\mu\lambda} + \delta\Gamma^{\alpha}_{\mu\nu} \bar{\Gamma}^{\lambda}_{\lambda\alpha} + \delta\Gamma^{\alpha}_{\alpha\lambda} \bar{\Gamma}^{\lambda}_{\mu\nu} - \delta\Gamma^{\alpha}_{\mu\lambda} \bar{\Gamma}^{\lambda}_{\nu\alpha} - \delta\Gamma^{\alpha}_{\nu\lambda} \bar{\Gamma}^{\lambda}_{\mu\alpha}. \tag{9.65}$$

Using the explicit expressions of the Christoffel symbols one finds,

$$\delta R_{00} = -\frac{1}{2a^2}\nabla^2 h_{00} - \frac{3}{2}H\dot{h}_{00} + \frac{1}{a^2}\partial_i \dot{h}_{i0} - \frac{1}{2a^2}\left[\ddot{h}_{ii} - 2H\dot{h}_{ii} - 2\dot{H}h_{ii}\right], \tag{9.66}$$

$$\begin{aligned}\delta R_{0i} =\delta R_{i0} &= -H\partial_i h_{00} - \frac{1}{2a^2}\left[\nabla^2 h_{i0} - \partial_i\partial_k h_{k0}\right] + (\dot{H} + 3H^2)h_{i0} \\ &- \frac{1}{2}\partial_0\left[\frac{1}{a^2}\left(\partial_i h_{kk} - \partial_k h_{ki}\right)\right]\end{aligned} \tag{9.67}$$

$$\begin{aligned}\delta R_{ij} =&\frac{1}{2}\partial_i\partial_j h_{00} + a^2(\dot{H} + 3H^2)\delta_{ij}h_{00} + \frac{1}{2}a^2 H\delta_{ij}\dot{h}_{00} \\ &- \frac{1}{2a^2}\left[\nabla^2 h_{ij} - \partial_k\partial_i h_{jk} - \partial_k\partial_j h_{kj} + \partial_i\partial_j h_{kk}\right] + \frac{1}{2}\ddot{h}_{ij} \\ &- \frac{1}{2}H\left[\dot{h}_{ij} - \delta_{ij}\dot{h}_{kk}\right] - H^2\left[-2h_{ij} + \delta_{ij}h_{kk}\right] + H\delta_{ij}\partial_k h_{k0} \\ &- \frac{1}{2}\left[\partial_i\dot{h}_{j0} + \partial_j\dot{h}_{i0}\right] + \frac{1}{2}H\left[\partial_i h_{j0} + \partial_j h_{i0}\right].\end{aligned} \tag{9.68}$$

We now have all the ingredients to explicitly calculate the perturbations of the Einstein equation for a FLRW metric. Using the specific forms of the metric perturbations one finds after some lines of calculations that the respectively $00, 0i, ii$ and ij-component, where in the last case $i \neq j$, of the perturbed Einstein equation (9.55) are,

$$4\pi G(\delta\rho + 3\delta p) = 6(\dot{H} + H^2)A + \frac{\nabla^2}{a^2}A + 3H\dot{A} + \nabla^2\dot{\sigma} + 2H\nabla^2\sigma - 3\ddot{\psi} - 6H\dot{\psi}, \tag{9.69}$$

$$-4\pi G(\bar{\rho} + \bar{p})a^2\partial_i v = H\partial_i A - \partial_i\dot{\psi}, \tag{9.70}$$

$$\begin{aligned}4\pi G(\delta p - \delta\rho) =& -\frac{1}{3}\frac{\nabla^2}{a^2}A + 2(\dot{H} + 3H^2)A + 3H\dot{A} - \frac{4}{3}\frac{\nabla^2}{a^2}\psi - \ddot{\psi} - 6H\dot{\psi} \\ &+ 2H\nabla^2\sigma + \frac{1}{3}\nabla^2\dot{\sigma},\end{aligned} \tag{9.71}$$

$$8\pi G\pi^{i\,S}_j = -\frac{1}{a^2}\partial_i\partial_j(A + \psi) - \partial_i\partial_j(\dot{\sigma} + 3H\sigma), \tag{9.72}$$

where we have defined $\sigma = F - \dot{K}$. We can rewrite this by combining expression (9.69) and (9.71) in Fourier space as,

$$4\pi G\delta\rho = \frac{k^2}{a^2} + Hk^2\sigma - 3H^2 A + 3H\dot{\psi}, \tag{9.73}$$

$$-4\pi G(\bar{\rho} + \bar{p})a^2 v = HA - \dot{\psi}, \tag{9.74}$$

$$4\pi G\delta p = -\frac{1}{3}\frac{k^2}{a^2}(A + \psi) + (2\dot{H} + 3H^2)A + H\dot{A} - \ddot{\psi} - 3H\dot{\psi} - \frac{1}{3}k^2(\dot{\sigma} + 3H\sigma), \tag{9.75}$$

$$-8\pi G\Pi = \frac{k^2}{a^2}(A + \psi) + k^2(\dot{\sigma} + 3H\sigma). \tag{9.76}$$

