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Research Article

Bianchi Types II, VIII, and IX String Cosmological Models in Brans-Dicke Theory of Gravitation

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Bianchi types II, VIII, and IX string cosmological models are obtained and presented in a scalar-tensor theory of gravitation proposed by Brans and Dicke (1961) for $\lambda + \rho = 0$. We also established the existence of only Bianchi type IX vacuum cosmological model for $\lambda = \rho$, where λ and ρ are tension density and energy density of strings, respectively. Some physical and geometrical features of the models are also discussed.

1. Introduction

Brans and Dicke [1] introduced a scalar-tensor theory of gravitation involving a scalar function in addition to the familiar general relativistic metric tensor. In this theory the scalar field has the dimension of inverse of the gravitational constant, and its role is confined to its effects on gravitational field equations.

Brans-Dicke field equations for combined scalar and tensor field are given by

$$\begin{aligned} G_{ij} &= -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) \\ &\quad - \phi^{-1}\left(\phi_{i;j} - g_{ij}\phi_{;k}{}^{,k}\right), \\ \phi_{;k}{}^{,k} &= 8\pi(3+2\omega)^{-1}T, \end{aligned} \tag{1.1}$$

where $G_{ij} = R_{ij} - (1/2)Rg_{ij}$ is an Einstein tensor, T_{ij} is the stress energy tensor of the matter, and ω is the dimensionless constant.

The equation of motion

$$T^{ij}{}_{,j} = 0 \tag{1.2}$$

is a consequence of the field equation (1.1).

Several aspects of Brans-Dicke cosmology have been extensively investigated by many authors. The work of Singh and Rai [2] gives a detailed survey of Brans-Dicke cosmological models discussed by several authors. Nariai [3], Belinskii and Khalatnikov [4], Reddy and Rao [5], Banerjee and Santos [6], Ram [7], Ram and Singh [8], Berman et al. [9], Reddy [10], Reddy and Naidu [11], Adhav et al. [12], and Rao et al. [13] are some of the authors who have investigated several aspects of this theory.

In recent years, there has been a considerable interest in cosmological models in Einstein's theory and in several alternative theories of gravitation with cosmic string source. Cosmic strings and domain walls are the topological defects associated with spontaneous symmetry breaking whose plausible production site is cosmological phase transitions in the early universe (Kibble [14]). The gravitational effects of cosmic strings have been extensively discussed by Vilenkin [15], Gott [16], Latelier [17], and Stachel [18] in general relativity. Relativistic string models in the context of Bianchi space times have been obtained by Krori et al. [19], Banarjee et al. [20], Tikekar and Patel [21], and Bhattacharjee and Baruah [22]. String cosmological models in scalar-tensor theories of gravitation have been investigated by Sen [23], Barros et al. [24], Banerjee et al. [25], Gundlach and Ortiz [26], Barros and Romero [27], Pradhan [28], Mohanty et al. [29], and others.

Bianchi type space-times play a vital role in understanding and description of the early stages of evolution of the universe. In particular, the study of Bianchi types II, VIII, and IX universes is important because familiar solutions like FRW universe with positive curvature, the de Sitter universe, the Taub-NUT solutions, and so forth correspond to Bianchi types II, VIII, and IX space-times. Chakraborty [30], Bali and Dave [31], and Bali and Yadav [32] studied Bianchi type IX string as well as viscous fluid models in general relativity. Reddy et al. [33] studied Bianchi types II, VIII, and IX models in scale covariant theory of gravitation. Shanthi and Rao [34] studied Bianchi types VIII and IX models in Lyttleton-Bondi universe. Also Rao and Sanyasiraju [35] and Sanyasirajuand Rao [36] have studied Bianchi types VIII and IX models in Zero mass scalar fields and self-creation cosmology. Rahaman et al. [37] have investigated Bianchi type IX string cosmological model in a scalar-tensor theory formulated by Sen [38] based on Lyra [39] manifold. Rao et al. [40–42] have obtained Bianchi types II, VIII, and IX string cosmological models, perfect fluid cosmological models in Saez-Ballester theory of gravitation, and string cosmological models in general relativity as well as self-creation theory of gravitation, respectively.

