Journal of Physics: Conference Series **343** (2012) 012028 doi:10.1088/1742-6596/343/1/012028

Spinless Duffin-Kemmer-Petiau oscillator in a Galilean non-commutative phase space

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Abstract. The Galilean covariant approach of field theory is based on a $(4+1)$-dimensional manifold using light-cone coordinates followed by a reduction to $(3+1)$ dimensions. We use this approach to investigate Galilean covariant linear wave equations in a non-commutative phase space. After a brief review of the Galilean covariance formalism, we construct the Galilean Duffin-Kemmer-Petiau equation for the harmonic oscillator in a non-commutative phase space. The exact wave functions and their energy levels are found, and we discuss the effects of non-commutativity.

1. Introduction

In this presentation, we shall exploit a formulation of Galilean covariance in an extended manifold in order to examine the non-relativistic Duffin-Kemmer-Petiau (DKP) oscillator for a spin-zero field in a non-commutative phase space. The DKP equation is similar to the Dirac equation but based on the so-called "DKP algebra" [1, 2]. The equivalence between the Klein-Gordon equation and the DKP equation, and the more complex algebraic structure of the latter, might explain why the DKP equation has not received much attention in the litterature [3, 4].

We will work on a space where coordinates do not commute, as first investigated in Ref. [5]. Our motivation is the connection between non-commutative coordinates and discrete space-time; a Galilean version should help describe non-relativistic lattice models. The Galilean covariance in an extended manifold can be seen as one more unifying tools (such as gauge invariance and spontaneous symmetry breaking) between particle physics and condensed matter physics.

A "Galilean covariant" theory is defined by the addition of an extra coordinate which defines a $(4+1)$ Minkowski manifold [6]. This five-dimensional manifold is described by the coordinates

$$x^\mu = (x^1, x^2, x^3, x^4, x^5) = (\mathbf{r}, t, s),$$

which transform under a Galilei boost as follows:

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \mathbf{v}t, \\ t' &= t, \\ s' &= s - \mathbf{r}\cdot\mathbf{v} + \tfrac{1}{2}\mathbf{v}^2 t. \end{aligned}$$

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This transformation leaves invariant the scalar product

$$(\mathbf{r}, t, s) \cdot (\mathbf{r}', t', s') \equiv \mathbf{r} \cdot \mathbf{r}' - ts' - t's,$$

so that we shall refer to the corresponding metric,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

as *Galilean metric.*

In order to illustrate this approach, let us retrieve the Galilean "magnetic" and "electric" limits of Maxwell's electromagnetism, obtained in 1973 by Le Bellac and Lévy-Leblond [7]. In the "magnetic limit", the Maxwell equations take the form

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0, \\ &\nabla \cdot \mathbf{E}_m = \tfrac{1}{\epsilon_0}\rho_m, \\ &\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \\ &\nabla \times \mathbf{E}_m = -\partial_t \mathbf{B}, \end{aligned}$$

(with the magnetic induction term missing) whereas in the "electric limit", we have

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0, \\ &\nabla \cdot \mathbf{E}_e = \tfrac{1}{\epsilon_0}\rho_e, \\ &\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0\partial_t \mathbf{E}_e, \\ &\nabla \times \mathbf{E}_e = \mathbf{0}, \end{aligned}$$

where the electric displacement term is absent. In Ref. [8], we observed that if we substitute

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & cB_3 & -cB_2 & E_{m1} & E_{e1} \\ -cB_3 & 0 & cB_1 & E_{m2} & E_{e2} \\ cB_2 & -cB_1 & 0 & E_{m3} & E_{e3} \\ -E_{m1} & -E_{m2} & -E_{m3} & 0 & a \\ -E_{e1} & -E_{e2} & -E_{e3} & -a & 0 \end{pmatrix}$$

and

$$J_\mu = (\mathbf{J}, -c\rho_m, -c\rho_e), \qquad A_\mu = (\mathbf{A}, -\phi_m, -\phi_e)$$

into

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\epsilon_0} J^\nu,$$

we obtain

$$c\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E}_m = -\nabla\phi_m - \frac{1}{c}\partial_t \mathbf{A}, \quad \mathbf{E}_e = -\nabla\phi_e.$$

