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Relativity and quantum mechanics

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Abstract. We first deduce Special Relativity from Quantum Mechanics. Then we discuss the implications and ramifications.

It is well known that both Special and General Relativity, and quantum theory had independent origins and development. This brand of quantum mechanics was the non-relativistic theory. It was only around 1930 that Paul Dirac could “unite” Special Relativity and Quantum Theory through his famous Dirac equation.

Can we go the reverse way? Let us see how Special Relativity can be “derived” from Quantum Theory. We first define a complete set of base states by the subscript i and $U(t_2, t_1)$ the time elapse operator that denotes the passage of time between instants t_1 and t_2 , t_2 greater than t_1 . We denote by, $C_i(t) \equiv \langle i | \psi(t) \rangle$, the amplitude for the state $|\psi(t)\rangle$ to be in the state $|i\rangle$ at time t , and [1, 2]

$$\langle i | U | j \rangle \equiv U_{ij}; U_{ij}(t + \Delta t, t) \equiv \delta_{ij} - \frac{i}{\hbar} H_{ij}(t) \Delta t.$$

We can now deduce from the superposition of states principle that,

$$C_i(t + \Delta t) = \sum_j [\delta_{ij} - \frac{i}{\hbar} H_{ij}(t) \Delta t] C_j(t) \quad (1)$$

and finally, in the limit $\Delta t \rightarrow 0$,

$$i\hbar \frac{dC_i(t)}{dt} = \sum_j H_{ij}(t) C_j(t) \quad (2)$$

where the matrix $H_{ij}(t)$ is identified with the Hamiltonian operator.

Starting from equation (2) Feynman had deduced a version of the Schrodinger equation using the probability amplitude for a particle to diffuse from one point to another. We had argued earlier at length that (2) leads to the Schrodinger equation [1, 2]. In the above we had taken the usual unidirectional time to deduce the non relativistic Schrodinger equation. If however we consider in (1) that time is oscillating between $t - \Delta t$ and $t + \Delta t$, then we will have to consider instead of (2)

$$C_i(t - \Delta t) + C_i(t + \Delta t) = \sum_j \left[2\delta_{ij} - \frac{i}{\hbar} H_{ij}(t) \right] C_j(t). \quad (3)$$



Equation (3) in the limit was seen to lead to the relativistic Klein-Gordon equation rather than the Schrodinger equation [3].

This follows if we consider a simple Taylor expansion:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \tag{4}$$

$$\begin{aligned} f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x) - f'(x-h)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - f(x) + f(x-h)}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \end{aligned} \tag{5}$$

so that by analogy

$$U(t + \Delta t, t) = 1 - \frac{i}{\hbar} H(t) \Delta t + \frac{1}{2} \left(\frac{i}{\hbar} H(t) \Delta t \right)^2 + 0(\Delta t^3) \tag{6}$$

$$C_i(t + \Delta t) + C_i(t - \Delta t) = \sum_j 2\delta_{ij} C_j(t) + \left[\left(\frac{i}{\hbar} H(t) \Delta t \right)^2 \right]_{ij} C_j(t) + 0(\Delta t^3) \tag{7}$$

$$C_i(t + \Delta t) + C_i(t - \Delta t) - 2C_i(t) = \sum_j \left[\left(\frac{i}{\hbar} H(t) \right)^2 \right]_{ij} C_j(t) \Delta t^2 + 0(\Delta t^3) \tag{8}$$

$$\frac{d^2}{dt^2} C_i(t) = \left[\left(\frac{i}{\hbar} H(t) \right)^2 \right]_{ij} C_j(t) \tag{9}$$