In this paper we will discuss Bianchi types II, VIII, and IX string cosmological models in a scalar-tensor theory proposed by Brans and Dicke [1].

2. Metric and Energy Momentum Tensor

We consider a spatially homogeneous Bianchi types II, VIII, and IX metrics of the form

$$ds^2 = dt^2 - R^2\left[d\theta^2 + f^2(\theta)d\phi^2\right] - S^2\left[d\psi + h(\theta)d\phi\right]^2, \tag{2.1}$$

where (θ, ϕ, ψ) are the Eulerian angles, R and S are functions of $\mathbf{t}$ only. It represents

Bianchi type II if $f(\theta) = 1$ and $h(\theta) = \theta$,

Bianchi type VIII if $f(\theta) = \cos h\theta$ and $h(\theta) = \sin h\theta$,

Bianchi type IX if $f(\theta) = \sin\theta$ and $h(\theta) = \cos\theta$.

The energy momentum tensor for cosmic strings [17] is

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j, \tag{2.2}$$

where u_i is the four-velocity of the string cloud, x_i is the direction of anisotropy, ρ and λ are the rest energy density and the tension density of the string cloud, respectively. The string source is along the Z-axis which is the axis of symmetry. Orthonormalisation of u^i and x^i is given as

$$u^i u_i = -x^i x_i = 1, \qquad u^i x_i = 0. \tag{2.3}$$

In the commoving coordinate system, we have from (2.2) and (2.3)

$$T_1^1 = T_2^2 = 0, \qquad T_3^3 = \lambda, \qquad T_4^4 = \rho, \qquad T_j^i = 0 \quad \text{for } i \neq j. \tag{2.4}$$

The quantities ρ, λ and the scalar field ϕ in the theory depend on t only.

3. Bianchi Types II, VIII, and IX String Cosmological Models in Brans-Dicke Theory of Gravitation

The field equations (1.1), (1.2) for the metric (2.1) with the help of (2.2), (2.3), and (2.4) can be written as

$$\begin{gathered}
\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + \frac{\dot{R}\dot{S}}{RS} + \frac{S^2}{4R^4} + \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{R}\dot{\phi}}{R\phi} + \frac{\dot{S}\dot{\phi}}{S\phi} = 0, \\
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + \delta}{R^2} - \frac{3S^2}{4R^4} + \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{2\dot{R}\dot{\phi}}{R\phi} = \frac{8\pi\lambda}{\phi}, \\
\frac{2\dot{R}\dot{S}}{RS} - \frac{S^2}{4R^4} + \frac{\dot{R}^2 + \delta}{R^2} - \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{2\dot{R}\dot{\phi}}{R\phi} + \frac{\dot{S}\dot{\phi}}{S\phi} = \frac{8\pi\rho}{\phi}, \\
\left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R}\right)\frac{h(\theta)\dot{\phi}}{\phi} = 0, \\
\ddot{\phi} + \dot{\phi}\left(\frac{2\dot{R}}{R} + \frac{\dot{S}}{S}\right) = \frac{8\pi}{3 + 2\omega}(\lambda + \rho), \\
\dot{\rho} + 2\rho\frac{\dot{R}}{R} + (\rho - \lambda)\frac{\dot{S}}{S} = 0,
\end{gathered} \tag{3.1}$$

where "." denotes differentiation with respect to "t".

When $\delta = 0, -1$ & $+1$, the field equation (3.1) correspond to the Bianchi types II, VIII, and IX universes, respectively.

