NONCOMMUTATIVE DIFFERENTIAL CALCULUS: QUANTUM GROUPS, STOCHASTIC PROCESSES AND ANTIBRACKET

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Abstract. We explore a differential calculus on the algebra of C^∞-functions on a manifold. The former is 'noncommutative' in the sense that functions and differentials do not commute, in general. Relations with bicovariant differential calculus on certain quantum groups and stochastic calculus are discussed. A similar differential calculus on a superspace is shown to be related to the Batalin-Vilkovisky antifield formalism.

Key words: Noncommutative geometry, quantum groups, stochastic differential equations, antifield formalism

1. Introduction

Since Connes' work on noncommutative geometry, the notion of differential calculus on algebras has entered the realm of physics through numerous publications. As the commutative algebra of ($\mathbb{C}$-valued) functions on a topological space carries all the information about the space in its algebraic structure, certain noncommutative algebras may be regarded as a generalization of the notion of a 'space'. If the algebra $\mathcal{A}$ is associative, one can enlarge it to a differential algebra, a kind of analogue of the algebra of differential forms on a differentiable manifold.

More precisely, this is a $\mathbb{Z}$-graded associative algebra $\bigwedge(\mathcal{A}) = \bigoplus_{r\geq 0} \bigwedge^r(\mathcal{A})$ where $\bigwedge^0 = \mathcal{A}$. The spaces $\bigwedge^r(\mathcal{A})$ of *r-forms* are generated as $\mathcal{A}$-bimodules via the action of an *exterior derivative* $d : \bigwedge^r(\mathcal{A}) \to \bigwedge^{r+1}(\mathcal{A})$ which is a linear operator acting in such a way that $d^2 = 0$ and $d(\omega\omega') = (d\omega)\,\omega' + (-1)^r \omega\, d\omega'$ (where ω and ω' are r- and r'-forms, respectively). Without further restrictions, $\bigwedge(\mathcal{A})$ is the so-called *universal differential envelope* of $\mathcal{A}$. It associates, for example, independent differentials with $f \in \mathcal{A}$ and f^2.

What we would rather like to have is a closer analogue of the algebra of differential forms on a manifold. In particular, if $\mathcal{A}$ is generated by a set of n elements (e.g., coordinate functions x^i on a manifold), we might want the space of 1-forms to be generated as a left- (or right-) $\mathcal{A}$-module by the differentials dx^i. In order to achieve this, one has to add commutation rules for functions and differentials to the differential algebra structure defined above. In case of the commutative algebra of C^∞-functions on a manifold, the ordinary calculus of differential forms simply assumes that 1-forms and functions commute. If, however, $\mathcal{A}$ is the algebra of functions on a discrete set, this assumption cannot be kept. The algebra of functions on a two-point set, for example, is generated by a function y such that $y^2 = 1$. Acting with d on this relation yields $y\,dy = -dy\,y$ and thus *anti*-commutativity. In this example the commutation relation is *not* an additional assumption, but follows from the general rules of differential calculus. This is a special feature of the two-point space. This example plays a crucial role in models of elementary particle physics [1]. Here we just take it to illustrate what we mean by 'noncommutative differential calculus', namely noncommutativity between functions and differentials.

Let $\mathcal{A}$ be the set of functions on $\mathbb{R}$ generated by a coordinate function x (and a unit element which we identify with $1 \in \mathbb{C}$). The simplest consistent deformation of the ordinary differential calculus is then determined by $[x, dx] = a\,dx$ where a is a positive real constant. If we define partial derivatives by $df = \overleftarrow{\partial} f\, dx = dx\, \overrightarrow{\partial} f$, they turn out to be (left- and right-) *discrete* derivatives. An integral is naturally associated with d and (for the higher-dimensional generalization of the calculus) it turns out that the deformation from $a = 0$ to $a > 0$ transforms continuum theories (like a gauge theory) to the corresponding lattice theory (where a plays the role of the lattice spacing) [2]. A simple coordinate transformation brings the above commutation relation into the form $y\,dy = q\,dy\,y$ with $q \in \mathbb{C}$, the differential calculus underlying q-calculus [3]. This noncommutative differential calculus is the best understood and most complete example so far. We can also introduce it on the space of functions on a lattice with spacings a instead of $\mathcal{A}$. More generally, differential calculus on discrete sets is supposed to be of relevance for approaches towards discrete field theory and geometry (see [4] and the references given there).