With $\phi_e = \mathbf{0}$, $\rho_e = 0$, we find

$$\nabla \cdot \mathbf{E}_m = \frac{1}{\epsilon_0}\rho_m, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \qquad \text{(magnetic limit)}$$

With $\phi_m = 0$, $\rho_m = 0$, we find

$$\nabla \cdot \mathbf{E}_e = \frac{1}{\epsilon_0}\rho_e, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0\partial_t \mathbf{E}_e. \qquad \text{(electric limit)}$$

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The equation

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0$$

leads to

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E}_m = -\partial_t \mathbf{B}, \qquad \text{(magnetic limit)}$$

and

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E}_e = \mathbf{0}. \qquad \text{(electric limit)}$$

In this paper, we follow the same lines as what was done in Ref. [9] for the relativistic situation. In order to describe the DKP equation on a non-commutative phase space, consider the usual position and momentum operators, with the canonical commutations relations:

$$[r_i, r_j] = 0, \qquad [p_i, p_j] = 0, \qquad [r_i, p_j] = \mathrm{i}\hbar\delta_{ij}.$$

Then a non-commutative space can be described by the operators:

$$\hat{r}_i = r_i - \frac{\Theta_{ij}}{2\hbar} p_j = r_i + \frac{(\mathbf{\Theta} \times \mathbf{p})_i}{2\hbar}, \tag{1}$$

$$\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = p_i - \frac{(\mathbf{\Omega} \times \mathbf{r})_i}{2\hbar}, \tag{2}$$

which satisfy the following commutation relations:

$$[\hat{r}_i, \hat{r}_j] = \mathrm{i}\Theta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = \mathrm{i}\Omega_{ij}, \qquad [\hat{r}_i, \hat{p}_j] = \mathrm{i}\hbar\Delta_{ij},$$

with $\Theta_{ij} = \epsilon_{ijk}\Theta_k$, $\Omega_{ij} = \epsilon_{ijk}\Omega_k$, where $\mathbf{\Theta} = (\Theta_1, \Theta_2, \Theta_3)$ and $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ are real parameters which describe the non-commutativity of the coordinates and momenta, respectively.

2. The DKP oscillator energy spectrum in a non-commutative space

We investigated the *commutative* Galilean covariant DKP wave equations in Refs. [10] and [11]. In these papers, we considered the following DKP Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\overline{\Psi}\beta^\mu\partial_\mu\Psi - \frac{1}{2}\partial_\mu\overline{\Psi}\beta^\mu\Psi - k\overline{\Psi}\Psi, \qquad \mu = 1, \ldots, 5, \tag{3}$$

with $\overline{\Psi}$ the adjoint of Ψ, defined by $\overline{\Psi} = \Psi^\dagger\eta$ and $\eta = \left(\beta^4 + \beta^5\right)^2 + 1$, where k is a constant, and β^μ are matrices that satisfy the DKP algebra [1, 2]

$$\beta^\mu\beta^\nu\beta^\rho + \beta^\rho\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\rho + g^{\rho\nu}\beta^\mu.$$

The Lagrangian in Eq. (3) leads to the Galilean DKP wave equation

$$\left(\beta^\mu\partial_\mu + k\right)\Psi = 0,$$
$$\overline{\Psi}\left(\beta^\mu \overleftarrow{\partial}_\mu - k\right) = 0.$$

With appropriate representations of the β-matrices, these equations describe spinless and spin-one fields. The β-matrices are given by representations of the Lie algebra so(5,1).

As in Refs. [11], we use the following 6-by-6 DKP matrices:

$$\beta^1 = e_{1,6} + e_{6,1},$$
$$\beta^2 = e_{2,6} + e_{6,2},$$
$$\beta^3 = e_{3,6} + e_{6,3},$$
$$\beta^4 = e_{4,6} - e_{6,5},$$
$$\beta^5 = e_{5,6} - e_{6,4}.$$

(The matrices e_{jk} are defined as $(e_{jk})_{mn} \equiv \delta_{jm}\delta_{kn}$.)