Using the transformation $R = e^{\alpha}$, $S = e^{\beta}$, $dt = R^2 S\, dT$, (3.1) reduce to

$$\alpha'' + \beta'' - \alpha'^2 - 2\alpha'\beta' + \frac{e^{4\beta}}{4} + \frac{\omega\phi'^2}{2\phi^2} - \frac{\alpha'\phi'}{\phi} + \frac{\phi''}{\phi} = 0, \tag{3.2}$$

$$2\alpha'' - \alpha'^2 - 2\alpha'\beta' + \delta e^{(2\alpha+2\beta)} - \frac{3}{4}e^{4\beta} + \frac{\omega\phi'^2}{2\phi^2} - \frac{\beta'\phi'}{\phi} + \frac{\phi''}{\phi} = \frac{8\pi\lambda}{\phi}e^{(4\alpha+2\beta)}, \tag{3.3}$$

$$2\alpha'\beta' + \alpha'^2 + \delta e^{(2\alpha+2\beta)} - \frac{1}{4}e^{4\beta} - \frac{\omega\phi'^2}{2\phi^2} + \frac{\beta'\phi'}{\phi} + \frac{2\alpha'\phi'}{\phi} = \frac{8\pi\rho}{\phi}e^{(4\alpha+2\beta)}, \tag{3.4}$$

$$(\alpha' - \beta')\frac{h(\theta)\phi'}{\phi} = 0, \tag{3.5}$$

$$\phi'' = \frac{8\pi}{3+2\omega}(\lambda + \rho)e^{(4\alpha+2\beta)}, \tag{3.6}$$

$$\rho' + 2\rho\alpha' + (\rho - \lambda)\beta' = 0, \tag{3.7}$$

where "$'$" denotes differentiation with respect to "T".

Since we are considering the Bianchi types II, VIII, and IX metrics, we have $h(\theta) = \theta$, $h(\theta) = \sinh\theta$, and $h(\theta) = \cos\theta$ for Bianchi types II, VIII, and IX metrics, respectively. Therefore, from (3.5), we will consider the following possible cases with $h(\theta) \neq 0$:

$$\begin{aligned} &(1) \quad \alpha' - \beta' = 0, \qquad \phi' \neq 0, \\ &(2) \quad \alpha' - \beta' \neq 0, \qquad \phi' = 0, \\ &(3) \quad \alpha' - \beta' = 0, \qquad \phi' = 0. \end{aligned} \tag{3.8}$$

Case 1 (for $\alpha' - \beta' = 0$ and $\phi' \neq 0$). Here, we get $\alpha = \beta + c$.

Without loss of generality by taking the constant of integration $c = 0$, we get

$$\alpha = \beta. \tag{3.9}$$

By using (3.9), (3.2) to (3.7) will reduce to

$$2\beta'' - 3\beta'^2 + \frac{e^{4\beta}}{4} + \frac{\omega\phi'^2}{2\phi^2} - \frac{\beta'\phi'}{\phi} + \frac{\phi''}{\phi} = 0, \tag{3.10}$$

$$2\beta'' - 3\beta'^2 + \delta e^{4\beta} - \frac{3}{4}e^{4\beta} + \frac{\omega\phi'^2}{2\phi^2} - \frac{\beta'\phi'}{\phi} + \frac{\phi''}{\phi} = \frac{8\pi\lambda}{\phi}e^{6\beta}, \tag{3.11}$$

$$3\beta'^2 + \delta e^{4\beta} - \frac{1}{4}e^{4\beta} - \frac{\omega\phi'^2}{2\phi^2} + \frac{3\beta'\phi'}{\phi} = \frac{8\pi\rho}{\phi}e^{6\beta}, \tag{3.12}$$

$$\phi'' = \frac{8\pi}{3+2\omega}(\lambda+\rho)e^{6\beta}, \tag{3.13}$$

$$\rho' + (3\rho - \lambda)\beta' = 0, \tag{3.14}$$

where "$'$" denotes differentiation with respect to "T".

Latelier [17] discussed, in general, the following equations of state:

$$\rho = \lambda \quad (\text{geometric string}), \qquad \rho = (1+\omega)\lambda \quad (p\text{-string}), \tag{3.15}$$

and Reddy [10] obtained inflationary string cosmological models in Brans-Dicke scalar-tensor theory of gravitation for

$$\rho + \lambda = 0 \quad (\text{Reddy string}). \tag{3.16}$$

Here we will present string cosmological models corresponding to $\rho + \lambda = 0$ and $\rho = \lambda$.