Another interesting example of a noncommutative differential calculus on a commutative algebra is the following [5, 6]. Let $\mathcal{A}$ be the algebra of C^∞-functions on a manifold $\mathcal{M}$ and let us assume the following commutation relations expressed in terms of local coordinates x^i:

$$[x^i, dx^j] = \gamma\, g^{ij}\, dt \tag{1.1}$$

where γ is a constant, g a real symmetric tensor (e.g., a metric) on $\mathcal{M}$, and t an 'external' (time) parameter. The above commutation relation is actually coordinate independent. The differential calculus based on it is related to quantum mechanics [5] and stochastics [6] (depending on whether γ is imaginary or real), and to 'proper time' (quantum) theories [5]. A generalization of (1.1) is obtained by replacing $\gamma\, dt$ by a 1-form τ, i.e.

$$[x^i, dx^j] = \tau\, g^{ij} \tag{1.2}$$

where τ should have the following properties,

$$[x^i, \tau] = 0 \quad , \quad \tau\,\tau = 0 \quad , \quad d\tau = 0\,. \tag{1.3}$$

This structure in fact shows up in the classical limit ($q \to 1$) of (bicovariant [7]) differential calculus on certain quantum groups [8]. For functions $f, h \in \mathcal{A}$, we have

$$[f, dh] = \tau\, (f,h)_g \quad , \quad (f,h)_g := g^{ij}\, \partial_i f\, \partial_j h \tag{1.4}$$

where $\partial_i := \partial/\partial x^i$. In sections 2-5, a brief introduction to various aspects of this differential calculus is given. Some of the results, in particular in sections 3 and 5, have not been published before.

Sections 6 and 7 present basically new results. We introduce a differential calculus on a superspace and show that the antibracket and the Δ-operator of the Batalin-Vilkovisky formalism [9] (developed for quantization of gauge theories) appear naturally in this framework. A corresponding generalization of gauge theory is also formulated. The differential calculus is a kind of superspace counterpart of the abovementioned differential calculus on manifolds.

Our work establishes relations between noncommutative differential calculus and various mathematical structures which play a role in physics. The latter are thus put into a new perspective which will hopefully contribute to an improved understanding and handling of these structures.

2. The classical limit of bicovariant differential calculi on the quantum groups $GL_q(2)$ and $SL_q(2)$

Let us denote the entries of a $GL(2)$-matrix as follows,

$$M = \begin{pmatrix} x^1 & x^2 \\ x^3 & x^4 \end{pmatrix}\,. \tag{2.1}$$

Let $\mathcal{A}$ be the algebra of polynomials in x^i. The quantum group $GL_q(2)$ is a noncommutative deformation of $\mathcal{A}$ as a Hopf algebra. The structure of a quantum group allows to narrow down the many possible differential calculi on it. This results in the notion of *bicovariant* differential calculus [7]. For $GL_q(2)$ there is a 1-parameter set of bicovariant differential calculi. In the classical limit $q \to 1$ they lead [8, 6] to the commutation relations (1.2) with

$$g^{ij} = (\det M)^{-1} x^i x^j + 4 (\delta_2^{(i} \delta_3^{j)} - \delta_1^{(i} \delta_4^{j)}) \tag{2.2}$$
$$\tau = s (dx^1 x^4 - dx^2 x^3 - dx^3 x^2 + dx^4 x^1) \tag{2.3}$$

where s is a free parameter. The ordinary differential calculus on $GL(2)$ is only obtained when $s = 0$.

