Singletons on AdS$_n$

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Abstract. We define the singletons for the invariance group $\overline{S}_n = \overline{SO}_0(2, n-1)$ of the AdS$_n$ space-time. We write down some of their important properties and characterizations. It is found that the tensor product of singletons of spin 0 or $1/2$ decomposes into representations that are a kind of massless representations of $\overline{S}_n$. Other kinds of massless representations, related to singletons, are also studied and a comparison is made. Various Gupta-Bleuler triplets are constructed for singletons and for massless representations.

Keywords: representations of Lie groups and Lie algebras; Minkowski and anti-de Sitter spaces; anti-de Sitter, conformal, and Poincaré groups; singletons; masslessness; Gupta-Bleuler triplets

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Dedicated to the memory of Moshé Flato

1. Introduction

Recently, mainly since the Maldacena conjecture which relates anti-de Sitter theories 'in the bulk' with conformal theories 'on the boundary at infinity', singletons became a widely popular subject in physics on space-times of arbitrary dimensions (see, e.g. [27, 11, 30, 17, 19, 1]). It is therefore important to know more about these representations and about massless representations of conformal groups. In this paper we describe some properties of singletons and characterize them after having given a rigorous definition. Since the singletons are related to masslessness it is important to know the nature of that relation and to compare with what happens in the classical 4-dimensional case.

The n-dimensional anti-de Sitter space-time (AdS$_n$) with (scalar) curvature $-\rho < 0$ is defined as the manifold $H_n^\rho = \{(y^a)_{-1\le a\le n-1} \in \mathbb{R}^{n+1} / \sum y^a y_a = 1/\rho\}$. Here $\sum y^a y_a \overset{\text{def}}{=} \sum y^a y^b \eta_{ab}$ where $\eta = (\eta_{ab})$ is the matrix $\left(\begin{array}{c|c} \mathbf{1}_2 & \\ \hline & -\mathbf{1}_n \end{array}\right)$. We assume throughout that $n \ge 3$. The invariance group of AdS$_n$ is the anti-de Sitter group $S_n = SO_0(2, n-1)$ and one has $H_n^\rho \simeq S_n/L_n$, where $L_n = SO_0(1, n-1)$ is the Lorentz group of both AdS$_n$ and Minkowski space-time $M_n = \mathbb{R}^{1,n-1}$.

Now the "time axis" of H_n^ρ is bounded: It is the S^1 (circle) part in $H_n^\rho \simeq S^1 \times \mathbb{R}^{n-1}$. But if one considers the universal covering $\overline{H}_n^\rho \simeq \mathbb{R} \times \mathbb{R}^{n-1}$ of H_n^ρ, then the time axis is no longer compact (of course there is no problem if one needs a physical theory with a cyclic time).

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Consequently, the invariance group is an infinite covering $\overline{S}_n$ of S_n. The best choice for $\overline{S}_n$ would be a covering which contains the quantum mechanical Lorentz group, i.e., the spinor covering group $\overline{L}_n$, such that $\overline{H}_n^\rho \simeq \overline{S}_n/\overline{L}_n$. Having in mind that an invariance group should be contained in the conformal group, such a choice is given by the universal covering of S_n if $n \geq 4$ and, for $n = 3$, by the infinite covering of $S_3 = SO_0(2,2)$, which is contained in, and has the same center as, the conformal group $\overline{G}_3 = \overline{SO}_0(2,3)$. Thus the fundamental group of $\overline{S}_n = \overline{SO}_0(2,n)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ for all $n \geq 3$. Another advantage in using this type of infinite coverings is that we can consider deformations of representations, a useful tool for constructing Gupta-Bleuler (GB) triplets.

Let $\overline{G}_n = \overline{SO}_0(2,n)$ the common conformal group [1] of $\overline{H}_n^\rho$ and M_n. Then $\overline{G}_n$ contains the Poincaré group $\overline{P}_n = \overline{SO}_0(1,n-1) \ltimes T_n$ ($T_n \simeq \mathbb{R}^n$), the anti-de Sitter group $\overline{S}_n = \overline{SO}_0(2,n-1)$, and the de Sitter group $\overline{L}_{n+1} = \overline{SO}_0(1,n)$. It is well known that the last two are deformations of the former, so that $\overline{S}_n$ and $\overline{L}_{n+1}$ can be contracted to $\overline{P}_n$. It follows that there is a contraction $\overline{H}_n^\rho \overset{\rho\to 0}{\longrightarrow} M_n$ which implies that a physical theory on AdS$_n$ cannot be independent of the corresponding one on M_n: It must be compatible, at least for $n = 4$, with physics in M_n. As a consequence, a massless particle on AdS$_n$ should correspond by contraction to a massless particle on M_n. This naturally leads to a first (weak) definition of masslessness (see Section 3).

Unfortunately this definition does not fix uniquely the notion of masslessness on AdS$_n$, even for $n = 4$: Additional conditions have to be introduced in order to make it unique. This was done by Flato and Frønsdal for $n = 4$ in the 80's (see [14]). Another way to avoid ambiguity is to consider less weak definitions of masslessness, introduced in Section 3, such as *conformal masslessness* or *composite masslessness*, both related to singletons and to gauge properties. The latter, as shown by Flato and Frønsdal in [13] (see also [23]), is expressed by the well-known property "*singleton* $\otimes$ *singleton* $= \oplus$ *massless representations*", while the former is (for $n \geq 3$) the property of unique extension from representations of the anti-de Sitter group to the corresponding conformal group. For $n = 4$, both notions coincide.

The paper is organized as follows. In Section 2, we define and characterize finite-dimensional (nonunitary) and infinite-dimensional (unitary) singleton representations of $\overline{SO}_0(2,n-1)$, along with a classification. We also obtain a generalization of the theorem of Flato and Frønsdal mentioned above. We define in Section 3 some notions of masslessness related to singletons, give their properties and discuss their differences. In section 4, we construct examples of Gupta-Bleuler triplets for the singletons and for almost all massless representations. We conclude the paper by a comparison between the 4-dimensional and higher-dimensional cases.

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[1] In fact, $\overline{G}_n$ is a covering of the actual conformal group.

2. Singletons of $\overline{SO}_0(2,n-1)$

2.1. The fundamental property of singletons

Let $\mathscr{G}_n = \mathrm{Lie}(\overline{G}_n)$, $\mathscr{S}_n = \mathrm{Lie}(\overline{S}_n)$, $\mathscr{L}_n = \mathrm{Lie}(\overline{L}_n)$, and $\mathscr{P}_n = \mathrm{Lie}(\overline{P}_n)$. Let $(e_a)_{-1\le a\le n}$ be the canonical basis of $\mathbb{R}^{2,n}$ which is endowed with the metric η. Then a set of generators of $\mathscr{G}_n$ is given by $\{M_{ab}\}_{-1\le a,b\le n}$, defined by:

$$M_{ab} = -M_{ba} \text{ and } M_{ab}y = y_b e_a - y_a e_b \quad \forall\, y = \sum_{a=-1}^{a=n} y^a e_a \in \mathbb{R}^{2,n}.$$

The following commutation relations are satisfied:

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} + \eta_{bd}M_{ca} - \eta_{ac}M_{bd} - \eta_{ad}M_{cb}. \tag{2.1}$$

$\mathscr{S}_n$, $\mathscr{L}_n$, and $\mathscr{P}_n$ are naturally embedded in $\mathscr{G}_n$. To obtain the generators of the first two Lie algebras one simply restricts the range of indices: $\{M_{\alpha\beta}\}_{-1\le\alpha,\beta\le n-1}$ and $\{M_{\mu\nu}\}_{0\le\mu,\nu\le n-1}$, respectively. $\mathscr{P}_n$ is the semi-direct sum of $\mathscr{L}_n$ and the Abelian Lie algebra $\mathscr{T}_n = \mathrm{Lie}(T_n)$, for which a set of generators can be given by $\{E_\mu = M_{-1,\mu} + M_{\mu,n}\}_{0\le\mu\le n-1}$. They satisfy the commutation relations:

$$[E_\mu, E_\nu] = 0 \quad \text{and} \quad [M_{\mu,\nu}, E_\rho] = \eta_{\nu\rho}E_\mu - \eta_{\mu\rho}E_\nu. \tag{2.2}$$

The notation used in physics is related to the present one by: $P_\mu = -\sqrt{-1}E_\mu$, $L_{\mu\nu} = \sqrt{-1}M_{\mu\nu}$, and so on.

