MATHEMATICAL THEORY OF POTENTIAL SCATTERING

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INTRODUCTION

Before we go into a detailed discussion of the potential scattering we would like to spend a few words on the reason potential scattering is interesting. We think that one of the main reasons of success of the potential model is that we can discuss it quite rigorously and that at the same time it gives a fairly intuitive picture of the scattering process and it provides in a way the language for a fully relativistic theory. We do not think that the potential model has been particularly satisfactory in explaining quantitatively the known experimental data, for instance the nucleon-nucleon scattering; yet we have good reasons to believe that at low energy any field theory will ultimately yield some sort of spin-dependent potential, containing spin orbit coupling and exchange terms. How this can be done and how far one has gone in this direction has nothing to do with the subject of these lectures which are merely concerned with the discussion of the solutions of the Schroedinger equation for a given class of potentials. That is, we assume from the very beginning that a potential exists although we do not know it or we know only broad features like the range and its analytic properties as function of the distance. For simplicity we do not deal with spin or exchange terms although they can be taken care of with little modifications. We just want to find those features of potential scattering which are to a large extent independent of the particular selection of the potential.

In so doing we shall need a large mathematical apparatus in order to derive those properties of the scattering amplitude which have been suggested by the general field theory, like dispersion relations. Unfortunately although it has not been possible to eliminate entirely from these lectures this apparatus, we have tried however to use as much as possible standard mathematical tricks and we have endeavoured to cover them with the largest amount of proofs. There are different mathematical approaches to the theory of potential scattering. Historically the first approach was developed by Heisenberg in his first attempts to create a theory of the S-matrix. But the most rigorous and extensive results on this particular subject were actually found by JOST and BARGMANN [1]. The starting point of their approach is the partial wave expansion of the wave function and of the scattering amplitude. Actually they did not derive any interesting feature of the full amplitude, but rather of the partial phase shifts only. The amount of work afterwards done on the properties of phase shifts as function of the energy has been considerable, and it has clarified the role of the potential in determining them.

This was not however the end of the story. When the first dispersion relations for fixed transmitted momentum were discovered in field theory, it was a natural question to ask whether these properties had a counterpart in potential scattering. This was found to be true by KHURI [2]. The paper of Khuri avoids entirely the use of partial waves and uses Fredholm's theory on the Green integral form of the Schroedinger equation written in full three-dimensional formalism. Alternative and simpler proofs then appeared in the literature [3, 4, 6]. The reason the partial wave expansion is totally unsuitable for this purpose is that it fails to converge in the interesting region where we want to prove analyticity in the energy. The advent of the Mandelstam conjecture of the double dispersion relation raised the question as to whether these relations were true for potential scattering. Mandelstam representation can be proved today for a special class of potentials (super-position of Yukawa potentials).

A proof of GOLDBERGER et al. [3] uses the pertubative expansion of the scattering amplitude as written in momentum space (as derived from the Lippman-Schwinger equation). They prove that each term of the expansion satisfies the Mandelstam representation, and they also succeeded in going around the question of uniform convergence. Incidentally, an incomplete proof, without uniform convergence, was given first by Bowcock and Martin. A paper by KLEIN also deals with this subject [4]. The partial wave expansion however can be used successfully in providing analytic properties in the momentum transfer for fixed energy. The usual form of it is apparently unsuitable for the job, but fortunately about fifty years ago WATSON [5] found a method of transforming it into an integral which is a highly flexible tool in these kinds of problem. With some care the Watson integral can be used to prove almost all of the analytic properties of the scattering amplitude, including those of Khuri's paper. It is for this reason that we decided to rest the whole theory on the partial wave expansion in the Watson form because we feel that in this way the whole structure of the lectures will be more homogeneous.

1. THE FORMALISM OF POTENTIAL SCATTERING. ELEMENTARY THEORY

The starting point of the theory is the Schroedinger equation:

$$\Delta \Psi(\vec{r}) + E\,\Psi(\vec{r}) = V\,\Psi(\vec{r}). \tag{1.1}$$

In this equation $\vec{r}$ is the position vector of the scattered particle, r its length, $\vec{r}$ has components x, y, z. We use natural units h = c = 1 and 2M = 1, where M is the mass of the scattered particle. The scattering of two particles of different mass m_1, m_2 can be treated by the same equation where M is now the reduced mass $m_1 m_2/(m_1 + m_2)$ of the system. In our units the energy has the dimension of an area^{-1}. The local potential V(r) depends on r only.

V(r) is supposed to be a short range potential; that is, we suppose it to decrease exponentially. Truly this is a rather restricted hypothesis; but if we have in mind a comparison with the field theoretical results, all interesting potentials satisfy this criterion apart from the Coulomb potential. We shall not examine here Coulomb-like potentials because there is no extensive and deep work done on this subject. Under these conditions [7] we may define the (total) scattering amplitude $f(E,\theta)$ once we know the solution of eq. (1.1) with the following asymptotic behaviour ($r \to \infty$):

$$\Psi \sim e^{i\vec{k}\cdot\vec{r}} + f(E,\theta)\,\frac{e^{ikr}}{r}. \tag{1.2}$$

This wave function represents a three-dimensional scattering process of a plane wave against a fixed scatterer. The plane wave is given by $e^{i\vec{k}\cdot\vec{r}}$, where $\vec{k}$ is the ingoing momentum. We have $(\vec{k})^2 = E$. The second contribution comes from the scattered waves and depends of course on the potential. The angle θ is the angle between $\vec{k}$ and the direction in which we take the asymptotic limit $r \to \infty$. In other words, we put $\vec{r} = r\vec{n}$ into $\Psi(\vec{r})$ and we let $r \to \infty$ while $\vec{n}$ is a fixed unit vector. Then $\vec{k}\cdot\vec{n} = k\cos\theta$. $d\Omega\,|f(E,\theta)|^2$ is then the probability of finding the particle scattered in the solid angle $d\Omega$ with the outgoing momentum $\vec{k} = k\vec{n}$.

There is no potential of the class considered by us for which eq. (1.1) is explicitly solvable. For any practical purpose of numerical evaluation one solves instead (1.1) with the method of the separation of variables due to D'Alembert. One tries to find the solution of (1.1) of the form

$$\Psi = \frac{\phi(r)}{r}\,Q(\theta,\phi). \tag{1.3}$$

It is well known then that Q has to be a spherical harmonic,

$$Q(\theta,\phi) = Y_\ell^m(\theta,\phi), \tag{1.4}$$

and that ϕ satisfies the ordinary differential equation

$$\phi_\ell'' + E\phi_\ell - \frac{\ell(\ell+1)}{r^2}\phi_\ell - V\phi_\ell = 0. \tag{1.5}$$

(1.5) depends on ℓ only and not on m. ϕ_ℓ must also satisfy the boundary condition of vanishing at the origin. More precisely, the analysis of (1.5) according to the Fuchsian classification of singularities shows that any solution of (1.5) behaves when r is small like

$$\phi_\ell \approx \alpha\, r^{\ell+1} + \beta\, r^{-\ell} \tag{1.6}$$

under some restrictive hypothesis on the potential to be examined closely later. If we want to avoid singularities at $r = 0$, we are forced to choose $\beta = 0$. In this case the ℓ-th partial wave function vanishes rapidly for small r. Physically we may interpret this fact as due to the repulsive centrifugal barrier $\ell(\ell+1)/r^2$ which becomes very large when the orbital momentum ℓ is also large. This barrier keeps the particle from approaching the origin.

This boundary condition defines each partial wave apart from a multiplicative factor. Take now r large. We have good reasons now to suppose that both $V(r)$ and $\ell(\ell+1)/r^2$ can be neglected in comparison with E so that (1.5) becomes

$$\phi_\ell'' + E\ \phi_\ell = 0. \tag{1.7}$$

This equation is trivially solved by oscillating exponentials ($E > 0$) and the corresponding asymptotic behaviour of ϕ_ℓ will be of the form

$$\phi_\ell \sim C_\ell \sin\left[kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right]. \tag{1.8}$$

We have introduced on purpose the term $\ell\pi/2$ in this asymptotic behaviour. Indeed, when $V = 0$, eq. (1.5) can be solved exactly in terms of Bessel functions of semi-integer order and the asymptotic behaviour at infinity explicitly evaluated. This behaviour corresponds to having $\delta_\ell(k) = 0$. The phase shift $\delta_\ell(k)$ therefore describes a cumulative effect of the potential on the wave function in the whole interval $0 \ldots\ldots\ \infty$. A large part of these lectures will be devoted to the investigation of the properties of $\delta_\ell(k)$. The importance of δ_ℓ is evident from the well-known Rayleigh-Faxen formula:

$$f(E,\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}\left(e^{2i\delta_\ell(k)} - 1\right)(2\ell+1)\, P_\ell(\cos\theta). \tag{1.9}$$

We shall refer to this fundamental formula as the expansion of the scattering amplitude in partial waves or more concisely as the RF expansion. A full account of (1.9) is contained in any elementary textbook on quantum mechanics and we shall not go into this matter further.

In (1.9) the functions $P_\ell(\cos\theta)$ are Legendre polynomials which form an orthogonal set normalized as follows:

$$\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx = \frac{2}{2\ell+1}\,\delta_{m\ell}.$$

The total cross-section is given by

$$\sigma(E) = \int d\Omega\, |f(E,\theta)|^2 = \frac{4\pi}{E}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell.$$

2. THE S WAVE

The S wave scattering is the first that has been extensively discussed, and this is for the simple reason that the mathematics of it is considerably simpler than that of the higher waves. A number of potentials have been produced which are explicitly solvable for the S wave and which give quite a number of clues concerning the general dependence of δ on the energy E. One such potential is of course the square well potential defined as

$$V(r) = A \quad \text{if} \quad r < R,$$

$$V(r) = 0 \quad \text{if} \quad r > R; \text{ we put } k_1 = \sqrt{E - A}.$$

It is simple exercise to find the corresponding phase shift:

$$e^{2i\delta(k)} = S(k) = e^{-2ikR} \frac{\cos k_1R + (ik/k_1)\sin k_1R}{\cos k_1R - (ik/k_1)\sin k_1R}. \tag{2.1}$$

Some features of this formula are the following:

(1) the formula is also valid when E is not positive. S(k) is the ratio of two holomorphic functions of k and is therefore meromorphic.

(2) $S(-k) = S^{-1}(k)$; that is, δ is an odd function of k. It is more convenient to study S(k) instead of δ because δ has logarithmic singularities at every pole or zero of S(k).

(3) $[S(k^*)]^* = S^{-1}(k)$. This implies that δ is real when k is real. We refer to this property as unitarity.

(4) $\lim S(k) = 1$ or $\lim \delta = 0$ when $k \to \infty$. How do we understand this result? If k (or E) is very large, the speed of the incoming particle, which in our units is given by 2k, also increases. The time of transit of the particle inside the potential well is of the order of R/k. Presumably the interaction is proportional to the transit time and the phase shift will be also of the order of magnitude of R/k or rather of the dimensionless parameter AR/k. Indeed, for large k we have from (2.1)

$$\delta \sim -\frac{AR}{2k} = -\frac{1}{2k} \cdot \int_0^\infty V(r)\,dr. \tag{2.2}$$

This result is naturally false in the relativistic region, and it is already different for the Dirac or Klein-Gordon equation. The limit is much more complicated if we move to infinity along any direction of the complex k plane. It must be pointed out that although k_1 is a two valued function of k it does not matter which value we use in (2.1). If $A > 0$, the potential is repulsive and pushes out the wave function. We expect δ to be negative in agreement with the asymptotic behaviour (2.2).

Take now eq. (1.5) when E is negative (and $\ell = 0$):

$$\phi'' + E\,\phi - V\,\phi = 0. \tag{2.3}$$

This equation does not describe any scattering state and it will have solutions which are bounded at infinity and at the origin only for special values of E. Putting $E = -b^2$ where b is real, we have for large r

$$\phi \sim \mu(b)\, e^{-br} + \nu(b)\, e^{br}.$$

If b is not restricted, we shall have an exploding exponential term at large distances. If however for a particular value of b we have $\nu(b) = 0$, the solution becomes square integrable and represents a bound state of the system. Great progress in the theory of bound states was achieved when it became clear that bound states correspond to poles of S(k); i.e. if a bound state of binding energy $-B^2$ occurs, there is a pole of S(k) in $k = iB$. Unfortunately it is not in general true that all poles of S(k) correspond to bound states. This makes it difficult to deduce the bound states from the analytic continuation of S(k), or at least it made it difficult before the advent of the modern ideas of dispersion theory. Before going into a detailed discussion of this connection, we point out that our statement can be verified directly on the explicit formula which we have just given for the square well potential. We leave this as an exercise for the reader. Other examples of soluble potential can be found in [1]. Jost defines a particular solution of eq. (2.3) with the boundary condition (the Jost solution)

$$f''(k,r) + E f(k,r) - V f(k,r) = 0,$$
$$f(k,r) \sim e^{-ikr}, \quad r \to \infty. \tag{2.4}$$

This solution will not satisfy in general the boundary conditions in $r = 0$; that is, $f(k,0) \neq 0$. Let us define the Jost function as $f(k) = f(k,0)$. If $f(k) = 0$, the Jost solution is regular in $r = 0$. Besides $f(k,r)$, $f(-k,r)$ also is a solution of (2.4); and since the Wronskian of these two functions does not depend on r and equals $-2ik$, they form a pair of independent solutions of (2.4). Take now the "regular" solution $\phi(k,r)$ defined by the boundary condition in $r = 0$:

$$\phi(k,0) = 0; \quad \phi'(k,0) = 1. \tag{2.5}$$

ϕ is not linearly independent of $f(k,r)$ and $f(-k,r)$ so that we have with some coefficients C, D

$$\phi(k,r) = C f(k,r) + D f(-k,r).$$

Now $W(\phi, f) = \phi' f - f' \phi$ is independent of r and we calculate it for $r = 0$:

$$W(\phi, f) = \phi'(k,0)\, f(k,0) - \phi(k,0)\, f'(k,0) = f(k).$$

On the other hand,

$$W(\phi, f) = C\, W[f(k,r),\ f(k,r)] + D\, W[f(-k,r),\ f(k,r)]$$
$$= D\, W[f(-k,r),\ f(k,r)] = 2ikD = f(k),$$

so that $D = f(k)/2ik$. Similarly $C = -f(-k)/2ik$. It follows that

$$\phi(k, r) = [f(k) f(-k, r) - f(-k) f(k, r)]/2ik = \phi(-k, r). \tag{2.6}$$

The asymptotic behaviour of ϕ is then

$$\phi(k, r) \sim [e^{ikr} f(k) - e^{-ikr} f(-k)]/2ik.$$

But from the definition of phase shift we have

$$\phi \sim \text{const. } \sin(kr + \delta) = \text{const. } (e^{ikr} e^{i\delta} - e^{-ikr} e^{-i\delta}).$$

By comparison we get

$$e^{2i\delta(k)} = f(k)/f(-k). \tag{2.7}$$

If $V = 0$, then $f(k, r) = e^{-ikr}$, $\phi(k, r) = (1/k) \sin kr$, $f(k) = 1$. If V is the already defined square well potential, we have

$$f(k) = e^{-ikR} (\cos k_1 R + i(k/k_1) \sin k_1 R). \tag{2.8}$$

In this case $f(k)$ turns out to be the entire function of k. Bargmann has investigated the general behaviour of $f(k)$ as a function of the complex variable k, paying special attention to the role of the range of the potential. His starting point is the integral equation for $f(k, r)$:

$$f(k, r) = e^{-ikr} + \frac{1}{k} \int_r^{\infty} V(x \;\sin k(x-r) f(k, x) dx.$$

We shall prove and discuss this equation in the next section.

3. THE ANALYTIC PROPERTIES OF JOST'S FUNCTION

In the last section we examined the integral equation for $f(k, r)$:

$$f(k, r) = e^{-ikr} + \frac{1}{k} \int_r^{\infty} V(x) \sin k(x-r) f(k, x) dx. \tag{3.1}$$

This equation can be proved as follows: Clearly we have

$$f'' + k^2 f = Vf,$$

$$\frac{d^2}{dx^2} \sin k(x-r) + k^2 \sin k(x-r) = 0. \tag{3.2}$$

Therefore,

$$Vf(k,x) \sin k(x-r) = f''(k,r) \sin k(x-r) - f(k,x) \frac{d^2}{dx^2} \sin k(x-r)$$

$$= \frac{d}{dx} \left[f'(k,x) \sin k(x-r) - f(k,x) \frac{d}{dx} \sin k(x-r) \right].$$

If we use the above form of the integrand in (3.1), the integration can be carried out explicitly and the result is

$$\int_r^\infty V(x) \sin k(x-r)\, f(k,x)\, dx = \int_r^\infty \frac{d}{dx} [f'(k,x) \sin k(x-r)$$

$$-f(k,x) \frac{d}{dx} \sin k(x-r)]\, dx = k\, [f(k,r) - e^{-ikr}].$$

QED.