The condition for the matrix M to be in $SL(2)$ is the quadratic equation

$$\det M = x^1 x^4 - x^2 x^3 = 1 . \tag{2.4}$$

Compatibility of the analogous condition for the quantum group $SL_q(2)$ with bicovariant differential calculus restricts the parameter s to only two values (both different from zero) [8]. There are thus only two bicovariant differential calculi on $SL_q(2)$ and for both the classical limit is not the ordinary differential calculus. We will only consider one of them here. In a cordinate patch where $x^1 \neq 0$ we can use x^a, $a = 1,2,3$, as coordinates. The differential calculus is then determined by (1.2) with

$$g^{ab} = x^a x^b + 4 \delta_2^{(a} \delta_3^{b)} \tag{2.5}$$
$$\tau = \frac{1}{3} (dx^1 x^4 - dx^2 x^3 - dx^3 x^2 + dx^4 x^1) \tag{2.6}$$

where $x^4 = (1 + x^2 x^3)/x^1$. Although we only have *three* independent coordinates in this case, the space of 1-forms (as a left or right $\mathcal{A}$-module) is *four*-dimensional since τ cannot be expressed as $\tau = \sum_{a=1}^3 dx^a f_a$ with $f_a \in \mathcal{A}$. What's going on here is explained in more detail in the following section, using a simple example.

3. Differential calculi on quadratic varieties

Let x^i, $i = 1, \ldots, n$, be real variables, α_{ij} a nondegenerate symmetric constant form with inverse α^{ij}. We want to construct a noncommutative differential calculus with (1.2) and (1.3), compatible with the quadratic relation

$$\alpha_{ij} x^i x^j = 1 . \tag{3.1}$$

The $SL(2)$-condition (2.4) provides us with a particular example. Acting with d on (3.1) and using (1.2), we obtain

$$\tau = dx^i (-2 \alpha_{ij} x^j / a) =: dx^i \tau_i \tag{3.2}$$

where we have assumed that $a := \alpha_{ij} g^{ij} \neq 0$. The condition $[x^i, \tau] = 0$ implies

$$g^{ij} \tau_j = 0 . \tag{3.3}$$

It is natural to look for an expression for g^{ij} in terms of α^{ij} and the coordinates x^i. We are then led to the following solution of the last equation:

$$g^{ij} = x^i x^j - \alpha^{ij} \, . \tag{3.4}$$

From this we find $a = 1 - n$. In the $SL(2)$ case, we recover (2.2) and (2.3) with the correct restriction on the parameter, i.e. $s = 1/3$.

Example: Consider two variables x, y subject to the quadratic relation

$$x\,y = 1 \, . \tag{3.5}$$

We thus have $n = 2$, $\alpha_{ij} = (1/2)(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1})$ and

$$(g^{ij}) = \begin{pmatrix} x^2 & -1 \\ -1 & y^2 \end{pmatrix} \, . \tag{3.6}$$

Furthermore, $\tau = dx\,y + dy\,x$. In the case under consideration, (1.2) is a system of four equations. Three of them are redundant, however, since they are consequences of

$$[x, dx] = \tau\, x^2 \, . \tag{3.7}$$

Although we have only one free coordinate (x), the 1-forms dx and τ are independent in the sense that $\tau = dx\,(1/x) - (1/x)\,dx$ cannot be expressed as $f(x)\,dx$ or $dx\,f(x)$. The space of 1-forms is therefore two-dimensional (as a left or right $\mathcal{A}$-module, where $\mathcal{A}$ is now the algebra of functions of x). We can use the expression for τ to eliminate τ from (3.7). This results in the equation $x\,dx - 2\,dx\,x + (1/x)\,dx\,x^2 = 0$ which is insufficient to transform the $\mathcal{A}$-bimodule of 1-forms into a left (or right) $\mathcal{A}$-module.

4. A generalized gauge theory and 'second order differential geometry'

It is rather straightforward to formulate a generalization of gauge theory and differential geometry using the 'deformed' differential calculus on $\mathcal{A} = C^\infty(\mathcal{M})$ with (1.2) and (1.3) (see also [5]). It should be noticed, however, that – as a consequence of the deformation – the differential of a function f is now given by

$$df = \tau\, \frac{1}{2}\, g^{ij}\, \partial_i \partial_j f + dx^i\, \partial_i f \tag{4.1}$$

and involves a *second* order differential operator. If a (space-time) metric is given, it is natural to identify it with g^{ij}.