Case 2 (for $\lambda + \rho = 0$ (Reddy string)). From (3.13), we get

$$\phi'' = 0, \tag{3.17}$$

then

$$\phi = aT + b. \tag{3.18}$$

Without loss of generality by taking the constants of integration $a = 1$ and $b = 0$, we get

$$\phi = T. \tag{3.19}$$

Now from (3.11), (3.12), and (3.19), we get

$$2\beta'' + 2\delta e^{4\beta} - e^{4\beta} + \frac{2\beta'}{T} = 0. \tag{3.20}$$

For Bianchi Type II Metric ($\delta = 0$)

From (3.20), we get

$$T\beta'' + \beta' = \frac{1}{2}Te^{4\beta}. \tag{3.21}$$

From (3.21), we get

$$e^{4\beta} = \frac{{c_1}^2}{T^2}\text{cosech}^2(c_1 \log T + c_2), \tag{3.22}$$

where c_1 and c_2 are integration constants.

Using (3.22) in (3.11) and (3.12), we have

$$8\pi\rho = -8\pi\lambda = \frac{T^2}{2c_1}\sinh(c_1 \log T + c_2). \tag{3.23}$$

The corresponding metric can be written in the form

$$\begin{aligned} ds^2 = {} & \frac{{c_1}^3}{T^3}\cos\, ech^3(c_1 \log T + c_2)dT^2 - \frac{c_1}{T}\cos\, ech(c_1 \log T + c_2)\left[d\theta^2 + d\phi^2\right] \\ & - \frac{c_1}{T}\cos\, ech(c_1 \log T + c_2)\left[d\psi + d\phi\right]^2. \end{aligned} \tag{3.24}$$

Thus, (3.24) together with (3.23) constitutes the Bianchi type II string cosmological model in Brans-Dicke theory of gravitation.

For Bianchi Type VIII Metric ($\delta = -1$)

From (3.20), we get

$$T\beta'' + \beta' = \frac{3}{2}Te^{4\beta}. \tag{3.25}$$

From (3.25), we get

$$e^{4\beta} = \frac{{c_3}^2}{3T^2}\cos \text{ech}^2(c_3 \log T + c_4), \tag{3.26}$$

where c_3 and c_4 are integration constants.

Using (3.26) in (3.11) and (3.12), we have

$$8\pi\rho = -8\pi\lambda = \frac{\sqrt{3}T^2}{c_3}\sinh(c_3 \log T + c_4). \tag{3.27}$$

The corresponding metric can be written in the form

$$\begin{aligned} ds^2 = {} & \frac{{c_3}^3}{9T^3}\cos\, ech^3(c_3 \log T + c_4)dT^2 - \frac{c_3}{\sqrt{3}T}\cos\, ech(c_3 \log T + c_4)\left[d\theta^2 + \cosh^2 d\phi^2\right] \\ & - \frac{c_1}{\sqrt{3}T}\cos\, ech(c_3 \log T + c_4)\left[d\psi + \sinh\theta\, d\phi\right]^2. \end{aligned} \tag{3.28}$$

Thus, (3.28) together with (3.27) constitutes the Bianchi type VIII string cosmological model in Brans-Dicke theory of gravitation.

For Bianchi Type IX Metric ($\delta = 1$)

From (3.20), we get

$$T\beta'' + \beta' = -\frac{1}{2}Te^{4\beta}. \tag{3.29}$$

From (3.29), we get

$$e^{4\beta} = \frac{{c_5}^2}{T^2}\sec h^2(c_5 \log T + c_6), \tag{3.30}$$

where c_5 and c_6 are integration constants.

Using (3.30) in (3.11) and (3.12), we have

$$8\pi\rho = -8\pi\lambda = \frac{T^2}{2c_5}\cosh(c_5 \log T + c_6). \tag{3.31}$$

The corresponding metric can be written in the form

$$\begin{aligned} ds^2 = {} & \frac{{c_5}^3}{T^3}\sec h^3(c_5 \log T + c_6)dT^2 - \frac{c_5}{T}\sec h(c_5 \log T + c_6)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \\ & - \frac{c_5}{T}\sec h(c_5 \log T + c_6)\left[d\psi + \cos\theta\, d\phi\right]^2. \end{aligned} \tag{3.32}$$

Thus, (3.32) together with (3.31) constitutes the Bianchi type IX string cosmological model in Brans-Dicke theory of gravitation.