We regard (3.1) as the proper definition of the Jost solution because it implies both the differential equation and the appropriate boundary conditions. Putting $f(k,r)\, e^{ikr} = g(k,r)$, we find

$$g(k,r) = 1 + \frac{1}{k} \int_r^\infty V(x) \left[\frac{1-e^{-2ik(x-r)}}{2i} \right] g(k,r)\, dx. \tag{3.3}$$

A formal solution of (3.3) is given by the perturbative expansion:

$$g(k,r) = \sum_n g_n(k,r); \quad g_0(k,r) = 1, \tag{3.4}$$

where

$$g_{n+1}(k,r) = \frac{1}{k} \int_r^\infty \frac{1-e^{-2ik(x-r)}}{2i}\, g_n(k,x)\, V(x)\, dx. \tag{3.5}$$

This expansion defines a solution of (3.3) only when it converges. In order to decide whether it really does so we need some preliminary bound on the kernel of (3.3). There is no real complication and much to be gained in supposing k complex. We put Im k = b. The proof and also the result are quite different for the cases $b > 0$ and $b < 0$. Let us first suppose $b < 0$, but $k \neq 0$. We have the bound (remember that $x > r$):

$$\left|\frac{1-e^{-2ik(x-r)}}{2ik}\right| < \left|\frac{1}{2ik}\right| + \left|\frac{e^{2b(x-r)}}{2ik}\right| < \frac{1}{|k|}. \qquad (3.6)$$

Obviously,

$$|g_{n+1}(k, r)| < \frac{1}{|k|} \int_r^\infty |V(x)|\, |g_n(k, x)|\, dx,$$

$$|g_1(k, r)| < \frac{1}{|k|} \int_r^\infty |V(x)|\, dx = \frac{M(r)}{|k|}.$$

A second iteration yields:

$$|g_2(k, r)| < \frac{1}{|k|^2} \int_r^\infty |V(x)|\, dx\, M(x) = -\frac{1}{|k|^2} \int_r^\infty \frac{dM(x)}{dx}\, M(x) dx = \frac{M^2(r)}{2|k|^2}.$$

This suggests that we have the following inequality for the general term:

$$|g_n(k, r)| < \frac{M^n(r)}{n\,|k|^n}. \qquad (3.7)$$

We prove it with the induction method; that is, it is supposed to be true for $g_n(k, r)$ and we deduce the result for $g_{n+1}(k, r)$. We have

$$|g_{n+1}(k, r)| < \frac{-1}{|k|^{n+1}\, n!} \int_r^\infty \frac{dM(x)}{dx}\, M^n(x)\, dx$$

$$= [M^{n+1}(r)]/|k|^{n+1}(n+1)!\,. \qquad (3.8)$$

QED.

By summing up all these inequalities we find

$$|g - 1| < e^{\frac{M(r)}{|k|}} - 1. \qquad (3.9)$$

What is the outcome of (3.9)? We have proved at least the following results:

(1) A solution exists for $b<0$, $k \neq 0$ if $\int_r^\infty |V(x)|\, dx < \infty$, because the perturbative expansion converges.

(2) Each term of the expansion is analytic in k as long as the corresponding integral converges; this is true by the above proof in $b<0$, $k \neq 0$. The sum is therefore also analytic because we have uniform convergence.

(3) We have the limit $g(k, r) \to 1$ when $k \to \infty$ in any direction in the lower half plane of k and along the real axis.

(4) Since clearly $g(k, 0) = f(k)$, points (1), (2), (3) also hold for the Jost function if $M(0) < \infty$.

A different condition can be obtained if we replace (3.6) by

$$\left| \frac{1-e^{-2ik(x-r)}}{2ik} \right| = \left| \int_0^{x-r} e^{-2ik\eta}\, d\eta \right| < |x-r| < x. \tag{3.10}$$

There is of course no difficulty in repeating the proof with the new bound and we find in lieu of (3.9)

$$|g - 1| < e^{N(r)} - 1; \;\; N(r) = \int_r^\infty x\,|V(x)|\,dx. \tag{3.11}$$

This last evaluation implies a slightly more stringent condition on $V(r)$ for large r, but it includes $k = 0$ and it relaxes the condition on $V(r)$ for small r. To this purpose we notice that for all short-ranged potentials both $M(r)$ and $N(r)$ exist but $M(0)$ diverges for the Yukawa potential.

We turn now to the case $b > 0$. Here we cannot use (3.6) or (3.10) but rather

$$\left| \left[1-e^{-2ik(x-r)} \right] / 2ik \right| < e^{2b(x-r)} / |k|. \tag{3.12}$$

We have correspondingly

$$|g_1(k, r)| < \frac{1}{|k|} e^{-2br} P(r); \;\; P(r) = \int_r^\infty |V(x)|\, e^{2bx} dx.$$

By induction we can similarly check that

$$|g_n(k, r)| < P(r) \frac{M^{n-1}(r)}{(n-1)!} \frac{1}{|k|^n} e^{-2br}. \tag{3.13}$$

This implies again analyticity in k if $P(r) < \infty$. (M converges if P converges.) This is by no means trivially satisfied, as we had before for M and N. If $V(r)$ decreases exponentially, we can always choose b large enough to have P diverging. If $V(r) \sim e^{-mr}/r$, we find $b < m/2$. If V is a Gaussian potential or a square well, then we have unrestricted convergence. But the interesting potentials are usually superposition of Yukawa potentials, and therefore we expect $f(k)$ to have singularities in the upper half-plane. With a slight modification of the proof the origin can be included in the analyticity domain.

Concluding: $f(k)$ is analytic in k in the half-plane $b < m/2$. Therefore $S(k) = f(k)/f(-k)$ is meromorphic in the strip $|b| < m/2$. This is BARGMANN's result [1]. In the above Bargmann's strip $S(k)$ can have poles only when $f(-k)$ vanishes. We shall discuss the significance of the poles of $S(k)$ in the next section. Here we just wish to give some kind of pictorial view of the analyticity of $f(k)$. As we said, $f(k, r)$ is that solution which behaves like e^{-ikr} for large r. As long as k is real, this is perfectly sufficient to define $f(k, r)$ from a physical point of view: if $k > 0$ (< 0), $f(k, r)$ represents a sink (source) in $r = 0$ which absorbs (emits) a set of stationary purely ingoing (outgoing) waves. If $b < 0$ the waves are damped at infinity. $f(-k, r)$ waves are exploding; there is no way of having $f(-k, r)$ waves accidentally mixed with $f(k, r)$, because for large r they would violently predominate. A damped wave is therefore quite uniquely determined. This in turn corresponds to the full solvability of the integral equation. If instead we take $b > 0$, there is apparently no safe way of defining an exploding wave because we are entitled to add to it any damped wave without disturbing the behaviour at infinity. It is possible to get round part of the difficulty by defining as a purely exploding wave $f(k, r)$, $b > 0$ in such a way that $f(k, r) - e^{-ikr}$ decreases faster than e^{ikr}. It is quite possible to do so for the potential well; in fact, there we have $f(k, r) - e^{-ikr} = 0$ identically outside the potential. But in general this procedure will meet some difficulty, because the potential tail perturbs the exploding wave by roughly the amount $e^{-mr} e^{-ikr}$. If this part is already larger than the damped wave, we have little chance of going further. The condition $|e^{-mr-ikr}| \ll |e^{ikr}|$ for large r is precisely $b < m/2$. This is Bargmann's condition. We went into some detail of this pictorial view of the analyticity proof, because with this kind of reasoning one often anticipates the final analyticity domain and paves the way to a rigorous proof.

4. POLOLOGY OF S(k)

We want now to discuss in detail the physical meaning of $S(k)$. If V is a real function (we wish to point out that Bargmann's proof holds even if V is not real), we have the following hermiticity properties:

$$
\begin{aligned}
f(k, r)^* &= f(-k^*, r), \\
f(k)^* &= f(-k^*), \\
S(k^*)^* &= S(k)^{-1}.
\end{aligned}
\tag{4.1}
$$

These properties can be broadly referred to as unitarity. They follow from the fact that $f(-k^*, r)^*$ satisfies exactly the same integral equation as $f(k, r)$.

Suppose now $f(-k_0) = 0$ within the Bargmann strip. From (2.6) we have $\phi(k_0, r) = f(k_0)\, f(-k_0, r)/2ik_0$. $f(-k_0, r)$ is therefore regular in $r = 0$. If $k_0 = ib$, b real > 0, $f(-k\ , r)$ behaves like e^{br} for larger r and is the wave function of a bound state. Therefore poles of $S(k)$ occuring on $k\ = ib$, $b > 0$ correspond to bound states.

The restriction of the Bargmann strip is essential; otherwise a pole of $S(k)$ could arise from a singularity of $f(k)$ and not from a zero of $f(-k)$.

This was regarded as a serious objection to the theory in the early times, and there were quite a number of attempts toward the elimination of these false poles. (Actually they discussed the zeros of S(k), but this is just the same by S(k) S(-k) = 1.) What about the other poles not lying on $k_0 = ib$, $b > 0$? If there is a pole in $k_0 = h + ib$, $b > 0$, we must have a pole in $-k_0^* = -h + ib$ by unitarity. By the same discussion used above both f(h - ib, r) and f(-h-ib, r) are square integrable solutions of our differential equation corresponding to different eigenvalues of the energy $E = (h \pm ib)^2$. They are orthogonal. This implies

$$\int_0^\infty dr\ f\,(h - ib, r)\ f\,(-h - ib, r) = 0. \tag{4.2}$$

This is clearly impossible because f(-h-ib, r) is the conjugate of f(h-ib, r) and the above integral is positive. Therefore, if $b > 0$, the only way out is $h = 0$. This proof is the usual quantum mechanical proof that a hermitian operator has real eigenvalues. The same proof breaks down if $b < 0$ because then the wave function is no longer square integrable. The $b < 0$ poles of S(k) occur either on $k = ib$, $b > 0$ or in pairs of conjugate poles. There is no commonly accepted name for the purely imaginary poles; either antibound states or virtual states have been used, and we suggest the first one. Numerical investigation on solvable examples [4] shows that they actually occur for reasonable choices of potentials. Experimentally they have no outstanding identity like the bound states; but, as we shall see, they can be seen as rather indirect effects on the low-energy cross-section. Indeed, suppose that an antibound state occurs with a small value of b. If k is small, we can expand f(-k) in powers of k-ib. We have

$$f(-k) \approx iC\,(k - ib).$$

C here is real because of unitarity. It follows

$$S(k) = e^{2i\delta} = f(k)/f(-k) \approx -\,(k + ib)/(k - ib). \tag{4.3}$$

At low energies the cross-section is almost entirely due to S-waves:

$$\sigma(E) = 4\pi\,\sin^2\delta/E.$$

In our approximation we have

$$\sigma(E) = 4\,\pi/(E + b^2). \tag{4.4}$$

If b is small, the cross-section should be abnormally large at $E = 0$. This is what we see in the singlet state of the proton-neutron system where we know that there is no bound state. Of course, since b is squared in (4.4), there is no way of telling from the cross-section whether we have a bound or anti-bound state. The pairs of conjugate poles are named (in [9] there is some disagreement with our convention) resonances. The reason is that they are quite visible in the cross-section if their b is small. Incidentally, we cannot have $b = 0$, because then also $f(k_0)$ would vanish and therefore also $\phi(k_0, r)$ and $\phi'(k_0, 0) = 1$, and this is contradictory. In order to see how the cross-section behaves near a resonance we calculate the phase shift for an energy which is very close to the location of the poles. If $f(-k)$ has a zero in $k = h + ib$, $b < 0$, $f(k)$ will have a zero in $k = h - ib$. Taking into account unitarity, we see that δ can be represented, when k is close to h, by the formula ($E_0 = h^2$, $\Gamma = -4bh$):

$$\delta \approx \eta + \operatorname{arc\,tg} \frac{\Gamma/2}{E_0 - E} \tag{4.5}$$

Suppose for simplicity $\eta = 0$. The cross-section will be given by

$$\sigma(E) = \frac{4\pi}{E} \sin^2 \delta = \frac{4\pi}{E} \frac{\Gamma^2/4}{(E - E_0)^2 + \Gamma^2/4}. \tag{4.6}$$

If we plot the phase shift as function of E in the neighbourhood of E_0, we find that it starts from the value η if $E_0 - E \gg \Gamma/2$ and it rapidly jumps up to $\eta + \pi/2$ when E passes through the value E_0. If $\eta = 0$, δ takes the value $\pi/2$ when $E = E_0$; this corresponds to a maximum of the cross-section, because $\sin^2\delta$ takes then the maximum value 1. The same behaviour is evident from (4.6) and shows up as a sharp peak in the plot of the cross-section. If of course b is not so small, the peak broadens and loses its identity by mixing up with nearby peaks. Eq. (4.6) is a simplified version of the Breit-Wigner one level formula. Correspondingly, the wave function for an energy close to E_0 is very small outside the range of the potential. This we see from (2.6). Indeed, if $k = h$, we know that both f(k) and f(-k) are nearly vanishing. As $\phi(k, r)$ inside the region of interaction is reasonably large, i.e. $\phi(k, r) \approx r$, if we normalize the solution from the asymptotic behaviour for large r by choosing a unit flux of ingoing and outgoing particles, the amplitude inside the potential will in turn become abnormally large. (Incidentally, we notice that $\phi(k, r)$ is normalized in the origin.) We may picture the process as follows: the incoming particles spend a long time inside the potential well before coming out. Their interaction is therefore quite strong, and this explains the occurrence of large cross-sections. Resonances are often called metastable states, and in several ways they can be approximately considered as states in the usual quantum mechanical sense, like bound states.

5. YUKAWIAN POTENTIALS. THE RESTRICTED CASE OF S WAVES

A potential will be named Yukawian if it can be written in the form:

$$V(r) = \int_m^\infty \sigma(\mu) \frac{e^{-\mu r}}{r} d\mu, \tag{5.1}$$

where $\sigma(\mu)$ is a suitable weight distribution. Yukawian potentials can be continued for complex values of r in the half-plane Re(r) > 0. This follows from the properties of Laplace transforms which are analytic in the half-plane of convergence. If a potential is Yukawian, then the Jost function has remarkable analyticity properties. The standard theory of differential equations tells us that, if the potential is analytic in some domain, then the wave function is also analytic in the same domain. The Jost solution can be continued then in the complex r domain Re(r) > 0. Take now ρ as a new variable in eq. (1.5) where $r = \rho e^{i\sigma}$ and σ is a fixed angle, $|\sigma| < \pi/2$. We have $(\ell = 0)$

$$\frac{d^2\psi}{d\rho^2} + e^{2i\sigma} E\psi = e^{2i\sigma} V(\rho e^{i\sigma})\psi. \tag{5.2}$$

This equation looks formally the same as (1.5) with a new distance ρ, a new wave function ψ, a new energy $E_1 = E e^{2i\sigma}$, a new (complex) potential $V_1(\rho) = V(\rho e^{i\sigma}) e^{2i\sigma}$. We are still able to define a new Jost solution $f_1(k_1, \rho)$ such that f_1 satisfies (5.2) and $f_1 \sim e^{-ik_1\rho}$ for large ρ. But $f(k, \rho e^{i\sigma})$ also satisfies (5.2) with the same boundary conditions* so that

$$f_1(k_1, \rho) = f(k, \rho e^{i\sigma}), \ k_1 = k e^{i\sigma}.$$

Now, f_1 is analytic in k_1 in the Bargmann domain $\mathrm{Im}\, k_1 < m_1/2$ where m_1 is related to the range of $V_1(\rho)$ just as m is related to the range of V(r). If V is given by (5.1), the lower limit in this integral has already been chosen to yield m as the correct value for the Bargmann proof; that is, P(r) of eq. (3.13) converges if $b < m/2$. Now, if r is large, the main contribution to V(r) from (5.1) is of the kind

$$V(r) \sim \sigma(m)\, e^{-mr}/r^2.$$

It follows that

$$|V_1(\rho)| \sim \sigma(m)\, e^{-m\rho\cos\theta}/\rho^2.$$

Clearly the correct value for m_1 is $m \cos\sigma$. f_1, and therefore f is analytic in $\mathrm{Im}\, k_1 < (m \cos\sigma)/2$. This domain is different from the original Bargmann domain of f(k, r). The union of all these domains for all $|\sigma| < \pi/2$ is the k plane with the cut $k = ib$, where $m/2 < b < \infty$.

S(k) is therefore analytic in the k plane with two cuts $k = ib$, with $M/2 < |b| < \infty$. There are of course different and more interesting ways of de-

* This result implies that the analytic continuation of the asymptotic behaviour of f coincides with the asymptotic behaviour of the analytic continuation of f. This seems to be evident but it is not, and it has to be justified It can best be proved using the Phragmen-Lindeloef lemma See CARTWRIGHT [8].

riving the same result. We did it here just because the result was very cheaply obtained. There is one case of Yukawian potential which can be solved exactly [9]: the Bethe potential $V(r) = -V_0 e^{-mr}$;

$$\psi'' + E\psi + V_0 e^{-mr}\psi = 0. \tag{5.3}$$

This equation can be reduced to the standard Bessel equation by going to the variable $\xi = 2(V_0^{1/2}/m)e^{-mr/2}$. We obtain for the Jost function

$$f(k) = e^{-(ik/m)\ln(V_0/m^2)}\,\Gamma(1 + 2ik/m)J_{2ik/m}(2V_0^{1/2}/m). \tag{5.4}$$

This example was considered by Jost in his discussion of the false poles. Our Jost function has namely an infinite set of false poles in the points $k = inm/2$, n integer > 1, these poles being what remains of the cut along the imaginary axis of k. For the pure Yukawa potential however there is a logarithmic singularity in $k = im/2$ and more complicated ones farther on. It is, however, much simpler to study these singularities with Martin's method, which shows the very interesting fact that higher perturbation terms produce singularities moving farther and farther away with increasing order of the term.