Let ψ be an element of $\mathcal{A}^n$ which transforms as $\psi \mapsto \psi' = U\,\psi$ under a representation of a Lie group G. For local transformations we can construct a covariant derivative in the usual way,

$$D\psi = d\psi + A\,\psi \, . \tag{4.2}$$

This is indeed covariant if the 1-form A transforms according to the familiar rule

$$A' = U\,A\,U^{-1} - dU\,U^{-1} \,. \tag{4.3}$$

In the following we will only consider the case where the coordinate differentials dx^i and the 1-form τ are linearly independent and form a basis of the space of 1-forms (as a left or right $\mathcal{A}$-module). A can then be written in a unique way as

$$A = \tau\,\frac{1}{2}\,A_\tau + dx^i\,A_i \,. \tag{4.4}$$

Inserting this expression in (4.3), we find that A_i behaves as an ordinary gauge potential and

$$A_\tau = g^{ij}\,(\partial_i A_j - A_i A_j) + M \tag{4.5}$$

where M is an arbitrary tensorial part ($M' = UMU^{-1}$). Since U depends on x^i, in general, it does not commute with dx^j. It is convenient to introduce the gauge-covariant differential $Dx^i := dx^i - \tau\,A^i$. The covariant derivative of ψ can now be written as

$$D\psi = \tau\,\frac{1}{2}\,(g^{ij}\,D_i D_j + M)\,\psi + Dx^i\,D_i\psi \tag{4.6}$$

where D_i denotes the ordinary covariant derivative (with A_i). The field strength of A is

$$F = dA + A^2 = \tau\,\frac{1}{2}\,(D^*F - DM) + \frac{1}{2}\,Dx^i\,Dx^j\,F_{ij} \tag{4.7}$$

where $D^*F = dx^i\,D^j F_{ji}$ involves the Yang-Mills operator (when g^{ij} is identified with the space-time metric). F_{ij} is the (ordinary) field strength of A_i.

If τ behaves as a scalar and g^{ij} as a contravariant tensor under coordinate transformations, the defining relations of our differential calculus – and in particular (1.2) – are coordinate independent [5, 6]. The coordinate differentials dx^i do not transform covariantly, however, since

$$dx'^k = \tau\,\frac{1}{2}\,g^{ij}\partial_i\partial_j x'^k + dx^\ell\,\partial_\ell x'^k \tag{4.8}$$

as a consequence of (4.1). For a vector field Y^i we introduce a (right-) covariant derivative

$$DY^i := dY^i + Y^j\,{}_j\Gamma^i \,. \tag{4.9}$$

This is indeed *right*-covariant iff the generalized connection ${}_j\Gamma^i$ is given by

$${}_j\Gamma^i = \tau\,\frac{1}{2}\,\left[g^{k\ell}(\partial_k \Gamma^i{}_{j\ell} + \Gamma^i{}_{mk}\Gamma^m{}_{j\ell}) + M^i{}_j\right] + dx^k\,\Gamma^i{}_{jk} \tag{4.10}$$

where ${\Gamma^i}_{jk}$ are the components of an ordinary linear connection on $\mathcal{M}$ and ${M^i}_j$ is a tensor. Let us introduce the *right*-covariant 1-forms

$$Dx^k := dx^k + \tau \, \frac{1}{2} \, {\Gamma^k}_{ij} g^{ij} \, . \tag{4.11}$$

(4.1) can now be rewritten as

$$df = \tau \, \frac{1}{2} \, g^{ij} \nabla_i \nabla_j f + Dx^i \, \partial_i f \tag{4.12}$$

where ∇_i denotes the ordinary covariant derivative. Also the covariant exterior derivative of Y^i can now be written in an explicitly right-covariant form,

$$DY^i = \tau \, \frac{1}{2} \, (g^{k\ell} \nabla_k \nabla_\ell Y^i + {M^i}_j Y^j) + Dx^j \, \nabla_j Y^i \, . \tag{4.13}$$

It is interesting that the (covariant) exterior derivative of a field contains in its τ-part the corresponding part of the field equation to which it is usually subjected in physical models. We refer to [5] for further results.