3.1. Physical and Geometrical Properties

The volume element V, expansion θ, and shear σ for the models (3.24), (3.28), and (3.32) are given by

$$\begin{aligned} V &= (-g)^{1/2} = \frac{{c_1}^{3/2}}{T^{3/2}}\cos ech^{3/2}(c_1 \log T + c_2), \\ \theta &= {u_{;i}}^{i} = \frac{3c_1}{2T}\coth(c_1 \log T + c_2), \\ \sigma^2 &= \frac{3{c_1}^2}{8T^2}\coth^2(c_1 \log T + c_2) \end{aligned} \tag{3.33}$$

for the Bianchi type II model,

$$V=(-g)^{1/2}=\frac{{c_3}^{3/2}}{T^{3/2}}\cos\ ech^{3/2}(c_3\log T+c_4)\cosh\theta,$$
$$\theta=u_{;i}{}^{i}=\frac{3c_3}{2T}\coth(c_3\log T+c_4), \tag{3.34}$$
$$\sigma^2=\frac{3{c_3}^2}{8T^2}\coth^2(c_3\log T+c_4)$$

for the Bianchi type VIII model, and

$$V=(-g)^{1/2}=\frac{{c_5}^{3/2}}{T^{3/2}}\sec h^{3/2}(c_5\log T+c_6)\sin\theta,$$
$$\theta=u_{;i}{}^{i}=\frac{3c_5}{2T}\tanh(c_5\log T+c_6), \tag{3.35}$$
$$\sigma^2=\frac{3{c_5}^2}{8T^2}\tanh^2(c_5\log T+c_6)$$

for the Bianchi type IX model.

Case 3 (for $\lambda=\rho$ (geometric string)). From (3.14), we get

$$\rho=c_1e^{-2\beta}. \tag{3.36}$$

From (3.11) and (3.12), we get

$$(\delta-1)=\frac{8\pi\rho}{\phi}e^{2\beta}. \tag{3.37}$$

From (3.37) and (3.36), we get

$$(\delta-1)=\frac{8\pi}{\phi}c_1. \tag{3.38}$$

For Bianchi Types II and VIII Metrics ($\delta=0$ *and* -1)

From (3.38), we have

$$\phi=\text{constant}\ (\text{say}\ c_2). \tag{3.39}$$

Using (3.39), the field equations (3.10) to (3.13) reduce to

$$2\beta'' - 3\beta'^2 + \frac{e^{4\beta}}{4} = 0, \tag{3.40}$$

$$2\beta'' - 3\beta'^2 + \delta e^{4\beta} - \frac{3}{4}e^{4\beta} = \frac{8\pi\rho}{c_2}e^{6\beta}, \tag{3.41}$$

$$3\beta'^2 + \delta e^{4\beta} - \frac{1}{4}e^{4\beta} = \frac{8\pi\rho}{c_2}e^{6\beta}, \tag{3.42}$$

$$0 = \frac{8\pi}{3+2\omega}(2\rho)e^{6\beta}. \tag{3.43}$$

From (3.43), we get

$$\rho = 0, \tag{3.44}$$

and, since $\lambda = \rho$, we will get $\lambda = 0$.

From (3.40) to (3.42), we get

$$2\beta'' + e^{4\beta} = 0. \tag{3.45}$$

From (3.45), we get

$$S^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T + n_1). \tag{3.46}$$

The corresponding metrics can be written in the form

$$\begin{aligned} ds^2 &= 8m_1{}^3 \sec h^3 2m_1(T+n_1)dT^2 - 2m_1 \sec h 2m_1(T+n_1)\left[d\theta^2 + d\phi^2\right] \\ &\quad - 2m_1 \sec h 2m_1(T+n_1)\left[d\psi + \theta\, d\phi\right]^2, \\ ds^2 &= 8m_1{}^3 \sec h^3 2m_1(T+n_1)dT^2 - 2m_1 \sec h 2m_1(T+n_1)\left[d\theta^2 + \cosh^2\theta\, d\phi^2\right] \\ &\quad - 2m_1 \sec h 2m_1(T+n_1)\left[d\psi + \sinh\theta\, d\phi\right]^2. \end{aligned} \tag{3.47}$$

Thus, (3.47) together with (3.44) constitutes an exact Bianchi types II and VIII vacuum cosmological models, respectively, in general relativity.