Conservation equations

Since the energy momentum tensor is conserved, $\nabla_\mu T^\mu_\nu = 0$, we can derive two conservation equations for $\nu = 0, i$. The perturbed conservation law is,

$$\delta\left(\nabla_\mu T^\mu_\nu\right) = \partial_\mu \delta T^\mu_\nu + \bar{\Gamma}^\mu_{\mu\lambda}\delta T^\lambda_\nu + \delta\Gamma^\mu_{\mu\lambda}\bar{T}^\lambda_\nu - \bar{\Gamma}^\mu_{\mu\nu}\delta T^\mu_\lambda - \delta\Gamma^\lambda_{\mu\nu}\bar{T}^\mu_\lambda = 0. \tag{9.77}$$

Using the expressions we found for the Christoffel symbols and energy momentum tensor one can derive after some lines of calculation the following conservation laws in Fourier space,

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) + 3\dot{\psi}(\bar{\rho} + \bar{p}) - (\bar{\rho} + \bar{p})k^2(v - \sigma) = 0, \tag{9.78}$$

$$\partial_0\left[(\bar{\rho} + \bar{p})a^2 v\right] + \delta p - \frac{2}{3}\Pi + 3H(\bar{\rho} + \bar{p})a^2 v + (\bar{\rho} + \bar{p})A = 0. \tag{9.79}$$

9.2.2 Curvature perturbation

We now have all the ingredients to calculate the amplitude of the curvature perturbation and derive a limit on the energy density. The intrinsic curvature of the spatial hypersurface $R^{(3)}$ is in our coordinate definition,

$$R^{(3)} = -\frac{4}{a^2}\nabla^2\psi. \tag{9.80}$$

Unfortunately this quantity is gauge dependent, because under the transformation $t \to t + \delta t$,

$$\psi \to \psi - H\delta t. \tag{9.81}$$

It will be more convenient to find a gauge invariant curvature parameter. We know that the energy density transforms as,

$$\delta\rho \to \delta\rho - \dot{\rho}\delta t. \tag{9.82}$$

We can therefore construct the gauge invariant curvature perturbation on uniform energy density hypersurfaces as [50][51],

$$\zeta = \psi - H\frac{\delta\rho}{\dot{\rho}}. \tag{9.83}$$

To find the evolution of this curvature perturbation we can use the first conservation law (9.78). Since we are working on a uniform energy density hypersurface $\delta\rho = 0$ and $\xi = \psi$. Therefore the first conservation law can be rewritten as,

$$3H\delta p + 3\dot{\zeta}(\bar{\rho} + \bar{p}) - (\bar{\rho} + \bar{p})k^2(v - \sigma) = 0. \tag{9.84}$$

The pressure perturbation can always be split in adiabatic and non-adiabatic part, as

$$\delta p = c_s^2 \delta\rho + \dot{p}\Gamma, \tag{9.85}$$

where $c_s^2 \equiv \dot{p}/\dot{\rho}$ is the adiabatic sound speed and $\delta p_{\rm nad} \equiv \dot{p}\Gamma$ is the non-adiabatic pressure, with $\Gamma = \delta p/\dot{p} - \delta\rho/\dot{\rho}$. Since we are on a uniform energy density hypersurface $\delta p = \delta p_{\rm nad}$. The evolution equation of the curvature perturbation is then,

$$\dot{\zeta} = -\frac{H}{\bar{\rho}+\bar{p}}\delta p_{\rm nad} + \frac{1}{3}k^2(v-\sigma), \tag{9.86}$$

which in the large scale limit reduces to,

$$\dot{\zeta} = -\frac{H}{\bar{\rho}+\bar{p}}\delta p_{\rm nad}. \tag{9.87}$$

In their paper Suyama and Yokoyama derive a more complicated evolution equation which has an extra term proportional to Π. This term originates from the term proportional to k^2 in expression (9.86). Since they also work in the large scale limit and later argue that the term proportional to Π is negligible compared to the other terms, we choose to neglect it from the start. The solution to the evolution equation is,