5. Stochastic differential calculus

When $\tau = \gamma \, dt$ as in (1.1), we may consider (smooth) functions $f(x^i, t)$ depending also on the parameter t. (4.1) then has to be replaced by

$$df = dt \, (\partial_t + \frac{\gamma}{2} \, g^{ij} \, \partial_i \partial_j) \, f + dx^i \, \partial_i f \, . \tag{5.1}$$

Such a formula is wellknown in the theory of stochastic processes (Itô calculus) [10] and suggests that our noncommutative differential calculus provides us with a convenient framework to deal with stochastic processes on manifolds. There is indeed a kind of translation [6] to the (Itô) calculus of stochastic differentials. This can be used to carry the *expectation* map from the latter over to our calculus. In this section, we introduce an expectation $\mathbf{E}$ on the (first order) differential calculus in a more formal way. It is then shown for a specific example, that our rules reproduce familiar results.

Let us consider the equation (1.1) in one dimension (for simplicity). We write it in the form

$$[X_t, dX_t] = dt \tag{5.2}$$

viewing X_t as a process on $\mathbb{R}$, a map $\mathbb{R} \times [0, \infty) \to \mathbb{R}$. $\mathcal{A}$ denotes the algebra of smooth functions of X_t and t, and $\mathcal{F}$ the subalgebra of functions of t only. Let $\mathbf{E}$ be an $\mathcal{F}$-linear map $\mathcal{A} \to \mathcal{F}$ which is the identity on $\mathcal{F}$. We extend it to 1-forms as an $\mathcal{F}$-linear map via

$$\mathbf{E}\, df_t = d(\mathbf{E} f_t) \quad , \quad \mathbf{E}(dX_t\, f_t) = 0 \qquad (\forall f_t \in \mathcal{A}) \,. \tag{5.3}$$

On the rhs of the first equation in (5.3), d is the ordinary exterior derivative. The second equation can be interpreted by saying that, given f_t, a further increment dX_t is statistically independent (i.e., f_t is 'nonanticipating'). Then, as a consequence of (5.2), $\mathbf{E}(f_t\, dX_t)$ does *not* vanish, in general. Here we should view f_t as evaluated *after* a time step dt with increment dX_t in X_t.

Example: (Ornstein-Uhlenbeck process)
Let us consider the differential equation

$$dY_t = -k\, dt\, Y_t + \sigma\, dX_t \tag{5.4}$$

with constants k, σ. For $\mathbf{E}Y_t$ we obtain from (5.4) the ordinary differential equation

$$d\mathbf{E}Y_t = -k\, \mathbf{E}Y_t\, dt \tag{5.5}$$

with the solution $\mathbf{E}Y_t = \mathbf{E}Y_0\, e^{-kt}$. Let us now show how to calculate higher moments. With

$$[Y_t, dY_t] = \sigma\, [Y_t, dX_t] = \sigma\, [X_t, dY_t] = \sigma^2\, dt \,. \tag{5.6}$$

we find

$$\begin{aligned} d(Y_t^2) &= dY_t\, Y_t + Y_t\, dY_t = 2\, dY_t\, Y_t + \sigma^2\, dt \\ &= 2\, \sigma\, dX_t\, Y_t + dt\, (\sigma^2 - 2k\, Y_t^2) \end{aligned} \tag{5.7}$$

and, using $\mathbf{E}(dX_t Y_t) = 0$, the ordinary differential equation

$$d(\mathbf{E}Y_t^2) = dt\, (\sigma^2 - 2\, k\, \mathbf{E}Y_t^2) \tag{5.8}$$

for the second moment. The solution is

$$\mathbf{E}Y_t^2 = e^{-2kt}\, \mathbf{E}Y_0^2 + \frac{\sigma^2}{2k}\, (1 - e^{-2kt}) \,. \tag{5.9}$$

If the moments $\mathbf{E}Y_0^n$ are given, we obtain in this way the moments $\mathbf{E}Y_t^n$, $t > 0$. The results are the same as if we treat (5.4) as an (Itô) stochastic differential equation, which is the Ornstein-Uhlenbeck equation (see [10], for example). We have used rather unusual techniques, however, namely a noncommutative differential calculus.