In order to describe the DKP wave equations in a *non-commutative* phase space, we express the equations in terms of the non-commutative coordinates and momenta, $\hat{r}_i$ and $\hat{p}_i$, and then apply Eqs. (1) and (2). The DKP equation with a non-minimal coupling $\mathbf{C}$ in a non-commutative space is written as

$$\left(\beta^{\mu}\pi_{\mu} - \mathrm{i}\hbar k\right)\Psi = 0, \tag{4}$$

where $\pi_{\mu} = (\hat{\mathbf{p}} + \mathbf{C}\eta, p_4, p_5)$ with $\mathbf{C} = \mathbf{C}(\hat{r})$. If we apply the operators P and P^{μ} to each term in Eq. (4), we obtain

$$\begin{aligned}
\mathrm{i}\hbar k P^{j}\Psi &= \left(\hat{p}^{j} - C^{j}\right)P\Psi, \\
\mathrm{i}\hbar k P^{4}\Psi &= -mP\Psi, \\
\mathrm{i}\hbar k P^{5}\Psi &= -EP\Psi, \\
\mathrm{i}\hbar k P\Psi &= \left(\left(\hat{p}_i + C_i\right)P^{i} + EP^{4} + mP^{5}\right)\Psi,
\end{aligned}$$

so that Eq. (4) becomes

$$EP\Psi = \frac{1}{2m}\left(\hat{\mathbf{p}}^2 - \mathbf{C}^2 + \left[\hat{p}_i,\ C_i\right]\right)P\Psi. \tag{5}$$

This is the wave equation for the scalar field $P\Psi$ in a non-commutative space with a general non-minimal coupling.

Now let us couple the scalar field to the three-dimensional harmonic oscillator in a non-commutative space, by utilizing Eq. (5) with

$$\mathbf{C} = \mathrm{i}m\omega\hat{\mathbf{r}}.$$

Then Eq. (5) reduces to

$$\begin{aligned}
EP\Psi = \frac{1}{2m}\Big[&\mathbf{p}^2 + m^2\omega^2\mathbf{r}^2 - 3m\hbar\omega - \frac{1}{\hbar}\left(\mathbf{\Omega} + m^2\omega^2\mathbf{\Theta}\right)\cdot\mathbf{L} + \\
&+\frac{1}{4\hbar^2}\left(\left(\mathbf{r}\times\mathbf{\Omega}\right)^2 + m^2\omega^2\left(\mathbf{p}\times\mathbf{\Theta}\right)^2\right) - \frac{m\omega}{2\hbar}\mathbf{\Theta}\cdot\mathbf{\Omega} + \hbar^2k^2\Big]P\Psi.
\end{aligned}$$

If we denote the field as $\psi \equiv P\Psi$ and choose the non-commutativity vectors to point in the z-direction,

$$\mathbf{\Theta} = (0, 0, \Theta)\,,\ \ \mathbf{\Omega} = (0, 0, \Omega)\,,$$

then Eqs. (1) and (2) take the explicit forms

$$\begin{aligned}
\hat{x} &= x - \frac{\Theta p_y}{2\hbar}, \qquad \hat{y} = y + \frac{\Theta p_x}{2\hbar}, \qquad \hat{z} = z, \\
\hat{p}_x &= p_x + \frac{\Omega y}{2\hbar}, \qquad \hat{p}_y = p_y - \frac{\Omega x}{2\hbar}, \qquad \hat{p}_z = p_z.
\end{aligned}$$