Let D be an irreducible representation (IR) of the AdS$_n$ group $\overline{S}_n = \overline{SO}_0(2,n-1)$ on a Banach space $\mathscr{H}$, not necessarily unitary. Let $K_n \simeq SO(2)\times SO(n-1)$ be the maximal compact subgroup of S_n and $\overline{K}_n \simeq \mathbb{R}\times \mathrm{Spin}(n-1)$ be the corresponding maximal essentially compact subgroup of $\overline{S}_n$. The common reductive Lie algebra $\mathfrak{k}_n$ is generated by $M_{-1,0}$ and $\{M_{ij}\}_{1\le i,j\le n-1}$, the latter generating the semi-simple part of $\mathfrak{k}_n$. The restriction $D|_{\overline{K}_n}$ is completely reducible, i.e., under the action of $D|_{\overline{K}_n}$ one has the direct sum decomposition:

$$\mathscr{H}^\infty = \oplus_\mu M(\mu)\otimes K(\mu), \tag{2.3}$$

where each μ is a highest weight (HW) relative to a given order of the roots of $\mathfrak{k}_n$, $K(\mu)$ is an irreducible $\mathfrak{k}_n$-module with weight μ and $M(\mu)$ is a trivial $\mathfrak{k}_n$-module the dimension of which is the multiplicity $m(\mu)$ of μ. $\mathscr{H}^\infty$ is the subspace of differentiable vectors. It is known that $\mathscr{H}^\infty$ is dense in $\mathscr{H}$. If π_μ is an IR of $\overline{K}_n$ with weight μ then the relation (2.3) may be written:

$$D|_{\overline{K}_n} = \oplus_\mu m(\mu)\pi_\mu. \tag{2.4}$$

Let us write $(\mu_1,\vec{\mu})$ for $\mu = (\mu_1,\mu_2,\ldots,\mu_r)$ where r is the rank of $\mathscr{S}_n^{\mathbb{C}}$ (r is the entire part of $\frac{n+1}{2}$). Let us call μ_1 the *energy part* (we choose $-\mu_1$ to be the *energy*) and $\vec{\mu}$ the *spinor part*. Then the diagram of such $\mathfrak{k}_n$'s HW μ has more than one dimension in general (recall that $n\ge 3$). Moreover the multiplicity of each weight can be different from 0 or 1. But it may also happen that this diagram is just one dimensional. It is indeed the case for Dirac singletons and the so-called ladder representations of the conformal group [3, 8, 26, 29] or the representations of $\overline{G}_n$ called C_n-massless in [25].

When that diagram is included in a line then each weight may be obtained from a fixed one by adding an integer multiple of a fixed root. Thus the convex envelope of the diagram is one dimensional if the representation is not trivial. So let us write down a definition of the singletons. This definition generalizes the one given by Dirac [8] in the 4-dimensional case to the representations $D(s+1,s)$, s being $1/2$ or 0, respectively called later on *Di* and *Rac* by Flato and Frønsdal (see for example [3]). More generally, the notation $D(E,\vec{\lambda})$ corresponds to the irreducible representation (up to equivalence) carried by the irreducible quotient $L(-E,\vec{\lambda})$ of the Verma module $M(-E,\vec{\lambda})$.

Definition 2.1 *An IR D of $\overline{S}_n$ is a* singleton representation*, or more simply a* singleton*, if D is not trivial and there exists a weight λ and a root α of $\mathfrak{k}_n$ such that*

$$D|_{\overline{K}_n} = \oplus_{l\in\mathbb{Z}}\pi_{\lambda+l\alpha}, \tag{2.5}$$

where π_μ is 0, if μ is not a weight of $D|_{\overline{K}_n}$.

An example is given by Dirac singletons $(n=4)$ $D(s+1,s)$ and their contragredients $\overline{D}(s+1,s)$, identified with $D(-(s+1),s)$. Here one has:

$$D(\pm(s+1/2),s)|_{\overline{K}_4} = \oplus_{l\in\mathbb{N}}\pi_{(\pm(s+1/2+l),s+l)}$$

and $K_4 \simeq SO(2)\times SO(3)$.

In Theorem 2.2, we shall give the fundamental mathematical property of singletons. It is a strong property of the enveloping algebra $\mathscr{U}$ of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(N)^{\mathbb{C}}$. We introduce N in order to treat both conformal and anti-de Sitter cases, hence $\mathfrak{g}^{\mathbb{C}}$ stands for $\mathscr{S}_{N-1}{}^{\mathbb{C}}$ or $\mathscr{G}_{N-2}{}^{\mathbb{C}}$. Before stating the theorem let us introduce some useful notation. Let $M_N(\mathscr{U})$ be the vector space of $N\times N$ matrices whose elements are in $\mathscr{U}$. $M_N(\mathscr{U})$ is also endowed with a natural $\mathscr{U}$-module structure. Let $\delta = (\eta_{ab}\mathrm{Id})_{1\le a,b\le N}$ and $M = (M_{ab})_{1\le a,b\le N}$ be two such matrices, Id and the M_{ab}'s being respectively the identity of $\mathscr{U}$ and the generators of the Lie algebra $\mathfrak{so}(N)^{\mathbb{C}}$. The commutation relations considered are those given by (2.1). Define $M^0 \stackrel{\mathrm{def}}{=} \delta$ and M^k for nonzero $k\in\mathbb{N}$ by $(M^k)_{ab} = \sum_{c=1}^{N}(M^{k-1})_{ac}M^c{}_b$ where $M^c{}_b \stackrel{\mathrm{def}}{=} \sum_{d=1}^{N}\eta^{cd}M_{db}$. If D is a representation of $\mathscr{U}$ and $A=(A_{ab})$ an element of $M_N(\mathscr{U})$, we write $D(A)$ for the matrix having entries $D(A_{ab})$. Finally let $\mathbf{C}_2 = \frac{1}{2}\mathrm{Tr}(M^2)$ be the Casimir operator.

Theorem 2.2 *A singleton representation D of $G=\overline{SO}(2,N-2)$ is a highest or lowest weight representation and it satisfies*[2]

$$D\big(M^2-\frac{N-2}{2}M-\frac{2}{N}\mathbf{C}_2\delta\big)=0. \tag{2.6}$$

Moreover D is unitarizable if and only if it is infinite dimensional.

Conversely a representation integrable on the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}=Lie(G)$ which sends the two-sided ideal spanned by the family $(F_{ab})_{a,b}$ to 0 has, as irreducible parts, all the singleton representations.

Proof. Case 1: D is finite dimensional. Suppose that D is a finite-dimensional representation of $\overline{SO}(N)$ of weight $\lambda=(\lambda_1,\dots,\lambda_r)$, r being the rank. Let us denote by D'_μ and D''_ν the finite-dimensional representations of respective weights μ and $\nu=(\nu_2,\dots,\nu_r)$ (up to equivalence) of $\overline{SO}(N-1)$ and $\overline{SO}(N-2)$ respectively. Let us write $\mu=(\mu_1,\dots,\mu_r)$ if N is odd and $\mu=(\mu_2,\dots,\mu_r)$ if N is even. Then it is known that

$$D|_{\overline{SO}(N-1)}=\begin{cases}\oplus_{\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\geq\lambda_{r-1}\geq\mu_r\geq|\lambda_r|}D'_\mu & \text{if } N \text{ is even,}\\ \oplus_{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_r\geq\mu_r\geq-\lambda_r}D'_\mu & \text{if } N \text{ is odd.}\end{cases} \tag{2.7}$$

The same decomposition holds for $D'_\mu|_{\overline{SO}(N-2)}$. Thus $(D|_{\overline{SO}(N-1)})|_{\overline{SO}(N-2)}$ decomposes into a sum of irreducible representations D''_ν such that $\lambda_1\geq\nu_2\geq\lambda_2$ and so on.

Now thanks to the preceding relations one sees that for each index i such that $1\leq i\leq r-1$, $\lambda_i>|\lambda_{i+1}|$ implies that the representation D'_{μ_i} occurs at least twice in the reduction $D|_{\overline{SO}(N-1)}$, thus D''_{ν_i} also occurs at least twice in $(D|_{\overline{SO}(N-1)})|_{\overline{SO}(N-2)}$. But if D is a singleton then what precedes must contain at most one component of D''_{ν_i} since after restriction to $\overline{K}$ (recall that $K\simeq SO(2)\times SO(N-2)$) one gets a sum of irreducible representations of the form $\chi(\nu_1)\otimes D''_\nu$ such that the multiplicities of ν_1 and ν are both 1. Thus one has necessarily $\lambda_1=\cdots=\lambda_{r-1}=|\lambda_r|$ and this is equivalent, as it is proved in [4], to the relation (2.6).