From this expression, the Klein-Gordon equation is motivated, and it is already evident that the generator of time translations $H(t)$ is the square root of the operator in the Klein-Gordon equation, in the limit rather than the Schrodinger equation. In other words considering a process in (1) that is time oscillating between $t - \delta t$ and $t + \delta t$ we reach the relativistic equation [3]. This has been discussed in great detail over the years, (cf [5] and references therein). In this case we have $(\Delta t)^2 = \beta \Delta x$ exactly as in a typical Wiener process. We should bear in mind, that here we are dealing with a non-differential spacetime manifold as in the Fokher-Planck theory (cf. [6]). This is the beginning of some modern Quantum Gravity approaches. More specifically, if we can neglect the fundamental scale \mathbf{a} which stands for a minimum scale as in (14) below, we have classical theory. If $\bar{\mathbf{a}}^2$ can be neglected, we have the usual Quantum Theory and Quantum Field Theory. But if $\bar{\mathbf{a}}^2$ has to be retained, we have the Quantum of area and the above approach which encompasses fractal spacetime.

As it is well known the Klein-Gordon equation is obtained from the energy momentum dispersion relation of Special Relativity

$$E^2 = p^2 + m^2 \quad (c = 1 = \hbar). \tag{10}$$

Using the usual Quantum Mechanical substitutions for energy and momentum, (10) becomes

$$\left(\frac{d^2}{dt^2} - \nabla^2 - m^2 \right) \psi = 0. \tag{11}$$

Conversely, from (11) we can deduce (10) with substitutions of the type

$$p^\mu = \frac{\hbar}{i} \nabla^\mu.$$

It is well known that the problem with (11) was that there is no fixed eigenvalue, that is loosely the negative energy states appear because of the second time derivative. On the other hand Dirac linearized the second time derivative in (11) to obtain his well-known equation

$$\{\gamma^\mu p_\mu - m\} \psi = 0 \quad (12)$$

where the symbols have their usual well known meanings. But there was a price to pay for this linearized relativistic Quantum Mechanical equation. Firstly we encounter spin half which had (and has) no place in the Classical Theory. Secondly, though (12) was linear in the time derivative, it still leads to the very same negative solutions. This then led Dirac to propose his Hole Theory.

We can now see the reason for the appearance of negative energy solutions in the derivation from (3): time as it were could run forward, this in the usual sense. But now we allow it to run backward (negative energy).

Let us now take another approach, starting from the coordinate for a Dirac electron as derived by Dirac himself [7] (cf. Appendix):

$$x = (c^2 p_1 H^{-1} t) + \frac{i}{2} c \hbar (\alpha_1 - c p_1 H^{-1}) H^{-1} \quad (13)$$

The first term in (13) is known by the usual (Hermitian) position. The imaginary part of (13) is of the order of the Compton wavelength.

It is at this stage that a proper physical interpretation begins to emerge. Dirac himself observed, that to interpret (13) meaningfully, it must be remembered that Quantum Mechanical measurements are really averaged over the Compton scale [7]: Within the scale there are the unphysical zitterbewegung effects represented by the second term of (13): for a point electron the velocity equals that of light.

Once such a minimum spacetime scale \mathbf{a} is invoked, then we have a noncommutative geometry as shown by Snyder [8, 9]:

$$[x, y] = (i\mathbf{a}^2/\hbar)L_z \quad [t, x] = (i\mathbf{a}^2/\hbar c)M_x \quad [x, p_x] = i\hbar[1 + (\mathbf{a}/\hbar)^2 p_x^2] \quad \text{etc.} \quad (14)$$

In this analysis, time too is a matrix (operator) — this is because we are allowing it to run backward and forward, or alternatively, we are allowing for the possibility of positive and negative energies.

The relations (14) are compatible with Special Relativity as shown by Snyder himself. Indeed such minimum spacetime models were studied for several decades, precisely to overcome the divergences encountered in Quantum Field Theory which originate from the point electron [2], [9]-[14], [15, 16].

Before proceeding further, it may be remarked that when \mathbf{a}^2 , which we will take to be the squared Compton wavelength (including the Planck scale, which is a special case of the Compton scale for a Planck mass viz., 10^{-5} gm), in view of the above comments can be neglected, then we return to ordinary Quantum Theory.