6. A differential calculus on superspace

So far we dealt with a commutative algebra generated by coordinate functions x^i, $i = 1, \dots, n$. In this section we enlarge it to an algebra $\mathcal{A}$ of functions on a superspace by adding odd variables ξ_i and η. Again, we associate with $\mathcal{A}$ a differential algebra $\bigwedge(\mathcal{A})$ via the action of an exterior derivative d. In the case of superalgebras a different version of the Leibniz rule is usually adopted [11],

$$d(\omega\,\omega') = d\omega\,\omega' + \hat{\omega}\,d\omega' \tag{6.1}$$

where the hat denotes the grading involution. This is defined on $\bigwedge(\mathcal{A})$ by $\hat{x}^i = x^i$, $\hat{\xi}_i = -\xi_i$, $\hat{\eta} = -\eta$, $\widehat{d\omega} = -d\hat{\omega}$, $\widehat{\omega\omega'} = \hat{\omega}\hat{\omega}'$ and linearity. In particular, the dx^i are odd and $d\eta$, $d\xi_i$ are even. In the even sector of $\mathcal{A}$, (6.1) coincides with our previous rule, however. We write $[\,,\,]$ for the graded commutator (i.e., $[\omega,\omega'] = \omega\omega' - \omega'\omega$ for ω even and $[\omega,\omega'] = \omega\omega' - \hat{\omega}'\omega$ for ω odd). The universal differential calculus is now restricted by the following relations,

$$[x^i, d\xi_j] = -[\xi_j, dx^i] = d\eta\,\delta^i_j\,. \tag{6.2}$$

The remaining graded commutators between superspace coordinates and their differentials are taken to be zero (so that we have the standard rules in the pure even and odd sectors). This defines a consistent differential calculus where the space of 1-forms is generated as a right (or left) $\mathcal{A}$-module by $dx^i, d\xi_j, d\eta$. The differential of a function f on the superspace can then be expressed as

$$df = d\eta\,\tilde{\partial}_\eta f + dx^i\,\tilde{\partial}_i f + d\xi_i\,\tilde{\zeta}^i f \tag{6.3}$$

where $\tilde{\partial}_\eta$, $\tilde{\partial}_i$, $\tilde{\zeta}^i$ are operators on $\mathcal{A}$. Using (6.1) and the basic commutation relations, we find

$$[dx^i, f] = -d\eta\,\tilde{\zeta}^i f \quad , \quad [d\xi_i, f] = -d\eta\,\tilde{\partial}_i f\,. \tag{6.4}$$

With the help of these relations, the Leibniz rule (6.1) for d now implies

$$\tilde{\partial}_i(fh) = (\tilde{\partial}_i f)\,h + f\,(\tilde{\partial}_i h) \quad , \quad \tilde{\zeta}^i(fh) = (\tilde{\zeta}^i f)\,h + \hat{f}(\tilde{\zeta}^i h) \tag{6.5}$$

$$\tilde{\partial}_\eta(fh) = (\tilde{\partial}_\eta f)\,h + \hat{f}\,(\tilde{\partial}_\eta h) + (\tilde{\zeta}^i f)\,\tilde{\partial}_i h + (\tilde{\partial}^i \hat{f})\,\tilde{\zeta}_i h\,. \tag{6.6}$$

Together with $\tilde{\partial}_i x^j = \delta^j_i = \tilde{\zeta}^j \xi_i$, $\tilde{\partial}_\eta \eta = 1$ (a consequence of (6.3)), this leads to

$$\tilde{\partial}_i = \partial_i := \frac{\partial}{\partial x^i}\,,\ \tilde{\zeta}^i = \zeta^i := \frac{\partial_{(\ell)}}{\partial \xi_i}\,,\ \tilde{\partial}_\eta = \partial_\eta + \Delta := \frac{\partial_{(\ell)}}{\partial \eta} + \zeta^i \partial_i \tag{6.7}$$

(where a subscript (ℓ) indicates that the derivative is taken from the left). Hence

$$df = d\eta\,(\partial_\eta f + \Delta f) + dx^i\,\partial_i f + d\xi_i\,\zeta^i f\,. \tag{6.8}$$

Using (6.4), we obtain

$$[f, dh] = d\eta\,(f, h) \tag{6.9}$$

where on the rhs appears the *antibracket* [9]

$$(f, h) := (\partial_i f)\,\zeta^i h + (\zeta^i \hat{f})\,\partial_i h = \Delta(\hat{f} h) - (\Delta \hat{f})\,h - f\,\Delta h\,. \tag{6.10}$$

The operator Δ satisfies $\Delta^2 = 0$.