$$\zeta(t) = -\int_{t_*}^{t} dt' \frac{H(t')}{\bar{\rho}(t')+\bar{p}(t')}\delta p_{\rm nad}(t'), \tag{9.88}$$

where t_* is at horizon crossing and we assume $\zeta(t_*) = 0$. To solve this equation we will assume that we have de Sitter inflation. Therefore $\dot{p} = -\dot{\rho}$ and H, $\bar{\rho}$ and $\bar{p}$ are constant during inflation. Since we are deriving an upper limit, we will assume that the entire anisotropic pressure is caused by the electromagnetic field, such that $\delta p_{\rm nad} = \delta p_{\rm nad-em}$. We saw in chapter 8 that $p_{\rm em} = 1/3\rho_{\rm em}$, so we can write,

$$\delta p_{\rm nad-em} = \frac{4}{3}\delta\rho_{\rm em}. \tag{9.89}$$

We can then write the solution as,

$$\begin{aligned}
\zeta(t) &= -\frac{4}{3}\frac{H}{\bar{\rho}+\bar{p}}\int_{t_*}^{t} dt'\delta\rho_{\rm em}(t'),\\
&= \frac{4H^2}{\dot{\bar{\rho}}}\int_{t_*}^{t} dt'\delta\rho_{\rm em}(t'),\\
&= \frac{16\pi G}{3}\frac{H}{\dot{H}}\int_{t_*}^{t} dt'\delta\rho_{\rm em}(t'),\\
&= -2\frac{H}{\epsilon}\frac{1}{\rho_{\rm inf}}\int_{t_*}^{t} dt'\delta\rho_{\rm em}(t').
\end{aligned} \tag{9.90}$$

where we used the fluid equation (2.18) in the second line, the first Friedmann equation (2.32) in the third and fifth line and $\epsilon = -\dot{H}/H^2$ is the first slow roll parameter. If we assume that also $\delta\rho_{\rm em}$ is constant during inflation the solution is simply,

$$\begin{aligned}\zeta &= -\frac{2}{\epsilon}\frac{\delta\rho_{\rm em}}{\rho_{\rm inf}}H(t-t_*),\\ &= -\frac{2N}{\epsilon}\frac{\delta\rho_{\rm em}}{\rho_{\rm inf}}\end{aligned} \tag{9.91}$$

To account for the fact that $\delta\rho_{\rm em}$ does change during inflation we can change this into the limit [48],

$$|\zeta| > \frac{1}{\epsilon}\left|\frac{\delta\rho_{\rm em}(t_{\rm end})}{\rho_{\rm inf}}\right|. \tag{9.92}$$

As mentioned before observations of the CMB show that the amplitude of the curvature perturbation is $|\zeta| \sim 10^{-5}$ [49], therefore,

$$\epsilon > 10^5\left|\frac{\delta\rho_{\rm em}(t_{\rm end})}{\rho_{\rm inf}}\right|. \tag{9.93}$$

The same observations show that [49],

$$\frac{\rho_{\rm inf}}{2\pi m_{\rm pl}^4}\frac{1}{\epsilon} \simeq 2.4\times 10^{-9}. \tag{9.94}$$

We can then replace ϵ in expression (9.93) and find,

$$\frac{\rho_{\rm inf}^2}{m_{\rm pl}^4} > 2.3\times 10^{-3}|\delta\rho_{\rm em}(t_{\rm end})|. \tag{9.95}$$

Recall that after inflation the electromagnetic field is described by Maxwell theory and therefore the energy density is $\rho_{\rm em} = \frac{1}{2}(E^2+B^2)$. Because of the high conductivity after inflation we saw in chapter 5 that $E=0$ and $B\propto a^{-2}$. We can therefore rewrite,

$$\frac{\rho_{\rm inf}^2}{m_{\rm pl}^4} > 1.2\times 10^{-3}B_0^2 a_{\rm end}^{-4}. \tag{9.96}$$

Using relation (2.51) and the fact that $\rho = M^4$, where M was the temperature during inflation, we find,

$$\frac{\rho_{\rm inf}}{m_{\rm pl}^4} > 1.2\times 10^{-3}B_0^2\left(\frac{g_{*S}(T_{\rm rad})}{g_{*S}(T_0)}\right)^{4/3}\frac{1}{T_0}. \tag{9.97}$$

Finally using the explicit expressions for g_{*S} and T_0 given in section 7.2 we find,

$$\rho_{\rm inf} > 1.4\times 10^{-25}m_{\rm pl}^4\left(\frac{B_0}{10^{-18}\ {\rm G}}\right)^2. \tag{9.98}$$

This is the same as having a limit on the temperature during inflation of $M > 10^{-6}m_{\rm pl}$, which is the same limit that we found for each of our models to be valid in chapter 7.