Martin's method works as follows: he defines

$$g(k, r) = e^{ikr} f(k, r)$$

and starts from the following Ansatz:

$$g(k, r) = 1 + \int_m^\infty \rho(k\cdot\alpha)\, e^{-\alpha r}\, d\alpha. \tag{5.5}$$

Inserting g(k, r) into the Schroedinger equation, one finds for it the differential equation,

$$g''(k\cdot r) - 2ikg'(k\cdot r) - V(r)\, g(k.r) = 0.$$

In this equation we replace g(k, r) by its integral representation (5.5), and we use for V(r) an expansion of the kind

$$V(r) = \int_m^\infty C(\mu)\, e^{-\mu r}\, d\mu.$$

Of course this representation for V(r) is just equivalent to (5.1) provided

$$\frac{d}{d\mu} C(\mu) = -\sigma(\mu).$$

We get then the integral equation,

$$\mu(\mu + 2ik)\rho(k\cdot\mu) = C(\mu) + \int_m^{\mu-m} C(\mu-\alpha)\rho(k\cdot\alpha)\,d\alpha. \qquad (5.6)$$

The main point about eq. (5.6) is that the value of $\rho(k,\mu)$ in a given interval $nm \le \mu \le (n+1)m$ can be calculated from the knowledge of the values of $\rho(k.\mu)$ when $\mu < nm$. This provides an interesting method of construction of $\rho(k\cdot\mu)$ since we know already that $\rho(k\cdot\mu) = 0$, $\mu \le m$ and $\rho(k\cdot\mu) = C(\mu)/\mu(\mu+2ik)$ for $m \le \mu \le 2m$. One can see the above situation also by saying that for values of μ lying in the interval nm. . . (n + 1)n the (n + 1) m perturbation term and the following one vanish identically so that the perturbation expansion always terminates. It is clear then that this also means that the support of the n-th terms moves away with increasing n. Martin has carefully examined this expansion, and a detailed account can be found in the Hercegnovi lectures.

6. THE HIGHER WAVES

All the results that we have so far derived for S waves can be extended to higher waves [9]. There is no simple method for doing this like the one we have for S waves. The reason is that the Green integral functions, which are used in order to define particular solutions of the wave equation, contain Bessel functions in their kernels and these are clumsy to handle.

We intend to quote here the corresponding results, and we also give a list of the most important functions used in the formalism of higher waves. The proof of these results actually does not teach anything newer than what we already know for S waves. A fairly complete review of this subject is in [9]. The reason we skip these lengthy mathematical proofs is that for Yukawian potentials Martin has much simpler methods.

Here follows a list of the most important functions of the theory:

(1) The Jost solution. It can be defined with an integral equation similar to (3.1) (see App. I):

$$f_\ell(k,r) = e^{-ikr} + \frac{1}{k}\int_r^\infty \sin k(x-r)\left[V(x) + \frac{\ell(\ell+1)}{x^2}\right] f_\ell(k,x)\,dx. \qquad (6.1)$$

If we put V = 0, we have,

$$f^0_\ell(k,r) = e^{-i\pi(\ell+1)/2}(\pi kr/2)^{1/2} H^{(2)}_{\ell+1/2}(kr). \qquad (6.2)$$

We have also the equation (see App. I)

$$f_\ell(k,r) = f_\ell^0(k,r) - i\,\frac{\pi}{4}\sqrt{r}\int_r^\infty \sqrt{\xi}\,[\,H^{(1)}_{\ell+1/2}(k\xi)\,H^{(2)}_{\ell+1/2}(kr)$$

$$- H^{(2)}_{\ell+1/2}(k\xi)\,H^{(1)}_{\ell+1/2}(kr)\,]\;V(\xi)\,f_\ell(k,\xi)\,d\xi. \qquad (6.3)$$

(2) The Jost function is defined as

$$f_\ell(k) = \lim_{r\to 0} r^\ell f_\ell(k,r). \qquad (6.4)$$

If V = 0, we have the free Jost function,

$$f_\ell^0(k) = \pi^{-1/2}\,e^{-i\ell\pi/2}\,\Gamma(\ell+1/2)\,(2/k)^\ell. \qquad (6.5)$$

The regular solution is defined as

$$\phi_\ell(k,r) = \frac{f_\ell(k)\,f_\ell(-k,r) - (f_\ell)(-k)\,f_\ell(k,r)}{2ik},$$

$$\phi_\ell(k,r) = \phi_\ell(-k,r);\;\; \phi_\ell(k,r) \approx r^{\ell+1},\; r\to 0. \qquad (6.6)$$

Comparing the asymptotic behaviour of this solution with the definition of phase shift, we find the formula,

$$S_\ell(k) = e^{2i\delta_\ell} = e^{i\pi\ell}\,\frac{f_\ell(k)}{f_\ell(-k)}. \qquad (6.7)$$

We quote here some results concerning the analyticity domain of these functions. All these analyticity proofs run exactly in the same way as for S waves (see App. II); that is, we place upper bounds on the perturbative expansion of $f_\ell(k,r)$, and we show that it converges uniformly in the Bargmann domain and that each term has the prescribed analytic properties:

(1) $k^\ell f_\ell(k,r)$ is analytic in $b < m/2$. If the potential is Yukawian the possible singularities lie on the cut $m/2 < b < \infty$, with $k = ib$.

(2) The same result holds for $k^\ell f_\ell(k)$. For large k in the lower half-plane we have $\lim_{k\to\infty} f_\ell(k)/f_\ell^0(k) = 1$.

(3) $S_\ell(k)$ is analytic in the cut k plane: $k = ib$, $m/2 < |b| < \infty$. The discussion of the poles of S(k) is exactly the same as the one we gave for $\ell = 0$.

(4) The only striking difference between higher waves and the S wave regards the behaviour of the phase shift at low energies. This property is linked with the so-called scattering length approximation. It asserts the validity of the expansion,

$$k^{2\ell+1} \operatorname{ctg} \delta_\ell(k) = a_0 + a_1 k^2 + \ldots\ldots \quad (6.8)$$

This expansion will be a byproduct of the complete theory of the properties of $S_\ell(k)$ as a function of both k and of (complex) ℓ which will be worked out in the next sections. Physically (6.8) has its origin in the existence of a repulsive centrifugal barrier which pushes the wave function out of the region of interaction. A parameter which decides the order of magnitude of the phase shift is the impact parameter (distance of closest classical approach) $T = \ell/k$. If k decreases while ℓ is kept constant, the wave function will scan the potential at increasing distances and the interaction will become negligible when $\ell/k \gg 1/m$.

(5) If we let ℓ increase while we keep k constant, we provide another mechanism which increases T and decreases the phase shift. The phase shift can be estimated for large ℓ with the following argument: We know that if $T \gg 1/m$ the bulk of the wave function lies almost totally outside the potential, and it is a good guess that the wave function is only slightly perturbed by the potential. We take now the exact formula,

$$\sin \delta_\ell(k) = -k \int_0^\infty V(r) \frac{\phi_\ell(k,r)}{|f_\ell(k)|} \frac{\phi_\ell^0(k,r)}{|f_\ell^0(k)|} dr,$$

and we replace $\phi_\ell(k,r)/|f_\ell(k)|$ by $\phi_\ell^0(k,r)/|f_\ell^0(k)|$. This yields the so-called Born approximation. The general reliability of the Born approximation has been repeatedly questioned, and now it is agreed that it gives at most only the order of magnitude of the scattering amplitude if blindly applied to low waves and it increases in accuracy at high energies. Anyway, if $T \gg 1/m$ and k is large, we can confidently use it. To us it is interesting just because it gives a reliable estimate of the phase shift for large ℓ, and we need it in order to discuss the convergence of the Rayleigh-Faxen expansion outside the physical range of $\cos\theta$. This argument can be made somehow more rigorous, but it then becomes so dull that we prefer not to interrupt our main flow of ideas with insipid mathematics. Anyway the general theory which we shall work out in the next lectures will bring new arguments to support our conclusions. We would just like to mention that Carter in an unfortunately unpublished thesis has proved rigorous equivalent results. (See [9], p. 333 or [3].) He states that for $\ell \to \infty$ the bound holds:

$$|\delta_\ell| < C |\delta_\ell|_{\text{Born}}, \quad (6.9)$$

where $C > 1$ is some constant. This is enough for our purposes.

7. LEVISON'S THEOREMS

There is a class of very elegant theorems which relate the number of bound states for a given partial wave to the total variation of the phase shift in the interval $0 < k < \infty$.

We know that for all reasonable potentials $\lim_{k\to\infty} S_\ell(k) = 1$ (k real). At infinity we can always choose

$$\delta_\ell(\infty) = 0.$$

Even within the Bargmann strip $\delta_\ell(k)$ is not analytic, because in general it has logarithmic branch points wherever $S_\ell(k)$ has poles or zeros. We define $\delta_\ell(0)$ as the value we get by continuing $\delta_\ell(k)$ analytically along the real k axis from $k = +\infty$. We know that, unless there is a bound state at $k = 0$, which we exclude for simplicity, $\sin\delta_\ell(0) = 0$ so that $\delta_\ell(0)$ is a multiple of π. Levinson's theorem then states that

$$\delta_\ell(0) - \delta_\ell(\infty) = n_\ell\pi, \tag{7.1}$$

when n_ℓ is the number of bound states of angular momentum ℓ. The proof we prefer here has been somewhat shortened (for a full discussion see [9], p. 332). Take the function

$$g_\ell(k) = f_\ell(k)/f_\ell^0(k). \tag{7.2}$$

(Incidentally $g_1(k)$ is named Jost function in [9] and written $f_\ell(k)$.) We know that in the lower half-plane of k we have $\lim_{k\to\infty} g_\ell(k) = 1$. By unitarity it is obvious that for real k

$$\delta_\ell(k) = \arg g_\ell(k).$$

Moreover $\delta_\ell(k)$ is an odd function of k. This semicircle in $b < 0$ ($b = \mathrm{Im}\, k$) is indented on the real axis of k. This semicircle encloses all zeros of $g_\ell(k)$ which correspond to bound states. We define $\arg g_\ell(0+\epsilon) = \delta_\ell(0) = n\pi$, where n is an integer which we do not identify yet with the number of bound states. We now move along the real axis of k until we meet the semicircle, here by definition $\arg g_\ell(k) = 0$. On the whole semicircle we also have $\arg g_\ell(k) = 0$. We move along the semicircle until we arrive on the real negative axis. If we move now toward $k = 0 - \epsilon$, we have the relation $\delta_\ell(-k) = -\delta_\ell(k)$. When we arrive at $k = 0$, we have $\delta_\ell(0-\epsilon) = -n\pi$ and $\delta_\ell(k)$ is clearly discontinuous in $k = 0$. The discontinuity arises from the fact that we have enclosed the bound states in the semicircle and we have gone clockwise around the zeros of $g_\ell(k)$. During the trip $\arg g_\ell(k)$ decreases by the amount $2\pi n_\ell$, where n_ℓ is the number of zeros of $g_\ell(k)$ inside the contour. But $\delta_\ell(k)$ has just decreased by $\delta_\ell(0+\epsilon) - \delta_\ell(0-\epsilon) = n\pi - (-n\pi) = 2n\pi$. Therefore $n = n_\ell$. QED.

8. THE TECHNIQUES OF COMPLEX ANGULAR MOMENTA

In the other lectures we have discussed the scattering amplitude for integer values of ℓ. This is easily understood because we cannot associate any direct physical meaning to unrestricted values of ℓ; ℓ came from the expansion in partial waves, and integer values of ℓ are a natural consequence of the quantization of angular momentum. Moreover, we apparently need to consider $\delta_\ell(k)$ when ℓ is integer only in order to know the scattering amplitude.

We want to oppose this general attitude and the reasons are the following:

(a) ℓ is quantized because spherical harmonics are considered on the sphere, that is for $|\cos\theta| < 1$, where θ is the scattering angle. Truly one can make experiments only when $|\cos\theta| < 1$, however, the crossing properties implied by the relativistic Mandelstam representation also mean that, for instance, the pion-nucleon scattering is directly related to the nucleon-antinucleon annihilation into two pions. In a way, therefore, the process, $N + \bar{N} \rightarrow \pi + \pi$ is simply the process $\pi + N \rightarrow \pi + N$ viewed in a region considered unphysical before. In other words, if we measure the first process we actually measure the second for $|\cos\theta| > 1$. Now the natural way of expanding a function of a hyperbolic angle is to use the set $P_{i\mu-1/2}(\cos\theta)$ which is the corresponding harmonics for a Lorentz invariant hyperboloid in an indefinite metric. Therefore, Mandelstam's representation is naturally associated with non-integer angular momenta. The potential scattering retains part of the full information of the original relativistic scattering, and there should be no surprise if unphysical angular momenta turn up.

(b) Even without the previous argument the technique has been used for years in the discussion of diffraction phenomena; a typical problem in this field was the theory of the rainbow or the theory of propagation of waves around the earth [5]. It is therefore a highly succesful tool in a wide range of problems.

The basic idea of the technique arises from a transformation, due to Watson, of the Rayleigh-Faxen formula:

$$f(E,\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)[S_\ell(k) - 1]\, P_\ell(\cos\theta). \tag{8.1}$$

This transformation is successful only if one succeeds in proving the existence of an analytic function $S(\lambda, k)$ of the complex variable λ which takes the values $S_\ell(k)$ of (8.1) when $\lambda = \ell + 1/2$. We use the variable λ because in the following it will have a more symmetrical role than ℓ and corresponds more closely to the classical angular momentum than ℓ. In this hypothesis (8.1) can be transformed into

$$f(E,\theta) = -\frac{1}{2k}\int_C \frac{\lambda\, d\lambda}{\cos\pi\lambda}\, P_{\lambda-1/2}(-\cos\theta)[S(\lambda,k)-1]. \tag{8.2}$$

The path C of integration encloses all the positive zeros of $\cos \pi\lambda$ but avoids the singularities of $S(\lambda, k)$. (See Fig. (1).) If we calculate the integral (8.2) with the contour method, we find the expansion (8.1).

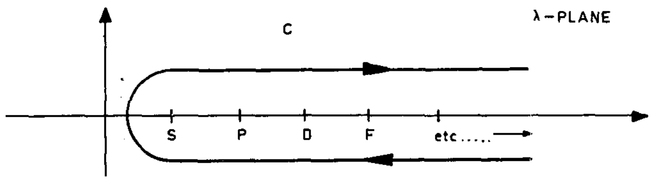

Fig. 1

The path of integration for the Watson integral

Eq. (8.2) contains all the information of eq. (8.1) and it has additional features of its own. Indeed, in constructing it we use properties of $S(\lambda, k)$ which depend rather critically on the potential. The path C of integration can be deformed in accordance with the analytic properties of $S(\lambda, k)$. The ensuing convergence domain of (8.2) depends on $P_{\lambda-1/2}$, and this allows us to extend the analytic properties beyond the Lehmann ellipse.

The necessary steps which we have to carry out in order to establish the validity of Watson's transformation are the following:

(1) Definition of $S(\lambda, k)$ for general values of λ and k. Analytic properties and asymptotic behaviour of $S(\lambda, k)$ for λ large. All these properties will be derived for the restricted case of Yukawian potentials.

(2) The convergence of (8.2) is investigated for the specific case of Yukawian potentials. The Mandelstam representation then follows for the total amplitude.

In order to achieve our goal we shall enlarge the definitions which we have used so far for S waves and higher waves. We think that those definitions are self-evident if one keeps an eye on the previous section.

The starting point of our theory is the partial wave Schroedinger equation:

$$\psi''(z) + k^2 \psi(z) - (\lambda^2 - \frac{1}{4}) z^{-2} \psi(z) - V(z) \psi(z) = 0. \qquad (8.3)$$

We shall use the attribute "physical" for the variables $\lambda = \ell + 1/2$ and k when is integer and k is real.

Moreover we assume the following conditions for the potential:

(a) $V(z)$ has the representation

$$V(z) = \int_{m>0} \sigma(\mu) \frac{e^{-\mu z}}{z} \, d\mu,$$

with a suitable weight distribution $\sigma(\mu)$.

(b) $V(z)$ can therefore be continued into the half-plane Re $z > 0$.
(c) On any ray $\arg z = \sigma$, $|\sigma| < \pi/2$, we have

$$\int^{\infty} |V(z)\, z|\, ds < M < \infty, \quad ds = |dz|.$$

We exclude the value $\sigma = \pi/2$ because the last condition would rule out the interesting case of the Yukawa potential. More refined assumptions will be made in order to derive special results if needed.

We intend to study eq. (8.3) and the associated quantities when λ and k are both complex. This programme has been partly carried out in previous papers [3,4,10,11], and we may group previous results into two classes:

(a) Analyticity in k when λ is physical;

(b) Analyticity in λ when k is physical.

We repeat here some of the already known definitions and formulas which will be used extensively in the lectures. Most of these definitions are purely formal since there are involved, for instance, variables defined through solutions of an integral equation, whose existence has only been proved in the cases (a) and (b). The proofs will be given in the next sections, and the formulas listed should be regarded rather as a framework for the parts to come. There are two ways of defining particular solutions of eq. (8.3).