It is interesting that starting from the Dirac coordinate in (13), we can deduce the noncommutative geometry (14), independently:

We observe that the first term on the right hand side is the usual Hermitian position. For the second term which contains α , we can easily verify from the commutation relations of the σ 's that

$$[x_i, x_j] = \beta_{ij} \cdot l^2 \quad (15)$$

where l is the Compton scale which comes from the factor $(c\hbar/H)$ in (13) and β_{ij} are suitable 2×2 matrices.

There is another way of looking at this. As can be seen the one dimensional coordinate in (13) is complex. We now try to generalize this complex coordinate to three dimensions. Then as is known we encounter a surprise - we end up with not three, but four dimensions,

$$(1, t) \rightarrow (I, \sigma),$$

where I is the unit 2×2 matrix and σ s are the Pauli matrices. We get the special relativistic Lorentz invariant metric at the same time. (In this sense, as noted by Sachs [17], Hamilton who made this generalization would have hit upon Special Relativity, if he had identified the new fourth coordinate with time).

That is,

$$x + iy \rightarrow Ix_1 + ix_2 + jx_3 + kx_4,$$

where (i, j, k) now represent the Pauli matrices; and, further,

$$x_1^2 + x_2^2 + x_3^2 - x_4^2$$

is invariant. Before proceeding further, we remark that special relativistic time emerges above from the generalization of the complex one dimensional space coordinate to three dimensions.

While the usual Minkowski four vector transforms as the basis of the four dimensional representation of the Poincare group, the two dimensional representation of the same group, given by the right hand side in terms of Pauli matrices, obeys the quaternionic algebra of the second rank spinors (cf [18, 19, 17] for details).

To put it briefly, the quaternion number field obeys the group property and this leads to a number system of quadruplets as a minimum extension. In fact one representation of the two dimensional form of the quaternion basis elements is the set of Pauli matrices. Thus a quaternion may be expressed in the form of the representation $SL(2, C)$,

$$Q = -i\sigma_\mu x^\mu = \sigma_0 x^4 - i\sigma_1 x^1 - i\sigma_2 x^2 - i\sigma_3 x^3 = (\sigma_0 x^4 + i\vec{\sigma} \cdot \vec{r})$$

This can also be written as

$$Q = -i \begin{pmatrix} ix^4 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & ix^4 - x^3 \end{pmatrix}.$$

It is well known that, there is a one to one correspondence between a Minkowski four-vector and Q [17]. The invariant is now given by $Q\bar{Q}$, where \bar{Q} is the complex conjugate of Q .

However, as is well known, there is a lack of spacetime reflection symmetry in this latter formulation. If we require reflection symmetry also, we have to consider the four dimensional representation,

$$(I, \vec{\sigma}) \rightarrow \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right] \equiv (\Gamma^\mu)$$

(cf also [20] for a detailed discussion). The motivation for such a reflection symmetry is that usual laws of physics, like electromagnetism do indeed show the symmetry.

We at once deduce spin and Special Relativity and the geometry (14) in these considerations. This is a transition that has been long overlooked [21]. It must also be mentioned that spin half itself is relational and refers to three dimensions, to a spin network in fact [22, 23].

While a relation like (15) above has been in use recently, in non-commutative models, we would like to stress that it has been overlooked that the origin of this non commutativity lies in the original Dirac coordinates.

This can be seen directly from the Dirac theory itself where we have [24] that the velocity operator is given by

$$\frac{c^2 \vec{p}}{H} - \frac{2i}{\hbar} \hat{x} H \quad (16)$$

In (16), the first term is the usual velocity. The second term is the extra contribution due to zitterbewegung.

In fact we can easily verify from (16) that

$$\tilde{p} = \frac{H^2}{\hbar c^2} \hat{x} \quad (17)$$

where \hat{x} has been defined in (16).