If we substitute the previous expressions into the representation (4) and reduce these equations into a single equation for ψ, we find, in cylindrical coordinates,

$$\begin{aligned}
E\psi = &\left[-\left(\frac{\hbar^2}{2m} + \frac{m\omega^2\Theta^2}{8\hbar^2}\right)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) + \left(\frac{1}{2}m\omega^2 + \frac{\Omega^2}{8m\hbar^2}\right)\rho^2\right]\psi + \\
&+\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2}m\omega^2z^2 - \frac{3}{2}\hbar\omega\right]\psi - \left(\frac{1}{2m\hbar}\left(\Omega + m^2\omega^2\Theta\right)L_3 + \frac{\omega}{4\hbar}\Theta\Omega - \frac{\hbar^2k^2}{2m}\right)\psi.
\end{aligned}$$

Next we write

$$\psi(\rho,\phi,z)=\chi(\rho)\Phi(\phi)\Xi(z), \qquad \Phi(\phi)=\exp\left(\mathrm{i}|m_l|\phi\right),$$

and

$$L_3\psi=m_l\hbar\psi,$$

and, after dividing each term by $\chi(\rho)\Phi(\phi)\Xi(z)$, the equation becomes

$$\begin{aligned} E &= -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right)\frac{1}{\chi}+\left(\frac{\hbar^2}{2m}+\frac{m\omega^2\Theta^2}{8}\right)\frac{m_l^2}{\rho^2}+ \\ &\quad +\left(\frac{1}{2}m\omega^2+\frac{\Omega^2}{8m\hbar^2}\right)\rho^2-\frac{m\omega^2\Theta^2}{8}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right)\frac{1}{\chi} \\ &\quad -\frac{\hbar^2}{2m}\frac{d^2\Xi}{dz^2}\frac{1}{\Xi}+\frac{1}{2}m\omega^2z^2+ \\ &\quad +\frac{3}{2}\hbar\omega-\frac{m_l}{2m}\left(\Omega+m^2\omega^2\Theta\right)+\frac{\omega}{4\hbar}\Theta\Omega+\frac{\hbar^2k^2}{2m}. \end{aligned} \tag{6}$$

By separating the variables, the third line of Eq. (6) gives

$$\frac{\hbar^2}{2m}\frac{d^2\Xi}{dz^2}+\left(E_{n_z}-\frac{1}{2}m\omega^2z^2\right)\Xi(z)=0. \tag{7}$$

The first two lines of Eq. (6) lead to

$$\left(\frac{\hbar^2}{2m}+\frac{m\omega^2\Theta^2}{8}\right)\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\chi}{d\rho}\right)+\left(E_\rho-\left(\frac{\hbar^2}{2m}+\frac{m\omega^2\Theta^2}{8}\right)\frac{m_l^2}{\rho^2}-\left(\frac{1}{2}m\omega^2+\frac{\Omega^2}{8m\hbar^2}\right)\rho^2\right)\chi(\rho)=0. \tag{8}$$

The constants E_{n_z} and E_ρ are related to the fourth line of Eq. (6) as follows:

$$E_{n_z}+E_\rho=E-\frac{3}{2}\hbar\omega+\frac{m_l}{2m}\left(\Omega+m^2\omega^2\Theta\right)-\frac{\omega}{4\hbar}\Theta\Omega-\frac{\hbar^2k^2}{2m}. \tag{9}$$

Clearly, Eq. (7) admits the simple harmonic oscillator solution:

$$\Xi(z)=2^{-n_z/2}\left(n_z!\right)^{-1/2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}z^2\right)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

where H_{n_z} is the Hermite polynomial of degree n_z, with the corresponding energy eigenvalue given by

$$E_{n_z}=\left(n_z+\frac{1}{2}\right)\hbar\omega. \tag{10}$$

Now let us rewrite Eq. (8) as

$$\left[\frac{\hbar^2}{2M}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m_l^2}{\rho^2}\right)+E_\rho-\frac{1}{2}M\overline{\omega}_{\Theta,\Omega}^2\rho^2\right]\chi(\rho)=0$$

with

$$\begin{aligned} M &= \frac{4m\hbar^2}{4\hbar^2+m^2\omega^2\Theta^2}, \\ \overline{\omega}_{\Theta,\Omega} &= \frac{1}{4m\hbar^2}\sqrt{\left(4m^2\hbar^2\omega^2+\Omega^2\right)\left(4\hbar^2+m^2\omega^2\Theta^2\right)} \end{aligned} \tag{11}$$