(i) We define $\phi(\lambda, k, z)$ as that solution which behaves like $z^{\lambda+1/2}$ when z is small. More rigorously, we define ϕ through the integral equation,

$$\phi(\lambda, k, z) = z^{\lambda+1/2} - \frac{1}{2\lambda} \int_0^z \left(\frac{\xi^{\lambda+1/2}}{z^{\lambda-1/2}} - \frac{z^{\lambda+1/2}}{\xi^{\lambda-1/2}} \right) [V(\xi) - k^2]\, \phi(\lambda, k, \xi)\, d\xi. \tag{8.4}$$

If ϕ_0 is the solution of eq. (8.3) when V = 0 (free solution), we have

$$\phi_0(\lambda, k, z) = \Gamma(\lambda+1)\, (2/k)^{\lambda} z^{1/2} J_\lambda(kz) \approx z^{\lambda+1/2}, \quad z \to 0. \tag{8.5}$$

Similarly,

$$\phi(\lambda, k, z) = z^{\lambda+1/2} - \frac{\pi\sqrt{z}}{2\sin(\pi\lambda)} \int_0^z \sqrt{\xi}\, [J_\lambda(k\xi) J_{-\lambda}(kz) - J_{-\lambda}(k\xi) J_\lambda(kz)]$$

$$\times V(\xi)\, \phi(\lambda, k, \xi)\, d\xi. \tag{8.6}$$

The derivation of these equations is quite simple (see App. I). Clearly $\phi(\lambda, k, z) = \phi(\lambda, -k, z)$. However, $\phi(-\lambda, k, z)$ is a new solution. If Re $\lambda > 0$, $\phi(+\lambda, k, z)$ will be regular at the origin, and any other independent solution will be irregular. On the line Re $\lambda = 0$, $\phi(\lambda, k, z)$ and $\phi(-\lambda, k, z)$ exchange their regularity roles and both have an oscillatory character. It is evident from this and other features of (8.3) that λ dominates the behaviour in the origin while k determines the behaviour at infinity. So far we have not committed ourselves to any theorem of existence of these solutions. In fact, unless one makes a special hypothesis on the potential, the region where both $\phi(\lambda, k, z)$ and $\phi(-\lambda, k, z)$ exist and are analytic is in general very limited. The line Re $\lambda = 0$ will be seen to belong to this region. Of some use is the Wronskian:

$$W[\phi(\lambda, k, z), \ \phi(-\lambda, k, z)]$$

$$= \phi(\lambda, k, z)\, \phi'(-\lambda, k, z) - \phi(-\lambda, k, z)\, \phi'(\lambda, k, z) = -2\lambda. \tag{8.7}$$

(ii) The second class of solutions is defined through the boundary conditions at infinity. Such a class of solutions was first introduced by Jost for S waves. We define $f(\lambda, k, z)$ as that solution which behaves like e^{-ikz} for large z. More rigorously,

$$f(\lambda, k, z) = e^{-ikz} + \frac{1}{k}\int_z^\infty \sin k(\xi - z)\left[V(\xi) + \frac{\lambda^2 - 1/4}{\xi^2}\right] f(\lambda, k, \xi)\, d\xi. \tag{8.8}$$

If $f_0(\lambda, k, z)$ is the free solution, we have

$$f_0(\lambda, k, z) = e^{i\pi(\lambda+1/2)/2}\, (\pi k z/2)^{1/2} H_\lambda^{(2)}(kz) \sim e^{-ikz}. \tag{8.9}$$

Similarly,

$$f(\lambda, k, z) = f_0(\lambda, k, z) - i\frac{\pi}{4}\sqrt{z}\int_z^\infty \sqrt{\xi}\,[H_\lambda^{(1)}(k\xi)\, H_\lambda^{(2)}(kz)$$

$$- H_\lambda^{(2)}(k\xi)\, H_\lambda^{(1)}(kz)]\, V(\xi)\, f(\lambda, k, \xi)\, d\xi. \tag{8.10}$$

We can also define $f(\lambda, ke^{-i\pi}, z)$. However, in general $f(\lambda, k, z)$ has a branch point in $k = 0$ and $f(\lambda, ke^{-i\pi}, z)$ will be different from $f(\lambda, k, z)$. This already happens for free solutions. For instance,

$$f_0(\lambda, ke^{-i\pi}, z) = e^{i\pi(\lambda+1/2)/2}(\pi kz/2)^{1/2} H_\lambda^{(1)}(kz) \sim e^{ikz}. \quad (8.11)$$

The Wronskian is uniquely defined:

$$W[f(\lambda, k, z), f(\lambda, ke^{-i\pi}, z)] = 2ik. \quad (8.12)$$

From the general theory of differential equations we know that the analyticity domains in z of $\phi(\lambda, k, z)$, $f(\lambda, k, z)$ and $V(z)$ are the same. If we take the conjugate of each of the previously written equations, we find (if $V(z)$ is real on the real positive axis of z):

$$\phi(\lambda, k, z) = \phi^*(\lambda^*, k^*, z), \qquad f(\lambda, k, z) = f^*(\lambda^*, -k^*, z). \quad (8.13)$$

The hermiticity requirement on the Hamiltonian needed for the above results will not be used in the proofs on the convergence of the perturbation expansions which we shall derive in the next section. This we do, not in view of possible application to absorbing potentials, but just as a mathematical artifice in order to extend the analytic properties. This will be apparent in the following.

9. THE JOST FUNCTIONS AND ANALYTIC PROPERTIES OF THE PARTIAL WAVE FUNCTIONS

Once we have defined the functions $\phi(\pm\lambda, k, z)$ and $f(\lambda, \pm k, z)$, we possess four solutions of the same differential equation. The Wronskian of any two solutions is of course a constant. We have already given such a Wronskian between two ϕ's and two f's in (8.7), (8.12); these two Wronskians do not carry any information about the potential, and they are therefore useful but trivial. A more useful quantity (the so-called Jost function) is

$$W[f(\lambda, k, z), \phi(\lambda, k, z)] = f(\lambda, k). \quad (9.1)$$

Besides $f(\lambda, k)$, we consider $f(-\lambda, k)$, $f(\lambda, -k)$ and $f(-\lambda, -k)$ too. The Jost function is interesting because, as we shall see, it is directly related to the scattering matrix. In order to show this let us first notice that, according to general principles, there is always a linear relation between any three solutions of (8.3). In particular, we must have

$$\begin{aligned} \phi(\lambda, k, z) &= Af(\lambda, k, z) + Bf(\lambda, -k, z), \\ \phi(-\lambda, k, z) &= Cf(\lambda, k, z) + Df(\lambda, -k, z). \end{aligned} \quad (9.2)$$

Here A, B, C, D are independent of z, but they are expected to be functions of λ and k. In order to evaluate them we introduce the formula (9.2) for $\phi(\lambda, k, z)$ into (9.1), thus finding:

$$2ikA_- = - f(\lambda, -k). \tag{9.3}$$

Similarly,

$$2ikB = f(\lambda, k), \; 2ikC = - f(-\lambda, -k), \; 2ikD = f(-\lambda, k).$$

These values can be reintroduced into (9.2), and we find

$$\phi(\lambda, k, z) = [f(\lambda, k) f(\lambda, -k, z) - f(\lambda, -k) f(\lambda, k, z)]/2ik. \tag{9.4}$$

Finally we calculate the Wronskian $W[\phi(\lambda, k, z), \phi(-\lambda, k, z)]$, using eq. (9.4), and we compare it with the known value,

$$f(\lambda, -k) f(-\lambda, k) - f(\lambda, k) f(-\lambda, -k) = 4i\lambda k. \tag{9.5}$$

This is an important identity*. From (9.5) and (9.4) we can find easily

$$\begin{aligned} f(\lambda, k, z) &= [f(-\lambda, k)\phi(\lambda, k, z) - f(\lambda, k)\phi(-\lambda, k, z)]/2\lambda, \\ f(\lambda, -k, z) &= [f(-\lambda, -k)\phi(\lambda, k, z) - f(\lambda, -k)\phi(-\lambda, k, z)]/2\lambda. \end{aligned} \tag{9.6}$$

The free Jost functions are given by the formula,

$$f_0(\lambda, k) = (2/\pi)^{1/2} 2^{\lambda} \Gamma(\lambda + 1) k^{-\lambda+1/2} e^{-i\pi(\lambda-1/2)/2}. \tag{9.7}$$

They are multivalued in k.

We now proceed to find the connection between the Jost functions and the scattering phase shifts. It is almost unnecessary to point out that what we shall define is actually a function which interpolates for unphysical values of λ (and k) the known and measurable phase shifts. It is also clear that there could be no other interpolations. The one we select is convenient merely because it retains part of the properties of the physical phases. Our definition starts from the known behaviour of the "regular" free solution $\phi_0(\lambda, k, z)$ at infinity:

$$\phi_0(\lambda, k, z) \sim e^{i\pi(\lambda-1/2)/2} \frac{1}{k} f_0(\lambda, k) \sin[kz - \pi(\lambda - 1/2)/2]. \tag{9.8}$$

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* This is identity (1.8) of [10]. The functions $C(\lambda)$ and $S(\lambda)$ are linear combinations of Jost functions.

This theory follows immediately from the theory of Bessel functions. We compare it with the behaviour of the perturbed regular solution (9.4):

$$\phi(\lambda, k, z) \sim [f(\lambda, k)e^{ikz} - f(\lambda - k)e^{-ikz}]/2ik$$

$$= e^{i\pi(\lambda-1/2)} e^{-i\delta(\lambda,k)} \frac{1}{k} f(\lambda, k) \sin[kz - \pi(\lambda - 1/2)/2 + \delta(\lambda, k)], \qquad (9.9)$$

where we have defined

$$S(\lambda, k) = e^{2i\delta(\lambda,k)} = [f(\lambda, k)/f(\lambda, -k)]e^{i\pi(\lambda-1/2)}. \qquad (9.10)$$

This formula we retain even when the comparison is no longer valid, in particular when one deals with exploding expotentials (k not real). So far this definition is purely formal, since we know very little about the existence and the analyticity of the Jost function when both k and λ are not physical. This will be discussed in the next sections.

We give here, for completeness, a relation that will be used later,

$$e^{i\delta(\lambda,k)} \sin\delta(\lambda, k) = -k \int_0^\infty V(z) \frac{\phi_0(\lambda, k, z)}{f_0(\lambda, k)} \frac{\phi(\lambda, k, z)}{f(\lambda, -k)} dz, \qquad (9.11)$$

and that has been deduced from

$$(\phi'\phi_0 - \phi_0'\phi)\Big|_0^\infty = -e^{i\pi(\lambda-1/2)} e^{-i\delta(\lambda,k)} \frac{1}{k} f_0(\lambda, k) f(\lambda, k) \sin\delta(\lambda, k)$$

$$= \int_0^\infty V(z)\phi_0(\lambda, k, z)\phi(\lambda, k, z)\, dz.$$

The existence theorems and the analytic properties of the partial wave functions are usually derived by the following method:

(a) We iterate the defining integral equation, and we define a formal perturbation expansion.

(b) The analytic properties of each term of the expansion must then be examined in order to find the analyticity domain of the solution.

(c) Bounds are placed on the solution in such a way that the series are seen to converge uniformly inside the analyticity domain. We give the practical calculations in Appendix II, and we merely state here the most important results:

(i) $\phi(\lambda, k, z)$ and $\phi'(\lambda, k, z)$ are integral functions of k (i.e., regular for all k with the exception of an essential singularity at $k = \infty$) and are analytic in λ for $\mathrm{Re}\,\lambda > 0$; the expansion also converges for $\mathrm{Re}\,\lambda = 0$ (in fact, we think it is possible to show that the analyticity region can be pushed inside the region $\mathrm{Re}\,\lambda < 0$ under very special assumptions on the potential). $\phi(\lambda, k, z)$ is analytic in both variables in the topological product of the k plane ($k = \infty$ excluded) with the half-plane $\mathrm{Re}\,\lambda > 0$ (and continuous for $\mathrm{Re}\,\lambda = 0$).

(ii) $f(\lambda, k, z)$ is analytic in the pair of variables λ, k in the topological product of the whole λ plane ($\lambda = \infty$ excluded) with the half-plane $\mathrm{Im}\,k < 0$ (and continuous for $\mathrm{Im}\,k = 0$). This allows one to define $f(\lambda, -k, z)$ as $f(\lambda, ke^{-i\pi}, z)$ unambiguously when k is real; in order to avoid confusion we shall retain the clearer notation $f(\lambda, ke^{-i\pi}, z)$. Correspondingly in (8.13) we have $f(\lambda, k, z) = f^*(\lambda^*, k^* e^{-i\pi}, z)$. It follows that $f(\lambda, k)$ is analytic in λ, k in the product on the half-planes $\mathrm{Re}\,\lambda > 0$, $\mathrm{Im}\,k < 0$ and is continuous on the boundaries $\mathrm{Re}\,\lambda = 0, \mathrm{Im}\,k = 0$. The branch point at $k = 0$ will be discussed later.

Under the stated assumptions on the potential it is possible to enlarge the analyticity domain of $f(\lambda, k, z)$ and consequently that of $f(\lambda, k)$. for this purpose let us consider eq. (8.3) along a prescribed direction in the complex $z = x + iy$ plane. Let therefore $z = \rho\, e^{i\sigma}$, where σ is a constant angle $|\sigma| < \pi/2$. Eq. (8.3) can be written in the variable ρ:

$$\frac{d^2\psi}{d\rho^2} - \frac{\lambda^2 - 1/4}{\rho^2}\psi + k^2 e^{2i\sigma}\psi - e^{2i\sigma} V(\rho e^{i\sigma})\psi = 0. \qquad (9.12)$$

This equation is still of the same kind as eq. (8.3), with a new wave number $k_1 = ke^{i\pi}$ and a new (complex) potential $V_1 = V(\rho e^{i\sigma})e^{2i\sigma}$. The previous analysis can be carried out on the new equation, and we shall arrive at a new set of wave functions $\phi_1(\lambda, k_1, \rho)$, $f_1(\lambda, k_1, \rho)$ and at a new Jost function $f_1(\lambda, k_1)$.

The Jost solution $f_1(\lambda, k_1, \rho)$ is defined as the solution with the following behaviour:

$$f_1(\lambda, k_1, \rho) \sim e^{-ik_1\rho}$$

for any value of σ. On the other hand, the Jost solution $f(\lambda, k, z)$, already defined for z real, may be continued analytically in the half-plane $\mathrm{Re}\,z > 0$ with the same boundary condition because of the conditions on the potential $V(z)$. So the analytic continuation of $f(\lambda, k, z)$ coincides with $f_1(\lambda, k_1, \rho)$, and we have

$$f_1(\lambda, k_1, \rho) = f(\lambda, k, z), \quad \phi(\lambda, k_1, \rho) = e^{-i\sigma(\lambda+1/2)}\phi(\lambda, k, z),$$

$$f_1(\lambda, k_1) = e^{-i\sigma(\lambda+1/2)} f(\lambda, k).$$

But the same general analysis used before for the variable z (for real values), if used for the variable ρ, implies that the new Jost function is analytic in $\mathrm{Im}\ k_1 < 0$ and $\mathrm{Re}\ \lambda > 0$ and that the old Jost function is also analytic in this domain, in view of the above relation. This domain depends on σ, where $|\sigma| < \pi/2$. The Jost function is therefore analytic in the union of all domains of the kind $\mathrm{Im}\ (ke^{i\sigma}) < 0$; this union is simply the k plane cut along the upper imaginary axis $k = i\eta$ ($\eta > 0$). Previous results (see Appendix II) actually state that, when λ is physical, the cut starts at $\eta = m/2$, m being the lower limit of integration in the integral defining $V(z)$.

Similarly, $f(\lambda, ke^{-i\pi})$ is holomorphic in the topological product of the whole k plane, cut along the lower imaginary axis (when l is integer, the cut starts at $\eta = -m/2$), with the half-plane $\mathrm{Re}\ \lambda > 0$.

Finally we discuss the branch point of the Jost functions at $k = 0$. From (8.9) it follows that

$$f_0(\lambda, ke^{-2i\pi}, z) = f_0(\lambda, k, z) + a(\lambda)\ f_0(\lambda, ke^{-i\pi}, z),$$

$$f_0(\lambda, ke^{-3i\pi}, z) = [1 + a^2(\lambda)] f_0(\lambda, ke^{-i\pi}, z) + a(\lambda)\ f_0(\lambda, k, z),$$

$$a(\lambda) = -2i \cos(\pi\lambda).$$

Introducing this relation now in definition (9.1), we have that the result of a circuit around the origin can be written as follows:

$$f_0(\lambda, ke^{-2i\pi}) = f_0(\lambda, k) + a(\lambda)\ f_0(\lambda, ke^{-i\pi}),$$

$$f_0(\lambda, ke^{-3i\pi}) = [1 + a^2(\lambda)]\ f_0(\lambda, ke^{-i\pi}) + a(\lambda)\ f_0(\lambda, k). \tag{9.13}$$

If we think of eq. (8.10) written for $f(\lambda, ke^{-i\pi}, z)$ and make the linear combination $f(\lambda, k, z) + a(\lambda)\ f(\lambda, ke^{-i\pi}, z)$, we find that, if one follows a path which, without crossing the $m/2 < \eta < \infty$ cut, encircles the origin, then, when λ is real, $f(\lambda, k)$ has exactly the same law of transformation as $f_0(\lambda, k)$.

Later on it proves convenient to use the function $F(\lambda, k) = f(\lambda, k)/f_0(\lambda, k)$. In terms of $F(\lambda, k)$ one writes

$$F(\lambda, ke^{-2i\pi}) = 2 \cos \pi\lambda\ e^{-i\pi\lambda} F(\lambda, ke^{-i\pi}) - e^{-2i\pi\lambda} F(\lambda, k)$$

or

$$F(\lambda, ke^{-2i\pi}) - F(\lambda, ke^{-i\pi}) = e^{-2i\pi\lambda}\left[F(\lambda, ke^{-i\pi}) - F(\lambda, k)\right].$$

One could argue that it would be easier to represent everything with a single cut starting from the origin. This is not true since we would lose the information that the branching properties at the origin do not depend on the potential and are purely kinematical. On the contrary, the other cut depends critically on the potential, and it is useful to separate the contributions.

For the S matrix (9.13) gives *

$$S(\lambda, ke^{-2i\pi}) = \frac{S(\lambda, k) - 2\cos(\pi\lambda)e^{i\pi\lambda}}{[1 - 4\cos^2(\pi\lambda)] + 2\cos(\pi\lambda)e^{-i\pi\lambda}S(\lambda k)}, \tag{9.14}$$

or

$$S(\lambda, k^{-i\pi}) = \frac{e^{2i\pi(\lambda-1/2)}}{S(\lambda, k) - 2i\cos(\pi\lambda)\, e^{i\pi(\lambda-1/2)}}. \tag{9.15}$$

It is useful to introduce a new function,

$$Z(\lambda, k) = ik^{2\lambda}[S(\lambda, k) - e^{2i\pi\lambda}]/[S(\lambda, k) - 1]. \tag{9.16}$$

The function Z can be linked to the so-called scattering length expansion. This expansion represents $k^{2\ell+1}\mathrm{ctg}\,\delta(\ell, k)$ at low energies as a power series. From this expansion it is evident that $\delta(\ell, k)$ tends to vanish, like $\approx k^{2\ell+1}$, when $k \to 0$ in the ℓ^{th} wave. Now, if λ is physical ($\lambda = \ell + 1/2$), we have $Z(\lambda, k) = k^{2\ell+1}\mathrm{ctg}\,\delta(\ell, k)$. This shows that $Z(\lambda, k)$ is the natural generalization of $k^{2\ell+1}\mathrm{ctg}\,\delta(\ell, k)$ because it retains the property of admitting a power series expansion in a neighbourhood of the origin. It must be noticed that $Z(\lambda, k)$ is not only regular in $k = 0$ but also an even function of k; its meromorphy domain is the same as that of $S(\lambda, k)$.