For Bianchi Type IX Metric ($\delta = 1$)

From (3.38), we have

$$c_1 = 0. \tag{3.48}$$

Using (3.48) in (3.36), we get

$$\rho = 0. \tag{3.49}$$

Using (3.49) the field equations (3.10) to (3.14) reduce to

$$2\beta'' - 3\beta'^2 + \frac{e^{4\beta}}{4} + \frac{\omega\phi'^2}{2\phi^2} - \frac{\beta'\phi'}{\phi} + \frac{\phi''}{\phi} = 0, \tag{3.50}$$

$$3\beta'^2 + \delta e^{4\beta} - \frac{1}{4}e^{4\beta} - \frac{\omega\phi'^2}{2\phi^2} + \frac{3\beta'\phi'}{\phi} = 0, \tag{3.51}$$

$$\phi'' = 0. \tag{3.52}$$

From (3.52), we get

$$\phi = c_3 T + c_4. \tag{3.53}$$

Without loss of generality by taking the constants of integration $c_3 = 1$ and $c_4 = 0$, we get

$$\phi = T. \tag{3.54}$$

From (3.50), (3.51), and (3.54), we get

$$2\beta'' + e^{4\beta} + \frac{2\beta'}{T} = 0, \tag{3.55}$$

that is,

$$T\beta'' + \beta' = -\frac{1}{2}Te^{4\beta}. \tag{3.56}$$

From (3.56), we get

$$e^{4\beta} = \frac{{c_1}^2}{T^2}\sec h^2(c_1 \log T + c_2). \tag{3.57}$$

The corresponding metric can be written in the form

$$\begin{aligned} ds^2 = {}& \frac{{c_1}^3}{T^3}\sec h^3(c_1 \log T + c_2)dT^2 - \frac{c_1}{T}\sec h(c_1 \log T + c_2)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \\ & - \frac{c_1}{T}\sec h(c_1 \log T + c_2)\left[d\psi + \cos\theta\, d\phi\right]^2. \end{aligned} \tag{3.58}$$

Thus, (3.58) together with (3.49) constitutes an exact Bianchi type IX vacuum cosmological model in Brans-Dicke theory of gravitation.

3.2. Physical and Geometrical Properties

The spatial volume V, expansion θ, and the shear σ for the models (3.47) and (3.58) are given by

$$\begin{aligned} V &= (-g)^{1/2} = 8{m_1}^3 \sec h^3 2m_1(T+n_1), \\ \theta &= 6m_1 \tanh 2m_1(T+n_1), \\ \sigma^2 &= 6m_1^2 \sec h^2 2m_1(T+n_1) \end{aligned} \tag{3.59}$$

for the Bianchi type II cosmological model ($\delta = 0$),

$$\begin{aligned} V &= (-g)^{1/2} = 8{m_1}^3 \sec h^3 2m_1(T+n_1)\cosh\theta, \\ \theta &= [2m_1 \tanh 2m_1(T+n_1)\cosh\theta + \sinh\theta], \\ \sigma^2 &= \left[2m_1^2 \tanh^2 2m_1(T+n_1)\cosh^2\theta + \sinh^2\theta + 4m_1\tanh 2m_1(T+n_1)\cosh\theta\sinh\theta\right] \end{aligned} \tag{3.60}$$

for the Bianchi type VIII cosmological model ($\delta = -1$), and

$$\begin{aligned} V &= (-g)^{1/2} = \frac{{c_1}^{3/2}}{T^{3/2}} \sec h^{3/2}(c_1 \log T + c_2), \\ \theta &= u_{;i}^{\;i} = \frac{3c_1}{2T}\tanh(c_1 \log T + c_2), \\ \sigma^2 &= \frac{3{c_1}^2}{8T^2}\tanh^2(c_1 \log T + c_2) \end{aligned} \tag{3.61}$$

for the Bianchi type IX cosmological model ($\delta = 1$).