By changing the variable from ρ to

$$y = \frac{M\varpi_{\Theta,\Omega}}{2\hbar}\rho^2,$$

the previous expression can be cast into the form

$$\left(y\frac{d^2}{dy^2} + \frac{d}{dy} - \frac{m_l^2}{4y} - y + \beta\right)\chi(y) = 0, \tag{12}$$

where

$$\beta = \frac{E_\rho}{\hbar\varpi_{\Theta,\Omega}}.$$

Define the function $\varphi(y)$:

$$\chi(y) = e^{-y}y^{|m_l|/2}\varphi(y),$$

that we substitute into equation (12), to obtain

$$\left[y\frac{d^2}{dy^2} + (\gamma - 2y)\frac{d}{dy} + \beta - \gamma\right]\varphi(y) = 0,$$

where $\gamma \equiv |m_l| + 1$. By taking $w \equiv 2y$ and $-2\alpha \equiv \beta - \gamma$, we finally obtain

$$w\frac{d^2\varphi}{dw^2} + (\gamma - w)\frac{d\varphi}{dw} - \alpha\varphi = 0.$$

This is Kummer's differential equation, with solution given by the confluent hypergeometric function (see Section 13.1.1 in Ref. [12]), so that

$$\varphi(w) = N \ \ {}_1F_1(\alpha;\gamma;w),$$

where N is a normalization constant, and

$${}_1F_1(\alpha;\gamma;w) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{w^n}{n!}$$

with the Pocchammer symbol defined as $(a)_n = \Gamma(a+n)/\Gamma(a)$.

From the boundary condition, $w \to \infty$ (which follows from $\rho \to \infty$), which implies $\varphi(w) \to 0$ (so that $\psi \to 0$), we obtain

$$\alpha = \frac{1}{2}\left(|m_l| + 1 - \frac{E_\rho}{\hbar\varpi_{\Theta,\Omega}}\right) = -n_\rho, \qquad n_\rho = 0, 1, 2, ...$$

so that

$$E_\rho = (2n_\rho + |m_l| + 1)\,\hbar\varpi_{\Theta,\Omega}. \tag{13}$$

To summarize, the energy eigenvalue, $E_{n_\rho m_l n_z}$, of the DKP oscillator is obtained by substituting Eqs. (10) and (13) into Eq. (9) and solving for E. If we absorb k within the energy, we find that

$$E_{n_\rho m_l n_z} = (n_z + 2)\,\hbar\omega + (2n_\rho + |m_l| + 1)\,\hbar\varpi_{\Theta,\Omega} - \frac{m_l}{2m}\left(\Omega + m^2\omega^2\Theta\right) + \frac{\omega}{4\hbar}\Theta\Omega,$$

where $\varpi_{\Theta,\Omega}$ is given in Eq. (11). The resulting energy spectrum is non-degenerate.

3. Normalized wave functions

In this section, we state some results, the details of which will appear in a separate paper. The total wave function $\psi(\rho,\phi,z)$ is written as

$$\psi(\rho,\phi,z)=\overline{N}\;\rho^{|m_l|}e^{\mathrm{i}|m_l|\phi}e^{-\frac{m\omega}{2\hbar}z^2-\frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}\;{}_1F_1(-n_\rho;|m_l|+1;\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2)H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),$$

where $\overline{N}$ is given by

$$\overline{N}=\sqrt{\frac{\frac{1}{\sqrt{\pi^3}}\frac{1}{2^{n_z/2+1}n_z!}\left(\frac{M\overline{w}}{\hbar}\right)^{|m_l|+1}\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}}{\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(|m_l|+n)!(a)_i(a)_{n-i}}{(b)_i(b)_{n-i}i!(n-i)!}}},\tag{14}$$

which was obtained by using the normalization condition:

$$\int(P\Psi)^\dagger(P\Psi)\,\rho d\rho d\phi=\int\psi^\dagger\psi\rho d\rho d\phi=1.\tag{15}$$