The following formula is also useful:

$$S(\lambda, k) = [Z(\lambda, k) - ik^{2\lambda}e^{2i\pi\lambda}]/[Z(\lambda, k) - ik^{2\lambda}]. \tag{9.17}$$

Finally, we wish to point out that eq. (9.5) implies

$$e^{-i\pi\lambda}S(\lambda, k) - e^{i\pi\lambda}S(-\lambda, k) = -4k\lambda/[f(\lambda, ke^{-i\pi})\, f(-\lambda, ke^{-i\pi})]. \tag{9.18}$$

This equation only holds when λ is imaginary; otherwise one of the two functions $f(\lambda, ke^{-i\pi})$, $f(-\lambda, ke^{-i\pi})$ is not defined. We also have from eqs. (8.13)

* In the following when we write $f(\lambda, k)$ we mean the Jost function on the sheet: $-3\pi/2 < \arg k < \pi/2$.

$$f(\lambda, k) = f^*(\lambda^*, k^* e^{-i\pi}), \quad S^*(\lambda, k) = S^{-1}(\lambda^*, k^* e^{-i\pi}). \tag{9.19}$$

If we define $Z(\lambda, k)$, when λ is not real, by eq. (9.16), we find that it is meromorphic in half-planes Re $k > 0$ and Re $k < 0$. There is at the moment no way of joining the left and right domains of $Z(\lambda, k)$, because there is no gap through the cut of either $f(\lambda, k)$ or $f(\lambda, ke^{-i\pi})$ unless λ is real. From Appendix II we can prove that actually the result holds for any real positive λ. Indeed, if λ is real positive,

$$\lim_{\epsilon \to 0} Z(\lambda, i\xi - \epsilon) - Z(\lambda, i\xi + \epsilon) = D(\lambda) = 0, |\xi| < m/2.$$

But we know that in general $D(\lambda)$ is analytic in λ for Re $\lambda > 0$. It follows that $D(\lambda) = 0$ for Re $\lambda > 0$ and $|\xi| < m/2$. This result enables us to join the right and left domains of meromorphy of $Z(\lambda, k)$ (and of course of $S(\lambda, k)$ and of related functions) through the gap $|\xi| < m/2$. This shows that actually the branch point of S in $k = 0$ is a purely kinematical one: that is, it does not depend on the potential.

10. THE ASYMPTOTIC BEHAVIOUR OF THE PHASE SHIFT

The behaviour for large values of λ and k of the phase shift can best be investigated with the help of the WKB method. In the current practice the use of this method has been limited for obvious reasons to the physical values of k and λ. We wish to point out, however, that the extension to the unphysical range of these variables does not add anything essentially new to the method and that the only difficulty is an increased complexity and variety in the classification and behaviour of the turning points. The most rigorous paper on this subject is certainly KEMBLE's paper [12], and we could almost quote his results with obvious changes. As Kemble's analysis is in some cases incomplete for our purposes or it becomes too complicated, it will not be reported here. A more realistic view of the situation has suggested that these details should be published elsewhere [16] and that we should discuss here the final results only.

The general idea of the WKB method is that of constructing a differential equation, which is very close to the Schroedinger equation, and whose solutions are well known. Such an equation is satisfied by the functions:

$$\frac{1}{\sqrt{p(x)}} \exp\left(\pm i \int^x p(z)dz\right),$$

$$p^2(z) = k^2 - \lambda^2/z^2 - V(z), \qquad p_0^2(z) = k^2 - \lambda^2/z^2. \tag{10.1}$$

The approximation is generally good on the whole complex z plane except in the neighbourhood of the points where p(z) vanishes. These points are usually named turning points T. If k and λ are very large, there is only one turning point in the domain Re $z > 0$ and this occurs very close to T_0 or $-T_0$, $T_0 = \lambda/k$, which are exactly the two turning points when $V = 0$. The choice between T_0 and $-T_0$ is dictated by the fact that only one of these points is on the good side Re $z > 0$ where V(z) is analytic. The turning points are branch points of p(z).

The main problem of the WKB is to connect the solution (10.1), which is good approximation at large distances. These solutions cannot in general be represented by the same formula because the approximation scheme fails near the turning point. An appropriate connection formula can be found in the literature [7].

The result of the above analysis is that when k and λ are large we have the following asymptotic formulas:

$$f(\lambda, k) \sim f_0(\lambda, k) \exp\left(-i \int_{0,\Gamma_\ell}^{\infty} [p_0(z) - p(z)] dz\right),$$
$$f(\lambda, ke^{-i\pi}) \sim f_0(\lambda, ke^{-i\pi}) \exp\left(i \int_{0,\Gamma_h}^{\infty} [p_0(z) - p(z)] dz\right). \quad (10.2)$$

The integration paths Γ_ℓ and Γ_h connect the origin with the infinity in the half-plane Re $z > 0$. Γ_ℓ passes below and Γ_h above T (see Figs. 2 and 3). The

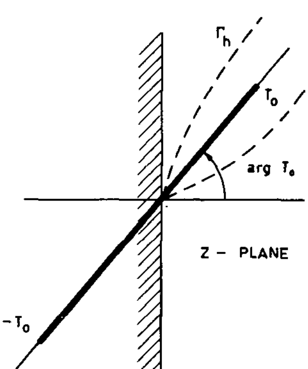

Fig. 2

Diagram for the asymptotic formulas

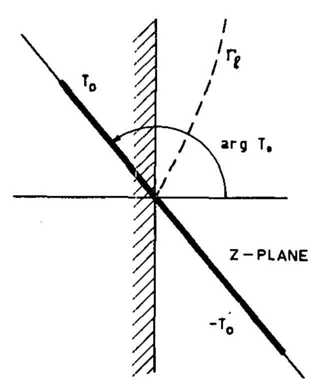

Fig. 3

Diagram for the asymptotic formulas

proposed formulas are valid under the restrictions that paths satisfying these criteria actually exist. If Re $T_0 > 0$, then we can obviously trace both paths. Suppose now that we let arg T_0 gradually increase toward $\pi/2$. When arg $T_0 = \pi/2$, the high path gets pinched between the turning point and the imaginary axis. For arg $T_0 > \pi/2$ the corresponding second formula is no longer valid. Only one formula therefore remains because it is still possible to define the low paths (see Fig. 3). Clearly, however, these low paths run high with respect to $-T_0$, which is now in the Re $z > 0$ plane. If in the formula we now replace k by $ke^{-i\pi}$, we see that the second formula has been replaced by the first. Conversely, if we let arg T_0 decrease toward $-\pi/2$, we find that the first formula is now meaningless and that the second one takes its place. An important complement to these formulas is that p(z) is made single-valued in Re $z > 0$ by cutting the z plane with a cut which joins T to the origin. On the opposite sides of this cut p(z) takes opposite values.

Let us now evaluate the asymptotic formula for the S function. We insert the expressions (10.2) into (9.10) and use (9.7). Thus we obtain

$$S(\lambda, k) \sim \exp\left(-i\left(\int_{0,\Gamma_\ell}^{\infty} + \int_{0,\Gamma_h}^{\infty}\right)[p_0(z) - p(z)]dz\right). \tag{10.3}$$

It is obvious that the sum of a high and a low integral can be reduced to a single complex integral which comes from infinity, passes across the cut of p(z) and goes back to infinity on the other sheet of the function p(z) after having encircled the point T. After this has been understood, it is clear that the WKB formula for the phase shift is just the one we already know from more elementary treatments:

$$\delta(\lambda, k) \sim -\int_{T\approx T_0}^{\infty}[p_0(z) - p(z)]dz. \tag{10.4}$$

The domain of validity of this formula is the intersection of the validity domain of the formulas (10.2); that is, Re $T_0 > 0$ (see Fig. 4). There is no

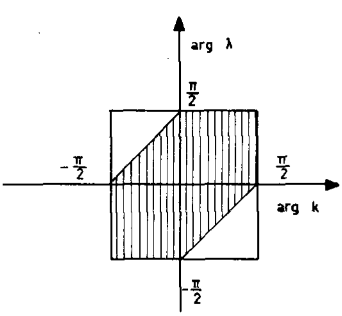

Fig. 4

The regions of validity of the WKB method (shaded) and of the bounds (10.5) (unshaded) point anyway in trying to use (10.4) when Re $T_0 < 0$, because the corresponding integral is in general meaningless (it implies the knowledge of V(z) when z is equal or at least very close to T, but V(z) is defined only when Re $z > 0$). From this formula it is apparent that $\lim S(\lambda, k) = 1$ when $|\lambda|, |k| \to \infty$, under the quoted restrictions. If Re $T < 0$, we have no proof of the validity of the above limit and we actually consider it not to be true. For our discussion it is necessary to know some upper bound on $S(\lambda, k)$. These bounds are derived in a paper by BOTTINO, LONGONI and REGGE [16]. They refer to the behaviour when λ is large and k is constant. (If λ is kept constant and k is large, $\lim S(\lambda, k) = 1$.) The desired bounds are

$$|S(\lambda, k)| < \mu(\lambda, k)\, e^{2(\arg T_0 + \pi/2)\mathrm{Im}\,\lambda}, \quad \arg T_0 < -\pi/2, \ \mathrm{Im}\,\lambda < 0$$

$$|S(\lambda, k)^{-1}| < \mu(\lambda, k) e^{2(\arg T_0 - \pi/2)\mathrm{Im}\,\lambda}, \quad \mathrm{Im}\,\lambda > 0, \ \arg T_0 > \pi/2. \tag{10.5}$$

The indicated domains of validity of these two bounds are the two unshaded regions in Fig. 4. $\mu(\lambda, k)$ is here a function which is bounded above by a constant independent of k and λ. Both bounds are equivalent to each other through the use of unitarity.

In using (10.4), one must always be aware that there is an error associated with it. If δ vanishes very rapidly with λ, the above formula becomes meaningless, because it can easily happen that the error, although small, is still larger than δ. The usefulness of the WKB method here is that it yields a proof that δ vanishes for large λ whenever Re $T_0 > 0$. This is already enough to obtain results concerning the analyticity in the variables $s = k^2$ and t (momentum transfer). Besides these asymptotic evaluations, we want to quote a more precise result which states that for large angular momenta the Born approximation (see (9.11)),

$$\delta \sim \delta_B = -k e^{-i\pi(\lambda - 1/2)} \int_0^\infty V(z) \left(\frac{\phi_0(\lambda, k, z)}{f_0(\lambda, k)} \right)^2 dz, \tag{10.6}$$

is a very reliable one. The reason for this is that the wave function for large λ lies totally outside the potential and is practically unaffected by it. Therefore, $\phi \sim \phi_0$ for large λ. This can be shown more exactly from the WKB analysis. For z fixed we find $\lim(\phi/\phi_0) = 1$. We would have to prove uniform convergence in order to derive $\lim(\delta/\delta_B) = 1$.

We do not want to cram the paper with an uninteresting proof*.

It is well known that the Born formula can be integrated for the class of Yukawa potentials and yields

$$\delta_B = -\frac{1}{2k} \int_m^\infty Q_{\lambda - \frac{1}{2}}(1 + \mu^2/2k^2)\, \sigma(\mu) d\mu.$$

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* In [3] it is stated that an equivalent rigorous proof has been obtained by D.S. Carter (Princeton thesis), but unfortunately this proof has not been published. Of course, a proof follows from the three-dimensional formalism and from the existence of the small Lehmann ellipse.

If λ is large, the asymptotic behaviour of δ is

$$\delta \sim -(\pi/2)^{1/2}\frac{\sigma(m)}{2m}\,k(\sin h\,a)^{1/2}\frac{e^{-a\lambda}}{\lambda^{3/2}};\ \cos h\,a = 1 + m^2/2k^2. \tag{10.7}$$

The standard WKB method yields instead

$$\delta = 0(e^{-m\lambda/k}).$$

The last evaluation is for our purposes too optimistic at low energies but becomes reliable at large energies.

11. THE POLES OF $S(\lambda, k)$

Earlier analysis of the poles of $S(\lambda, k)$ have been carried out in the following cases:

(1) λ physical, k complex. The current names given to these poles are
   - (a) bound states if $k = i\eta$ (η real > 0),
   - (b) anti-bound states or virtual states if $k = -i\eta$,
   - (c) resonances if Im $k < 0$.

The resonances occur in pairs of conjugate poles. Except for bound states, the region Im $k > 0$ is forbidden to poles. It is evident from the existing literature that the anti-bound states and metastable states (resonances) are not states in the accepted frame of definition of quantum mechanics because their wave functions are not square integrable. However, they share many of the properties of ordinary states.

(2) k physical, λ complex. The poles occur only when $\lambda > 0$. They have been named shadow states in [11] *. In the full complex domain of k and λ shadow states and resonances are particular intersections of the same singular surface of $S(\lambda, k)$. For we remember that analytic functions of two variables are never singular on isolated points but always on analytic surfaces (of dimension 2). In [11] a number of inequalities was derived concerning the distribution of the shadow states.

The discussion will now be extended to complex λ and k. Roughly speaking, there are two kinds of limitations on the position of the poles: the first follows from the equation of continuity and applies equally well, under very weak conditions, to any kind of potential; the second uses special properties of V(x) like limitations on the depth and width of V(x) and analyticity.

The continuity equation can be used as follows: We suppose that, for a particular set of values of λ and k, $\lambda = \lambda_0$, $k = k_0$, inside its meromorphy domain, $S(\lambda, k)$ has a simple pole. Then clearly $f(\lambda_0, k_0 e^{-i\pi}) = 0$. Under this hypothesis,

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* Recently we have found a paper [13] where the name "spiralling states" has been adopted.

$$\phi(\lambda_0, k_0, z) = \frac{f(\lambda_0, k_0)}{2ik_0} f(\lambda_0, k_0 e^{-i\pi}, z). \tag{11.1}$$

If Re $\lambda_0 > 0$ and Im $k_0 > 0$, the above function vanishes as a function of real z at zero and at infinity. Its complex conjugate ϕ^* will also vanish in the same points; ϕ^* satisfies the conjugate equation (z real):

$$\phi^{*''} + k_0^{*2}\phi^* - \frac{\lambda_0^{*2} - 1/4}{z^2}\phi^* - V(z)\phi^* = 0. \tag{11.2}$$

It follows that:

$$(\phi'\phi^* - \phi^{*'}\phi)' = (k_0^{*2} - k_0^2)|\phi|^2 - (\lambda_0^{*2} - \lambda_0^2)\frac{|\phi|^2}{z^2}. \tag{11.3}$$

This identity can be integrated from zero to infinity. The contribution of the first term vanishes with ϕ and ϕ^* at both ends. What is left yields the equation,

$$\text{Im } k_0 \text{ Re } k_0 \int_0^\infty |\phi|^2 dz - \text{Im } \lambda_0 \text{ Re } \lambda_0 \int_0^\infty \frac{|\phi|^2}{z^2} dz = 0. \tag{11.4}$$

From (11.4) it is clear that, where Re k_0 and Im λ_0 have opposite signs, poles do not occur. Therefore we obtain two domains of holomorphy of

$$\begin{cases} \text{Re } k_0 > 0 \\ \text{Im } \lambda_0 < 0 \end{cases} \qquad \begin{cases} \text{Re } k_0 < 0 \\ \text{Im } \lambda_0 > 0 \end{cases} \tag{11.5}$$

having a common boundary where Re $k_0 = 0$, Im $\lambda_0 = 0$.

A complete discussion of domain of analyticity beyond what is stated in (11.5) is contained in a paper by BOTTINO and LONGONI [17]. A preliminary discussion can be found in [11]. We just notice that, while (11.5) holds for any of the potentials considered by us, any other inequalities will contain some more detailed information on V(r). Particularly interesting are the upper bounds on Reλ_0 when k is real, because they insure a finite number of subtractions in the scattering amplitude. If, for instance,

$$|V(i\gamma)| < \frac{M}{|\gamma|}, \text{ then Re } \lambda < \frac{M}{k}.$$

12. THE TOTAL AMPLITUDE AND THE LEHMANN ELLIPSE

We have recalled so far a number of properties of the partial wave amplitudes. The next task is to relate them to the properties of the total scattering amplitude. After Mandelstam's work it has become fashionable to use the notations s = E and $t = -\Delta^2 = -2E\,(- \cos\theta)$. We define f(s, t) through (1.9) or the equivalent transforms.

The property of the total amplitude which we shall discuss is the existence of the so-called small Lehmann ellipse.

The mathematical theory of Legendre polynomials teaches us that any expansion in these polynomials;

$$F(\cos\theta) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta), \tag{12.1}$$

converges in the cos θ plane within an ellipse of foci ± 1. It may happen that the ellipse of convergence reduces to the segment joining -1 to 1. It always happens that the function represented by (12.1) is analytic within the convergence region. This is quite analogous to the corresponding theorem for power series where we have circles instead of ellipses. The magnitude of the ellipse of convergence must be such that the sum of the expansion does not have singularities inside the ellipse. Therefore the singularities which are nearest to the foci are those which dominate the convergence. Without an attempt to make our arguments rigorous but only the suggestion that they are reasonable, all the above results can be understood from the asymptotic behaviour of P_ℓ (cos θ) when ℓ is large and fixed. This behaviour is of the kind

$$P_\ell(\cos\theta) \sim \left(\frac{2}{\pi \ell \sin\theta}\right)^{1/2} \cos[(\ell + 1/2)\theta - \pi/4]. \tag{12.2}$$

If cos θ is complex and ℓ is large and real, P_0 (cos θ) will be dominated by Im θ. P_ℓ (cos θ) is therefore always exploding for high real ℓ, unless θ is real, in which case it is oscillating.