Case 2: D is infinite dimensional. The Cartan decomposition of $\mathfrak{g}=\mathfrak{so}(2,N-2)$ writes $\mathfrak{k}+\mathfrak{p}$ and the triangular one is given by $\mathfrak{g}^{\mathbb{C}}=\mathfrak{n}^-+\mathfrak{h}+\mathfrak{n}^+$. A common Cartan subalgebra $\mathfrak{h}$ to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ is generated by $H_1=-\sqrt{-1}M_{-1,0}$ and, for j running from 2 to the rank r, $H_j=\sqrt{-1}M_{2j-3,2j-2}$.We write $(\varepsilon_j)_{1\leq j\leq r}$ for the dual basis of $(H_j)_{1\leq j\leq r}$, such that the roots are given by $\Delta=\Delta^+\cup\Delta^-$ where Δ^+ and Δ^- are respectively the sets of positive and negative roots for the lexicographic order, i.e., $\Delta^-=-\Delta^+$ and

$$\Delta^+=\{\varepsilon_j\pm\varepsilon_k,1\leq j<k\leq r\}\cup\{\varepsilon_j,1\leq j\leq r\}^{N-2r},$$

where $E^0=\emptyset$ and $E^1=E$ for any set E. A more appropriate basis of $\mathfrak{g}^{\mathbb{C}}$ is given by the family $(X_{ij})_{-r\leq i,j\leq r}$, such that:

$$X_{ij}=-X_{ji} \text{ for } -r\leq i,j\leq r,\quad H_j=X_{-j,j} \text{ for } 1\leq j\leq r,$$

[2] We use the same notation D for the corresponding representations of $\mathscr{U}$ and $\mathfrak{so}(2,N-2)$.

and satisfying the following commutation relations:

$$[H, \pm X_{\pm j,\pm\sigma k}] = \pm(\varepsilon_j + \sigma\varepsilon_k)(H)(\pm X_{\pm j,\pm\sigma k}), \quad \forall H \in \mathfrak{h}, \tag{2.8}$$
$$[X_{j,\sigma k}, -X_{-j,-\sigma k}] = H_j + \sigma H_k, \tag{2.9}$$

where $|j| \neq |\sigma k|$, $1 \leq j,k \leq r$, and

$$\sigma \in \begin{cases} \{-1,1\} \text{ if } N \text{ is even,} \\ \{-1,0,1\} \text{ if } N \text{ is odd.} \end{cases}$$

In this way one sees that when $i, j \geq 1$, X_{ij} corresponds to the root $\varepsilon_i + \varepsilon_j$, $X_{-i,j}$ to the root $-(\varepsilon_i - \varepsilon_j)$, and so on. The set Δ_c^+ of positive compact roots is obtained by restricting the indices $i, j, \ldots$ of the roots to $\{-r, \ldots, r\} \setminus \{-1,1\}$. The remaining positive roots are the noncompact ones, the set of which we write Δ_n^+.

Now let D be a singleton representation of G realized on a Banach space $\mathscr{H}$. Then one can write, under the action of $\mathfrak{k}$:

$$\mathscr{H}^\infty = \oplus_\mu K(\mu), \tag{2.10}$$

each $K(\mu)$ being an irreducible $\mathfrak{k}$-module. The action of $\mathfrak{p}^\pm = \mathfrak{p}^\mathbb{C} \cap \mathfrak{n}^\pm$, which is generated by the family $(X_{\pm 1,j})_{|j|\neq 1}$, sends a nonzero $\mathfrak{k}$-module $K(\mu_1, \vec{\mu})$ to an irreducible one $K(\mu_1 \pm 1, \vec{\mu}')$ for some $\vec{\mu}'$, since D is a singleton. Suppose that $K(\mu) \neq \{0\}$ but $X_{\pm 1,2}K(\mu) = \{0\}$. Then the second component of the weights of $\mathscr{U}(\mathfrak{p}^\pm)K(\mu)$ are bounded from above by μ_2. Two cases arise: $N > 5$ or $N = 4$ (since $n \geq 3$), the first one being the only one for which $\mathfrak{k}$ is semi-simple. Let $N \geq 5$. What precedes implies that $\mathscr{U}(\mathfrak{p}^\pm)K(\mu)$ is finite dimensional because each weight is a Δ_c^+-dominant integer. Indeed, some power of $X_{\pm 1,j}$, $|j| \notin \{0,1\}$, is zero on $K(\mu)$. If N is odd, some power of $X_{\pm 1,0}$ is also vanishing, thanks to the relation $[X_{20}^m, X_{\pm,-2}^m] = m! X_{\pm,0}^m$.

Thus for an infinite-dimensional D, there exists $\varepsilon \in \{-1,1\}$ such that the powers of $X_{\varepsilon,2}$ are not vanishing on nonzero $\mathfrak{k}$-modules. Without loss of generality, we shall consider from now on that $\varepsilon = -1$, i.e., $X_{-1,2}K(\mu) \neq \{0\}$ for each nonzero $K(\mu)$. It follows that $D|_{\overline{K}_n} = \oplus_{l\in\mathbb{Z}} \pi_{\lambda + l(\varepsilon_1 - \varepsilon_2)}$, hence $D|_{\overline{K}_n}$ is a highest weight representation, i.e., there exist a weight $\lambda^{(0)}$ such that $D|_{\overline{K}_n} = \oplus_{l\in\mathbb{N}} \pi_{\lambda^{(0)} - l(\varepsilon_1 - \varepsilon_2)}$. We shall write $D_{\lambda^{(0)}}$ or $D(-\lambda_1^{(0)}, \vec{\lambda^{(0)}})$ for such a representation.

Let $K(\mu) \neq \{0\}$. Since $[\mathfrak{k}^\mathbb{C} \cap \mathfrak{n}^+, X_{-1,2}] = \{0\}$, $X_{-1,2}^2 K(\mu)$ is an irreducible $\mathfrak{k}$-module of weight $\mu - 2(\varepsilon_1 - \varepsilon_2) = (\mu_1 - 2, \mu_2 + 2, \mu_3, \ldots, \mu_r)$. The element $Y_{-1,-1} = \sum_{|k|\neq 1} X_{-1,k} X_{-1,-k}$ of $\mathscr{U}^\mathbb{C}$ satisfies $[\mathfrak{k}^\mathbb{C}, Y_{-1,-1}] = \{0\}$, thus $Y_{-1,-1}K(\mu)$ is also an irreducible $\mathfrak{k}$-module of weight $\mu - 2\varepsilon_1 = (\mu_1 - 2, \mu_2, \mu_3, \ldots, \mu_r)$. As D is an infinite-dimensional singleton, $X_{-1,2}^2 K(\mu) \neq \{0\}$ and the multiplicity of $\mu_1 - 2$ is 1. Then one has necessarily $Y_{-1,-1}K(\mu) = \{0\}$. It follows that $Y_{-1,-1} = 0$ on $\mathscr{H}^\infty$. Finally the application of the adjoint representation on $Y_{-1,-1}$ yields the following relation on $\mathscr{H}^\infty$:

$$\sum_{i=-r}^{r} X_{j,-i}X_{-k,i} + \frac{N-2}{2}X_{-k,j} + \frac{2}{N}\delta_{jk}\mathbf{C}_2 = 0, \quad \forall j,k \in \{-r,\dots,r\}. \tag{2.11}$$

The $N=4$ case is more simple since $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to a direct sum of two copies of $\mathfrak{sl}(2)^{\mathbb{C}}$ and D is a singleton if and only if its restriction to one, and only one, of the two copies is trivial, but this property is equivalent to the relation (2.11).

Finally the fundamental relation (2.6) follows for D as a representation of $\mathfrak{g}$ and it has been proved in [4] that each nontrivial representation that satisfies (2.6) is a singleton (finite or infinite dimensional).

The following result is a characterization of the infinite-dimensional singletons of the n-anti-de Sitter group (or $(n-1)$-conformal group) $\overline{SO}_0(2,n-1)$.

Corollary 2.3 *Let D an infinite-dimensional IR of* $\overline{S}_n = \overline{SO}_0(2,n-1)$, $n \geq 3$. *The following conditions are equivalent:*

i) D is a singleton;

ii) The restriction $D|_{\overline{L}_n}$ *of D to the n-Lorentz group* $\overline{SO}_0(1,n-1)$ *is a UIR;*

iii) The restriction $D|_{\overline{P}_{n-1}}$ *of D to the* $(n-1)$*-Poincaré group* $\overline{SO}_0(1,n-2) \ltimes T_{n-1}$ *is a UIR.*

Proof. It has been proved in [4] that an irreducible infinite-dimensional representation which satisfies the fundamental relation (2.6) is irreducible when restricted to the n-Lorentz group. This proves the implication i) $\Rightarrow$ ii). The implication iii) $\Rightarrow$ i) has been proved in the same paper and the proof of ii) $\Rightarrow$ iii) is in [25].

Remark 2.4 The restriction of an infinite-dimensional singleton of the n-anti-de Sitter group to the $(n-1)$-anti-de Sitter one (in other words one restricts from the conformal group to the anti-de Sitter one) is not irreducible in general. In fact there is only one case for which it is not irreducible, but it is a sum of two irreducible ones, as shown in theorem 3.4.

Let U a nontrivial UIR of the $(n-1)$-Poincaré group $\overline{P}_{n-1}$, the invariance group of the $(n-1)$-dimensional Minkowski space M_{n-1}. Then it is proved in [4] that if U extends to $\overline{S}_n = \overline{G}_{n-1}$, the conformal group of M_{n-1}, then the extension is a singleton uniquely defined by U.