The complete spinor Ψ, given in Eq. (4), is rewritten in terms of the P^μ algebra in the form

$$\Psi=\frac{1}{\mathrm{i}\hbar k}\left({}^\mu P+P^\mu\right)\pi_\mu\Psi,$$

with P^μ and ${}^\mu P$ given by $P^\mu\Psi=-\frac{1}{\hbar k}\partial^\mu P\Psi$ and $\overline{\Psi}^\mu P=\frac{1}{\hbar k}\partial^\mu\overline{\Psi}P$. Thus, if we use the 6×6 representation mentioned earlier, we can express the spinor Ψ as follows:

$$\mathrm{i}\hbar k\Psi=\begin{pmatrix}G_{11}\\G_{21}\\G_3\\E\\m\\1\end{pmatrix}{}_1F_1(-n_\rho;|m_l|+1;\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2)+\begin{pmatrix}G_{12}\\G_{22}\\0\\0\\0\\0\end{pmatrix}{}_1F_1(1-n_\rho;|m_l|+2;\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2).$$

The functions $G_{11},G_{12},G_{21},G_{22}$ and G_3 are given by

$$\begin{aligned}G_{11}&=\overline{N}\left[-\mathrm{i}\hbar\left(\cos\phi+\mathrm{i}\frac{m\omega\Theta}{2\hbar}\sin\phi\right)\left(|m_l|+\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)\rho^{-1}\right.\\&\quad\left.+\left(-\frac{\hbar|m_l|}{\rho}\sin\phi+\frac{\mathrm{i}\hbar|m_l|}{\rho}\cos\phi+\frac{\Omega}{2\hbar}\rho\sin\phi-\mathrm{i}m\omega\rho\cos\phi\right)\right]\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),\end{aligned}$$

$$\begin{aligned}G_{12}&=-2\mathrm{i}\overline{N}M\overline{\omega}_{\Theta,\Omega}\left(\cos\phi+\mathrm{i}\frac{m\omega\Theta}{2\hbar}\sin\phi\right)\rho\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),\\G_{21}&=\overline{N}\left[-\mathrm{i}\hbar\left(\sin\phi-\mathrm{i}\frac{m\omega\Theta}{2\hbar}\cos\phi\right)\left(|m_l|+\frac{M\overline{\omega}_{\Theta,\Omega}}{\hbar}\rho^2\right)\rho^{-1}\right.\\&\quad\left.+\left(\frac{\hbar|m_l|}{\rho}\cos\phi+\frac{\mathrm{i}\hbar|m_l|}{\rho}\sin\phi+\frac{\Omega}{2\hbar}\rho\sin\phi-\mathrm{i}m\omega\rho\cos\phi\right)\right]\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),\\G_{22}&=-2\mathrm{i}\overline{N}M\overline{\omega}_{\Theta,\Omega}\left(\sin\phi-\mathrm{i}\frac{m\omega\Theta}{2\hbar}\cos\phi\right)\Lambda H_{n_z}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),\\G_3&=-2\mathrm{i}\sqrt{\frac{m\omega}{\hbar}}\overline{N}\Lambda H_{n_z-1}\left(\sqrt{\frac{m\omega}{\hbar}}z\right),\end{aligned}$$

where the symbol Λ is a short-hand for

$$\Lambda=\rho^{|m_l|}e^{\mathrm{i}|m_l|\phi}e^{-\frac{m\omega}{2\hbar}z^2-\frac{M\overline{\omega}_{\Theta,\Omega}}{2\hbar}\rho^2}.$$

Acknowledgments
We acknowledge partial support by the Natural Sciences and Engineering Research Council (NSERC) of Canada (MdeM) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) of Brazil (ESS). MdeM thanks the organizers of the conference *Quantum Theory and Symmetries 7*, held in Prague, from August 7-13, 2011.

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