If we consider the expansion (12.1) and we suppose it to be convergent for a given value of θ, it follows that the general term of it must vanish for large ℓ:

$$\lim_{\ell\to\infty} a_\ell e^{\ell|\operatorname{Im}\theta|} = 0, \text{ or } a_\ell < Ce^{-\ell|\operatorname{Im}\theta|}. \tag{12.3}$$

The general term is therefore dominated by a decreasing geometric progression. Clearly the expansion also converges for smaller values of Im θ, and it represents there an analytic function because it is a uniform convergent series of analytic functions.

In the cos θ plane the curve Im θ = const. is an ellipse. Suppose namely that $z = \cos\theta = x + iy$ and $\theta = \sigma + i\mu$. We have

$$\begin{aligned} x &= \cos\sigma\cosh\mu, \\ y &= -\sin\sigma\sinh\mu. \end{aligned} \tag{12.4}$$

From these equations we deduce easily

$$(x^2/\cosh^2\mu) + (y^2/\sinh^2\mu) = 1; \quad (x^2/\cos^2\sigma) - (y^2/\sin^2\sigma) = 1. \tag{12.5}$$

The first of these equations does not depend on σ and represents the locus of all points in the z plane which have the same Im $\theta = \mu$. This locus is evidently an ellipse. The other equation is the locus of the points where Re $\theta = \sigma$ = const.

This locus is obviously a hyperbola with foci ± 1. The sets of ellipses and hyperbolas are mutually orthogonal. The hyperbola which corresponds to $\sigma = 0$ degenerates into the upper and lower limit of the cos $\theta > 1$, the one with $\sigma = \pi$ into the line cos $\theta < -1$. Any value of σ between these extremes corresponds to half a hyperbola; the other half obviously comes from $\pi - \sigma$. The whole z plane can be mapped into the strip $0 < \sigma < \pi$ of the θ plane. However, it is better to map it into $-\pi < \sigma < \pi$ and $\mu > 0$. A given value of σ is then associated with a quarter of a hyperbola. By taking all the combinations $\pm\sigma$ and $\pm\pi \pm\sigma$ within the interval $(-\pi, \pi)$, we get all quarters of the hyperbola. The line μ = const. is then a full ellipse. This kind of mapping is very similar to the usual polar co-ordinates where μ plays the role of a radius and σ the role of the polar angle. We prefer this mapping also because it is the natural one when we want the asymptotic behaviour of the Legendre functions when the index ℓ is large. As long as ℓ remains an integer, there is no doubt about the meaning of (12.2) because it is unessential which determination we take of Re $\theta = \sigma + n\pi$ when cos $\theta = z$ is given. But if ℓ is no longer an integer, we are forced to specify the value of n. This turns out to be the one of our mapping. This fact is very important when used with Watson's integral.

The size of the ellipse can now be estimated for large ℓ with the help of (10.7) and (12.3). The partial wave expansion clearly converges if Im$\theta = \mu < a$, where $\cosh a = 1 + m^2/2k^2$. We refer to the ellipse $\mu = a$ as to the small Lehmann ellipse.

The term large Lehmann ellipse is commonly used instead for the analytic continuation of the imaginary part of f(s, t). We define it in the physical region as

$$F(s, t) = \text{Im } f(s, t). \tag{12.6}$$

We take s real and $0 < -t < 4s$. We consider then the analytic continuation of F(s, t) when s is kept fixed and t is complex. People refer to F(s, t) somewhat improperly as the imaginary part of f(s, t), but this is true only under the stated conditions. The partial wave expansion of F(s, t) is then

$$F(s,t) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2\delta_\ell (k) P_\ell (\cos\theta). \tag{12.7}$$

This expansion converges in an ellipse which is larger than the small Lehmann ellipse, because the general term contains $\sin^2 \delta$ and vanishes more rapidly. This new ellipse is given by $\mu = 2a$. This fact could have been deduced from unitarity directly if the corresponding result for f(s, t) were known, without passing through the partial wave expansion for F(s, t).

13. ANALYTICITY IN t FOR FIXED s

In this section we want to explore with new techniques the full domain of analyticity of f(s, t) in the t plane. We already know of the existence of Lehmann s ellipse, but we must go much further in order to prove the analogue of the Mandelstam representation for potential scattering. For the sake of simplicity we work on the assumption that for real positive k

$$\lim_{\lambda\to\infty} [S(\lambda, k) - 1] = 0 \tag{13.1}$$

in every direction of the Re $\lambda > 0$ half plane, including (and this is really an additional hypothesis) the imaginary axis of λ. We know that for Yukawian potentials the result holds in any direction within the above region; we know also that little can be said when $\lambda = ia$. The proof which follows could be carried out without this additional hypothesis, but there is nothing interesting to be gained and the formal machinery would be much more complicated. Under this simplification we apply Watson's transform and we obtain the formula:

$$f(s,t) = -\frac{1}{2k}\int_{-i\infty}^{i\infty} \frac{e^{2i\delta(\lambda, k)} - 1}{\cos\pi\lambda} P_{\lambda - 1/2} (-\cos\theta)\lambda d\lambda$$

$$+ i\frac{\pi}{k}\sum_n S_n P_{\ell_n} (-\cos\theta) \frac{2\ell_n + 1}{\sin\pi\ell_n}. \tag{13.2}$$

The path C has now been deformed into the line $\lambda = ia$. The extra terms arise from the poles of $S(\lambda, k)$ which we know to exist in the upper half-plane of λ only. We examine, separately, the contributions of the integral and of the poles. S_n is the residue of $S(\lambda, k)$ at the pole $\lambda = \ell_n + 1/2$. The convergence of the integral is now determined uniquely by $\cos\theta$. If $\lambda = ia$ is large, we have

$$\left|\frac{1}{\cos \pi\lambda}\right| \sim 2\, e^{-\pi|a|};\ P_{\lambda-1/2}(\cos\theta) \sim 0(e^{|a||\pi-\sigma|}). \qquad (13.3)$$

The integral therefore converges if $|\pi - \sigma| < \pi$. The asymptotic behaviour of $P_{\lambda-1/2}(-\cos\theta) = P_{\lambda-1/2}[\cos(\pi-\theta)]$ for large λ has been evaluated by keeping the condition $|\pi - \sigma| < \pi$ in accordance with the discussion of section 12, so that $0 < \sigma < 2\pi$ is the range of σ. This includes the whole z plane with the cut z real > 1. The terms $P_{\ell_n}(-\cos\theta)$ have the same cut. The cut in the z plane actually starts outside the small Lehmann ellipse at the point $z = 1 + m^2/2k^2$. (This concerns the cut of f(s, t) which also includes the contribution of the poles.) In the t plane this cut is mapped into the cut:

$$m^2 < t < \infty. \qquad (13.4)$$

This is actually the full result to be expected from the Mandelstam representation. Our discussion obviously holds also when $[s(\lambda, k) - 1]$ does not vanish along $\lambda = ia$ but grows at most like a power of a.

What about the behaviour when t or z is large? The usual partial wave expansion is really unsuitable, because it breaks down long before we need to use it and anyway its accuracy decreases with μ. Eq. (13.2) can still be used and yields the interesting result that this behaviour is actually controlled by the poles of $s(\lambda, k)$. Indeed if we now consider $P_{\lambda-1/2}(-\cos\theta)$ when λ is fixed and $\cos\theta$ is now variable and large, we find

$$P_{\lambda-1/2}(-\cos\theta) \sim 0(z^{\lambda-1/2}). \qquad (13.5)$$

This term is growing provided Re $\lambda > 1/2$ and is at the same time oscillating if λ is complex, as expected. If z is very large, then what counts is the pole with the larger Re λ. What about the integral? This is easily disposed of because it is the superposition of decreasing terms with strongly oscillating factors when $|a| = |\lambda|$ is large. We expect it to vanish for large z. Concluding, we are led to the behaviour:

$$f(s, t) \sim 0(t^{a(s)}), \qquad (13.6)$$

where $a(s) = \ell_>(s)$, $\ell_>(s)$ being the ℓ_n with the largest real part. This behaviour is energy-dependent.

What is the physical interpretation of these poles? We expect a(s) to be an analytic function of s in some region which we do not need to specify now in detail. We suppose such a pole to exist for $s = s_0$ with a small Im λ and Re λ almost half-integral (physical). This means that for some value s_0 of s

$$a(s_0) = \ell + \epsilon(s_0) + i\eta(s_0);\ \epsilon \ll \ell,\ \eta \ll \ell. \qquad (13.7)$$

If we now exploit the fact that a(s) is analytic in s in a sufficiently large region around s_0, we can expand a(s) in a power series in $s - s_0$:

$$a(s) = \ell + \epsilon(s_0) + i\,\eta(s_0) + (s - s_0)\left.\frac{da}{ds}\right|_{s = s_0} + \ldots\ldots \qquad (13.8)$$

We can choose however s equal to

$$s = s_0 - \left(\epsilon(s_0) + i\eta(s_0)\right)\left(\left.\frac{ds}{da}\right|_{s = s_0}\right)^{-1.} \qquad (13.9)$$

in order to make a(s) = 1. It is clear now that, if there is a shadow pole, we expect a pole to appear when ℓ is integer and s is almost real, this pole being the same complex singularity in the variables λ and k (or s) intersecting the many-fold $\lambda - 1/2$ = integer. This pole can only be interpreted as a resonance according to the discussion of section 4 or section 12. Resonances are therefore responsible for the high t behaviour of f(s, t).

In [11] quite a number of inequalities has been derived for a(s) for a large class of potentials, including the pure Yukawa potential. We wish to point out that it is not at all impossible to choose potentials such that there is an infinite set of shadow poles and, even worse, such that there is no upper bound on Re ℓ_n. f(s, t) in this case shows an extremely complex behaviour for large t, and one needs an infinite number of subtractions in order to write the Mandelstam representation. It is a good feature that we can rule out this trouble for the most interesting potentials, i.e. those we can form by choosing for $\sigma(\mu)$ in (5.1) a distribution with no higher singularities than Dirac's functions (positively no derivatives of it).

14. THE RESULTS OF KHURI

In the previous section we have investigated the analytic properties of f(s, t) when s was held fixed and t was varying. A more difficult task in our formalism is to prove analytic properties in s when t is fixed. We now keep t fixed and real negative. None of the previously proposed representations for f(s, t) seems to be working now because they all diverge. We now use instead

$$f(s, t) = \frac{1}{2k}\int_C \frac{e^{-i\pi(\lambda + 1/2)}}{\cos \pi\lambda}[\,S(\lambda, k) - 1]\,P_{\lambda - 1/2}(\cos\theta)\lambda\,d\lambda. \qquad (14.1)$$

The integration path C is the same as in Fig. 1. The validity of (14.1) can be first of all proved when $z < 1$ or inside the small Lehmann ellipse. In particular, if $-1 < z < 1$, then t is negative: $0 < -t < 4s$.

Here the WKB method holds: we have $S(\lambda, k) - 1 \to 0$ when $\lambda \to \infty$. Secondly, $P_{\lambda-1/2}(\cos\theta) \sim 0(e^{\pm i\lambda\theta})$, whichever choice is larger. If

$$\operatorname{Im}\lambda \to +\infty,\ \frac{e^{-i\pi(\lambda+1/2)}}{\cos\pi\lambda} \to -2i \text{ and if } \operatorname{Im}\lambda \to -\infty,\ \left|\frac{e^{-\pi(\lambda+1/2)}}{\cos\pi\lambda}\right| \to e^{-2|\operatorname{Im}\lambda|\pi}$$

In this last case the above factor provides a strong cut-off which makes the integral easily converging for $\operatorname{Im}\lambda \to -\infty$. If $\operatorname{Im}\lambda < 0$, we can move the path C along the lower imaginary axis of λ. In so doing, even if $P_{\lambda-1/2}(\cos\theta)$ now diverges like $e^{|\operatorname{Im}\lambda||\sigma|}$, we still have convergence since $|\sigma| < \pi$.

We now move k into the domain $R_e\, k > 0$, $\operatorname{Im} k > 0$. Now the WKB formula breaks down for $\lambda = ia$, $a \to -\infty$, but there we have no trouble since by the formula (10.5) $S(\lambda, k) - 1$ is bounded in this domain by $e^{|\operatorname{Im}\lambda|\pi}$. When $\lambda \to +\infty$, we have to be careful. The factors here which decide the convergence are $P_{\lambda-1/2}(\cos\theta) \sim 0(e^{\pm i\lambda\theta})$ and $s - 1 \to 0$. Recalling now that $\theta = 1 + t/2s$ and that t is real and < 0, we see that, if s is complex, then $\cos\theta$ and θ are also complex. We expect $e^{\pm i\lambda\theta}$ to diverge in any direction of the λ plane with the sole exception of $\arg\lambda = n\pi - \arg\theta$ where n is integer. Is it possible to choose $\arg\lambda$ in $0 < \arg\lambda < \pi/2$ such that this happens? The answer is yes because, when k is moved from the real axis to the imaginary axis, σ and μ vary in the range $\pi < \sigma < 0$, $\mu > 0$. Arg θ is therefore always in the range $\pi/2 < \arg\theta < \pi$. We get the desired result by taking $\arg\lambda = \pi - \arg\theta$. Our integral representation is convergent in the upper quadrant $\operatorname{Re} k > 0$. If $\operatorname{Re} k < 0$, we simply use the fact that, if k is real and t real negative, then $f^*(s + i\epsilon, t) = f(s - i\epsilon, t)$ so that by analytic continuation we have in the whole cut s plane $f^*(s, t) = f(s^*, t)$. This cut plane maps into the upper half plane of k. This equality is quite adequate for definition of an analytic continuation of $f(s, t)$ in the quadrant $\operatorname{Re} k < 0$, $\operatorname{Im} k > 0$.

We are left with the points of the imaginary k axis (negative s axis). Here apparently a new singularity appears, which is not caused by any failure of (14.1) to converge but rather by the fact that $S(\lambda, k)$ has singularities along the imaginary axis of k. However, when we are close to the imaginary axis of k, the WKB formula holds along $\lambda = ia$, $a > 0$. We can deform C into the imaginary axis of the λ plane, because λ and k are imaginary and the integral converges. Now,

$$\frac{\lambda}{\cos\pi\lambda} P_{\lambda-1/2}(\cos\theta)$$

is an odd function of λ, and therefore what counts in the integral is only the odd part of $e^{-i\pi(\lambda+1/2)}[S(\lambda, k) - 1]$. But if we use identity (9.18), this odd part can be written as

$$\sin\pi\lambda\left(1 - \frac{f_0(\lambda, -k)\, f_0(-\lambda, -k)}{f(\lambda, k)\, f(-\lambda, -k)}\right). \tag{14.2}$$

Upon substitution into (14.1) we find

$$f(s,t) = \frac{1}{2k}\int_{-i\infty}^{i\infty}\left(1 - \frac{f_0(\lambda,-k)\,f_0(-\lambda,-k)}{f(\lambda,-k)\,f(-\lambda,-k)}\right) \mathrm{tg}\,\pi\lambda P_{\lambda-1/2}(\cos\theta)\lambda d\lambda \quad (14.3)$$

+ the contribution of poles. But now the function (14.2) is analytic in the whole upper half k-plane, and there is no discontinuity associated with $S(\lambda, k)$ on the dynamical cut $k = ib$, $m/2 < b < \infty$. This happens because $e^{i\pi\lambda} S(\lambda, k)$ and $e^{-i\pi\lambda} S(-\lambda, k)$ have the same discontinuity and when the odd part is taken, it disappears. (14.3) can therefore be used in defining $f(s,t)$ in a region containing the imaginary axis of k. We have now joined the right and left part of $\mathrm{Im}\, k > 0$, because the $f(s,t)$ defined in (14.3) clearly satisfies $f^*(s,t) = f(s^*, t)$. Indeed,

$$f^*(s,t) = \frac{1}{2k^*}\int_{-i\infty}^{i\infty}\left(1 - \frac{f_0(\lambda,-k)\,f_0(-\lambda,-k)}{f(\lambda,-k)\,f(-\lambda,-k)}\right)^* \mathrm{tg}\,\pi\lambda^* P_{\lambda^*-1/2}\left(1 + \frac{t}{2s^*}\right)\lambda^* d\lambda^*$$

+ (the contribution of poles)*. But $\lambda^* = -\lambda$, $P_{-\lambda-1/2}(\cos\theta) = P_{\lambda-1/2}(\cos\theta)$ and $f(\lambda,-k)^* = f(-\lambda, k^*)$ so that $f^*(S(k), t) = f(S(-k^*), t) = f(s^*, t)$. Clearly k and $-k^*$ are both in the upper half-plane. Formula (14.3) therefore defines an analytic function of s in the neighbourhood of the real negative axis of s (apart from the contribution of the poles, which we shall discuss later). For we notice that according to the WKB formula

$$1 - \frac{f_0(\lambda,-k)\,f_0(-\lambda,-k)}{f(\lambda,-k)\,f(-\lambda,-k)}$$

decreases exponentially for large λ. This is necessary in order to have analyticity in a neighbourbood of the imaginary axis of the k plane rather than convergence on a line only. The actual size and form of this domain is unimportant once we have the full analyticity domain.