Case 4 (for $\alpha' - \beta' \neq 0$ and $\phi' = 0$). In this case, we get Bianchi types II, VIII, and IX string cosmological models in general relativity as obtained and presented by Rao et al. [42].

Case 5 (for $\alpha' - \beta' = 0$ and $\phi' = 0$). Here, we get $\alpha = \beta + c$.

Without loss of generality by taking the constant of integration $c = 0$, we get

$$\begin{aligned} \alpha &= \beta, \\ \phi &= \text{constant (say } c_1). \end{aligned} \tag{3.62}$$

Using (3.62), the field equations (3.2) to (3.7) will reduce to

$$2\beta'' - 3\beta'^2 + \frac{e^{4\beta}}{4} = 0, \tag{3.63}$$

$$2\beta'' - 3\beta'^2 + \delta e^{4\beta} - \frac{3}{4}e^{4\beta} = \frac{8\pi\lambda}{c_1}e^{6\beta}, \tag{3.64}$$

$$3\beta'^2 + \delta e^{4\beta} - \frac{1}{4}e^{4\beta} = \frac{8\pi\rho}{c_1}e^{6\beta}, \tag{3.65}$$

$$0 = \frac{8\pi}{3+2\omega}(\rho+\lambda)e^{6\beta}, \tag{3.66}$$

$$\rho' + (3\rho - \lambda)\beta' = 0. \tag{3.67}$$

From (3.66), we get

$$\rho + \lambda = 0. \tag{3.68}$$

From (3.63) to (3.65) and (3.68), we have

$$2\beta'' + 2\delta e^{4\beta} - e^{4\beta} = 0. \tag{3.69}$$

For Bianchi Type II Metric ($\delta = 0$)

From (3.69), we get

$$e^{\beta} = (aT+b)^{-1/2}, \quad \text{where } a^2 = 1. \tag{3.70}$$

From (3.64), (3.65), and (3.70), we have

$$8\pi\lambda = \frac{-c_1(aT+b)}{2}, \qquad 8\pi\rho = \frac{c_1(aT+b)}{2}. \tag{3.71}$$

From (3.71) we get $\lambda + \rho = 0$.

The corresponding metric can be written in the form

$$ds^2 = (aT+b)^{-3}dT^2 - (aT+b)^{-1}\left[d\theta^2 + d\phi^2\right] - (aT+b)^{-1}\left[d\psi + \theta\, d\phi\right]^2. \tag{3.72}$$

Thus, (3.72) together with (3.71) constitutes an exact Bianchi type II string cosmological model in general theory of relativity.

For Bianchi Type VIII Metric ($\delta = -1$)

From (3.69), we get

$$e^{\beta} = (aT+b)^{-1/2}, \quad \text{where } a^2 = 3. \tag{3.73}$$

From (3.64) and (3.65), we have

$$8\pi\lambda = -c_1(aT+b), \qquad 8\pi\rho = c_1(aT+b). \tag{3.74}$$

Therefore, from (3.74), we have

$$\lambda + \rho = 0. \tag{3.75}$$

The corresponding metric can be written in the form

$$ds^2 = (aT+b)^{-3}dT^2 - (aT+b)^{-1}\left[d\theta^2 + \cosh^2\theta\, d\phi^2\right] - (aT+b)^{-1}\left[d\psi + \sinh\theta\, d\phi\right]^2. \tag{3.76}$$

Thus, (3.76) together with (3.74) constitutes an exact Bianchi type VIII string cosmological model in general theory of relativity.