The relation (6.9) is very much analogous with the relation (1.4). Of course, we may consider both deformations of the ordinary differential calculus on the superspace simultaneously. In a sense, η is the odd counterpart of t in (1.1).

7. Generalized gauge theory on superspace

We consider again the superspace differential calculus introduced in the preceeding section. Let ψ transform under the action of a (super) group G according to $\psi \mapsto \psi' = U\psi$. With respect to local transformations on the superspace, an exterior covariant derivative can be defined in the usual way as

$$D\psi := d\psi + A\,\psi \tag{7.1}$$

with a connection 1-form A. It is indeed covariant, i.e. $D'\psi' = \hat{U}\,D\psi$, if

$$A' = \hat{U}\,A\,U^{-1} - dU\,U^{-1}\,. \tag{7.2}$$

Inserting the decomposition

$$A = d\eta\,\alpha + dx^i\,A_i + d\xi_i\,\Lambda^i \tag{7.3}$$

we find

$$A'_i = U\,A_i\,U^{-1} - (\partial_i U)\,U^{-1} \quad , \quad \Lambda'^i = \hat{U}\,\Lambda^i\,U^{-1} - (\zeta^i U)\,U^{-1} \tag{7.4}$$

and

$$\mu' = \hat{U}\,\mu\,U^{-1} - (\partial_\eta U)\,U^{-1} \quad , \quad \mu := \alpha + \hat{A}_i\,\Lambda^i - \zeta^i A_i\,. \tag{7.5}$$

In order to read off gauge covariant components from covariant (generalized) differential forms, we need the following covariantized differentials (cf also section 4),

$$Dx^i := dx^i - d\eta\,\Lambda^i \quad , \quad D\xi_i := d\xi_i - d\eta\,\hat{A}_i\,. \tag{7.6}$$

Their transformation rule is

$$D'x^i = \hat{U}\,Dx^i\,U^{-1} \quad , \quad D'\xi_i = \hat{U}\,D\xi_i\,\hat{U}^{-1}\,. \tag{7.7}$$

Now we find

$$D\psi = d\eta\,(D_\eta\psi + \Gamma^i D_i\psi) + Dx^i\, D_i\psi + D\xi_i\, \Gamma^i\psi \tag{7.8}$$

where

$$D_\eta := \partial_\eta + \mu \quad , \quad D_i := \partial_i + A_i \quad , \quad \Gamma^i := \zeta^i + \Lambda^i\, . \tag{7.9}$$

The operator $\Gamma^i D_i$ (the covariantized Δ) which appears in (7.8) is a ***generalization of the Dirac operator***. If a metric tensor g^{ij} is given and $\zeta^i U = 0$, we can choose $\Lambda^i = g^{ij}\xi_j = \xi^i$ so that $\Gamma^i = \zeta^i + \xi^i$ and

$$\Gamma^i\Gamma^j + \Gamma^j\Gamma^i = 2\,g^{ij} \tag{7.10}$$

which is the Clifford algebra relation. In this case, $\Gamma^i D_i$ is indeed the Dirac operator.

More generally, we have the following relations between transformation properties and exterior covariant derivatives,

$$\begin{array}{lllll}
\psi \mapsto U\psi & \Rightarrow & D\psi = d\psi + A\,\psi & \mapsto & \hat{U}\, D\psi \\
\psi \mapsto \hat{U}\psi & \Rightarrow & D\psi = d\psi - \hat{A}\,\psi & \mapsto & U\, D\psi \\
\psi \mapsto \psi U^{-1} & \Rightarrow & D\psi = d\psi - \hat{\psi}\, A & \mapsto & D\psi\, U^{-1} \\
\psi \mapsto \psi\hat{U}^{-1} & \Rightarrow & D\psi = d\psi + \hat{\psi}\,\hat{A} & \mapsto & D\psi\,\hat{U}^{-1}\, .
\end{array} \tag{7.11}$$

The curvature 2-form of the connection A is given by

$$F := dA - \hat{A}\,A\, . \tag{7.12}$$

We will leave the further investigation of this calculus to a separate work.

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