Let D a finite-dimensional singleton of $\overline{SO}(N)$, $N \geq 4$. Then the restriction of D to $\overline{SO}(N-1)$ is irreducible if and only if N is even. If it is odd, the restriction is a sum of two irreducible ones.

2.2. Classification of the singletons

The following result is a corollary of the theorem 2.2. We write again D_μ or $D(-\mu_1,\vec{\mu})$ for a highest weight irreducible representation with weight $(\mu_1,\vec{\mu})$, of the anti-de Sitter group. The corresponding common Cartan subalgebra of $\mathscr{S}_n$ and of its maximal compact subalgebra $\mathfrak{k}_n$ is the one introduced in the preceding section. An infinite-dimensional highest weight representation is thus, with this choice, a positive minimal energy representation, $-\mu_1$ being the energy.

Theorem 2.5 *Let D a singleton representation of $G=\overline{SO}(2,N-2)$. Then, for a certain order of the roots, D is a highest weight IR such that:*

If D is finite dimensional, D is equivalent to one of the following series:

$$D(-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}), \qquad \text{if } N \text{ is odd}, \tag{2.12}$$

$$D(-s,s,\dots,s,\varepsilon s), \qquad 2s\in\mathbb{N}\setminus\{0\} \text{ and } |\varepsilon|=1, \text{ if } N \text{ is even}. \tag{2.13}$$

If D is infinite-dimensional, then it is a unitary representation equivalent to one of the following series:

$$D(s+\frac{N-4}{2},s,\dots,s), \qquad s\in\{0,\frac{1}{2}\}, \text{ if } N \text{ is odd}, \tag{2.14}$$

$$D(s+\frac{N-4}{2},s,\dots,s,\sigma s), \qquad 2s\in\mathbb{N} \text{ and } |\sigma|=1, \text{ if } N \text{ is even}. \tag{2.15}$$

Proof. Assume $N\geq 5$. Since D is a weight representation, one can assume that it is a HW one, with weight $\lambda^{(0)}$. Let $v\in\mathscr{H}^\infty$ be a maximal vector, i.e., such that $\mathfrak{n}^+v=\{0\}$ and $Hv=\lambda^{(0)}(H)v$ for all $H\in\mathfrak{h}$. Then applying the fundamental relation (2.11) to v yields

$$(\lambda_i^{(0)}-\lambda_{i+1}^{(0)})(\lambda_i^{(0)}+\lambda_{i+1}^{(0)}+\frac{N-4}{2}+1-i)=0, \text{ where } 1\leq i\leq r-1, \tag{2.16}$$

and, if N is odd,

$$\sum_{i=1}^{r}\lambda_i^{(0)}=-\frac{2}{N}\mathbf{C}_2. \tag{2.17}$$

Now writing $s=\lambda_2^{(0)}$, (2.16) gives the desired result when N is even, no matter if D is finite dimensional or not. If N is odd one has also, thanks to (2.17), $s(s-1/2)=0$.

If $N=4$, then one has $\lambda^{(0)}=(\lambda_1^{(0)},\lambda_2^{(0)})$ and thanks to the fundamental relation one gets $|\lambda_1^{(0)}|=|\lambda_2^{(0)}|$, which means that D is trivial on one of the two copies of $\mathfrak{so}(3)^{\mathbb{C}}$ (see the Remark below).

Remark 2.6 Let $\mathbf{J}_2$ be the (second order) Casimir operator of the "spin" subalgebra $\mathfrak{so}(N-2)$ of $\mathfrak{g}$. Then the relation (2.6) is equivalent to

$$\mathbf{J}_2-H_1^2=\frac{N-4}{N}\mathbf{C}_2.$$

In particular, if $N=4$ one has $H_1^2 = H_2^2$, since $\mathbf{J}_2 = H_2^2$.

On the lowest energy level of the singleton D the corresponding representation of $\mathfrak{so}(N-2)$ is itself a (finite-dimensional) singleton. If, conversely, one starts with a singleton representation of $\mathfrak{so}(N-2)$, then this can be the ground state of two singleton representations of $\mathfrak{g}$: one being finite dimensional, the other infinite dimensional.

2.3. A remarkable property of singletons

Here we present a generalization of the theorem of Flato and Frønsdal [13]: " *singleton* $\otimes$ *singleton* $= \oplus$ *massless representations* " proved in 1978 for the case $N=4$.

Let $\mathrm{Rac} = D(\frac{N-4}{2}, 0, \dots, 0)$ and $\mathrm{Di}^{\pm} = D(\frac{1}{2}+\frac{N-4}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \varepsilon\frac{1}{2})$, where $|\varepsilon| = 1$ and $\pm$ is the sign of ε. Then the former is the spin 0 unitary singleton while the latter is one of the two unitary singletons (resp. the unitary singleton) with spin $\frac{1}{2}$ if N is even (resp. odd). If N is odd, $\varepsilon = 1$. We just write Di when $\varepsilon = 1$, N odd or even.

Theorem 2.7

$$\mathrm{Rac} \otimes \mathrm{Rac} = \oplus_{s=0}^{\infty} D(s+N-4, s, 0, \dots, 0), \tag{2.18}$$

$$\mathrm{Rac} \otimes \mathrm{Di}^{\pm} = \oplus_{s-\frac{1}{2}=0}^{\infty} D(s+N-4, s, \frac{1}{2}, \dots, \frac{1}{2}, \varepsilon\frac{1}{2}). \tag{2.19}$$

Proof. Let D' and D'' two unitary singletons of $\overline{G}$, both with the same energy sign, i.e., both highest weight representations or both lowest ones. We are interested in reducing the product $D' \otimes D''$. Thus if D is an irreducible representation contained in this product then $D(M_{ab}) = D'(M_{ab}) \otimes 1 + 1 \otimes D''(M_{ab})$ for all a and b. For simplicity of the proof we use the notation introduced just before Theorem 2.2 and we write M', M'', and M instead of $D'(M) \otimes 1$, $1 \otimes D''(M)$, and $D(M)$, respectively. Then one has, since D' and D'' satisfy the fundamental relation (2.6),

$$\begin{aligned} M'^2 &= \frac{N-2}{2} M' + \frac{2}{N} \mathbf{C}'_2 \delta, \\ M''^2 &= \frac{N-2}{2} M'' + \frac{2}{N} \mathbf{C}''_2 \delta. \end{aligned} \tag{2.20}$$

Then one gets, because of the relation $M^2 = M'^2 + M'M'' + M''M' + M''^2$:

$$M^2 = \frac{N-2}{2} M + \frac{2}{N} (\mathbf{C}'_2 + \mathbf{C}''_2) \delta + K^2, \tag{2.21}$$

where $K^2 = M'M'' + M''M'$. More generally we define K^k, $k \geq 2$, by

$$K^k = \overbrace{M'M''M'M''\cdots}^{k \text{ terms}} + \overbrace{M''M'M''M'\cdots}^{k \text{ terms}}.$$

Multiplying both sides of (2.21) by M and using (2.20), one finds

$$M^3 = \frac{N-2}{2}M^2 + \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)M + \frac{N-2}{2}K^2 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M') + K^3. \tag{2.22}$$

Let $D' = D'' = \text{Rac}$. Then $\mathbf{C}'_2 = \mathbf{C}''_2 = -\frac{N(N-4)}{4}$, if one identifies the Casimir operators with their scalar values. It follows that $\mathbf{C}'_2 M'' + \mathbf{C}''_2 M' = -\frac{N(N-4)}{4}M$. The expression of K^2 follows from (2.21). It remains to calculate K^3. For let us define the symmetrizer S by $S(M_{ab}M_{cd}) = \frac{1}{2}(M_{ab}M_{cd} + M_{cd}M_{ab})$ and consider the element of the enveloping algebra $\Lambda^2_{abcd} = S(M_{ab}M_{cd}) + S(M_{bc}M_{ad}) + S(M_{ca}M_{bd})$. Then Λ^2_{abcd} is completely skew-symmetric in the indices a,b,c,d and is sent to zero by Rac, what we may write $\Lambda'^2_{abcd} = 0$. This implies, for each a, d:

$$\begin{aligned}\sum_{bc}\Lambda'^2_{abcd}M''^{bc} =& 2\sum_{bc}M'_{ab}M''^{bc}M'_{cd} - \sum_{bc}M'_{ab}M''^{bc}\eta_{cd} \\ &+ \sum_{bc}M'_{bc}M''^{bc}(M'_{ad} - \eta_{cd}) = 0,\end{aligned} \tag{2.23}$$

hence $M'M''M' = M'M'' + \frac{1}{2}\text{Tr}(M'M'')(M' - \delta)$. A similar formula holds for $M''M'M''$. Thus, after adding them, one gets