We now give some approximate argument about the behaviour of $f(s,t)$ when t is held fixed and negative and $|s| \to \infty$ in the cut s plane which maps into the upper half k-plane. We use the WKB formula for $f(\lambda, k)$ and eq.(14.3). We put

$$\cos\theta = 1 - \Delta^2/2k^2,\ k = i\xi, \lambda = i\eta, t = -\Delta^2, \quad (14.4)$$

and we obtain

$$f(s,t) = -\frac{1}{\xi}\int_0^\infty \left(1 - \frac{f_0(i_\eta, -i\xi)\, f_0(-i\eta, -i\xi)}{f(i\eta, -i\xi)\, f(-i\eta, -i\xi)}\right) \operatorname{tgh}(\pi\eta) P_{i\eta - 1/2}\left(1 + \frac{\Delta^2}{2\xi^2}\right)\eta d\eta.$$

Using the formula $P_\ell(\cos\theta) \approx J_0[(\ell + 1/2)\theta] = J_0(\lambda\theta)$, which is valid for large ℓ, $\theta \ll 1$, and taking into account that $\cos\theta \approx 1-\theta^2/2$, we have

$$P_{i\eta-1/2}(1 + \Delta^2/2\xi^2) \approx J_0(T_0\Delta), \quad T_0 = \lambda/k.$$

The WKB formulas (10.2) give us

$$\frac{f_0(\lambda, -k)\, f_0(-\lambda, -k)}{f(\lambda, -k)\, f(-\lambda, -k)} \sim e^{-2i\int_{T_0}^{\infty}(p_0 - p)dz}.$$

If λ, k are large, we deduce approximately

$$1 - e^{2i\int_{T_0}^{\infty}(p-p_0)\,dz} \sim \frac{1}{\xi}\int_{T_0}^\infty \frac{V(z)dz}{(1-T_0^2/z^2)^{1/2}}.$$

It follows that

$$f(s,t) \sim \int_0^\infty T\,dT J_0(T\Delta) \int_T^\infty \frac{V(z)dz}{\left(1 - T^2/z^2\right)^{1/2}} \int_0^\infty dz\; V(z) \int_0^z \frac{TJ_0(T\Delta)dT}{\left(1 - T^2/z^2\right)^{1/2}}.$$

Putting $T = z\sin\phi$, $dT = \cos\phi d\phi$, we obtain [14]

$$\int_0^z \frac{TJ_0(T\Delta)}{\left(1 - T^2/z^2\right)^{1/2}}\, dT = z^2 \int_0^{\pi/2} J_0(z\Delta\sin\phi)\sin\phi d\phi = \sin(z\Delta).$$

Finally we get the Born approximation:

$$f(s,t) \sim \frac{1}{\Delta}\int_0^\infty z\sin(z\Delta)\, V(z)dz.$$

This result is independent of s and can be obtained directly from KHURI's approach [2]. We frankly admit that the above argument is not rigorous. However, there is no point in being choosy about it, because rigorous proofs exist abundantly and whoever wants them has only to look for them in the quoted literature. Here we show it just for completeness.

What about the contribution of the poles of $S(\lambda, k)$ in the formula (14.3)? They give extra contributions to $f(s,t)$ of the sort;

$$\sum_n \frac{C_n(s)}{\sin \pi \ell_n(s)} P_{\ell_n}(\cos\theta)\, e^{i\pi \ell_n(s)} ,$$

where $C_n(s)$ are some s-dependent constants. This contribution has a singularity when some of $\ell_n(s)$ become integral. This happens on the upper imaginary k axis when Im $k > 0$, according to our general discussion in sections 4 and 6, and these poles represent bound states. $f(s,t)$ is therefore analytic in Im $k > 0$ with the exception of a finite number of bound state poles. All these properties can be condensed into the single formula:

$$f(s,t) = f(t) + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, f(s',t)}{s'-s}\, ds' + \sum_n \frac{C_n(t)}{s-s_n},$$

where $-s_n > 0$ are the binding energies of the bound states, $f(t)$ is the Born approximation. $C_n(t)$ are polynomials in t. This result is due to KHURI [2].

15. EXTENSIONS AND GENERALIZATION OF THE THEORY OF COMPLEX ANGULAR MOMENTA

A number of papers dealing with an interesting generalization and application of the idea of complex angular moments has appeared since the first draft of these notes was first published. Remaining in the frame of potential scattering, one has tried to do away with potentials bounded by a power $A/r^{2-\epsilon}$, $\epsilon > 0$ in the neighbourhood of the origin. In particular, one has allowed $V(r)$ to have a strong repulsive core at small r. As is well known, attractive cores require very disturbing boundary conditions, and it is generally agreed that, if anything can be called physics in the frame of potential scattering, this has nothing to do with attractive cores, which produce systems where there are for instance no ground states but there are states of arbitrarily low energies.

With repulsive cores, however, FIVEL and others [18, 19, 20, 21] have shown that a peculiar fact occurs in the angular momentum plane, that is, that the scattering amplitude can be continued in the Re $\lambda < 0$ plane by virtue of a simple reflection property:

$$e^{-i\pi\lambda} S(\lambda, k) = e^{i\pi\lambda} S(-\lambda, k).$$

This property formally follows from (9. 18) when the Jost function is allowed to be infinite. As a matter of fact, this is in a way to be expected; because, if we try to calculate f(λ, k) with the usual perturbation expansion, we find diverging integrals. The analyticity of (15. 1) makes it natural to ask whether we can postulate it in field theory. So far we have no evidence either in favour of or against it apart from its logical simplicity.

Other work has been carried out on the many channel problems, mainly by CHARAP and SQUIRES [21, 22]. They show that, as far as we are concerned with angular momentum properties, all previous results extend in a straightforward manner. Particularly interesting, however, is the extension of Clebsch-Gordan coefficients for the composition of angular momenta to complex values of the indices. I feel that we shall hear more of these properties in the future as soon as the necessity of studying more complicated systems urges us. In fact, just the interaction of a resonance with an elementary particle (if there are any) or with another resonance is already confronting us with such a problem. They also produce some results on the wave functions of the symmetrical top, and this is natural because they adopt in their second paper the helicity formalism of Jacob and Wick. Incidentally, properties of the many channel amplitudes as functions of the energy and transmitted momentum were discussed in [23].

Particularly interesting in regard to its immediate application to field theory is the so-called factorization theorem for the many channel problem. This theorem was first suggested by Gell-Mann and proved by Charap-Squires.

It states, that barring accidental degeneracy, the residuum of the scattering amplitude matrix at a pole in the angular momentum is a matrix $\Omega_{\alpha\beta}$ of characteristics zero; that is, all minors of the determinant of the matrix vanish. This implies that $\Omega_{\alpha\beta}$ factorizes as

$$\Omega_{\alpha,\beta} = A_\alpha B_\beta,$$

where α, β label the channels. This of course happens for resonances in the energy variable.

Another type of problem which has excited the phantasy of many, me included, is how to continue the amplitude for Re $\lambda < 0$. My personal philosophy is in favour of course of the symmetry (15.1), but there are some who would like to see what happens for ordinary potentials. Well, this problem has been completely solved by two papers by Froissart and Mandelstam. Froissart solves it for all potentials, and he finds indeed a lot of singularities; in particular, there are singularities about any time the analytic continuation of the Mellin transform of V;

$$M(\lambda) = \int_0^i r^{2\lambda} V(r) dr, \tag{15. 2}$$

is singular in λ. There are other sources of singularities, but we stick to (15. 2) just to exemplify. Clearly we can produce almost anything by a judicious choice of $V(r)$, including a natural boundary of Re $\lambda = 0$. Moreover, small variations in V do not correspond to small variations in $M(\lambda)$, and in fact $M(\lambda)$ is completely unstable in Re $\lambda < 0$. So no definite V-independent conclusion can be deduced from this analysis. Mandelstam solves the Yukawa potentials in a very elegant way, which is used later by Lovelace in order to carry out numerical calculations on the trajectories, that is, on the function $\lambda_0(s)$. The Mandelstam method reduces to the time-honoured Schroedinger method of solving the hydrogen atom where the Yukawa potential reduces to a Coulomb potential.

Numerical calculations have been performed in large amounts, but unfortunately much effort has been wasted in calculating trajectories for negative Re λ, where, as stated, their physical interpretation is doubtful and where in fact they do crazy things. These calculations show a definite pattern in Re $\lambda > 0$ which can be sketched as follows: We know that for negative real energies the trajectories lie on the real axis and move forward with increasing energies. Where E = 0, the pole leaves the real axis forward if in that point $\lambda > \frac{1}{2}$, at $\frac{\pi}{2}$ angle if $\lambda = \frac{1}{2}$ (s waves) and backwards if $\lambda < \frac{1}{2}$. The pole then eventually swings backwards into the Re $\lambda < 0$ region.

If we let the range m^{-1} of the Yukawa potential grow to infinity, that is, we carry out the transition to Coulomb potential, the pole leaves the real axis at very large angular momenta. Therefore, it crosses the integer values several times, and many bound states arise. The swing-back loop is then very large, and in the limit $m = 0$ it plunges into infinity. We have then an infinite number of bound states.

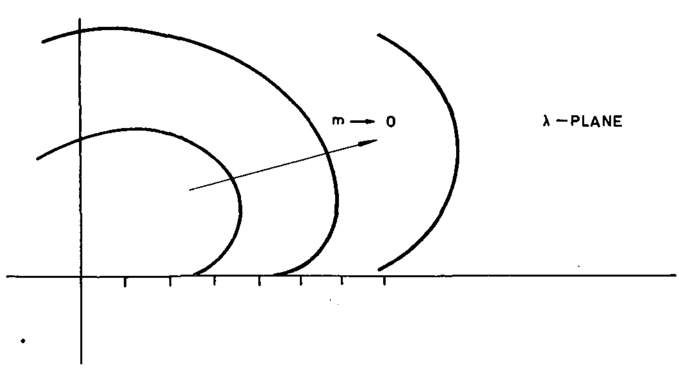

Fig. 5

Swing back loop

APPENDIX I

In this appendix we deduce all the integral equations appearing in these lecture notes. The scheme by which they can be derived is summarized in Table I.

TABLE I

D	$\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2}$	$\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2} + k^2$	$\frac{d^2}{dz^2} + k^2$	$\frac{d^2}{dz^2} - \frac{\lambda^2 - 1/4}{z^2} + k^2$
h	$V(z) - k^2$	$V(z)$	$V(z) + \frac{\lambda^2 - 1/4}{z^2}$	$V(z)$
Behaviour of ψ at $z \to z_1 = 0$	$z^{\lambda+1/2}$	$z^{\lambda+1/2}$		
Behaviour of ψ at $z \to z_1 = \infty$			e^{-ikz}	e^{-ikz}
ψ_1	$z^{\lambda+1/2}$	$(2/k)^{\lambda}\Gamma(\lambda+1) \times z^{1/2} J_\lambda (kz)$	e^{ikz}	$e^{i\pi(\lambda+1/2)/2} \times \left(\frac{\pi k z}{2}\right)^{1/2} H_\lambda^{(1)} (kz)$
ψ_2	$z^{-\lambda+1/2}$	$(2/k)^{-\lambda}\Gamma(-\lambda+1) \times z^{1/2} J\ (kz)$	e^{-ikz}	$e^{-i\pi(\lambda+1/2)/2} \times \left(\frac{\pi k z}{2}\right)^{1/2} H_\lambda^{(2)} (kz)$
$W[\psi_1, \psi_2]$	-2λ	-2λ	$-2ik$	$-2ik$

Let us consider a differential equation of this kind:

$$D(\lambda, k, z)\psi(\lambda, k, z) \equiv \left[\frac{d^2}{dz^2} + g(\lambda, k, z)\right]\psi(\lambda, k, z) = h(\lambda, k, z)\ \psi(\lambda, k, z). \tag{A1.1}$$

As is well known, the integral equation equivalent to (A 1.1) is

$$\psi(\lambda,k,z) = \lim_{z \to z_1} \psi(\lambda,k,z) + \frac{1}{W[\psi_1,\psi_2]} \int_{z_1}^{z} [\psi_1(z')\psi_2(z) - \psi_2(z')\psi_1(z)]\, h(\lambda, k, z')\psi(\lambda,k,z')dz',$$

where ψ_1 and ψ_2 are two independent solutions of the "free" equation

$$D(\lambda, k, z)\psi(\lambda, k, z) = 0,$$

and

$$W[\psi_1, \psi_2] = \psi_1 \psi_2' - \psi_1' \psi_2.$$

APPENDIX II

Here we want to give the majorizations of the integral equations deduced in App. I in order to deduce analytic properties of the functions $\phi(\lambda, k, z)$ and $f(\lambda, k, z)$. The integral equations we are dealing with can be written in this general form:

$$g(\lambda, k, z) = g_0(\lambda, k, z) + \int_{z_1}^{z} L(\lambda, k, z')\, g(\lambda, k, z')dz'. \qquad \text{(A2.1)}$$

Then

$$|g(\lambda, k, z)| \leq |g_0(\lambda, k, z)| + \int_{z_1}^{z} |L(\lambda, k, z')g(\lambda, k, z')dz'|.$$

It is useful to introduce the notations

$$|g_0(\lambda, k, z)| \leq M(\lambda, k, z), \quad G(\lambda, k, z) = \frac{g(\lambda, k, z)}{M(\lambda, k, z)}$$

in order to get

$$|G(\lambda, k, z)| \leq 1 + \int_{z_1}^{Z} |K(\lambda, k, z')\, G(\lambda, k, z')dz'|,$$

where

$$K(\lambda, k, z') = \frac{M(\lambda, k, z')}{M(\lambda, k, z)}\, L(\lambda, k, z').$$

By using TITCHMARSCH's lemma [15], we obtain

$$|g(\lambda, k, z)| \leq M(\lambda, k, z) \exp\left(\int_{z_1}^{z} |K(\lambda, k, z')dz'|\right).$$

Let us write the solution of (A 2.1) in the following way:

$$g(\lambda, k, z) = \sum_{n=0}^{\infty} g_n(\lambda, k, z).$$

Then Titchmarsch's lemma assures the convergence of this series if we put an upper bound to the integral:

$$\int_{z_1}^{z} |K(\lambda, k, z')dz'| . \qquad (A2.2)$$

The common region of analyticity of all terms g_n represents the analyticity domain of $g(\lambda, k, z)$. We give in the following the majorizations of the integral (A 2.2) for the integral equations previously written:

$$\phi(\lambda, k, z) = z^{\lambda+1/2} - \frac{1}{2\lambda}\int_0^z \left(\frac{z'^{\lambda+1/2}}{z^{\lambda-1/2}} - \frac{z^{\lambda+1/2}}{z'^{\lambda+1/2}}\right)[V(z') - k^2]\phi(\lambda, k, z')dz',$$

$$|g_0(\lambda, k, z)| = |z^{\lambda+1/2}| \equiv M(\lambda, k, z),$$

$$\left|\frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}}\right|\left|\frac{z'^{\lambda+1/2}}{z^{\lambda-1/2}} - \frac{z^{\lambda+1/2}}{z'^{\lambda-1/2}}\right| \le \left|\frac{z'^{2\lambda+1}}{z^{2\lambda}} - z'\right| \le 2z', \ \mathrm{Re}\,\lambda \le 0,$$

$$|V(z') - k^2| \le Hz'^{\epsilon-2} + N \le Rz'^{\epsilon-2},$$

where k is from any finite domain of the k-plane, where the upper limit of k^2 is N. H and R are constants.

$$\int_0^z |K(\lambda, k, z')dz'| = \frac{1}{2|\lambda|}\int_0^z \left|\frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}}\right|\left|\frac{z'^{\lambda+1/2}}{z^{\lambda+1/2}} - \frac{z^{\lambda+1/2}}{z'^{\lambda-1/2}}\right||V(z')-k^2|\,dz'$$

$$\le \frac{R}{|\lambda|}\int_0^z z'^{\epsilon-2}\cdot z'dz' \le \frac{R}{\epsilon|\lambda|} .$$

It is now apparent that $\phi(\lambda, k, z)$ is an integral function of k, holomorphic in the half-plane $\mathrm{Re}\,\lambda > 0$ (continuous for $\mathrm{Re}\,\lambda = 0$).

$$f(\lambda, k, z) = e^{-ikz} + \frac{1}{2ik}\int_z^\infty \left[e^{ik(z'-z)} - e^{-ik(z'-z)}\right]\left[V(z') + \frac{\lambda^2 - 1/4}{z'^2}\right]f(\lambda, k, z')dz',$$

$$|g_0(\lambda, k, z)| = |e^{-ikz}| \equiv M(\lambda, k, z),$$

$$\int_z^\infty |K(z')dz'| = \frac{1}{z|k|}\int_z^\infty |e^{-ik(z'-z)}||e^{ik(z'-z)} - e^{-ik(z'-z)}|\left|V(z') + \frac{\lambda^2 - 1/4}{z'^2}\right|dz',$$

$$|1 - e^{-2ik(z'-z)}| \le N = \mathrm{const.}, \ \mathrm{Im}\,k \le 0.$$

Then

$$\int_z^\infty |K(z')dz'| \le \frac{N}{2|k|}\int_z^\infty \left|\frac{H}{z'^{2-\epsilon}} + \frac{\lambda^2-1/2}{z'^2}\right| dz' \le \text{Const.}$$

Therefore, $f(\lambda, k, z)$ is an integral function of λ, holomorphic in the half-plane $\mathrm{Im}\, k < m/2$ (continuous for $\mathrm{Im}\, k = 0$).

For real λ and Yukawian potentials this analyticity domain of $f(\lambda, k, z)$ can be extended to $\mathrm{Im}\, k < m/2$, and it is continuous for $\mathrm{Im}\, k = m/2$. This can be shown by treating the integral equation

$$f(\lambda, k, z) = f_0(\lambda, k, z)$$

$$-\frac{i\pi}{4} z^{1/2}\int_z^\infty z'^{1/2}\left(H_\lambda^{(2)}(kz')H_\lambda^{(2)}(kz) - H_\lambda^{(2)}(kz')H_\lambda^{(2)}(kz)\right)V(z')f(\lambda, k, z')dz',$$

in a way similar to that used before.