For Bianchi Type IX Metric ($\delta = 1$)

From (3.69), we get

$$2\beta'' + e^{4\beta} = 0. \tag{3.77}$$

From (3.77), we get

$$S^2 = e^{2\beta} = 2m_1 \sec h 2m_1(T+n_1). \tag{3.78}$$

From (3.63) and (3.64), we get

$$(\delta - 1) = \frac{8\pi\lambda}{c_1}e^{2\beta}. \tag{3.79}$$

From (3.66) & (3.79), we get

$$\lambda = \rho = 0. \tag{3.80}$$

The corresponding metric can be written in the form

$$\begin{aligned} ds^2 &= 8{m_1}^3 \sec h^3 2m_1(T+n_1)dT^2 - 2m_1 \sec h 2m_1(T+n_1)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right] \\ &\quad - 2m_1 \sec h 2m_1(T+n_1)\left[d\psi + \cos\theta\, d\phi\right]^2. \end{aligned} \tag{3.81}$$

Thus, (3.81) together with (3.80) constitutes an exact Bianchi type IX vacuum cosmological model in general theory of relativity.

3.3. Physical and Geometrical Properties

The spatial volume V, expansion θ, and the shear σ for the models (3.72), (3.76), and (3.81) are given by

$$V = (-g)^{1/2} = (aT+b)^{-3/2}, \qquad \theta = \frac{-3a}{(aT+b)}, \qquad \sigma^2 = \frac{3a^2}{2(aT+b)^2} \tag{3.82}$$

for the Bianchi type II model,

$$\begin{gathered} V = (-g)^{1/2} = (aT+b)^{-3/2}\cosh\theta, \\ \theta = \tanh\theta(aT+b)^{-3/2} - \frac{3a}{(aT+b)^{-1/2}}, \\ \sigma^2 = \tanh^2\theta(aT+b)^{-3} + 3a^2(aT+b) - \frac{a\tanh\theta}{(aT+b)} \end{gathered} \tag{3.83}$$

for the Bianchi type VIII model, and

$$\begin{aligned} V &= (-g)^{1/2} = 8{m_1}^3 \sec h^3 2m_1(T+n_1)\sin\theta, \\ \theta &= {u_{;i}}^{i} = [2m_1\tanh 2m_1(T+n_1)\sin\theta + \sin 2\theta], \\ \sigma^2 &= \frac{\left[2{m_1}^2\tanh^2 2m_1\mathfrak{H} + 2\cos^2\theta + 4m_1\tanh 2m_1\mathfrak{H}\cos\theta\right]}{3}, \end{aligned} \tag{3.84}$$

where $\mathfrak{H} = (T+n_1)$ for the Bianchi type IX model.

4. Conclusions

In view of the importance of Bianchi types II, VIII, and IX space times and cosmic strings in the study of relativistic cosmology and astrophysics, in this paper we have studied and presented Bianchi types II, VIII, and IX string cosmological models in Brans-Dicke theory of gravitation.

In case of (1.1), for the equation of state $\lambda + \rho = 0$, the models (3.24), (3.28), and (3.32) represent, respectively, Bianchi types II, VIII, and IX string cosmological models in Brans-Dicke theory of gravitation. The spatial volume of the models (3.24), (3.28), and (3.32) are decreasing as $T \to \infty$; that is, the models are contacting with the increase of time. Also, the models have no initial singularity.

In Case of 3, for the equation of state $\lambda = \rho$, we will get only Bianchi type IX vacuum cosmological model in Brans-Dicke theory of gravitation. Also, in this case, we established the nonexistence of Bianchi types II and VIII geometric string cosmological models in Brans-Dicke theory of gravitation and hence presented only vacuum cosmological models of general relativity. The volume of all the models is decreasing as $T \to \infty$, and also the models are free from singularities.

In Case 5, we obtained only Bianchi types II and VIII string cosmological models of general relativity with $\lambda + \rho = 0$ and also got Bianchi type IX vacuum cosmological model of general relativity, since the scalar field ϕ is constant. The spatial volume of the models (3.72), (3.76), and (3.81) are decreasing as $T \to \infty$; that is, the models are contracting with the increase of time. Also the models (3.72) and (3.76) have initial singularity at $T = -b/a$, $a \neq 0$, and the model (3.81) has no initial singularity.

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