We finally investigate the angular momentum $\sim \vec{x} \times \vec{p}$ — that is, the angular momentum at the Compton scale. We can easily show that [25]

$$(\vec{x} \times \vec{p})_z = \frac{c}{E} (\vec{\alpha} \times \vec{p})_z = \frac{c}{E} (p_2 \alpha_1 - p_1 \alpha_2) \quad (18)$$

where E as usual in the continuum case [26] is the eigenvalue of the Hamiltonian operator H . Equation (18) shows that the usual angular momentum but in the context of the minimum Compton scale cut off, leads to the “mysterious” Quantum Mechanical spin. In the above approach, spin emerges from the Compton scale [27].

In the above considerations, we started with the Dirac equation and deduced the underlying non commutative geometry of spacetime.

Interestingly, starting with Snyder’s non commutative geometry, based solely on Lorentz invariance and a minimum spacetime length, which we have taken to be the Compton scale, (14), it is possible to deduce the relations (18), (17) and the Dirac equation itself.

We have thus established the correspondence between considerations starting from the Dirac theory of the electron and Snyder’s (and subsequent) approaches based on a minimum or fuzzy spacetime interval and Lorentz covariance.

Special Relativity however gets “modified” due to Quantum Effects. In fact the relation [28]

$$(\Delta x)^2 - (\Delta t)^2 = 0 \quad (19)$$

goes over to, as is well known [29],

$$(\Delta x)^2 - (\Delta t)^2 \leq \frac{1}{m^2} \quad (\hbar = 1 = c) \quad (20)$$

where term on the right side arises from the uncertainty in the Compton wavelength region as discussed in detail by Weinberg [10]. Effectively, the velocity of light $c(= 1)$ becomes c' , given by from (20),

$$c'^2 \leq 1 + \frac{1}{m^2(\Delta t)^2} \leq 2 \quad (21)$$

as $\Delta t \geq \frac{1}{m}$, the Compton time. Alternatively, the energy momentum relation gets modified.

So, the usual energy $E = mc^2$ becomes E' where

$$E < E' \leq 2E. \quad (22)$$

There is an increase in energy in the extreme relativistic case. The extra energy is the contribution of the background dark energy as argued by the author elsewhere [29].

The actual situation is that within the Compton time there are superluminal “velocities” or more realistically the Zitterbewegung effects. These effects are interpreted as the appearance of anti particles [30].

There is however another perspective: Wigner and Salecker [31] have argued that no physical measurement is possible within the Compton scale. In other words all our physics originates outside the Compton scale. There would thus be a minimum spacetime interval within which there would be no physics (cf [5] for a detailed description). This is exactly what we were discussing earlier. Such minimum spacetime theories are now in vogue in Quantum Gravity approaches, including the author’s own.

Interestingly this interplay between Special Relativity and Quantum Mechanics manifests itself in graphene [32, 33]. As is known graphene is a two dimensional form of carbon with a honeycomb structure. The particles in graphene obey the equation

$$v_F \vec{\sigma} \cdot \vec{\nabla} \psi(r) = E \psi(r) \quad (23)$$

This is the two dimensional Dirac equation that describes neutrinos, except that $v_F \sim 10^6$ m/s replaces the velocity of light c . A finite graphene sheet cannot be expected to be Lorentz invariant, by the very nature of its finiteness. However we have argued that an infinite sheet reproduces Special Relativistic effects [34]. In fact in this case we have

$$v_F^2 = \left(\frac{\hbar \pi}{m} \right)^2 \cdot \frac{1}{A} \quad (24)$$

where A the area is $\sim l^2$, l the inter lattice distance.

Whence we get

$$\frac{m^2 v_F^2}{\hbar^2} \cdot l^2 \sim 0(1) \quad (25)$$

We can see from (25) that if v_F tends to the velocity of light c , h/mv_F tends to the Compton wavelength consistently. That is spacetime can be modelled consistently with an infinite lattice sheet.

Indeed following this train of thought it has been argued [35] that the Hall Effect parallels the Lorentz Force of electromagnetic theory.