$$K^3 = K^2 + \frac{1}{2}\text{Tr}(M'M'')(M - 2\delta).$$

Since $\mathbf{C}_2 = \frac{1}{2}\text{Tr}(M^2) = \frac{1}{2}\text{Tr}(M'^2 + M'M'' + M''M' + M''^2)$ one has $\text{Tr}(M'M'') = \mathbf{C}_2 - (\mathbf{C}'_2 + \mathbf{C}''_2)$ thanks to the relations above and, after factorizing, it follows that

$$[M^2 - (N-3)M - \frac{1}{2}\mathbf{C}_2\delta](M - 2\delta) = 0. \tag{2.24}$$

Let us determine explicitly the corresponding representations D. Since Rac is a highest weight representation, D is a also a highest weight one, say λ. The lowest energy of Rac being $\frac{N-4}{2}$, one has necessarily $\lambda_1 \leq -(N-4)$. To determine the possible values of λ, let us complexify $\mathfrak{g}$ and consider the (X_{ij})-basis version of the relation (2.24):

$$[X^2 + (N-3)X - \frac{2}{2}\mathbf{C}_2\delta](X + 2\delta) = 0. \tag{2.25}$$

Then the application of this relation on the maximal vector (ground state) v_λ of D yields:

$$[\lambda_1^2 + (N-3)\lambda_1 - \frac{1}{2}\mathbf{C}_2](\lambda_1 + 2) = 0.$$

Since $\lambda_1 \leq -(N-4)$, if $N \geq 5$ it follows:

$$\lambda_1^2 + (N-3)\lambda_1 - \frac{1}{2}\mathbf{C}_2 = 0. \tag{2.26}$$

If $N = 4$, we shall see that $\lambda_1 = -2$ is a solution of Equation (2.26). Thus we can work only with the latter. For the second component of λ one gets, since this weight is a Δ_c^+-dominant integer:

$$\lambda_2^2 + \lambda_1 + (N-4)\lambda_2 - \frac{1}{2}\mathbf{C}_2 = 0. \tag{2.27}$$

Subtracting this equation from the preceding one gives:

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + N - 4) = 0,$$

from which follows $\lambda_1 = -(\lambda_2 + N - 4)$. This condition implies that the other components are zero, i.e., $\lambda = (-s - N + 4, s, 0, \ldots, 0)$ where $s = \lambda_2$. Then the integrability conditions imply that s is an integer. The necessary condition is thus proved. To prove that this is sufficient consider the following vectors, where $s \in \mathbb{N}$, v' is the maximal vector of D' and v'' the maximal one of D'' (see Remark 2.8 below):

$$v_s = \sum_{l=0}^{s} \frac{s!(-1)^l}{(s-l)!l!} \frac{\Gamma(-\frac{N-4}{2} - l + 1)\Gamma(-\frac{N-4}{2} - s - l + 1)}{\Gamma(-\frac{N-4}{2} + 1)^2} (X_{-1,2}^{s-l} v') \otimes (X_{-1,2}^{l} v''). \tag{2.28}$$

Then $\mathscr{U}(\mathfrak{g})v_s$ carries an irreducible highest weight representation of $\mathfrak{g}$ with weight $\lambda = (-s - N + 4, s, 0, \ldots, 0)$. This finishes the proof for the Rac $\otimes$ Rac part.

To prove the other part let again $D' = \text{Rac}$ but $D'' = \text{Di}$. Then $\mathbf{C}''_2 = -\frac{N(N-3)}{8} \neq \mathbf{C}'_2$ and Λ''^2_{abcd} is no longer zero. But considering the fourth degree of M and using the relations (2.21) and (2.22) one gets:

$$\begin{aligned} M^4 =& (N-2)M^3 + \big[-(\frac{N-2}{2})^2 + \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)\big]M^2 \\ &- (N-2)\frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)M - (\frac{2}{N})^2(\mathbf{C}'_2 + \mathbf{C}''_2)^2\delta + (K^2)^2. \end{aligned} \tag{2.29}$$

After some calculations, we find

$$(K^2)^2 = K^4 + \frac{N-2}{2}\big[K^3 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M')\big] + \frac{8}{N}\mathbf{C}'_2\mathbf{C}''_2\delta.$$

From Equation (2.22) one may write $K^3 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M')$ in terms of M, M^2, and M^3. Thus it remains to calculate K^4. After lengthy calculations one arrives to

$$S(K^4) = [N - 2 + \mathrm{Tr}(M'M'')][M^2 - \frac{N-2}{2}M - \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)\delta].$$

This relation, together with the others above yields after other lengthy calculations an expression for $S(M^4)$ from which one gets the factorized relation:

$$\big[M^2 - (N-3)M - \frac{1}{2}\big(\mathbf{C}_2 - \frac{(N-4)(N-5)}{8}\big)\delta\big]\big(M - \frac{N-1}{2}\delta\big)\big(M - \frac{3}{2}\delta\big) = 0. \tag{2.30}$$

Again, in order to identify explicitly the representations contained in $\mathrm{Rac}\otimes\mathrm{Di}$, we write the preceding relation in terms of the basis (X_{ij}) of $\mathfrak{g}^{\mathbb{C}}$. Then one gets:

$$\big[X^2+(N-3)X-\frac{1}{2}\big(\mathbf{C}_2-\frac{(N-4)(N-5)}{8}\big)\delta\big](X+\frac{N-1}{2}\delta)(X+\frac{3}{2}\delta)=0. \tag{2.31}$$

Using similar arguments as for the case $\mathrm{Rac}\otimes\mathrm{Rac}$ yields

$$\lambda_1^2+(N-3)\lambda_1-\frac{1}{2}\big(\mathbf{C}_2-\frac{(N-4)(N-5)}{8}\big)=0, \tag{2.32}$$

and, for λ_2, the weight being a Δ_c^+-dominant integer:

$$\lambda_2^2+\lambda_1+(N-4)\lambda_2-\frac{1}{2}\big(\mathbf{C}_2-\frac{(N-4)(N-5)}{8}\big)=0. \tag{2.33}$$

After subtracting this equation from the preceding one it follows:

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+N-4)=0,$$

from which finally one gets $\lambda=(-s-N+4,s,\frac{1}{2},\dots,\frac{1}{2})$, where $s=\lambda_2$ and $s-\frac{1}{2}\in\mathbb{N}$. Now to obtain the sufficient condition we consider again the vectors v_s of (2.28) but with some changes: We replace in the right-hand side s by $s-\frac{1}{2}$. Evidently v'' is now a maximal vector for Di. Then again $\mathscr{U}(\mathfrak{g})v_s$ carries an irreducible highest weight representation of $\mathfrak{g}$, but now with weight $\lambda=(-s-N+4,s,\frac{1}{2},\dots,\frac{1}{2})$.

The case $D''=\mathrm{Di}^-$ is similar. The only change is $\lambda_r=-\frac{1}{2}$ instead of $\lambda_r=\frac{1}{2}$.

Remark 2.8 One may reduce the product $\mathrm{Di}\otimes\mathrm{Di}$ in the same manner as in Theorem 2.7, i.e., by seeking for an ideal which is sent to zero by each representation contained in $\mathrm{Di}\otimes\mathrm{Di}$. Another way to reduce this product is to consider one of the Di's as a summand in the product $\Sigma\otimes\mathrm{Rac}$, Σ being the spinor representation, and consider the product $\Sigma\otimes\mathrm{Rac}\otimes\mathrm{Di}$.

The vectors v_s appearing in (2.28) can be realized explicitly in a very simple manner. For this, let us realize the Rac on the cone $\{y\in\mathbb{R}^{2,N-2}\mid y^2=0\}$ in the usual way. Then one may choose v' to be the function $y\mapsto(x_1)^{-\frac{N-4}{2}}$ where $x_1=\frac{\sqrt{-1}y_{-1}-y_0}{\sqrt{2}}$. For the other Rac, v'' is defined in a similar manner but with primes on the variables. For the Di, if v'' is the maximal vector, one can choose the map $y'\mapsto({x'}_1)^{-\frac{N-3}{2}}w_{\frac{1}{2}}$, where $w_{\frac{1}{2}}$ is the maximal vector of Σ. Now let $\sigma=0$ or $\frac{1}{2}$, $w_0=1$, and define $x_2=\frac{y_1+\sqrt{-1}y_2}{\sqrt{2}}$ and x'_2 by a similar formula, but with primes on the variables. D'' can be either a Rac or a Di. Then the map

$$v_s:(y,y')\mapsto(x_1x'_1)^{-\frac{N-4}{2}-s}(x_1x'_2-x_2x'_1)^{s-\sigma}w_\sigma,$$

where $s-\sigma\in\mathbb{N}$, is a maximal vector of the representation $D(s+N-4,s,\sigma,\dots,\sigma)$.