This method could have been used to derive the same analytic properties for the prime derivatives of the solutions considered.

APPENDIX III

We know from standard textbooks the most important properties of Legendre functions. It is well known that Legendre functions are particular cases of hypergeometric functions with singularities located at ± 1 and ∞. Therefore, the only singularities of $P_\ell(x)$ and $Q_\ell(x)$ lie on ± 1 or ∞.

From the general theory of Legendre equations one finds out at once that in ± 1 the solutions either are regular or have a logarithmic singularity. It is always possible, however, to choose the parameters in the general integral of the equations in such a way as to make the solution regular in a given point. In particular, $P_\ell(x)$ is regular in $x = 1$ and $P_\ell(1) = 1$ and $Q_\ell(x)$ is regular at $x = \infty$ provided $\mathrm{Re}\,(\ell) + 1/2 \ge 0$. Since ℓ enters in the differential equation under the form $\ell(\ell+1)$ and since the boundary conditions for $P_\ell(x)$ are ℓ-independent, it follows from a general theorem of Poincaré that $P_\ell(x)$ is an entire function of ℓ for x fixed and that $P_{\ell-1}(x) = P_\ell(x)$ because $\ell(\ell+1)$ is invariant under the substitution $\ell \to -\ell-1$. Also, if $\lambda = \ell + 1/2$,

$$P_{\lambda-1/2}(x) = P_{-\lambda-1/2}(x). \quad \text{(A 3.1)}$$

$P_\ell(z)$ has a cut between -1 and $-\infty$. It is otherwise regular in z. Its asymptotic behaviour for large λ is given by

$$P_{\lambda-1/2}(\cosh\alpha) \simeq \frac{1}{\sqrt{2\pi\lambda}}\,\frac{1}{\sinh\alpha}\left(e^{\alpha\lambda} + i\,e^{-\alpha\lambda}\right)\left[1 + 0\left(\frac{1}{\lambda}\right)\right]. \quad \text{(A3.2)}$$

For large z $P_\ell(z) = 0(z^\ell)$. $Q_\ell(z)$ is instead defined through its behaviour for large z; that is,

$$Q_{\lambda-1/2}(z) = 0(z^{-\ell-1}). \qquad (A\,3.3)$$

If Re $\ell + 1/2 > 0$, this is the only solution which does so apart from a multiplicative factor. We have also for large λ

$$Q_{\lambda-1/2}(\cosh \alpha) = \sqrt{\frac{\pi \sinh \alpha}{2\lambda}}\, e^{-\alpha\lambda}\left[1 + 0\left(\frac{1}{\lambda}\right)\right]. \qquad (A\,3.4)$$

$Q_\ell(z)$ has singularities in both 1 and -1. Moreover,

$$Q_{-\lambda-1/2}(z) = Q_{\lambda-1/2}(z) + \pi \,\mathrm{tg}\, \pi\lambda\; P_{\lambda-1/2}(z). \qquad (A\,3.5)$$

This relation says that $Q_{\lambda-1/2}(z)$ has poles in Re $\ell < 0$ at the negative half-integer points. In these points the residuum of $Q_{\lambda-1/2}$ is given by the corresponding $P_{\lambda-1/2}$, which turn out to be polynomials. From the pre-existing literature one knows already that $Q_{\lambda-1/2}$ is regular in Re $\ell > 0$.

We have already listed the symmetries arising from the reflection $\lambda \to -\lambda$ or $\ell \to -\ell-1$. But the Legendre equation turns out also to be symmetric under the exchange $z \to -z$. The consequences of this fact are

$$Q_{\lambda-1/2}(z\,e^{\mp i\pi}) = \pm\, i\, e^{\pm i\pi\lambda} Q_{\lambda-1/2}(z). \qquad (A\,3.6)$$

There is ambiguity in taking $e^{\pm i\pi}$ because $Q_{\lambda-1/2}(z)$ has a cut $-1 \geq z \geq -\infty$. It has to be remarked that, in encircling anticlockwise both points ± 1, $Q_{\lambda-1/2}(z)$ is multiplied by the factor $e^{-2i\pi(\lambda+1/2)}$ so that $Q_{\lambda-1/2}(z)\, z^{\lambda+1/2}$ is left unaffected.

We also have:

$$P_{\lambda-1/2}(e^{\mp i\pi} z) = \pm\, i\, e^{\mp i\pi\lambda} P_{\lambda-1/2}(z) + \frac{2}{\pi}\cos \pi\lambda\; Q_{\lambda-1/2}(z); \qquad (A\,3.7)$$

if λ is half integer, it reduces simply to

$$P_\ell(-x) = (-)^\ell P_\ell(x). \qquad (A\,3.8)$$

Mehler has found the following interesting inversion formula when λ is imaginary (conical functions): If

$$F(\lambda) = \mathrm{tg}\, \pi\lambda \int_1^\infty P_{\lambda-1/2}(\omega)\, G(\omega)\, d\omega, \qquad (A\,3.9)$$

then

$$G(\omega) = i \int_0^{i\infty} \lambda d\lambda\; P_{\lambda-1/2}(\omega)\; F(\lambda),$$

valid under conditions similar to those of the Fourier transform. They can be written as

$$\int_{-i\infty}^{i\infty} \lambda\, d\lambda\ \mathrm{tg}\ \pi\lambda\ P_{\lambda-1/2}(\xi)\, P_{\lambda-1/2}(\eta) = -2i\ \delta(\xi - \eta). \qquad (A\,3.10)$$

This is the prototype of many integrals to be derived. From (A3.5) we have

$$\int_{-i\infty}^{i\infty} \lambda d\lambda\ P_{\lambda-1/2}(\xi)\ Q_{\pm\lambda-1/2}(\eta) = \pm i\,\pi\,\delta(\xi - \eta). \qquad (A\,3.11)$$

We notice that easily

$$\int_{-i\infty}^{i\infty} \lambda d\lambda\ Q_{\pm\lambda-1/2}(z_1)\ Q_{\pm\lambda-1/2}(z_2) \ldots\ldots Q_{\pm\lambda-1/2}(z_n) = 0, \qquad (A\,3.12)$$

where all $\pm$ are correlated.

(A 3.11) follows from the fact that the integrand is analytic in the right (left) hand plane and it vanishes there at large distances.

Take now Heine's formula:

$$\sum_{\ell} (2\ell + 1)\ Q_{\ell}(\xi)\ P_{\ell}(\eta) = 1/(\xi - \eta), \qquad (A\,3.13)$$

which holds for $\mathrm{Im}\,\alpha > \mathrm{Im}\,\beta$ where $\cos\alpha = \xi$, $\cos\beta = \eta$. Take ξ, η real > 1 and apply to it the Watson-Sommerfeld transform. We get

$$\int_{-i\infty}^{i\infty} \lambda\, d\lambda\ Q_{\lambda-1/2}(\xi)\ P_{\lambda-1/2}(\eta)\ \ \mathrm{tg}\pi\lambda = i/(\xi - \eta); \qquad (A\,3.14)$$

and using (A 3.5),

$$\int_{-i\infty}^{i\infty} \lambda\, d\lambda\ Q_{\lambda-1/2}(\xi)\ Q_{-\lambda-1/2}(\eta) = i\pi/(\xi - \eta). \qquad (A\,3.15)$$

We can have more complicated identities as follows: Take the addition theorem for Legendre functions (ℓ integer):

$$P_{\ell}(x)\ P_{\ell}(y) = P_{\ell}(xy + \sqrt{1 - x^2}\sqrt{1 - y^2}\cos\psi)$$

$$+ 2\sum_{m=1}^{\infty} (-)^m \frac{\Gamma(\ell - m + 1)}{\Gamma(\ell + m + 1)}\ P_{\ell}^m(x)\ P_{\ell}^m(y)\ \cos m\psi; \qquad (A\,3.16)$$

$$\mathrm{Re}\ x > 0;\ \mathrm{Re}\ y > 0;\ |\arg(x - 1)| < \pi;\ |\arg(y - 1)| < \pi.$$

Let us integrate this on ψ between 0 and π. All terms containing $\cos m\psi$ vanish, and we have

$$P_\ell(x)\, P_\ell(y) = \frac{1}{\pi}\int_0^\pi P_\ell\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\cos\psi\right) d\psi. \qquad \text{(A 3.17)}$$

Let

$$z = xy + \sqrt{1-x^2}\sqrt{1-y^2}\cos\psi$$

be a new variable instead of ψ.
We have

$$d\psi = \frac{d\psi}{dz}\, dz = dz\, \frac{1}{dz/d\psi} = -\frac{dz}{\sqrt{1-x^2}\sqrt{1-y^2}\sin\psi},$$

but

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-(z-xy)^2/(1-x^2)(1-y^2)}$$

$$= \sqrt{1-z^2-x^2-y^2+2xyz}\Big/\sqrt{(1-x^2)(1-y^2)}$$

so that

$$d\psi = -\,dz\Big/\sqrt{1-z^2-x^2-y^2+2xyz}.$$

It is easily seen that the limits of integration in z are the points where $1-z^2-x^2-y^2+2xyz$ vanishes. It follows that

$$P_\ell(x)\, P_\ell(y) = \frac{1}{\pi}\int_{-1}^{1} \frac{dz\, \Theta(1-z^2-x^2-y^2+2xyz)}{\sqrt{1-z^2-x^2-y^2+2xyz}}\, P_\ell(z). \qquad \text{(A 3.18)}$$

From this it is evident that ($x, y, z < 1$ and real)

$$\sum_{\ell=0}^{\infty}(2\ell+1)\, P_\ell(x)\, P_\ell(y)\, P_\ell(z)$$

$$= (2/\pi)\, \Theta(1-z^2-x^2-y^2+2xyz)\Big/\sqrt{1-z^2-x^2-y^2+2xyz}$$

$$= (2/\pi)\, K_1(x, y, z) \qquad \text{(A 3.19)}$$

TABLE II

$f(\lambda)_{\alpha,\beta,\gamma\geq 1}$	$\int_{-i\infty}^{i\infty} F(\lambda)\,\lambda\, d\lambda$	
$Q_{\pm\lambda-1/2}(\alpha)\, Q_{\pm\lambda-1/2}(\beta)$	0	(B2.30)
$Q_{\lambda-1/2}(\alpha)\, Q_{-\lambda-1/2}(\beta)$	$i\pi(\alpha-\beta)^{-1}$	(B2.31)
$Q_{\pm\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)$	$\pm i\pi\delta(\alpha-\beta)$	(B2.32)
$Q_{\pm\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)\, \mathrm{tg}\pi\lambda$	$i(\alpha-\beta)^{-1}$	(B2.33)
$P_{\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)\, \mathrm{tg}\pi\lambda$	$-2i\delta(\alpha-\beta)$	(B2.34)
$Q_{\pm\lambda-1/2}(\alpha)\, Q_{\pm\lambda-1/2}(\beta)\, Q_{\pm\lambda-1/2}(\gamma)$	0	(B2.35)
$Q_{\pm\lambda-1/2}(\alpha)\, Q_{\pm\lambda-1/2}(\beta)\, Q_{\pm\lambda-1/2}(\gamma)$	$\pm i\pi \int_1^\infty d\omega\, K(\omega;\alpha,\beta)\,(\omega-\gamma)^{-1}$	(B2.36)
$Q_{\pm\lambda-1/2}(\alpha)\, Q_{\pm\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)$	$\pm i\pi K(\gamma;\alpha,\beta)$	(B2.37)
$Q_{\lambda-1/2}(\alpha)\, Q_{\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)$	$i\pi\left\{K(\alpha;\beta,\gamma) - K(\beta;\alpha,\gamma)\right\}$	(B2.38)
$Q_{\pm\lambda-1/2}(\alpha)\, Q_{\pm\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)\mathrm{tg}\pi\lambda$	$i\int_1^\infty d\omega\, K(\omega;\alpha,\beta)\,(\omega-\gamma)^{-1}$	(B2.39)
$Q_{\lambda-1/2}(\alpha)\, Q_{-\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)\mathrm{tg}\pi\lambda$	$-i\int_1^\infty \left(\frac{K(\omega;\alpha,\gamma)}{\omega-\beta} - \frac{K(\omega;\beta,\gamma)}{\omega-\alpha}\right)d\omega$	(B2.40)
$Q_{\pm\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)$	$\pm i K_1(\alpha,\beta,\gamma)$	(B2.41)
$Q_{\pm\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)\mathrm{tg}\pi\lambda$	$i H(\alpha;\beta,\gamma)$	(B2.42)
$P_{\lambda-1/2}(\alpha)\, P_{\lambda-1/2}(\beta)\, P_{\lambda-1/2}(\gamma)\mathrm{tg}\pi\lambda$	$(2/\pi i)\, K_1(\alpha,\beta,\gamma)$	(B2.43)

by definition. This remarkable formula came to our knowledge through Prof. Goldberger and does not seem to appear anywhere in the literature. It generalizes the usual

$$\sum_{0}^{\infty}(2\ell+1)\, P_{\ell}(x)\, P_{\ell}(y) = 2\,\delta(x-y). \qquad \text{(A 3.20)}$$

We could try the Watson transform directly on (A 3.18), but it would be of no use because it does not converge. A better way is to multiply (A 3.18) by $1/(\xi - x)$ and integrate on x between ± 1. The result is

$$\sum_{\ell}(2\ell+1)\, Q_{\ell}(\xi)\, P_{\ell}(y)\, P_{\ell}(z) = 1\Big/\sqrt{\xi^2+y^2+z^2-2yz\xi-1}. \qquad \text{(A 3.21)}$$

(A 3.21) is valid also when ξ, y, z are complex, while in (A 3.19) they had to be real. The only condition is that, if $\cos\beta = y$, $\cos\gamma = z$, $\cos\alpha = \xi$, then $\operatorname{Im}\xi > \operatorname{Im}\beta + \operatorname{Im}\gamma$. Applying to (A 3.21) the Watson transform, we get

$$\int_{-i\infty}^{i\infty}\lambda\, d\lambda\ \operatorname{tg}\pi\lambda\ Q_{\lambda-1/2}(\xi)\, P_{\lambda-1/2}(\eta)\, P_{\lambda-1/2}(\zeta)$$
$$= i\Big/\sqrt{\xi^2+\eta^2+\zeta^2-2\zeta\eta\xi+1}.$$

Using identity (A 3.11), (A 3.5) repeatedly, one arrives at a large number of integrals. We skip here a detailed proof and limit ourselves to giving a table of them (Table II). Here K_1 is defined by (A 3.19) and

$$K(\xi;\ \eta,\ \zeta) = \Theta(\xi - \xi_>)\Big/\sqrt{\xi^2+\zeta^2+\eta^2-2\eta\zeta\xi-1},$$

where $\xi_>$ is the largest root of the denominator;

$$H(\xi;\eta,\zeta) = K(\xi;\eta,\zeta) - K(\eta;\zeta,\xi) - K(\zeta;\xi,\eta).$$

Many other identities can be written, but they would take much more space and we refer the reader to a coming paper by V. de Alfaro, T. Regge and G. Rossetti to be published in Nuovo Cimento.

REFERENCES

[1] JOST, R., Helv. Phys. Acta 20 (1947) 256; BARGMANN, V., Rev. Mod. Phys. 21 (1949) 488.
[2] KHURI, N., Phys. Rev. 107 (1957).
[3] BLANKENBECL, R., GOLDBERGER, M.L., KHURI, N. and TREIMAN, S.B., Ann. of Phys. 10 (1960) 62.
[4] KLEIN, A., J. Math. Phys. 1 (1960) 41.
[5] SOMMERFELD, A., Partielle Differential Gleichungen der Physik, Leipzig (1943) 285.

[6] KLEIN, A. and ZEMACH, C., Ann. of Phys. 7 (1959) 440; Nuovo Cimento 10 (1958) 1078.
[7] SCHIFF, L., Quantum Mechanics, McGraw-Hill, New York.
[8] CARTWRIGHT, M. L., Integral Functions (1956) 28.
[9] NEWTON, R. G., J. Math. Phys. 1 (1960) 319.
[10] REGGE, T., Nuovo Cimento 14 (1959) 951.
[11] REGGE, T., Nuovo Cimento 18 (1960) 947.
[12] KEMBLE, E. C., Phys. Rev. 48 (1935) 549; KEMBLE, E. C., Fundamental Principles of Quantum Mechanics, McGraw-Hill (1937).
[13] FORD, K. W. and WHEELER, J. A., Ann. of Phys. 7 (1959) 259.
[14] Higher Transcendental Functions (Bateman Manuscript Project) II, McGraw-Hill, New York (1953) 46.
[15] TITCHMARSCH, E. C. Eigenfunction Expansions, Clarendon Press, Oxford (1946) 97.
[16] BOTTINO, A., LONGONI, A. M. and REGGE, T., Nuovo Cimento 23 (1962) 954.
[17] BOTTINO, A. and LONGONI, A. M., Nuovo Cimento 24 (1962) 353.
[18] JOST, R. and PAIS, A., Phys. Rev. 82 (1951) 840.
[19] FIVEL, D., Phys. Rev. 125 (1962) 1085.
[20] PREDAZZI, E. and REGGE, T., Nuovo Cimento 24 (1962) 518.
[21] CHARAP, J. M. and SQUIRES, reprint UCRL 10209 - 10115 - 10209 - 10138.
[22] FAVELLA, L. and REINERI, M. T., Nuovo Cimento 23 (1962) 616.
[23] FONDA, L., RADICATI, L. and REGGE, T., Ann. Phys. 12 (1961) 68.