UNBOUNDED DERIVATIONS OF C* ALGEBRAS

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1. Introduction

Symmetry groups play an important role in physical theories. In quantum-mechanical theories of a finite number of particles symmetry groups are traditionally given by unitary group representations on Hilbert space. These representations are usually continuous and the notion of infinitesimal generator can be introduced. The infinitesimal generators themselves have direct physical significance; the generator of space translations is the momentum operator, the generator of time translations is the Hamiltonian or energy operator, and the angular momentum operator generates rotations. The question of when an operator is an infinitesimal generator of a unitary group often arises and in particular one often asks whether certain operators are suitable as Hamiltonians. The answer to this kind of question is well-known. An operator H on a Hilbert space $\mathcal{H}$ generates a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$ if, and only if, the operator is self-adjoint. Various criteria for self-adjointness have been given in terms of deficiency spaces, sets of analytic vectors, positivity etc., and these criteria have played a useful role in such contexts as scattering theory and statistical mechanics.

In theories of infinite systems it appears both useful and necessary to interpret symmetries in a more general fashion. The basic observables of the theory can be taken to form a C* algebra $\mathfrak{A}$ and the symmetries enter as groups of automorphism of $\mathfrak{A}$. If the automorphism group is continuous in a suitable sense one can again introduce the notion of an infinitesimal generator and such generators will be symmetric derivations of $\mathfrak{A}$, i.e operators δ defined on a dense *subalgebra $D(\delta)\subseteq\mathfrak{A}$ with the properties

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1. $\delta(AB) = \delta(A)B + A\delta(B)$ $\quad A,B \in D(\delta)$.

2. $\delta(A^*) = -\delta(A)^*$ $\quad A \in D(\delta)$.

3. If $\mathfrak{A}$ contains an identity element $\mathbb{1}$ then $\mathbb{1} \in D(\delta)$ and $\delta(\mathbb{1}) = 0$.

In general δ will be an unbounded operator on $\mathfrak{A}$ with a precise physical interpretation. It is known that δ is bounded, i.e

$$\|\delta(A)\| \leq \text{constant}\|A\|,\ A \in D(\delta),$$

if, and only if, $D(\delta) = \mathfrak{A}$; bounded derivations have been extensively studied (for a review see, for example, [7] Chapter 4). The analysis of unbounded derivations is at a much more embryonic stage and it is only in the last year that a significant number of results concerning such derivations have appeared (see Bratteli and Robinson [1], Powers and Sakai [3][4], and Sakai [8]; earlier results were derived in Robinson [5], Sinai and Helemskii[2]) Naturally one of the important questions concerning unbounded derivations is the analogue of the Hilbert space problem previously mentioned; under what conditions does a derivation generate a strongly continuous one-parameter group of *automorphisms of $\mathfrak{A}$. In this talk we announce and describe various new results which characterize infinitesimal generators [6] and review some of the general results given in [1] [3] [4] [8]. Although these results have not as yet had any striking application to physical theories we are hopeful that this theory will eventually play the same useful role that the Hilbert space theory plays.

2. Infinitesimal Generators

Let $\mathfrak{A}$ denote a C* algebra and

$$\tau\ ;\ A \in \mathfrak{A} \mapsto \tau_t(A) \in \mathfrak{A},\ t \in \mathbb{R}$$

a one-parameter group of * automorphisms of the C*-algebra satisfying the strong continuity condition

$$\lim_{t\to 0} \|\tau_t(A) - A\| = 0,\ A \in \mathfrak{A}.$$

Next define δ by

$$\delta(A) = \lim_{t\to 0} [\tau_t(A) - A]/it$$

for the set $D(\delta)$ of $A \in \mathfrak{A}$ such that the limit exists. It is easily checked that δ

is a derivation of $\mathfrak{A}$, e.g. the automorphic property

$$\tau_t(AB) = \tau_t(A)\ \tau_t(B)$$

leads to the first derivation property listed in the previous section,

$$\tau_t(A)^* = \tau_t(A^*)$$

the second, and

$$\tau_t(\mathbb{1}) = \mathbb{1}$$

the third. The density of $D(\delta)$ follows from consideration of certain 'regularized' elements of $\mathfrak{A}$ and is a standard part of semi-group theory.

A derivation arising in the above manner will be called the infinitesimal generator of the group τ. It is of primary interest to characterize those derivations which generate groups. This is a problem analogous to the characterization of the symmetric operators on Hilbert space which are actually self-adjoint. The following result gives a characterization simlar to the Stone-von Neumann self-adjointness criterion.

Theorem 1[1] let δ be a derivation of a C^* algebra $\mathfrak{A}$. The following conditions are equivalent

1. δ is the infinitesimal generator of a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$

2. δ is closed, $R(\delta\pm i) = \mathfrak{A}$, and

$$\|\delta(A) + zA\| \geq |\mathrm{Im}\,z|\,\|A\| \qquad (1)$$

In the foregoing statement $R(\delta \pm i)$ is the range of $\delta \pm i$, i.e.

$$R(\delta\pm i) = \{B;\ B = \delta(A)\pm iA,\ A\in D(\delta)\},$$

and the assumption that δ is closed means that if $\|A_n\| \to 0$ and $\|\delta(A_n) - B\| \to 0$ then B must be identically zero.