3. Masslessness

Since the anti-de Sitter space-time H_n^ρ contracts to the Minkowski space-time M_n, it is reasonable to demand that the notion of masslessness on H_n^ρ should correspond under the contraction to masslessness on M_n, where the square of the mass operator (usually denoted by $\sum P_\mu P^\mu$) is sent to 0 by massless representations of the Poincaré group $\overline{P}_n$. Such representations, when they do not have continuous spin (we will say *discrete spin*)[3], are induced by a unitary finite dimensional representation of a semi-direct product of the Euclidean subgroup $\overline{E}(n-2)$ (contained in $\overline{L}_n$) by the subgroup of translations T_n. Their restrictions to the translations T_{n-2} of $\overline{E}(n-2)$ are trivial. The subgroup $\overline{E}(n-2)$ is such that the isotropic cone of M_n is homeomorphic to $\overline{L}_n/\overline{E}(n-2)$. Contractibility may be used as a criterion for masslessness on the anti-de Sitter space. In the definition we will give below, by *natural contraction* of a representation of the anti-de Sitter group to a representation of the Poincaré group shall mean a contraction which leaves the restriction to the Lorentz group L_n (contained in both of them) invariant up to equivalence. It is thus compatible with the contraction $H_n^\rho \stackrel{\rho\to 0}{\longrightarrow} M_n$. For example, the minimal energy representation $U = D(E_0, \vec{\lambda})$ may be contracted to a representation of the Poincaré group. In terms of the curvature, $\sqrt{\rho}E_0$ is sent to the mass. If E_0 is fixed, for example equal to $\frac{n-3}{2}$, then the resulting mass is 0. But the contracted representation is massless for the Poincaré group if it is not trivial on the translations.

Definition 3.1 *A unitary representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* massless *if U contracts naturally to a discrete spin massless representation of the Poincaré group $\overline{P}_n$.*

An immediate consequence is that singletons of $\overline{S}_n$ are not massless representations. Indeed such a singleton contracts to a representation of the Poincaré group $\overline{P}_n$ which is trivial when restricted to the subgroup of translations T_n [25]. But singletons are not massive particles either, in the sense that a massive particle on the anti-de Sitter space-time must be described by a representation of $\overline{S}_n$ which contracts to a massive representation of the Poincaré group $\overline{P}_n$, thus necessarily nontrivial on the translations. Hence singletons have no analog in Minkowski spaces M_n. This was already pointed out by Flato and Frønsdal for the $n = 4$ case (see for example [15]).

The notion of masslessness is not unique since several nonequivalent representations of $\overline{S}_n$ may be contracted to a massless representation of $\overline{P}_n$. Below we shall consider two notions of masslessness, both closely related to singletons. To distinguish them we give the following Definitions.

[3] When the space-time dimension n is even helicity is easily defined: It is a straightforward generalization of the notion of helicity in the 4-dimensional case. Thus a discrete spin representation is nothing but discrete helicity representation when n is even (see Remark 7 in [25]).

Definition 3.2 *A massless representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* conformal massless *if there exists a singleton D of the conformal group $\overline{G}_n$ such that $U \sim D|_{\overline{S}_n}$.*

This is the classical definition of masslessness given by Flato and Frønsdal when $n = 4$ (see for example [14]). A remarkable fact is that a conformal massless representation $U \sim D|_{\overline{S}_n}$ contracts naturally to the restriction $D|_{\overline{P}_n}$ of the singleton D to the Poincaré group. $D|_{\overline{P}_n}$ is irreducible (see Corollary 2.3) and is a discrete spin massless representation of $\overline{P}_n$ (see [4]).

It is important to notice that since the unitarity of U is required, only infinite-dimensional (unitary) singletons D may be used in the preceding definition.

Next, we present a second notion of masslessness. We shall see that it does not coincide with the first notion given, if $n \geq 5$.

Definition 3.3 *A massless representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* composite massless *if U occurs in the reduction of the tensor product $D_1 \otimes D_2$ where D_1 and D_2 are irreducible weight representations, with the same energy sign, equivalent to a* Rac *or a* Di*.*

This definition means that a composite massless representation describing a massless particle on the anti-de Sitter space is composed of two subparticles, the singletons, in the same manner as nucleons are composed of quarks, except that singletons are unobservable for kinematical reasons while the unobservability of quarks is due to their confinement [15]. The representations appearing in the right-hand side of Theorem 2.7 are composite massless representations. They were considered for $n = 5$ by Ferrara and Frønsdal as the massless ones in [11, 12]. Irreducibility and Gupta-Bleuler (GB) quantization are almost always possible, but there are few exceptions. Some results concerning GB quantization are given in Section 4. Here we list the conformal massless representations.

Theorem 3.4. ([25]) *Let U be a conformal massless representation of the anti-de Sitter group $\overline{S}_n$. Then for a certain order of the roots:*

$$n \text{ even} \Longrightarrow \begin{cases} U \sim D(s+\frac{n-2}{2}, s, \ldots, s), \text{ for } s \neq 0 \text{ such that } 2s \in \mathbb{N} \quad \text{or} \\ U \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0), \end{cases} \tag{3.1}$$

$$n \text{ odd} \Longrightarrow \begin{cases} U \sim D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \quad \text{or} \\ U \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0). \end{cases} \tag{3.2}$$

In particular there is no conformal massless representations with spin different from 0 or $\frac{1}{2}$ in odd-dimensional anti-de Sitter spaces. It can be seen that this is still true in odd-dimensional Minkowski spaces if one defines conformal masslessness in a similar fashion, i.e., by restricting a singleton of the conformal group to the Poincaré group.

The following questions naturally arise: Are the conformal massless representations composite massless? What about the converse? The answer is practically negative. In fact, we have:

Theorem 3.5. **([25])** *A conformal massless representation is composite massless if and only if* $n = 3$ *or* $n = 4$. *More precisely, there is no tensor product of unitary weight representations with the same energy sign containing such a representation.*

Thus for $n \geq 5$, conformal invariance is not compatible with singleton composed of massless particles, provided that those singletons have the same energy sign. Though other type of composite particles are allowed. For example one can find conformal massless representations contained in the tensor product of a singleton by some multipleton or in the product of some two multipletons (see Remark 1 in [25]). Here we call a multipleton (or m-ton, for a certain m) a representation for which the diagram of maximal weights is included in m parallel lines: a 1-ton is a singleton, a 2-ton is a doubleton, and so on. Multipletons are generally not unitary; it is the case of the multipletons concerned by the "compositeness" of conformal massless representations.

4. Gupta-Bleuler quantization

For simplicity, we consider in this section that $n \geq 4$.

The method used to construct GB triplets is the following: Suppose that the IR $D(E,\vec{\lambda})$ is unitary if $E \geq E_0$ (E_0 is the *limit of unitarity*). Then usually when $E \to E_0$ ($E \neq E_0$), even for $E < E_0$, the IR $D(E,\vec{\lambda})$ becomes indecomposable. More precisely one obtains a (non-direct) sum of $D(E_0,\vec{\lambda})$ with another representation. The (physical) representation $D(E_0,\vec{\lambda})$ is realized as a quotient (by the gauge representation). Then using a third (scalar) representation together with some conditions usually satisfied one may construct a GB triplet.

The physical and gauge representations are usually minimal or maximal energy representations and are related to the Verma modules in the following way. Let $\mathfrak{g}$ a noncompact semi-simple Lie algebra, the typical example being $\mathfrak{so}(2,N-2)$. Let $\mathfrak{k}$ the (reductive) maximal compact subalgebra of $\mathfrak{g}$; $\mathfrak{k} \simeq \mathbb{R} \oplus \mathfrak{so}(N-2)$ for $\mathfrak{so}(2,N-2)$. Let $K(\lambda)$ a finite-dimensional simple $\mathfrak{k}$-module with weight λ. As usual the energy is $E = -\lambda_1$. Then there exists E_0 depending on $\vec{\lambda}$ such that the $\mathfrak{g}$-module $N(\lambda) = \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^+)} K(\lambda)$ is not unitarizable if $E = |\lambda_1| < E_0$ [2, 9]. In particular, if $\mathfrak{g} = \mathfrak{so}(2,N-2)$, this module is unitarizable if and only if $E \geq E_0$, $E_0 = \lambda_2 + k$, where k a constant depending on $\vec{\lambda}$. $N(\lambda)$ is not always irreducible. If $\mathfrak{g} = \mathfrak{so}(2,N-2)$, $N(\lambda)$ is irreducible (i.e., $N(\lambda) = L(\lambda)$) if and only if $E > E_0$. If $E = E_0$ then $N(\lambda)$ is not irreducible. It contains a maximal submodule I generated by relations similar to those of (2.6), the fundamental relation of singletons, or to those of (2.24) and (2.30), satisfied by the composite massless representations, obtained in Section 2. The irreducible quotient $L(\lambda) = N(\lambda)/I$ corresponds to the physical space and carries the representation $D(E_0,\vec{\lambda})$ while an irreducible quotient of I corresponds to the gauge space. To construct a GB triplet one needs a third space, endowed with an indefinite metric, and a representation conjugated to the gauge defined on a quotient space, the so-called scalar space. This is the method for constructing GB triplets which we shall call *natural* in the remaining.