CHAPTER 10

Conclusion

Observations of galaxies at high redshifts show that magnetic fields were already present in galaxies at early times. This suggests that the origin of these magnetic fields lies in the early universe. Inflation seems to be a good candidate for their origin, since it could explain their large coherence scale and the fact that they are also present in the intergalactic medium.

We have evaluated the magnetic field strength, assuming that the field was generated during inflation, using four different models. The first model was Maxwell electrodynamics. However, the conformal invariance of the theory led to vanishing magnetic fields and therefore Maxwell electrodynamics alone is not able to explain this phenomenon. To solve this problem and to still be able to maintain the assumption that the fields were generated during inflation we broke the conformal invariance in the other three models.

In the first model the electromagnetic field was coupled to the gravitational field through the coupling $(R/m^2)^n F^2$. We saw that this model could generate the intergalactic magnetic fields when $n \geq 6$ in the case $m^2 = R_{\rm rad}$, where rad denotes the value at the end of reheating. We also found that for this model to work the temperature during inflation must have been at least $10^{-6} m_{\rm pl}$.

The next model we reviewed contained a coupling of the electromagnetic field to a scalar field. We took the example of the a inflaton field ϕ in Power-Law inflation with a coupling of the form $I(\phi) \propto \exp[(1+3\gamma)/(2\sqrt{3(1+\gamma)})\, b\phi]$. We argued that this was equivalent to having a coupling $I(\eta) = \eta^b$. We found that intergalactic magnetic fields could be generated when $3.4 < b < 4$ and there was a minimum temperature during inflation of $10^{-6} m_{\rm pl}$.

Finally we looked at the model where the electromagnetic field was coupled to a pseudo-scalar field ϕ, through the coupling $I(\phi)F\tilde{F}$. We looked at an example where the coupling was of the form $I(\phi) \propto \phi$. We showed that in the case of a massive pseudo-scalar field or a massless pseudo-scalar field that is non-minimally coupled to gravity, with respective potentials $m^2\phi^2$ and $\xi R\phi^2$, that this corresponded to taking the Fourier transform of the coupling $I_{\mathbf{k}} = c(2\pi)^3\delta(\mathbf{k})(-k\eta)^b$. Calculations showed that only strong enough fields could be generated in the case $b = -1$, which corresponds to $m^2 = -4H^2$ and $\xi = 1/3$ depending on the model used. Again a minimum temperature during inflation of $10^{-6}m_{\mathrm{pl}}$ was needed.

Besides reviewing these models we looked at the classical evolution of a magnetic field in a spatially curved FLRW metric. We found that for a spatially open universe the magnetic field decays proportional to a^{-1}, which is slower than the rate in flat spacetime (a^{-2}). This shows that an open universe could lead to stronger magnetic fields.

We also reviewed two suggestions to derive a bound on the temperature during inflation from the fact that intergalactic magnetic fields exist. We showed that the derivation of an upper limit by Fujita and Mukohyama [47] was not valid in our framework and we calculated a new limit of $M < 10^{16}m_{\mathrm{pl}}$, which is well above the limit from graviton production and therefore not interesting. The derivation of a lower limit by Suyama and Yokoyama [48] was more promising and we found a limit of $M > 10^{-6}m_{\mathrm{pl}}$, which is equivalent to the limits found in the evaluated models.

Our results show that all three models in which the conformal invariance is broken are able to generate the required fields. It would therefore be interesting to conduct further research to be able to single out one of these models. This further research could both be observational and theoretical. One way to single out one of the models is to get a stronger observational limit on either intergalactic fields, which gives a stronger restriction on the models, or the helicity in the CMB, which could distinguish between the last model and the rest. Theoretically it would be interesting to evaluate the different models for a non-instantaneous reheating stage. This would influence the strength of the fields and might therefore be able to single out one of the models.

APPENDIX A

Conventions

In this thesis we will use the following conventions:

- We will use units in which $c = \hbar = k_B = 1$.
- We will use Gaussian units, which means that $\epsilon_0 = 1/(4\pi)$, where ϵ_0 is the vacuum permittivity. In these units the electron charge is equal to $e = \sqrt{\alpha} \propto \sqrt{1/137}$. For this reason the magnitude of a magnetic field may be expressed in Gauss (G).
- We use Greek indices to indicate a four dimensional space and Latin indices for the spacelike parts.
- We define:

$$\dot{x} = \frac{dx}{dt},$$
$$x' = \frac{dx}{d\eta},$$

where t is the normal time coordinate and η is the conformal time.

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