Theorem 1 should be compared to the Stone-von Neumann criterion; a symmetric operator H on a Hilbert space $\mathcal{H}$ is the infinitesimal generator of a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$ if, and only if, H is closed and $R(H\pm i) = \mathcal{H}$, where now we have

$$R(H\pm i) = \{\psi\in\mathcal{H};\ \psi = (H\pm i)\phi,\ \phi\in D(H)\}$$

Thus the two results differ principally because of the extra lower, bound assumption.

In the Hilbert space case the symmetry of H allows one to immediately conclude that

$$\|(H \pm i)\phi\|^2 \geq \|\phi\|^2$$

This inequality together with the assumption that $R(H\pm i) = \mathcal{H}$ proves that the resolvent operators

$$R(\pm i) = \frac{1}{H\pm i}$$

are everywhere defined and have norm smaller than one. Exploitation of this fact by the Hille-Yosida *theory of semigroups* allows the construction of a group of unitaries with H as infinitesimal generator. In the algebraic case this estimate is not necessarily true as the following example shows.

Example (Brattel i) Let $\mathfrak{A} = C([0,1])$ the C*algebra of continuous functions over the interval [0,1] and define the derivation δ by

$$\delta(f)(x) = i\,\frac{df}{dx}(x)$$

where $D(\delta)$ is the set of absolutely continuous functions over [0.1].

It follows that δ is closed, $R(\delta\pm i) = \mathfrak{A}$ but

$$(\delta + z)\,(e^{izx}) = 0$$

and consequently δ is not an infinitesimal generator.

The similarity of Theorem 1 and the Stone-von Neumann theorem suggests that other theorems concerning symmetric operators on Hilbert space might lift to theorems about derivations on C* algebras. A typical example would be Nelson's theorem on analytic vectors.

Let δ be a derivation. It is natural to define an analytic (entire) element of δ as an element $A\varepsilon D(\delta^n)$, n=1, 2, 3, $\cdots$. Such that the function

$$z\varepsilon\mathbb{C} \longmapsto e_z(A) = \sum_{n\geq 0} \frac{z^n}{n!}\,\delta^n(A) \;\varepsilon\; \mathfrak{A}$$

exists and is analytic in some neighbourhood of the origin (is entire). An analogue of Nelson's theorem would state that δ is an infinitesimal generator if, and only if δ is closed δ possesses a dense set of analytic (entire) elements, and estimate (1) is valid. It is unclear whether this theorem is true but a weakened form of it may be established in terms of geometric elements of δ. A geometric element of δ is defined to be an element $A\in D(\delta^n)$, $n=1, 2, \ldots,$ such that

$$\|\delta^n(A)\|^{1/n} \leq C_A$$

where C_A is independent of n. This is equivalent to demanding that

$$e^{-\lambda|t|} \sum_{n\geq 0} \frac{|t|^n}{n!} \|\delta^n(A)\| \xrightarrow[|t|=\infty]{} 0$$

for λ sufficiently large, i.e. A is required to be an entire element of δ with a certain restriction on the growth properties of $e_z(A)$. The following is now true.

Theorem 2[6] Let δ be a derivation of a C* algebra $\mathfrak{A}$. The following conditions are equivalent

1. δ is the infinitesimal generator of a strongly continuous one-parameter group of * automorphisms of $\mathfrak{A}$.

2. δ is closed, δ possesses a dense set of geometric elements.

$$\|\delta(A) + zA\| \geq |\mathrm{Im} z|\,\|A\|$$

Although this result is weaker than the analytic element conjecture it does have at least one interesting consequence.

Theorem 3[6] Let δ be a derivation of a C* algebra $\mathfrak{A}$ and suppose that δ is the infinitesimal generator of a strongly continuous one parameter group of *-automorphisms τ of $\mathfrak{A}$.

If $D\subset D(\delta)$ is a dense *-subalgebra of $\mathfrak{A}$ with the property that

$$\tau_t(D) \subset D, \qquad t\in\mathbb{R},$$

then it follows that D is a core for δ, i.e. the closure $\overline{\delta|_D}$ of the restriction of δ to D satisfies

$$\overline{\delta|_D} = \delta.$$

3 Closed Derivations

One of the basic properties that a derivation must have to qualify as an infinitesimal generator is the property of being closed. A symmetric operator H on a Hilbert space $\mathcal{H}$ always has the property of being closeable (H^* and H^{**} are closed extensions of H); the analogy between symmetric operators on $\mathcal{H}$ and derivations δ on a C* algebra $\mathfrak{A}$ automatically leads to the conjecture that all derivations are closeable. This conjecture is, however, false by the following result of Bratteli and Robinson [1].

Theorem 4 Let $\mathfrak{A}$ be the CAR algebra and B_n an increasing sequence of $2^n x 2^n$ full matrix algebras generating $\mathfrak{A}$.