Note that GB triplets do not require unitarity of the three representations; only the physical one has to be unitary.

In what follows we will give some examples of GB triplets for all singletons and almost all of the massless representations. Futher details can be find in [25].

The representations we shall consider are those of the anti-de Sitter group $\overline{S}_n$. The *Lorentz conditions*, i.e., the conditions which define the space $\mathscr{H}_2$ (the physical space is $\mathscr{H}_2/\mathscr{H}_3$), follow directly from the relations defining the submodule I (e.g., see (2.6)).

4.1. Singletons: spin 0

Let $\mathscr{H}_1^{(0)}$ the space of square-integrable (with respect to the Riemannian measure) positive energy solutions of $(\partial^2)^2\varphi = 0$ and $y\cdot\partial\varphi = -\frac{n-3}{2}\varphi$, $\mathscr{H}_2^{(0)} = \{\varphi \in \mathscr{H}_1^{(0)} \mid \partial^2\varphi = 0\}$ and $\mathscr{H}_3^{(0)} = \{\varphi \in \mathscr{H}_2^{(0)} \mid \varphi \text{ has the form } y^2\phi\}$, where $y \in H^+ = \cup_{\rho>0} H_n^\rho$, $\partial^2 = \sum \partial_a\partial^a$, and $y\cdot\partial = \sum y^a\partial_a$. Then these spaces realize the GB triplet:

$$D(\frac{n+1}{2},0,\dots,0) \to D(\frac{n-3}{2},0,\dots,0) \to D(\frac{n+1}{2},0,\dots,0),$$

i.e., $\mathscr{H}_1^{(0)}/\mathscr{H}_2^{(0)}$ and $\mathscr{H}_3^{(0)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n+1}{2},0,\dots,0)$ while the quotient $\mathscr{H}_2^{(0)}/\mathscr{H}_3^{(0)}$ (physical space) carry the singleton $D(\frac{n-3}{2},0,\dots,0)$. Since $\lim_{y^2\to 0}\varphi = 0$ if $\varphi \in \mathscr{H}_3^{(0)}$ one may realize the singletons in $\mathscr{H}_2^{(0)}$ by taking $\lim_{y^2\to 0}\varphi(y)$ or, equivalently, at the "boundary" of the space-time, i.e., by considering $\lim_{R\to\infty} R^{\frac{n-3}{2}}\varphi(y)$, where $R = \sqrt{\sum_{a=1}^{n-1}(y^a)^2}$.

4.2. Singletons: spin 1/2

Let Σ be the spinor representation on the spinor module V_Σ (not irreducible if $n-1$ is even), (γ_a) the Dirac matrices such that $\{\gamma_a,\gamma_b\} = 2\eta_{ab}$ (if $n-1$ is odd then γ_{n-1} is a multiple of the product of the others). As usual $\not{y} = \sum y^a\gamma_a$, $\not{\partial} = \sum\partial^a\gamma_a$. To realize the GB triplet explicitly we reduce the tensor product $D(E,\vec{0})\otimes\Sigma$, $E > \frac{n-1}{2}$, to realize the irreducible representation $D(E-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$ and then we let $E \to \frac{n-1}{2}$ to get an indecomposable representation from which we realize the following GB triplet. The spinors we are using are the maps $\Psi : H^+ \to V_\Sigma$.

Let $\mathscr{H}_1^{(1/2)}$ be the space of spinors which are square-integrable positive energy solutions of $\partial^2\Psi = 0$, $y\cdot\partial\Psi = -\frac{n-2}{2}\Psi$ and $(\not{y}\not{\partial})^2\Psi = 0$. We shall write $\mathscr{H}_2^{(1/2)} = \{\Psi \in \mathscr{H}_1^{(1/2)} \mid \not{\partial}\Psi = 0\}$ and $\mathscr{H}_3^{(1/2)} = \{\Psi \in \mathscr{H}_2^{(1/2)} \mid \Psi \text{ has the form } \not{y}\Phi\}$. Let

$\nu = 0$, if n is even, and 1 otherwise. Then these spaces realize the GB triplet:

$$D(\frac{n}{2}, \frac{1}{2}, \dots, \frac{1}{2}, (-1)^\nu \frac{1}{2}) \to D(\frac{n-2}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \to D(\frac{n}{2}, \frac{1}{2}, \dots, \frac{1}{2}, (-1)^\nu \frac{1}{2}),$$

i.e., $\mathscr{H}_1^{(1/2)}/\mathscr{H}_2^{(1/2)}$ and $\mathscr{H}_3^{(1/2)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n}{2}, \frac{1}{2}, \dots, \frac{1}{2}, (-1)^\nu \frac{1}{2})$ while the quotient $\mathscr{H}_2^{(1/2)}/\mathscr{H}_3^{(1/2)}$ (physical space) carries the singleton $D(\frac{n-2}{2}, 1/2, \dots, 1/2)$. Since $\lim_{y^2\to 0} \not{y}\Psi = 0$ whenever $\Psi \in \mathscr{H}_3^{(1/2)}$ one can realize the singletons in $\mathscr{H}_2^{(1/2)}$ by taking $\lim_{y^2\to 0} \not{y}\Psi(y)$ or, equivalently, at the "boundary" by $\lim_{R\to\infty} R^{\frac{n-1}{2}}\Psi(y)$.

4.3. Singletons: spin $s \geq 1$

Here $n-1$ is necessarily even. The tensor product we use is $D(E, \vec{0}) \otimes (\Sigma)^{\otimes 2s}$, with E close to $\frac{n-3}{2} + 2s$. The corresponding multispinors are the maps $\Psi : H^+ \to \Sigma^{\otimes 2s}$. Define $\gamma_a^{(t)}$ by $\gamma_a^{(t)} v_1 \otimes \cdots v_t \cdots v_{2s} = v_1 \otimes \cdots \gamma_a v_t \cdots v_{2s}$, $\not{y}^{(t)} = \sum y^a \gamma_a^{(t)}$, $\not{\partial}^{(t)} = \sum \partial^a \gamma_a^{(t)}$, τ_{ij} the transposition $i \leftrightarrow j$ and

$$\mathscr{Y} = \frac{1}{2s}\Big[\sum_{1\leq t\leq 2s} \tau_{(t,2s)}\Big]\not{y}^{(2s)}.$$

Let $\mathscr{H}_1^{(s)}$ be the space of spinors which are square-integrable and positive energy solutions of $\partial^2\Psi = 0$, $y\cdot\partial\Psi = (-\frac{n-3}{2} - 2s)\Psi$, and $(\sum_{t=1}^{2s} \not{y}^{(t)}\not{\partial}^{(t)})^2\Psi = 0$. Let $\mathscr{H}_2^{(s)} = \{\Psi \in \mathscr{H}_1^{(s)} \mid \not{\partial}^{(t)}\Psi = 0, \forall t\}$ and $\mathscr{H}_3^{(s)} = \{\Psi \in \mathscr{H}_2^{(s)} \mid \Psi$ has the form $\mathscr{Y}\Phi\}$. Then these spaces realize the GB triplet:

$$D(\frac{n-1}{2} + s, s, \dots, s-1) \to D(\frac{n-3}{2} + s, s, \dots, s) \to D(\frac{n-1}{2} + s, s, \dots, s-1),$$

i.e., $\mathscr{H}_1^{(s)}/\mathscr{H}_2^{(s)}$ and $\mathscr{H}_3^{(s)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n-1}{2} + s, s, \dots, s-1)$ while the quotient $\mathscr{H}_2^{(s)}/\mathscr{H}_3^{(s)}$ (physical space) carries the singleton $D(\frac{n-3}{2} + s, s, \dots, s)$. Since $\lim_{y^2\to 0}\sum_t \not{y}^{(t)}\Psi = 0$ whenever $\Psi \in \mathscr{H}_3^{(s)}$, as before one can realize the singletons in $\mathscr{H}_2^{(s)}$ as the limit $\lim_{y^2\to 0}\sum_t \not{y}^{(t)}\Psi(y)$ or, again, at the "boundary" as the limit $\lim_{R\to\infty} R^{(\frac{n-3}{2}+2s)}\Psi(y)$.

4.4. Conformal massless representations: spin $s \geq 1$

Define ε by $|\varepsilon| = 1$ if $n-1$ is even and $\varepsilon = 1$ if $n-1$ is odd. The limit of unitarity of the IR $D(E_0, s, \dots, s, \varepsilon s)$ is $E_0 = \frac{n-2}{2} + s$ if n is even (or $n-1$ is odd) and $s \geq 1$, $E_0 = \frac{n-3}{2} + s$ if not, i.e., n odd (or $n-1$ even) or $s \in \{0, \frac{1}{2}\}$. Thus the construction of natural GB triplets for conformal massless representations is only possible in even-dimensional space-time, i.e., $n-1$ odd, and for spin greater or equal to 1. So let $s \geq 1$ and n even. We use the same tensor product as in the preceding subsection, but with E close to $\frac{n-2}{2} + 2s$.