Finally, it may be mentioned that graphene resembles a wind tunnel in the sense that there are Reynold number type scalings that lead to 3D Minkowski space [34]. Specifically, $L \sim 10^3 l$, $c = 300v_F$ and $m_{\text{graphene}} \sim 10^{-2}m$ where m is the electron mass.

This suggests that graphene can be used as a test bed for High Energy experiments in view of the fact that it can be used to harvest protons, as confirmed by Andre Geim [36]. We must also remember that its mysterious minimum conductivity gives a source of protons themselves.

Appendix (Based on *The Principles of Quantum Mechanics* by P.A.M. Dirac [7])

The x_1 -component of the velocity is

$$\dot{x}_1 = [x_1, H] = c\alpha_1. \quad (26)$$

The \dot{x}_1 given by (26) has as eigenvalues $\pm c$, corresponding to the eigenvalues ± 1 of α_1 . That is a measurement of a component of the velocity of a free electron is $\pm c$. This conclusion is easily seen to hold also when there is a field present. We must remember that the velocity in above is the velocity at one instant of time while observed velocities are always average velocities through appreciable time intervals. Actually the velocity is not at all constant, but oscillates rapidly about a mean value which agrees with the observed value. This can also be seen from an application of the Uncertainty Principle.

To measure the velocity we must measure the position at two slightly different times and then divide the change of position by the time interval. "...In order that our measured velocity may approximate to the instantaneous velocity, the time interval between the two measurements of position must be very short and hence these measurements must be very accurate. The great accuracy with which the position of the electron is known during the time-interval must give rise, according to the principle of uncertainty, to an almost complete indeterminacy in its momentum. This means that almost all values of the momentum are equally probable, so that the momentum is almost certain to be infinite. An infinite value for a component of momentum corresponds to the value $\pm c$ for the corresponding component of velocity.

Let us now examine how the velocity of the electron varies with time. We have

$$i\hbar\dot{\alpha}_1 = \alpha_1 H - H\alpha_1.$$

Now since α_1 anticommutes with all the terms in H except $c\alpha_1 p_1$,

$$\alpha_1 H + H\alpha_1 = \alpha_1 c\alpha_1 p_1 + c\alpha_1 p_1 \alpha_1 = 2cp_1,$$

and hence

$$i\hbar\dot{\alpha}_1 = 2\alpha_1 H - 2cp_1 = -2H\alpha_1 + 2cp_1. \quad (27)$$

Since H and p_1 are constants, it follows from the first of equations (27) that

$$i\hbar\dot{\alpha}_1 = 2\dot{\alpha}_1 H. \quad (28)$$

This differential equation in $\dot{\alpha}_1$ can be integrated immediately, the result being

$$\dot{\alpha}_1 = \dot{\alpha}_1^0 e^{-2iHt/\hbar}, \quad (29)$$

where $\dot{\alpha}_1^0$ is a constant, equal to the value of $\dot{\alpha}_1$ when $t = 0$. The factor $e^{-2iHt/\hbar}$ must be put to the right of the factor $\dot{\alpha}_1^0$ in (29) on account of the H occurring to the right of the $\dot{\alpha}_1$ in (28). The second of equations (27) leads in the same way to the result

$$\dot{\alpha}_1 = e^{2iHt/\hbar} \dot{\alpha}_1^0.$$

We can now easily complete the integration of the equation of motion for x_1 . From (29) and the first of equations (27)

$$\alpha_1 = \frac{1}{2} i \hbar \dot{\alpha}_1^0 e^{-2iHt/\hbar} H^{-1} + c p_1 H^{-1}, \quad (30)$$

and hence the time-integral of equation (26) is

$$x_1 = -\frac{1}{4} c \hbar^2 \dot{\alpha}_1^0 e^{-2iHt/\hbar} H^{-2} + c^2 p_1 H^{-1} t + a_1, \quad (31)$$

a_1 being a constant."

This is the desired result.

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