There exists a non-zero derivation δ of $\mathfrak{A}$ such that

1. Every B_n is in the domain $D(\delta)$ of δ
2. δ restricted to each B_n is zero.

Hence δ is not closeable.

The existence of abelian algebras with non-closeable derivations is established as a by-product of the construction used to prove Theorem 5.

The foregoing result proves that the property of closeability of derivations is a real restriction in contrast to the situation with symmetric operators. We next consider the problem of characterizing closeable derivations. This is an algebraic problem and the first criterion for closeability is given by a functional analytic property of the domain of the derivation

Theorem 5 Let δ be a derivation of a C* algebra

If δ is such that $A^{1/2} \varepsilon D(\delta)$ whenever $0 \leq A \varepsilon D(\delta)$ then δ is closeable.

Conversely if δ is closed and $A \varepsilon D(\delta)$ is positive and invertible than $A^{1/2} \varepsilon D(\delta)$.

The first statement of the theorem is given by Powers and Sakai [4] the second statement occurs in Bratteli and Robinson [1]. In fact the latter authors develop a more detailed functional-analytic description of the domains of closed derivations. The essential point is that if δ is closed and $A = A^* \varepsilon D(\delta)$ then the resolvent $(\lambda - A)^{-1}$ is also in $D(\delta)$ whenever λ is not in the spectrum $\sigma(A)$ of A.

Note that the foregoing result is not a good characterization of closed derivations because the converse statement places the extra requirement of invertibility.

In general the domain of a derivation is not closed under the square root operation. If $\mathfrak{A} = C_o(\mathbb{R})$ and δ is the infinitesimal generator of translations then

$$f(x) = |x|^{3/2} e^{-x^2}$$

is such that $f \in D(\delta)$ but $f^{1/2} \notin D(\delta)$.

It remains an open question whether a derivation δ whose domain $D(\delta)$ is invariant under the formation of resolvents, i.e. $A \in D(\delta)$ implies $(\lambda - A)^{-1} \in D(\delta)$ for $\lambda \in \sigma(A)$, is automatically closeable.

A second criterion for closeability can be given in terms of an invariance condition. Assume for the moment that δ is the infinitesimal generator of an automorphism group τ and that ω is a state over $\mathfrak{A}$ which is invariant under τ, i.e.

$$\omega(\tau_t(A)) = \omega(A)$$

for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$. This invariance condition is equivalent to the following condition expressed in terms of δ

$$\omega(\delta(A)) = 0$$

for all $A \in D(\delta)$. The following result considers derivations with faithful invariant states.

Theorem 6 Let δ be a derivation of a C^* algebra $\mathfrak{A}$. Assume that $\mathfrak{A}$ possesses a state ω which generates a faithful cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and also satisfies the invariance condition

$$\omega(\delta(A)) = 0$$

for all $A \in D(\delta)$.

It follows that

1. δ is closeable
2. There exists a symmetric operator H_δ on $\mathcal{H}_\omega$ such that

$$D(H_\delta) = \{\psi;\ \psi = \pi_\omega(A)\Omega_\omega,\ A \in D(\delta)\}$$

$$\pi_\omega(\delta(A))\psi = [H_\delta,\ \pi_\omega(A)]\psi$$

for all $A \in D(\delta)$ and $\psi \in D(H_\delta)$.

The theorem as stated occurs in Bratteli and Robinson [1]; Powers and Sakai [4] give a special version of the theorem for UHF algebras.

It remains unclear whether Theorem 6 has a converse. Is it true that for each closed derivation of a C*-algebra $\mathfrak{A}$ ($\mathfrak{A}$) there exists a state ω such that

$$\omega(\delta(A)) = 0$$

for all $A \in D(\delta)$? This result has been established in [1] for special algebras, C* algebras acting on a Hilbert space $\mathcal{H}$ and containing the C* algebra $\mathcal{L}\mathcal{C}(\mathcal{H})$ of compact operators on $\mathcal{H}$ as subalgebra. It is also true if $\mathfrak{A} \ni \mathbb{1}$ and δ is an infinitesimal generator by a simple compactness and fixed point argument.

ACKNOWLEDGEMENTS The author's interest in the foregoing subject developed through collaboration with O. Bratteli. Part of the material of Section 2 was developed while the author was a guest of the Physics Dept. and the Maths. Dept. at the University of California, Berkeley. I am indebted to E. Wichmann and O.E. Lanford for their hospitality during this period.
Research support inpart by the National Science Foundation, under Grant # GP-42249X.

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Discussion

Doplicher (Comment): It seems that the difference between the Hilbert space situation and the derivation situation is related to the different definition of adjoints: the adjoint of a Hilbert space linear operator can be also made to correspond to the transpose of a linear operator between Banach spaces; if the transpose is densely defined the initial operator is closable. For the sake of * automorphism groups you use rather the "skew adjointness" $\delta(A^*)^* = -\delta(A)$, which as you say does not force δ to be closable.