Let $\mathscr{H}_1^{(s)}$ be the space of multispinors which are square-integrable positive energy solutions of $\partial^2\Psi = 0, y\cdot\partial\Psi = (-\frac{n-2}{2} - 2s)\Psi$ and $(\sum_{t=1}^{2s} \not{y}^{(t)}\not{\partial}^{(t)})^2\Psi = 0$. Let $\mathscr{H}_2^{(s)} = \{\Psi \in \mathscr{H}_1^{(s)} \mid \not{\partial}^{(t)}\Psi = 0,\ \forall\ \mathrm{t}\}$, $\mathscr{H}_3^{(s)} = \{\Psi \in \mathscr{H}_2^{(s)} \mid \Psi$ has the form $\mathscr{Y}\psi\}$, where

$$\mathscr{Y} = \frac{1}{2s(2s-1)}\Big[\sum_{1\le t\le 2s-1}\tau_{(t,2s-1)} + \sum_{1\le t<t'\le 2s-1}\tau_{(t,2s)}\tau_{(t',2s-1)}\Big]\big[\not{y}^{(2s-1)} - \not{y}^{(2s)}\big].$$

Then these spaces realize the GB triplet:

$$D(\frac{n-1}{2}+s,s,\ldots,s-1) \to D(\frac{n-3}{2}+s,s,\ldots,s) \to D(\frac{n-1}{2}+s,s,\ldots,s-1),$$

i.e., $\mathscr{H}_1^{(s)}/\mathscr{H}_2^{(s)}$ and $\mathscr{H}_3^{(s)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n-1}{2}+s,s,\ldots,s-1)$ while the quotient $\mathscr{H}_2^{(s)}/\mathscr{H}_3^{(s)}$ (physical space) carries the conformal massless one $D(\frac{n-3}{2}+s,s,\ldots,s)$.

4.5. Composite massless representations: spin $s\geq 1$

If ε is defined in the same way as in the preceding subsection and if σ is the fractional part of s, then the limit of unitarity of the IR $D(E_0,s,\sigma,\ldots,\sigma,\varepsilon\sigma)$, is $n-3+s$ if $s \geq 1$, and $\frac{n-3}{2}+s$ otherwise. Thus natural GB triplets for the composite massless representations of this form can be obtained only for $s \geq 1$, regardless to the parity of n. So let $s \geq 1$. Then the corresponding GB triplets are realized in a somewhat known fashion (see for example [6, 7, 20, 22]).

First suppose that $s\in\mathbb{N}$. We reduce the tensor product $D(E,\vec{0})\otimes D(-s,\vec{0})$, with E close to $s+n-3$. $D(-s,\vec{0})$ is a finite-dimensional representation realized on the space of polynomials in the variables $z_{-1},\ldots,z_{n-1}$. The GB triplet is realized on the space of functions $(y,z)\mapsto\phi(y,z)$ with the usual conditions on the variable $y\in H^+$. This is equivalent to realizing the representation on the space of symmetric tensor fields of rank s on H^+. The generators of the Lie algebra are $M_{ab} = y_a\partial_b - y_b\partial_a + z_a\delta_b - z_b\delta_a$, $\delta_c = \frac{\partial}{\partial z^c}$. Once the irreducible representation $D(E,s,0,\ldots,0)$ is realized for $E \neq s+n-3$, then one obtains an indecomposable representation after taking the limit $E\to s+n-3$. From there one constructs the GB triplet:

$$D(s+n-2,s-1,0,\ldots,0) \to D(s+n-3,s,0,\ldots,0) \to D(s+n-2,s-1,0,\ldots,0).$$

As above one needs some Lorentz conditions to fix the space $\mathscr{H}'^{(s)}_2$ which defines the physical situation: Its elements are the tensor fields φ which satisfy $\partial^2\varphi(y,z) = 0$, $y\cdot\partial\varphi(y,z) = -(s+n-3)\varphi(y,z)$ (homogeneity), $z\cdot\delta\varphi(y,z) = s\varphi(y,z)$, $\delta^2\varphi(y,z) = 0$ (φ is traceless), $\partial\cdot\delta\varphi(y,z) = 0$ (φ is divergenceless) and $y\cdot\delta\varphi(y,z) = 0$ (φ is transverse). The physical representation $D(s+n-3,s,0,\ldots,0)$ is realized on the quotient $\mathscr{H}'^{(s)}_2/\mathscr{H}'^{(s)}_3$ where the gauge space $\mathscr{H}'^{(s)}_3$ is the subspace of elements $\varphi\in\mathscr{H}'^{(s)}_2$ of the form $\varphi(y,z) = \big[y^2 z\cdot\partial + (n-3+2s)y\cdot z\big]\phi(y,z)$.

Now suppose that s is a half-integer, i.e., $s-\frac{1}{2}\in\mathbb{N}$. The tensor product we consider is $D(E,\vec{0})\otimes D(-(s-\frac{1}{2}),\vec{0})\otimes\Sigma$, with the same material as above but with $s-\frac{1}{2}$ instead of s for the first two representations. The desired representations act on tensor-spinor fields Ψ on H^+. The generators of the Lie algebra are $M_{ab}=y_a\partial_b-y_b\partial_a+z_a\delta_b-z_b\delta_a+\frac{1}{4}[\gamma_a,\gamma_b]$. To get the needed indecomposable representation we let $E\to s+n-3+\frac{1}{2}$. Then one gets the GB triplet:

$$D(s+n-2,s-1,\frac{1}{2},\dots,\varepsilon\frac{1}{2})\to D(s+n-3,s,\frac{1}{2},\dots,\varepsilon\frac{1}{2})\to \\ \to D(s+n-2,s-1,\frac{1}{2},\dots,\varepsilon\frac{1}{2}).$$

The value of ε, ±1, depends as usual on the parity of n and on the irreducible component of Σ used (Σ is irreducible only if $n-1$ is odd).

$\mathscr{H}'^{(s)}_2$ is defined from the Lorentz conditions: $\partial^2\Psi(y,z)=0$, $y\cdot\partial\Psi(y,z)=-(s+n-3+\frac{1}{2})\Psi(y,z)$, $z\cdot\delta\Psi(y,z)=(s-\frac{1}{2})\Psi(y,z)$, $\delta^2\Psi(y,z)=0$, $\partial\cdot\delta\Psi(y,z)=0$, $y\cdot\delta\Psi(y,z)=0$, $\not{y}\not{\partial}\Psi(y,z)=0$, and $\not{z}\delta\Psi(y,z)=0$. Finally the physical representation $D(s+n-3,s,\frac{1}{2},\dots,\varepsilon\frac{1}{2})$ is realized on the quotient $\mathscr{H}'^{(s)}_2/\mathscr{H}'^{(s)}_3$ where the gauge space $\mathscr{H}'^{(s)}_3$ is the subspace of elements $\Psi\in\mathscr{H}'^{(s)}_2$ of the form $\Psi(y,z)=\left[y^2z\cdot\partial+(n-3+2s-1)y\cdot z+\not{y}\not{z}\right]\Phi(y,z)$.

5. The unreasonable effectiveness of the 4-dimensional space-time

From what precedes one sees that singletons of the anti-de Sitter group are well defined for $n\geq3$. They are defined for all half-integer spin if the space-time dimension n is odd and only for spin 0 or $1/2$ if n is even. Moreover singleton theory is always quantizable in the sense of Gupta and Bleuler and the resulting gauge theory is topological in the sense that singletons appear at spatial infinity, since in the Lorentz condition one lets the "radius" R of space-time tend to infinity. Apparently there is no noticeable difference with the $n=4$ dimensional case.

However masslessness behaves differently. Indeed if one needs conformal invariance one has to work with conformal masslessness, but then the space-time dimension must be even if massless particles with spins other than 0 or $1/2$ are necessary. Furthermore the corresponding massless particles cannot be composed of subparticles like singletons if $n>4$. Obviously the conformal invariance of masslessness is not always needed, in which case there is in general no unique way to define the masslessness notion, even in n-dimensional Minkowski space-time. This is due to the fact that if n is sufficiently large, the rank of the maximal compact subalgebra $\mathfrak{k}$ is greater than 2, thus there is no unique way to choose the right spinor part of the weight of the representation. But if the composite aspect of singletons is important, as they are in some theories, then composite masslessness becomes more appropriate, since for every spin and for every parity of the space-time dimension composite massless representations occur in the reduction of the tensor product of two singletons, i.e., the corresponding massless particles are composed of two singletons. Unfortunately the two notions of masslessness we have at hand are not compatible if $n > 4$: A physical theory on the anti-de Sitter space-time such that masslessness is conformal invariant and composite (the particle-anti-particle case being excluded) does not exist for $n > 4$.

Thus the $n = 4$ anti-de Sitter space-time appears to be the only one for which masslessness is well defined and is conformal invariant and composite, if one considers that an n-dimensional physical space-time satisfies necessarily $n \geq 4$.

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